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Construction and Classification of Generalized Hadamard Codes over Eisenstein Local Rings $\mathbb{Z}_{2^s}[\omega]$

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Abstract. The research paper examines the design principles and structural features of Generalized Hadamard (GH) codes that operate within Eisenstein local rings $\mathbb{Z}_{2^s}[\omega]$, utilizing a primitive cube root of unity ω that satisfies the relation $\omega^2 + \omega + 1 = 0$. The paper first introduces an algebraic Eisenstein integer framework before developing an appropriate Gray mapping to examine binary-domain representations of these codes. We establish the essential criteria and necessary checks for determining the linear properties of GH codes based on $\mathbb{Z}_{2^s}[\omega]$ structures. This research defines the kernel structure of these codes together with their rank specification and an evaluation of their structural properties. A classification system for $\mathbb{Z}_{2^s}[\omega]$-linear Hadamard codes is presented in the final part of the paper, based on their algebraic and combinatorial characteristics. Future studies on coding techniques within algebraic integer rings can build upon this work, as our research expands the understanding of code theory over non-traditional rings.

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1. Introduction

Modulator–Demodulator, as we are all accustomed to it, is one of the cornerstones of contemporary digital communication, thanks to which we are able to detect and correct errors in transmitted data. It is mainly a theory for the construction of structured codes that can efficiently handle errors while remaining data-intuitive. It laid the foundations

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for several critical concepts in this area, including linear codes, cyclic codes, and BCH codes, which have seen practical applications in data storage, satellite communication, and cybersecurity [1–4]. The use of algebraic structures, such as rings and fields, to define codes with robust or stable properties has led to the development of more general and efficient error-correcting techniques [5–7]. These advances were complemented by Gray maps, isometries, and duality concepts, linking binary code representations with algebraic tools [7–9]. In recent years, this algebraic framework has been extended to non-field structures, such as finite rings, group rings, and number-theoretic rings like Gaussian and Eisenstein integers [4, 10–12].

Many algebraic domains have studied Hadamard codes—a class of codes with a simple structure and an optimal minimum distance. These codes, originally built from Hadamard matrices, are a backbone of many applications due to their strong error detection and simple structure. Researchers have explored the rank and kernel properties in the binary and $\mathbb{Z}_4$-linear settings to understand whether the algebraic complexity of these problems impacts code dimension. The study of Hadamard codes over $\mathbb{Z}_{2^s}$, $\mathbb{Z}_{p^s}$, and mixed modules has allowed a deeper examination of structural invariants and equivalence classes of such codes [4, 7, 9]. These studies highlight the influence of the underlying ring on linearity, decoding, and code equivalence, motivating extensions to code constructions over Eisenstein and quaternion integers [9, 13, 14].

Recently, much effort has been directed toward classifying and constructing generalized Hadamard (GH) codes over various algebraic structures. Bhunia et al. [15, 16] developed a framework for $\mathbb{Z}_{p^s}$-linear GH codes in terms of kernel, linearity, and equivalence. This builds upon previous work by Dougherty, Villanueva, and Rifà [6, 17, 18], and other contributions [5, 19, 20] that focused on rank and kernel of codes over $\mathbb{Z}_{2^s}$ and related rings. These investigations have greatly enhanced the understanding of code structure, Gray maps, and their connection to classification theory. The significance of $\mathbb{Z}_4$-linear codes was demonstrated by Carlet [5] and Hammons et al. [7], who revealed that $\mathbb{Z}_4$-linear codes underpin other well-known nonlinear codes such as Kerdock and Preparata codes. Further studies into other ring-based codes include extended perfect codes and duality over $\mathbb{Z}_{2^k}$, as explored by Krotov [21, 22]. More recent work by Shi et al. [9, 23] has focused on additive codes over mixed rings, duality principles, and classification criteria, demonstrating the algebraic depth and practical relevance of such constructions.

The other significant direction has been the exploration of Hadamard and generalized Hadamard (GH) codes over number-theoretic rings. In particular, Sajjad et al. have made substantial contributions to coding theory in the context of Gaussian and Eisenstein integers [10, 12], including BCH code constructions and alternant codes with applications in modern technology [24]. Modified Berlekamp–Massey algorithms, along with included algebraic tools, have been employed in their decoding frameworks, which generalize classical coding theory into broader algebraic domains [4, 11]. Additionally, the work of Villanueva and Zinoviev [25, 26] on Hadamard matrix construction has influenced generalized code design across diverse metrics and algebraic rings.

Despite the fact that the theory of GH codes over classical rings like $\mathbb{Z}_{2^s}$ is fairly well established, there exists a significant gap in the literature concerning their generalization over Eisenstein integers and corresponding rational domains. Eisenstein integers offer a rich algebraic structure with complex arithmetic, which has already proven useful in BCH and alternant code design. It is anticipated that the promising results of Sajjad et al. [10, 12, 24] in robust code construction and error correction using Eisenstein integers will yield analogous benefits in the context of GH codes. Furthermore, intellectual stimulation arises from the complexity of such a non-trivial ring system, where understanding linearity, the behaviour of the Gray map, and the structure of the kernel becomes crucial. As the foundational works on ring-based and non-binary Hadamard codes [6, 15, 17, 27] already offer natural starting points for generalization, the Eisenstein local ring $\mathbb{Z}_{2^s}[\omega]$, where $\omega^2 + \omega + 1 = 0$, is chosen as the natural candidate for extension.

Finally, this study helps to bridge a critical gap between classical GH code theory and a novel algebraic domain, with potential applications in secure communications and high-reliability systems.

This article contributes the following:

- It demonstrates how GH codes can be constructed over Eisenstein local rings $\mathbb{Z}_{2^s}[\omega]$, where $\omega^2 + \omega + 1 = 0$.
- It defines and analyzes a Gray map suitable for Eisenstein local rings with binary image representation.
- It obtains linearity conditions for GH codes with respect to elements in $\mathbb{Z}_{2^s}[\omega]$.
- It investigates the kernel and rank structures of these codes, leading to the determination of some of their algebraic invariants.
- It partially classifies $\mathbb{Z}_{2^s}[\omega]$-linear Hadamard codes in terms of their structural and combinatorial properties.
- It distinguishes classical GH code theory from modern algebraic settings such as Eisenstein rings.

2. Eisenstein Integers [12, 24]

Let $\omega = \frac{-1+i\sqrt{3}}{2}$ be a primitive cube root of unity. Then the identity

$$1 + \omega + \omega^2 = 0$$

implies that $\omega^2 = -\omega - 1$.

The set of Eisenstein integers consists of complex numbers that can be written in the form $a + \omega b$, where $a, b \in \mathbb{Z}$. Mathematically, this set is defined as

$$\mathbb{E} = \{a + \omega b \mid a, b \in \mathbb{Z}\}.$$

The set $\mathbb{E}$ forms a ring.

The conjugate of $z = a + \omega b \in \mathbb{E}$ is given by

$$z^* = a + \omega^2 b.$$

For example, $z^* = 1 + 2\omega^2$ is the conjugate of $z = a + b\omega$ in the set $\mathbb{E}$.

Every Eisenstein integer z has a norm denoted by $N(z)$, which is defined as

$$N(z) = zz^* = a^2 - ab + b^2.$$

Moreover, the norm is multiplicative:

$$N(z_1 z_2) = N(z_1)N(z_2), \quad \text{for all } z_1, z_2 \in \mathbb{E}.$$

Proposition 2.1 [12, 24]: For $\omega = \frac{-1+i\sqrt{3}}{2}$, the ring $\mathbb{E}$ is a Euclidean domain.
The units in the ring $\mathbb{E}$ of Eisenstein integers are $\pm 1, \pm\omega, \pm\omega^2$.
The primes in $\mathbb{E}$ include:

- Rational primes p satisfying $p \equiv 2 \pmod 3$,
- Eisenstein integers z such that $N(z) = p$, where p is a prime.

The quotient ring $\mathbb{E}/n\mathbb{E}$ is canonically isomorphic to the ring

$$\mathbb{E}_n = \{a + b\omega \mid a, b \in \mathbb{Z}_n\},$$

which is the ring of Eisenstein integers modulo n. This ring is also a principal ideal ring.

Lemma 2.1 [24]: Let $z = a + \omega b \in \mathbb{E}_n$. Then z is a unit in $\mathbb{E}_n$ if and only if $N(z)$ is a unit in $\mathbb{Z}_n$.

Since our main focus is on local rings of Eisenstein integers, we consider $n = p^s$, where p is a prime integer and s is a positive integer.

Note that $\mathbb{E}_{p^s}$ is not always a local ring, unlike $\mathbb{Z}_{p^s}$. Similarly, it $\mathbb{E}_p$ is not always a field even when p is prime.

Theorem 2.1 [12, 24]: For $p \geq 3$, the ring $\mathbb{E}_{p^s}$ is local if and only if $p \equiv 2 \pmod 3$ or $p = 2$.

3. Gray Map and Related Results [6, 16, 17]

In this section, we provide the definition of the generalized Gray map for $\mathbb{Z}_{2^s}[\omega]$ GH codes. Then we establish some properties of the Gray map for $\mathbb{Z}_{2^s}[\omega]$, based on the results given in Section 2 of [6, 16, 17].

Let ϕ_s be Carlet's Gray map from $\mathbb{Z}_{2^s}[\omega]$ to $\mathbb{Z}_2^{2^{2(s-1)}}[\omega]$, defined as

$$\phi_s(h) = (h_{s-1}, h_{s-1}, \ldots, h_{s-1}) + (h_0, \ldots, h_{s-2})Y_{s-1},$$

where $h \in \mathbb{Z}_{2^s}[\omega]$, and $[h_0, h_1, \ldots, h_{s-1}]_2$ is the binary (2-ary) expansion of h, i.e.,

$$h = \sum_{i=0}^{s-1} 2^i h_i,$$

with $h_i \in \mathbb{Z}_2[\omega]$. Here, Y_{s-1} is an $(s-1) \times 2^{2(s-1)}$ matrix whose columns are all distinct elements of $\mathbb{Z}_2^{s-1}[\omega]$.

Now we extend the map ϕ_s to a component-wise map:

$$\varphi_s : \mathbb{Z}_{2^s}^n[\omega] \longrightarrow \mathbb{Z}_2^{n \cdot 2^{2(s-1)}}[\omega].$$

The matrix Y_{s-1} can be obtained recursively, starting from

$$Y_1 = \begin{bmatrix} 00 & 01 & 10 & 11 \end{bmatrix},$$

and for $s > 1$,

$$Y_s = \begin{pmatrix} Y_{s-1} & Y_{s-1} & Y_{s-1} & Y_{s-1} \\ \mathbf{00} & \mathbf{01} & \mathbf{10} & \mathbf{11} \end{pmatrix}.$$

Example 3.1. Let $s = 2$. Take $h \in \mathbb{Z}_4[\omega]$, where $h = h_0 + 2h_1$ is the 2-ary expansion of h. Then ϕ_2 is a Gray map from $\mathbb{Z}_4[\omega]$ to $\mathbb{Z}_2^4[\omega]$, which is given in Table 1. As $h \in \mathbb{Z}_4[\omega]$ with binary representation $[h_0, h_1]_2$. The Carlet's generalized Gray map ϕ_2 is defined as:

$$\phi_2(h) = (h_1, h_1, h_1, h_1) + h_0 Y_1 \in \mathbb{Z}_2^4[\omega],$$

where $Y_1 = (00, 01, 10, 11)$.

Table 1: Gray map from $\mathbb{Z}_4[\omega]$ to $\mathbb{Z}_2^4[\omega]$

$h \in \mathbb{Z}_4[\omega]$	$[h_0, h_1]_2$	$\phi_2(h)$	$h \in \mathbb{Z}_4[\omega]$	$[h_0, h_1]_2$	$\phi_2(h)$
00	$[00, 00]_2$	(00, 00, 00, 00)	20	$[00, 10]_2$	(10, 10, 10, 10)
01	$[01, 00]_2$	(00, 11, 01, 10)	21	$[01, 10]_2$	(10, 01, 11, 00)
02	$[00, 01]_2$	(01, 01, 01, 01)	22	$[00, 11]_2$	(11, 11, 11, 11)
03	$[01, 01]_2$	(01, 10, 00, 11)	23	$[01, 11]_2$	(11, 00, 10, 01)
10	$[10, 00]_2$	(00, 01, 10, 11)	30	$[10, 10]_2$	(10, 11, 00, 01)
11	$[11, 00]_2$	(00, 10, 11, 01)	31	$[11, 10]_2$	(10, 00, 01, 11)
12	$[10, 01]_2$	(01, 00, 11, 10)	32	$[10, 11]_2$	(11, 10, 01, 00)
13	$[11, 01]_2$	(01, 11, 10, 00)	33	$[11, 11]_2$	(11, 01, 00, 10)

Now some results for the above-defined Carlet's generalized Gray map are presented below.

Let e_k be the vector that has 1 in the kth position and 0 elsewhere. Let $u, v \in \mathbb{Z}_{2^s}[\omega]$ and $[u_0, u_1, \ldots, u_{s-1}]_2$, $[v_0, v_1, \ldots, v_{s-1}]_2$ be the 2-ary expansions of u and v, respectively, i.e.,

$$u = \sum_{i=0}^{s-1} 2^i u_i, \quad v = \sum_{i=0}^{s-1} 2^i v_i.$$

Now define the operation $\oplus_2$ for elements of $\mathbb{Z}_{2^s}[\omega]$ as:

$$u \oplus_2 v = \sum_{i=0}^{s-1} r_i 2^i, \quad \text{where } r_i = u_i + v_i \mod 2 \text{ in } \mathbb{Z}_2[\omega],$$

and the operation $\odot_2$ as:

$$u \odot_2 v = \sum_{i=0}^{s-1} t_i 2^i,$$

where

$$t_i = \begin{cases} 10 & \text{if } u_i + v_i \geq 2, \\ 00 & \text{otherwise.} \end{cases}$$

The 2-ary expansion of $u \odot_2 v$ is $[t_0, t_1, \ldots, t_{s-1}]_2$, where $t_i \in \{00, 10\}$.
Lemma 3.1: Let $u \in \mathbb{Z}_{2^s}[\omega]$ and $\mu \in \mathbb{Z}_2[\omega]$. Then

$$\phi_s(u + \mu 2^{s-1}) = \phi_s(u) + (\mu, \mu, \ldots, \mu).$$

Proof: Since

$$u + \mu 2^{s-1} = u_1 + \mu_0 2^{s-1} + \mu 2^{s-1} = u_1 + (\mu_0 + \mu) 2^{s-1},$$

where Let $u_1 \in \left\{00, \ldots, 0 \cdot 2^{s-1} - 1, \ldots, 2^{s-1} - 1 \cdot 0, \ldots, 2^{s-1} - 1 \cdot 2^{s-1} - 1\right\}$ and $u_0 \in \mathbb{Z}_2[\omega]$. Then, by the definition of the Gray map ϕ_s, we have:

$$\begin{aligned} \phi_s(u + \mu 2^{s-1}) &= \phi_s(u_1) + (\mu_0 + \mu, \ldots, \mu_0 + \mu) \\ &= \phi_s(u_1) + (\mu_0, \ldots, \mu_0) + (\mu, \ldots, \mu) \\ &= \phi_s(u_1) + (\mu, \ldots, \mu). \end{aligned}$$

Corollary 3.1: Let $\lambda, \mu \in \mathbb{Z}_2[\omega]$. Then,

$$\phi_s(\lambda \mu 2^{s-1}) = \lambda \phi_s(\mu 2^{s-1}) = \lambda \mu \phi_s(2^{s-1}).$$

Proof: By the definition of the Gray map ϕ_s, we have $\phi_s(\mu 2^{s-1}) = (\mu, \ldots, \mu)$. Then,

$$\phi_s(\lambda \mu 2^{s-1}) = (\lambda \mu, \ldots, \lambda \mu) = \lambda(\mu, \ldots, \mu) = \lambda \phi_s(\mu 2^{s-1}) = \lambda \mu \phi_s(2^{s-1}).$$

Proposition 3.1: Let $u, v \in \mathbb{Z}_{2^s}[\omega]$. Then

$$\phi_s(u) + \phi_s(v) = \phi_s(u \oplus_2 v).$$

Proof: Let $[u_0, u_1, \ldots, u_{s-1}]_2$ and $[v_0, v_1, \ldots, v_{s-1}]_2$ be the 2-ary expansions of u and v, respectively. Let $\boldsymbol{y}_i$ be the $(i+1)$-th row of Y, for $0 \leq i \leq s-2$. Then,

$$\phi_s(u) = (u_{s-1}, u_{s-1}, \ldots, u_{s-1}) + \sum_{i=0}^{s-2} u_i \boldsymbol{y}_i,$$

$$\phi_s(v) = (v_{s-1}, v_{s-1}, \ldots, v_{s-1}) + \sum_{i=0}^{s-2} v_i \boldsymbol{y}_i.$$

Therefore,

$$\phi_s(u) + \phi_s(v) = (r_{s-1}, r_{s-1}, \ldots, r_{s-1}) + \sum_{i=0}^{s-2} r_i \boldsymbol{y}_i = \phi_s(u \oplus_2 v),$$

where $r_i = u_i + v_i$ in $\mathbb{Z}_2[\omega]$ for $0 \leq i \leq s-1$.

Proposition 3.2: Let $u, v \in \mathbb{Z}_{2^s}[\omega]$. Then

$$u \oplus_2 v = u + v - 2(u \odot_2 v).$$

Proof: Let $[u_0, u_1, \ldots, u_{s-1}]_2$, $[v_0, v_1, \ldots, v_{s-1}]_2$ be the 2-ary expansions of u and v, respectively. Note that $0 \leq u_i + v_i \leq 2$. By the division algorithm,

$$u_i + v_i = 2t_i + r_i,$$

where $t_i = 1$ if $u_i + v_i \geq 2$, and $t_i = 0$ otherwise; also $0 \leq r_i \leq 1$. Then we have:

$$\begin{aligned} u + v &= \sum_{i=0}^{s-1} (u_i + v_i) 2^i \\ &= \sum_{i=0}^{s-1} (2t_i + r_i) 2^i \\ &= 2 \sum_{i=0}^{s-1} t_i 2^i + \sum_{i=0}^{s-1} r_i 2^i \\ &= 2(u \odot_2 v) + u \oplus_2 v. \end{aligned}$$

Therefore,

$$u \oplus_2 v = u + v - 2(u \odot_2 v).$$

Corollary 3.2: Let $u, v \in \mathbb{Z}_{2^s}[\omega]$. Then,

$$\phi_s(u) + \phi_s(v) = \phi_s(u + v - 2(u \odot_2 v)).$$

Proof: From Proposition 3.1, $\phi_s(u) + \phi_s(v) = \phi_s(u \oplus_2 v)$. From Proposition 3.2, $u \oplus_2 v = u + v - 2(u \odot_2 v)$. So,

$$\phi_s(u) + \phi_s(v) = \phi_s(u \oplus_2 v) = \phi_s(u + v - 2(u \odot_2 v)).$$

Corollary 3.3: Let $u, v \in \mathbb{Z}_{2^s}[\omega]$. Then,

$$2^{s-1}(u \oplus_2 v) = 2^{s-1}(u + v).$$

Proof: Let $[u_0, u_1, \ldots, u_{s-1}]_2$, $[v_0, v_1, \ldots, v_{s-1}]_2$ be the binary expansions of u and v, respectively. Then $[0, 0, \ldots, 0, u_0]_2$ is the binary expansion of $2^{s-1}u$, so $2^{s-1}u \odot_2 v$ is 2^{s-1} if $u_0 + v_{s-1} \geq 2$, and 0 otherwise. In any case, $2(2^{s-1}u \odot_2 v) = 0$. Hence, by Proposition 3.2, the result follows.

Corollary 3.4: Let $u \in \mathbb{Z}_{2^s}[\omega]$ and $[u_0, u_1, \ldots, u_{s-1}]_2$ be its binary expansion. Then, for any $i \in \{0, \ldots, s-2\}$,

$$\phi_s(u) + \phi_s(2^i) = \phi_s(u + 2^i - 2^{i+1}t_i),$$

where

$$t_i = \begin{cases} 1 & \text{if } u_i \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 3.5: Let $v \in \mathbb{Z}_{2^s}[\omega]$. Then,

$$\phi_s(2^{s-1} + v) = \phi_s(2^{s-1}) + \phi_s(v).$$

Corollary 3.6: Let $u, v \in \mathbb{Z}_{2^s}[\omega]$. Then,

$$\phi_s(2^{s-1}u + v) = \phi_s(2^{s-1}u) + \phi_s(v).$$

Lemma 3.2: Let $u \in \{(01)2^{s-2}, (03)2^{s-2}, \ldots, (31)2^{s-1}, (33)2^{s-1}\} \subset \mathbb{Z}_{2^s}[\omega]$. Then,

$$\phi_s(u) + \phi_s(0 \cdot 2^{s-2}) = \phi_s(u + 0 \cdot 2^{s-2} + 0 \cdot 2^{s-1}).$$

Corollary 3.7: Let $v \in \{(01)2^{s-2}, (03)2^{s-2}, (21)2^{s-2}, (23)2^{s-2}\}$ and $U = \{(01)2^{s-2}, (03)2^{s-2}, \ldots, (31)2^{s-1},$ $\mathbb{Z}_{2^s}[\omega]$. Then,

$$\phi_s(u) + \phi_s(v) = \begin{cases} \phi_s(u + v + 0 \cdot 2^{s-1}) & \text{if } u \in U, \\ \phi_s(u + v) & \text{if } u \in \mathbb{Z}_{2^s}[\omega] \setminus U. \end{cases}$$

Lemma 3.3: Let $u \in \{(10)2^{s-2}, \ldots, (13)2^{s-2}, (30)2^{s-1}, \ldots, (33)2^{s-1}\} \subset \mathbb{Z}_{2^s}[\omega]$. Then,

$$\phi_s(u) + \phi_s(2^{s-2} \cdot 0) = \phi_s(u + 2^{s-2} \cdot 0 + 2^{s-1} \cdot 0).$$

Corollary 3.8: Let $v \in \{(10)2^{s-2}, (12)2^{s-2}, (30)2^{s-2}, (32)2^{s-2}\}$ and let

$$U' = \{(10)2^{s-2}, \ldots, (13)2^{s-2}, (30)2^{s-1}, \ldots, (33)2^{s-1}\} \subset \mathbb{Z}_{2^s}[\omega].$$

Then,

$$\phi_s(u) + \phi_s(v) = \begin{cases} \phi_s(u + v + 2^{s-1} \cdot 0) & \text{if } u \in U', \\ \phi_s(u + v) & \text{if } u \in \mathbb{Z}_{2^s}[\omega] \setminus U'. \end{cases}$$

Corollary 3.9: Let $v \in \{(11)2^{s-2}, (13)2^{s-2}, (31)2^{s-2}, (33)2^{s-2}\}$, and define

$$U_1 = \{(01)2^{s-2}, (03)2^{s-2}, (21)2^{s-1}, (23)2^{s-1}\},$$
$$U_2 = \{(10)2^{s-2}, (12)2^{s-2}, (30)2^{s-1}, (32)2^{s-1}\},$$
$$U_3 = \{(11)2^{s-2}, (13)2^{s-2}, (31)2^{s-1}, (33)2^{s-1}\}.$$

Then,

$$\phi_s(u)+\phi_s(v)=\begin{cases}\phi_s(u+v+02^{s-1}) & \text{if } u\in U_1,\\ \phi_s(u+v+2^{s-1}\cdot 0) & \text{if } u\in U_2,\\ \phi_s(u+v+2^{s-1}\cdot 2^{s-1}) & \text{if } u\in U_3,\\ \phi_s(u+v) & \text{if } u\in \mathbb{Z}_{2^s}[\omega]\setminus (U_1\cup U_2\cup U_3).\end{cases}$$

Lemma 3.4: Let $\mu_k \in \mathbb{Z}_2[\omega],\ k\in\{0,\dots,s-2\}$. Then,

$$\sum_{k=0}^{s-2}\mu_k\phi_s(2^k)=\phi_s\left(\sum_{k=0}^{s-2}\mu_k 2^k\right),\quad \text{where } 2^k\in\mathbb{Z}_{2^s}[\omega].$$

Proof: Let y_k be the k-th row of matrix Y. By definition, we have

$$\sum_{k=0}^{s-2}\mu_k\phi_s(2^k)=\sum_{k=0}^{s-2}\mu_k e_{k+1}Y=\sum_{k=0}^{s-2}\mu_k y_{k+1}=\boldsymbol{\mu}Y,$$

where $\boldsymbol{\mu}=(\mu_0,\dots,\mu_{s-2})$. Since $[\mu_0,\dots,\mu_{s-2},0]$ is the binary expansion of $\sum_{k=0}^{s-2}\mu_k 2^k$, we conclude that

$$\boldsymbol{\mu}Y=\phi_s\left(\sum_{k=0}^{s-2}\mu_k 2^k\right).$$

Proposition 3.3. Let $u,v\in\mathbb{Z}_{2^s}[\omega]$. Then,

$$\phi_s(u)+\phi_s(v)=\phi_s(u-v)=(\mu,\dots,\mu)\quad \text{if } u-v=\mu 2^{s-1}\in 2^{s-1}\mathbb{Z}_{2^s}[\omega]\setminus\{00\},$$

and

$\phi_s(u)-\phi_s(v)$ contains each element of $\mathbb{Z}_2[\omega]$ exactly $2^{2(s-2)}$ times if $u-v\in\mathbb{Z}_{2^s}[\omega]\setminus 2^{s-1}\mathbb{Z}_{2^s}[\omega]$.

Proof: If $u-v=\lambda 2^{s-1}\in 2^{s-1}\mathbb{Z}_{2^s}[\omega]\setminus\{0\}$, then by Lemma 3.1, $\phi_s(u)=\phi_s(v)+(\lambda,\dots,\lambda)$, so

$$\phi_s(u)-\phi_s(v)=(\lambda,\dots,\lambda)=\phi_s(\lambda 2^{s-1})=\phi_s(u-v).$$

Now assume that $u-v\in\mathbb{Z}_{2^s}[\omega]\setminus 2^{s-1}\mathbb{Z}_{2^s}[\omega]$. Without loss of generality, either $u\in 2^{s-1}\mathbb{Z}_{2^s}[\omega]$, $v\in\mathbb{Z}_{2^s}[\omega]\setminus 2^{s-1}\mathbb{Z}_{2^s}[\omega]$ or $u,v\in\mathbb{Z}_{2^s}[\omega]\setminus 2^{s-1}\mathbb{Z}_{2^s}[\omega]$.

For the first case, $\phi_s(u)=(\lambda_1,\dots,\lambda_1)$ and $\phi_s(v)=\phi_s(v_1)+(\lambda_2,\dots,\lambda_2)$, where $v_1\in\{01,\dots,2^{s-1}-1,\dots,(2^{s-1}-1)(2^{s-1}-1)\}$, $\lambda_1,\lambda_2\in\mathbb{Z}_2[\omega]$. Note that $\phi_s(v_1)$ is a nonzero row of the GH matrix $H(2^2,2^{2(s-2)})$ corresponding to the GH code $\phi_s(\mathbb{Z}_{2^s}[\omega])$. Therefore, $\phi_s(v_1)$ contains each element of $\mathbb{Z}_2[\omega]$ exactly $2^{2(s-2)}$ times and hence $\phi_s(u)-\phi_s(v)$ contains each element of $\mathbb{Z}_2[\omega]$ exactly $2^{2(s-2)}$ times.

For the second case, $\phi_s(u)=\phi_s(u_1)+(\lambda_1,\dots,\lambda_1)$ and $\phi_s(v)=\phi_s(v_1)+(\lambda_2,\dots,\lambda_2)$, where $u_1,v_1\in\{01,\dots,2^{s-1}-1,\dots,(2^{s-1}-1)(2^{s-1}-1)\}$ and $\lambda_1,\lambda_2\in\mathbb{Z}_2[\omega]$. Note that both $\phi_s(u_1)$ and $\phi_s(v_1)$ are nonzero rows of $H(2^2,2^{2(s-2)})$, so they contain each element of

$\mathbb{Z}_2[\omega]$ exactly $2^{2(s-2)}$ times, and hence $\phi_s(u)-\phi_s(v)$ contains each element of $\mathbb{Z}_2[\omega]$ exactly $2^{2(s-2)}$ times.

Proposition 3.4. Let $u, v \in \mathbb{Z}_{2^s}[\omega]$. Then,

$$d_H(\phi_s(u), \phi_s(v)) = \mathrm{wt}_H(\phi_s(u-v)).$$

Proof: If $u = 0$ or $v = 0$, the result is trivially true. Assume $u \neq 0$ and $v \neq 0$, and consider three cases. First, if $u = v$, the result is trivially true.

Second, if $u - v \in 2^{s-1}\mathbb{Z}_{2^s}[\omega] \setminus \{0\}$, then by Proposition 3.3, $\phi_s(u) - \phi_s(v) = \phi_s(u-v)$, and hence

$$d_H(\phi_s(u), \phi_s(v)) = \mathrm{wt}_H(\phi_s(u-v)).$$

Finally, assume that $u, v \in \mathbb{Z}_{2^s}[\omega] \setminus 2^{s-1}\mathbb{Z}_{2^s}[\omega]$. By Proposition 3.3, $\phi_s(u) - \phi_s(v)$ contains each element of $\mathbb{Z}_2[\omega]$ exactly $2^{2(s-2)}$ times, and hence

$$d_H(\phi_s(u), \phi_s(v)) = 3 \cdot 2^{2(s-2)} = \mathrm{wt}_H(\phi_s(u-v)).$$

4. Construction of GH Codes over $\mathbb{Z}_{2^s}[\omega]$

Let

$$T_i = \{jk \cdot 2^{i-1} : j, k \in \{0, 1, \ldots, 2^{s-i+1} - 1\}\} \quad \text{for all } i \in \{1, \ldots, s\}.$$

Note that

$$T_1 = \{00, 01, \ldots, 2^s - 1\ 2^{s-1} - 1\}.$$

Let $t_1, t_2, \ldots, t_s$ be nonnegative integers with $t_1 \geq 1$. Consider the matrix

$$A_2^{(t_1,\ldots,t_s)}$$

whose columns are exactly all the vectors of the form $\boldsymbol{z}^{\boldsymbol{t}}$, where

$$\boldsymbol{z} \in \{0\} \times T_1^{t_1-1} \times T_2^{t_2} \times \cdots \times T_s^{t_s}.$$

Let

$$\mathbf{00}, \mathbf{01}, \ldots, \mathbf{2^s - 1\ 2^s - 1}$$

be the vectors having the same element $00, 01, \ldots, 2^s - 1$ from $\mathbb{Z}_{2^s}[\omega]$ in all coordinates, respectively.

Any matrix $A_2^{(t_1,\ldots,t_s)}$ can also be obtained by applying the recursive construction given below. Start with the matrix

$$A_2^{(1,0,\ldots,0)} = (1\ 0).$$

If we have a matrix $A_2^{(t_1,\ldots,t_s)}$, then for any $i \in \{1, \ldots, s\}$, we can construct the matrix

$$A_i = \begin{bmatrix} A & A & \cdots & A & \cdots & A & \cdots & A \\ \mathbf{2^{i-1}} \cdot 00 & \mathbf{2^{i-1}} \cdot 01 & \cdots & \mathbf{2^{i-1}} \cdot 0(2^{s-i+1}-1) & \cdots & \mathbf{2^{i-1}} \cdot (2^{s-i+1}-1)0 & \cdots & \mathbf{2^{i-1}} \cdot (2^{s-i+1}-1)(2^{s-i+1}-1) \end{bmatrix}$$

Finally, by permuting the rows of A_i, we obtain a matrix

$$A_2^{(t'_1,\ldots,t'_s)}$$

where $t'_j = t_j$ for $j \neq i$. Note that by permuting the columns of A_i, another matrix $A_2^{(t'_1,\ldots,t'_s)}$ can also be obtained.

To construct matrices recursively, starting from the base matrix $A_2^{(1,0,\ldots,0)}$, proceed in the following way. First, to obtain matrix $A_2^{(t_1,0,\ldots,0)}$, we add $t_1 - 1$ rows of order 2^s, then t_2 rows of order 2^{s-1}, and so on, up to generate $A_2^{(t_1,t_2,\ldots,0)}$; and finally we add t_s rows of order 2 to achieve

$$A_2^{(t_1,\ldots,t_s)}.$$

Let $\widetilde{H}_2^{(t_1,\ldots,t_s)}$ be the $\mathbb{Z}_{2^s}[\omega]$-additive code of type

$$(n, t_1, \ldots, t_s)$$

generated by the matrix $A_2^{(t_1,\ldots,t_s)}$, where $t_1, t_2, \ldots, t_s$ are nonnegative integers with $t_1 \geq 1$.

Let

$$n = 2^{2(t-s+1)}, \quad \text{where } t = \left(\sum_{i=1}^{s} (s-i+1)t_i \right) - 1.$$

The code $\widetilde{H}_2^{(t_1,\ldots,t_s)}$ has length n, and the corresponding $\mathbb{Z}_{2^s}[\omega]$-linear code

$$H_2^{(t_1,\ldots,t_s)} = \varphi_s\left(\widetilde{H}_2^{(t_1,\ldots,t_s)}\right)$$

is a Generalized Hadamard code of length 2^{2t}.

Example 4.1: For $s = 2$, we have the following matrices which generate codes over $\mathbb{Z}_4[\omega]$.

$$A^{1,1} = \begin{bmatrix} 10 & 10 & 10 & 10 \\ 00 & 02 & 20 & 22 \end{bmatrix}$$

$$A^{1,2} = \begin{bmatrix} 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 \\ 00 & 02 & 20 & 22 & 00 & 02 & 20 & 22 & 00 & 02 & 20 & 22 & 00 & 02 & 20 & 22 \\ 00 & 00 & 00 & 00 & 02 & 02 & 02 & 02 & 20 & 20 & 20 & 20 & 22 & 22 & 22 & 22 \end{bmatrix}$$

$$A^{2,0} = \begin{bmatrix} 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 \\ 00 & 01 & 02 & 03 & 10 & 11 & 12 & 13 & 20 & 21 & 22 & 23 & 30 & 31 & 32 & 33 \end{bmatrix}$$

Example 4.2: For $s = 3$, the following are generator matrices.

$$A^{1,0,1} = \begin{bmatrix} 10 & 10 & 10 & 10 \\ 00 & 04 & 40 & 44 \end{bmatrix}$$

$$A^{1,1,0} = \begin{bmatrix} 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 \\ 00 & 02 & 04 & 06 & 20 & 22 & 24 & 26 & 40 & 42 & 44 & 46 & 60 & 62 & 64 & 66 \end{bmatrix}$$

Example 4.3: The additive code $\bar{\mathcal{H}}^{(1,0,\ldots,0)}$ is generated by

$$A_2^{(1,0,\ldots,0)} = (10),$$

so $\bar{\mathcal{H}}^{(1,0,\ldots,0)} = \mathbb{Z}_{2^s}[\omega]$. This additive code has length $n = 1$, cardinality 2^{2s}, and minimum distance 1. Thus,

$$\mathcal{H}^{(1,0,\ldots,0)} = \varphi_s(\bar{\mathcal{H}}^{(1,0,\ldots,0)})$$

has length $N = 2^{2(s-1)}$, cardinality $4N = 2^{2s}$, and minimum distance 3. Since

$$\frac{N}{4} = 3 \cdot 2^{2(s-2)},$$

it is clearly a binary generalized Hadamard code and also a linear code.

Example 4.4: For $\lambda = 1$, the normalized GH matrix is given as:

$$H(4,1) = \begin{bmatrix} 00 & 00 & 00 & 00 \\ 00 & 01 & 10 & 11 \\ 00 & 11 & 01 & 10 \\ 00 & 10 & 11 & 01 \end{bmatrix}$$

Then,

$$\mathcal{F}_H = \{(00,00,00,00),\ (00,01,10,11),\ (00,11,01,10),\ (00,10,11,01)\},$$

and

$$\mathcal{C}_H = \bigcup_{\alpha \in \mathbb{Z}_2[\omega]} (\mathcal{F}_H + \alpha \cdot 10).$$

Here, $\mathcal{C}_H$ is a linear GH code over $\mathbb{Z}_2[\omega]$ of length 4, and

$$\mathcal{C}_H = \mathcal{H}^{1,0} = \varphi_s(\bar{\mathcal{H}}^{1,0}) = \varphi_s(\mathbb{Z}_4[\omega]),$$

where $\bar{\mathcal{H}}^{1,0}$ is generated by $A_2^{1,0} = (10)$.

Theorem 4.1: Let $t_1, \ldots, t_s$ be non-negative integers with $t_1 \geq 1$. The $\mathbb{Z}_{2^s}[\omega]$-linear code $\mathcal{H}^{(t_1,\ldots,t_s)}$ of type $(n, t_1, \ldots, t_s)$ is a generalized Hadamard (GH) code over $\mathbb{Z}_{2^s}[\omega]$ of length 2^{2t}, where

$$t = \left(\sum_{i=1}^{s} (s-i+1) \cdot t_i\right) - 1 \quad \text{and} \quad n = 2^{2(t-s+1)}.$$

Proof. Let $\bar{\mathcal{H}} = \bar{\mathcal{H}}^{(t_1,\ldots,t_s)}$ be the $\mathbb{Z}_{2^s}[\omega]$-additive code of length n. Consider the matrix $A = A_2^{(t_1,t_2,\ldots,t_s)}$ as its generator. The additive code can be written as

$$\bar{\mathcal{H}} = \bigcup_{\lambda\in\mathbb{Z}_2[\omega]} \left(\mathcal{A}_{\bar{\mathcal{H}}} + \lambda \cdot \mathbf{2^{s-1}}\right),$$

where

$$\mathcal{A}_{\bar{\mathcal{H}}} = \{\boldsymbol{h} \bmod 2^{s-1} : \boldsymbol{h} \in \bar{\mathcal{H}}\} \quad \text{and} \quad \mathcal{A}_{\bar{\mathcal{H}}} + \lambda \cdot \mathbf{2^{s-1}} = \{\boldsymbol{h} + \lambda \cdot \mathbf{2^{s-1}} : \boldsymbol{h} \in \bar{\mathcal{H}}\}.$$

Then by Lemma 3.1,

$$\mathcal{H} = \varphi_s(\bar{\mathcal{H}}) = \bigcup_{\lambda\in\mathbb{Z}_2[\omega]} \left(\varphi_s(\mathcal{A}_{\bar{\mathcal{H}}}) + \lambda \cdot \mathbf{10}\right).$$

The code $\mathcal{H}$ has length $2^{2t} = n \cdot 2^{2(s-1)}$ and cardinality $2^{2(t+1)} = n \cdot 2^{2s}$.

It is enough to show that $\varphi_s(\mathcal{A}_{\bar{\mathcal{H}}})$ is the set of rows of a generalized Hadamard matrix $H(2^2, 2^{2(s-2)}n)$.

Let us consider two distinct elements $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{A}_{\bar{\mathcal{H}}}$. We need to show that $\varphi_s(\boldsymbol{u}) - \varphi_s(\boldsymbol{v})$ contains each element of $\mathbb{Z}_2[\omega]$ exactly $2^{2(s-2)}n$ times.

We analyze two scenarios based on the order of $\boldsymbol{u} - \boldsymbol{v}$:

Case 1: If $\mathrm{ord}(\boldsymbol{u} - \boldsymbol{v}) = 2$, then by the established construction, $\boldsymbol{u} - \boldsymbol{v}$ includes all elements of $2^{s-1} \cdot \mathbb{Z}_{2^s}[\omega]$ exactly $n/4$ times. Therefore,

$$\varphi_s(\boldsymbol{u} - \boldsymbol{v}) \text{ includes all elements of } \mathbb{Z}_2[\omega] \text{ exactly } \frac{2^{2(s-1)} \cdot n}{4} = 2^{2(s-2)}n \text{ times.}$$

By Proposition 3.3, $\varphi_s(\boldsymbol{u} - \boldsymbol{v}) = \varphi_s(\boldsymbol{u}) - \varphi_s(\boldsymbol{v})$, and hence $\varphi_s(u) - \varphi_s(v)$ comprises all elements of $\mathbb{Z}_2[\omega]$ exactly $2^{2(s-2)}n$ times.

Case 2: If $\mathrm{ord}(\boldsymbol{u} - \boldsymbol{v}) \geq 2$, then following the construction, $\boldsymbol{u} - \boldsymbol{v}$ includes all elements of $2^{s-1} \cdot \mathbb{Z}_{2^s}[\omega]$ exactly α times ($\alpha \geq 0$), and the remaining $n - 4\alpha$ coordinates are from $\mathbb{Z}_{2^s}[\omega] \setminus 2^{s-1} \cdot \mathbb{Z}_{2^s}[\omega]$.

Then by Proposition 3.3, we have:

$$\varphi_s(\boldsymbol{u}) - \varphi_s(\boldsymbol{v}) \text{ includes all elements of } \mathbb{Z}_2[\omega] \text{ exactly } \alpha \cdot 2^{2(s-1)} + (n-4\alpha) \cdot 2^{2(s-2)} = 2^{2(s-2)}n \text{ times.}$$

Therefore, $\mathcal{H}$ is a generalized Hadamard code over $\mathbb{Z}_{2^s}[\omega]$.

5. Linearity of $\mathbb{Z}_{2^s}[\omega]$-Linear GH Codes

This section establishes several results concerning the linearity of $\mathbb{Z}_{2^s}[\omega]$-linear GH codes by generalizing the results given in Section 4.

Theorem 5.1: The $\mathbb{Z}_{2^s}[\omega]$-linear Hadamard codes $H^{(1,0,\ldots,0)}$ and $H^{(1,0,\ldots,0,1,0)}$, with $s > 2$, are linear.

Proof: By Example 4.4, we know that $H^{(1,0,\ldots,0)}$ is linear. Now, we examine $\widetilde{H} = \widetilde{H}^{(1,0,\ldots,0,1,0)}$ and $H = \varphi_s(\widetilde{H})$. Recall that the code H of length 16 is constructed from:

$$A_2^{(1,0,\ldots,0,1,0)} = \begin{pmatrix} 10 & 10 & \cdots & 10 & 10 \\ 00 & 01 \cdot 2^{s-2} & \cdots & 32 \cdot 2^{s-2} & 33 \cdot 2^{s-2} \end{pmatrix}$$

Let $\boldsymbol{\alpha_i} = (2^i\,0, 2^i\,0, \ldots, 2^i\,0, 2^i\,0)$ for $0 \leq i \leq s-1$,

$$\boldsymbol{\alpha_s} = (00, 0\,2^{s-1}, 00, 0\,2^{s-1}, 2^{s-1}\,0, 2^{s-1}\,2^{s-1}, 2^{s-1}\,0, 2^{s-1}\,2^{s-1}, 00, 0\,2^{s-1}, 00, 0\,2^{s-1}, 2^{s-1}\,0, 2^{s-1}\,2^{s-1}, 2^{s-1}\,0, 2^{s-1}\,2^{s-1})$$

$$\boldsymbol{\alpha_{s+1}} = (00, 01 \cdot 2^{s-2}, \ldots, 33 \cdot 2^{s-2})$$

Suppose C represents the linear code constructed from

$$B = \{\varphi_s(\boldsymbol{\alpha_i}) : 0 \leq i \leq s+1\}.$$

We now prove that $C \subseteq H$. Let $c = \sum_{i=0}^{s+1} \lambda_i \varphi_s(\boldsymbol{\alpha_i}) \in C$, where $\lambda_i \in \mathbb{Z}_2[\omega]$.

By Corollary 3.5, it is sufficient to observe:

$$\boldsymbol{c'} = \lambda_{s+1}\varphi_s(\boldsymbol{\alpha_{s+1}}) + \sum_{i=0}^{s-2} \lambda_i \varphi_s(\boldsymbol{\alpha_i}) \in H.$$

If $\lambda_{s+1} = 0$, then $\boldsymbol{c'} \in H$ since

$$\sum_{k=0}^{s-2} \lambda_i \varphi_s(\boldsymbol{\alpha_i}) = \varphi_s\left(\sum_{k=0}^{s-2} \lambda_i \boldsymbol{\alpha_i}\right).$$

If $\lambda_{s+1} = 10$, then:

$$\boldsymbol{c'} = \varphi_s(00, 01 \cdot 2^{s-2}, \ldots, 33 \cdot 2^{s-2}) + \phi_s(u, u, \ldots, u),$$

where $u = \sum_{k=0}^{s-2} \lambda_i 2^i\,0$.

Let us define:

$$\begin{aligned} U &= \{(01) \cdot 2^{s-2}, (03) \cdot 2^{s-2}, \ldots, (31) \cdot 2^{s-1}, (33) \cdot 2^{s-1}\} \\ U' &= \{(10) \cdot 2^{s-2}, \ldots, (13) \cdot 2^{s-2}, (30) \cdot 2^{s-1}, \ldots, (33) \cdot 2^{s-1}\} \\ U_1 &= \{(01) \cdot 2^{s-2}, (03) \cdot 2^{s-2}, (21) \cdot 2^{s-1}, (23) \cdot 2^{s-1}\} \\ U_2 &= \{(10) \cdot 2^{s-2}, (12) \cdot 2^{s-2}, (30) \cdot 2^{s-1}, (32) \cdot 2^{s-1}\} \\ U_3 &= \{(11) \cdot 2^{s-2}, (13) \cdot 2^{s-2}, (31) \cdot 2^{s-1}, (33) \cdot 2^{s-1}\} \end{aligned}$$

Then, by Corollaries 3.7, 3.8, and 3.9:

$$\boldsymbol{c'} = \varphi_s(00, 01 \cdot 2^{s-2}, \ldots, 33 \cdot 2^{s-2}) + \varphi_s(u, u, \ldots, u) + \alpha_s.$$

If $u \in U \cup U'$, then

$$\boldsymbol{c'} = \varphi_s(00, 01 \cdot 2^{s-2}, \ldots, 33 \cdot 2^{s-2}) + \phi_s(u, u, \ldots, u),$$

otherwise. In both cases, $\boldsymbol{c'} \in H$.

For $\lambda_{s+1} = 01$ and $\lambda_{s+1} = 11$, it can be proven similarly using Corollaries 3.7, 3.8, and 3.9.

Since $|C| = |H| = 2^{2(s+2)}$, it follows that $C = H$, and therefore, H exhibits linearity.

Theorem 5.2: The codes $H^{(1,0,\ldots,0,1,t_s)}$ and $H^{(1,0,\ldots,0,t_s)}$, with $s > 2$ and $t_s \geq 0$, are the only $\mathbb{Z}_{2^s}[\omega]$-linear Hadamard codes that exhibit linearity.

Proof: Initially, we prove the linearity of these codes using induction on t_s. By Theorem 5.1, the codes $H^{(1,0,\ldots,0)}$ and $H^{(1,0,\ldots,0,1,0)}$ exhibit linearity.

We hypothesize that $H = \varphi_s(\bar{H})$, where $\bar{H} = \mathcal{H}^{(1,0,\ldots,0,t_{s-1},t_s)}$, $t_{s-1} \in \{0,1\}$ and $t_s \geq 0$, is linear.

Now, we prove that $H_s = H^{(1,0,\ldots,0,t_{s-1},t_s+1)}$ is linear. Through iterative construction,

$$H_s = \left\{\varphi_s\left((\boldsymbol{h},\boldsymbol{h},\boldsymbol{h},\boldsymbol{h}) + \lambda(\mathbf{00},(01)\mathbf{2^{s-1}},(10)\mathbf{2^{s-1}},(11)\mathbf{2^{s-1}})\right) : \boldsymbol{h} \in \bar{H}, \lambda \in \mathbb{Z}_2[\omega]\right\},$$

which simplifies to:

$$\left\{\left(\varphi_s(\boldsymbol{h}), \varphi_s(\boldsymbol{h} + \lambda(01)\mathbf{2^{s-1}}), \varphi_s(\boldsymbol{h} + \lambda(10)\mathbf{2^{s-1}}), \varphi_s(\boldsymbol{h} + \lambda(11)\mathbf{2^{s-1}})\right) : \boldsymbol{h} \in \bar{H}, \lambda \in \mathbb{Z}_2[\omega]\right\}.$$

By Corollaries 3.1 and 3.6, this equals:

$$\left\{(\boldsymbol{h'}, \boldsymbol{h'} + \lambda \cdot \mathbf{01}, \boldsymbol{h'} + \lambda \cdot \mathbf{10}, \boldsymbol{h'} + \lambda \cdot \mathbf{11}) : \boldsymbol{h'} \in H, \lambda \in \mathbb{Z}_2[\omega]\right\}.$$

Thus, it is possible to partition H_s into 4-blocks as follows:

$$\begin{aligned}
H_{s00} &= \{(\boldsymbol{h'},\boldsymbol{h'},\boldsymbol{h'},\boldsymbol{h'}) : \boldsymbol{h'} \in H\},\\
H_{s01} &= \{(\boldsymbol{h'},\boldsymbol{h'} + 01 \cdot \mathbf{01}, h' + 01 \cdot \mathbf{10}, \boldsymbol{h'} + 01 \cdot \mathbf{11}) : \boldsymbol{h'} \in H\},\\
H_{s10} &= \{(\boldsymbol{h'},\boldsymbol{h'} + 10 \cdot \mathbf{01}, h' + 10 \cdot \mathbf{10}, \boldsymbol{h'} + 10 \cdot \mathbf{11}) : \boldsymbol{h'} \in H\},\\
H_{s11} &= \{(\boldsymbol{h'},\boldsymbol{h'} + \mathbf{11} \cdot \mathbf{01}, h' + 11 \cdot \mathbf{10}, \boldsymbol{h'} + 11 \cdot \mathbf{11}) : \boldsymbol{h'} \in H\}.
\end{aligned}$$

Given that H is linear, it is clear that the sum of any two vectors from H_s will lie in one of the blocks $H_{s00}, H_{s01}, H_{s10}, H_{s11}$. Therefore, H_s is linear.

We now demonstrate the nonlinearity of $H = \varphi_s(\bar{H})$, where $\bar{H} = \mathcal{H}^{(1,0,\ldots,0,2,0)}$. Let

$$\boldsymbol{r} = (\mathbf{00}, \mathbf{012^{s-2}}, \ldots, \mathbf{332^{s-2}}).$$

H has length 256 and is constructed from

$$A_2^{(1,0,\ldots,0,2,0)} = \begin{pmatrix} \mathbf{10} & \mathbf{10} & \mathbf{10} & \cdots & \mathbf{10} & \mathbf{10} \\ \boldsymbol{r} & \boldsymbol{r} & \boldsymbol{r} & \cdots & \boldsymbol{r} & \boldsymbol{r} \\ \mathbf{00} & (\mathbf{01})\mathbf{2^{s-2}} & (\mathbf{02})\mathbf{2^{s-2}} & \cdots & (\mathbf{32}) & (\mathbf{33})\mathbf{2^{s-2}} \end{pmatrix}.$$

By Corollaries 3.5, 3.7, 3.8, and 3.9, we have:

$$\varphi_s(\boldsymbol{r},\boldsymbol{r},\ldots,\boldsymbol{r},\boldsymbol{r}) + \varphi_s(\mathbf{00},(\mathbf{01})\mathbf{2^{s-2}},\ldots,(\mathbf{33})\mathbf{2^{s-2}}) = \varphi_s(\boldsymbol{z}),$$

where

$$\boldsymbol{z} = (\boldsymbol{r},\boldsymbol{r},\ldots,\boldsymbol{r}) + (\mathbf{00},(\mathbf{01})\mathbf{2^{s-2}},\ldots,(\mathbf{33})\mathbf{2^{s-2}}) + \boldsymbol{p},$$

with

$$\boldsymbol{p} = (\mathbf{0}, \boldsymbol{u_1}, \mathbf{0}, \boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_2}, \boldsymbol{u_3}, \mathbf{0}, \boldsymbol{u_1}, \mathbf{0}, \boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_2}, \boldsymbol{u_3}),$$

and

$$\begin{aligned} u_1 &= (00, 2^{s-1}, 00, 2^{s-1}, \ldots, 00, 2^{s-1}), \\ u_2 &= (00, 00, 00, 00, 2^{s-1}0, \ldots, 2^{s-1}0), \\ u_3 &= (00, 2^{s-1}, 00, 2^{s-1}, 2^{s-1}0, 2^{s-1}2^{s-1}, \ldots, 2^{s-1}2^{s-1}). \end{aligned}$$

Since $\varphi_s(\boldsymbol{p}) = 28 \cdot 2^{2(s-1)} < \frac{3N}{4}$, where N is the length of H, we have $\varphi_s(\boldsymbol{p}) \notin H$, and thus $\varphi_s(\boldsymbol{z}) \notin H$. Therefore, $H = H^{(1,0,\ldots,0,2,0)}$ is nonlinear.

Let $H = \varphi_s(\bar{H})$, where $\bar{H} = \mathcal{H}^{(t_1,\ldots,t_s)}$. For any $i \in \{1, \ldots, s\}$, define $H_i = \varphi_s(\bar{H}_i)$, where $\bar{H}_i = \mathcal{H}^{(t'_1,\ldots,t'_s)}$, $t'_i = t_i + 1$ and $t'_j = t_j$ for $j \neq i$.

We consider that $H = \varphi_s(\bar{H})$, where $\bar{H} = \mathcal{H}^{(1,0,\ldots,0)}$. Now, we establish the nonlinearity of H_i for every $i \in \{1, \ldots, s-2\}$. The generator matrix of $\bar{H}_i$ contains two nonzero rows:

$$\boldsymbol{w_1} = 10, \quad \boldsymbol{w_2} = 2^{i-1}(00, \ldots, 0, 10, \ldots, 12^{s+1-i}, \ldots).$$

Let w_{2j} be the j-th coordinate of $\boldsymbol{w_2}$ and $[(w_{2j})_0, (w_{2j})_1, \ldots, (w_{2j})_{s-1}]_p$ its p-ary expansion. By Corollary 3.4,

$$\phi_s(w_{2j}) + \phi_s(2^{i-1}) = \phi_s(w_{2j} + 2^{i-1} - z_j),$$

where $z_j = 2^i$ if $(w_{2j})_{i-1} \geq 1$, and 0 otherwise. Then,

$$\varphi_s(\boldsymbol{w_2}) + \varphi_s(\boldsymbol{2^{i-1}}) = \varphi_s(\boldsymbol{w_2} + \boldsymbol{2^{i-1}} - \boldsymbol{z}),$$

where $\boldsymbol{z} = (z_1, z_2, \ldots, z_{2^{2(s+1-i)}}) \in \mathbb{Z}_{2^s}^{2^{2(s+1-i)}}$ and $z_j = 2^i$ for even $k \in \{2, 4, \ldots, 2^{s+1-i}\}$ and $z_j = 0$ otherwise.

We just need to show $\boldsymbol{z} \notin \bar{H}_i$. Note that $\mathrm{wt}_H(\varphi_s(\boldsymbol{z})) = 2^{2(s-i)} \cdot \mathrm{wt}_H(\varphi_s(2^i))$. If $i \in \{1, \ldots, s-2\}$, then

$$\mathrm{wt}_H(\varphi_s(\boldsymbol{z})) = 6 \cdot 2^{2(2s-i-2)}.$$

But the code H_i has minimum distance

$$3 \cdot 2^{2(2s-i-1)} > \mathrm{wt}_H(\varphi_s(z)),$$

therefore, $\varphi_s(\boldsymbol{z}) \notin H_i$, for $i \in \{1, \ldots, s-2\}$.

Finally, in general, for $H = \varphi_s(\widehat{H})$, where $\widehat{H} = \widehat{H}^{(t_1,\ldots,t_s)}$, we demonstrate that whenever H is nonlinear, H_i remains nonlinear for all $i \in \{1, \ldots, s\}$.

Let us assume that H_i does not exhibit linearity. Then, by considering iterative construction, for any $\boldsymbol{u}, \boldsymbol{v} \in \widehat{H}$, we have that

$$(\boldsymbol{u}, \ldots, \boldsymbol{u}), (\boldsymbol{v}, \ldots, \boldsymbol{v}) \in \widehat{H}_i.$$

Moreover, since H_i is linear,

$$\varphi_s(\boldsymbol{u},\ldots,\boldsymbol{u})+\varphi_s(\boldsymbol{v},\ldots,\boldsymbol{v})$$
$$=\varphi_s(\boldsymbol{a},\ldots,\boldsymbol{a})+\lambda\cdot 2^{i-1}\big(0,\ldots,0,2^{s-i+1}-1,\ldots,2^{s-i+1}-1,0,\ldots,2^{s-i+1}-1,2^{s-i+1}-1\big)\in H_i$$

where $\boldsymbol{a}\in\widehat{H}$ and $\lambda\in\mathbb{Z}_{2^s}[\omega]$.
Therefore,

$$\varphi_s(\boldsymbol{u})+\varphi_s(\boldsymbol{v})=\varphi_s(\boldsymbol{a})\in H.$$

So, H is linear.

6. Kernel of $\mathbb{Z}_{2^s}[\omega]$-Linear GH Codes

The method to find the kernel of codes over $\mathbb{Z}_{2^s}$ is given in Section 4 of [6, 17]. This section is devoted to establishing several results related to the kernel of $\mathbb{Z}_{2^s}[\omega]$-linear codes.

Assume $A^{(t_1,\ldots,t_s)}$ represents the generator matrix of $\widehat{H}^{(t_1,\ldots,t_s)}$ and denote $\boldsymbol{w_i}$ as the i-th row vector of $A^{(t_1,\ldots,t_s)}$. By established construction, $\boldsymbol{w_1}=\mathbf{1}$ and $\mathrm{ord}(\boldsymbol{w_i})\leq\mathrm{ord}(\boldsymbol{w_j})$ if $i>j$.

We introduce $\sigma\in\{1,\ldots,s\}$ as the integer satisfying the condition that $\mathrm{ord}(\boldsymbol{w_2})=2^{s+1-\sigma}$. Note $\sigma=1$ if $t_1>1$, and

$$\sigma=\min\{i:t_i>0,\ i\in\{2,\ldots,s\}\}\quad\text{if } t_1=1.$$

In this case, if $\sigma=s$, the code is $\widehat{H}^{(1,0,\ldots,0,t_s)}$, which is linear.

Let $\boldsymbol{u}=(u_1,\ldots,u_n)\in\mathbb{Z}_{2^s}^n[\omega]$ and

$$[u_{j,0},u_{j,1},\ldots,u_{j,s-1}]_2$$

be the 2-ary expansion of u_j, where $j\in\{1,\ldots,n\}$.

Assume i is an integer such that $i\in\{1,\ldots,s-1\}$. Then $\boldsymbol{u^i}$ denotes the vector in which the j-th coordinate corresponds to the i-th element of the 2-ary expansion of $\boldsymbol{u_j}$, that is,

$$\boldsymbol{u^i}=(u_{1,i},\ldots,u_{n,i})\in\mathbb{Z}_2^n[\omega].$$

Proposition 6.1 Let $\widehat{H}=\widehat{H}^{(t_1,\ldots,t_s)}$ be the $\mathbb{Z}_{2^s}[\omega]$-additive Hadamard code of type $(n;t_1,\ldots,t_s)$ such that $\varphi_s(\widehat{H})$ is nonlinear. Define $\widehat{H}_b$ as the subcode of $\widehat{H}$ consisting of all codewords of order two. Let

$$B=\begin{cases}\{\mathbf{2^p}\}_{p=0}^{\sigma-2} & \text{if } \sigma\geq 2,\\ \emptyset & \text{if } \sigma=1.\end{cases}$$

Then,

$$\langle\varphi_s(\widehat{H}_b),\ \varphi_s(B),\ \varphi_s\big(\sum_{i=0}^{s-2}\mathbf{2^i}\big)\rangle\subseteq K(\varphi_s(\widehat{H}))$$

and

$$\ker(\varphi_s(\widehat{H})) \geq \sigma + \sum_{i=1}^{s} t_i.$$

Proof: Let $H = \varphi_s(\widehat{H})$ and $r = \sum_{i=1}^{s} t_i$. Let

$$Q = \{\mathrm{ord}(\boldsymbol{w_j}/2) \cdot \boldsymbol{w_j}\}_{j=0}^{r}.$$

Since $\widehat{H}_b$ includes all elements of $\widehat{H}$ with order two, the set $\varphi_s(Q)$ serves as a basis for the binary linear subcode $H_b = \varphi_s(\widehat{H}_b)$ of H.

By Corollary 3.6, for all $\boldsymbol{b} \in \widehat{H}_b$ and $\boldsymbol{u} \in \widehat{H}$, we have

$$\varphi_s(\boldsymbol{b}) + \varphi_s(\boldsymbol{u}) = \varphi_s(\boldsymbol{b} + \boldsymbol{u}) \in H,$$

and therefore, $H_b \subseteq K(H)$.

Assume $\sigma \geq 2$. Now, we prove that $\varphi_s(\boldsymbol{2^p}) \in K(H)$ for all $p \in \{0, \ldots, \sigma - 2\}$. Equivalently, we show that

$$\varphi_s(\boldsymbol{2^p}) + \varphi_s(\boldsymbol{u}) \in H \quad \text{for all } \boldsymbol{u} \in \widehat{H}.$$

If $\boldsymbol{u} \in \widehat{H}$, then $\boldsymbol{u} = \mu \cdot \boldsymbol{1} + \boldsymbol{u'}$, where $\mu \in \mathbb{Z}_{2^s}[\omega]$ and $\mathrm{ord}(\boldsymbol{u'}) \leq \mathrm{ord}(\boldsymbol{w_2}) = 2^{s+1-\sigma}$.

Let $\boldsymbol{u} = (u_1, \ldots, u_n) \in \mathbb{Z}_{2^s}^n[\omega]$ and

$$[u_{i,0}, u_{i,1}, \ldots, u_{i,s-1}]_2$$

be the binary expansion of u_i, $i \in \{1, \ldots, n\}$.

Let

$$[\mu_0, \mu_1, \ldots, \mu_{s-1}]_2$$

be the binary expansion of $\mu \in \mathbb{Z}_{2^s}[\omega]$.

Note that if $\boldsymbol{v} \in \mathbb{Z}_{2^s}[\omega]$ is of order 2^i, then its binary expansion is of the form

$$[0, \ldots, 0, v_{s-i}, v_{s-i+1}, \ldots, v_{s-1}]_2.$$

Since $p \in \{0, \ldots, \sigma - 2\}$ and $\mathrm{ord}(\boldsymbol{u'}) \leq 2^{s+1-\sigma}$, we have

$$\boldsymbol{u^{(p)}} = (u_{1,p}, \ldots, u_{n,p}) = (\mu_p, \ldots, \mu_p).$$

By Corollary 3.4, we have

$$\varphi_s(\boldsymbol{2^p}) + \varphi_s(\boldsymbol{u}) = \varphi_s(\boldsymbol{2^p} + \boldsymbol{u} - 2^{p+1}\boldsymbol{t_p}),$$

where

$$\boldsymbol{t_p} = \begin{cases} \boldsymbol{1} & \text{if } \mu_p \geq 10, \\ \boldsymbol{0} & \text{otherwise.} \end{cases}$$

Therefore, $2^{p+1}\boldsymbol{t_p}$ is either $\boldsymbol{0}$ or $\boldsymbol{2^{p+1}}$. In both cases, $2^{p+1}\boldsymbol{t_p} \in \widehat{H}$, so

$$\varphi_s(\boldsymbol{2^p}) + \varphi_s(\boldsymbol{u}) = \varphi_s(\boldsymbol{2^p} + \boldsymbol{u} - 2^{p+1}\boldsymbol{t_p}) \in H.$$

Next, it can be easily verified that

$$\varphi_s\left(\sum_{i=0}^{s-2}\mathbf{2^i}\right) \in K(H).$$

Finally, it remains to verify that the elements of the set

$$\{\varphi_s(\widehat{H}_b), \varphi_s(B), \varphi_s\left(\sum_{i=0}^{s-2}\mathbf{2^i}\right)\}$$

are linearly independent.

Due to the block upper-triangular structure of the generator matrix, it is straightforward to verify that the codewords in $\varphi_s(Q)$ are linearly independent from the codewords in

$$\{\varphi_s(B), \varphi_s\left(\sum_{i=0}^{s-2}\mathbf{2^i}\right)\}.$$

Note $\sigma \leq s$ since H is nonlinear. Thus, by applying Lemma 3.4, we readily conclude that the codewords in

$$\{\varphi_s(B), \varphi_s\left(\sum_{i=0}^{s-2}\mathbf{2^i}\right)\}$$

are linearly independent, which implies that the dimension of their linear span is $\sigma + r$, so

$$\ker(H) \geq \sigma + r.$$

Lemma 6.1: Let $v, \mu \in \mathbb{Z}_{2^s}[\omega]$. Then,

$$v \odot_2 \mu = \sum_{i=0}^{s-1}\left(v \odot_2 \mu_i 2^i\right),$$

where $\left[\mu_0, \ldots, \mu_{s-1}\right]_2$ is the 2-ary expansion of μ.

Proof: Let $v \in \mathbb{Z}_{2^s}[\omega]$ and $\left[v_0, \ldots, v_{s-1}\right]_2$ be its 2-ary expansion. From the definition, we have

$$v \odot_2 \mu = v \odot_2 \sum_{i=0}^{s-1}\mu_i 2^i = \sum_{i=0}^{s-1} t_i 2^i,$$

where

$$t_i = \begin{cases} 1 & \text{if } v_i + \mu_i \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$t_i 2^i = v \odot_2 \mu_i 2^i,$$

so

$$v \odot_2 \sum_{i=0}^{s-1} \mu_i 2^i = \sum_{i=0}^{s-1} \left(v \odot_2 \mu_i 2^i \right).$$

Lemma 6.2: Let $\mathcal{H} = \mathcal{H}^{(t_1,\ldots,t_s)}$ be the $\mathbb{Z}_{2^s}[\omega]$-additive Hadamard code of type $(n; t_1, \ldots, t_s)$. Define

$$\mathcal{N} = \left\{ \sum_{i=0}^{s-1} \mu_i \mathbf{2}^{\boldsymbol{i}} : \mu_i \in \mathbb{Z}_2[\omega] \right\} \setminus \left\{ \sum_{i=0}^{s-1} \mathbf{2}^{\boldsymbol{i}} \right\} \quad \text{if } \sigma \leq s-1.$$

Then,

$$\varphi_s(\mathcal{N}) \cap K\big(\varphi_s(\mathcal{H})\big) = \{\mathbf{0}\}.$$

Proof: Let $H = \varphi_s(\mathcal{H})$. Suppose

$$\boldsymbol{u} = \sum_{i=0}^{s-1} \mu_i 2^i \in \mathcal{N} \quad \text{such that} \quad \varphi_s(\boldsymbol{u}) \in K(H).$$

We want to show that $\boldsymbol{u} = \mathbf{0}$.

Based on the construction, the second row $\boldsymbol{w_2}$ of $A^{(t_1,\ldots,t_s)}$ is a $2^{t-2s+\sigma}$-fold replication of

$$\boldsymbol{v} = 2^{\sigma-1}\big(00, \ldots, 0, 2^{s+1-\sigma} - 1, 10, \ldots, 2^{s+1-\sigma} - 1, 2^{s+1-\sigma} - 1\big),$$

and $\mathrm{ord}(\boldsymbol{w_2}) = 2^{s+1-\sigma}$.

By Corollary 3.2, we have

$$\varphi_s(\boldsymbol{w_2}) + \varphi_s(\boldsymbol{u}) = \varphi_s\big(\boldsymbol{w_2} + \boldsymbol{u} - 2(\boldsymbol{w_2} \odot_2 \boldsymbol{u})\big).$$

Since $\varphi_s(\boldsymbol{u}) \in K(H)$, it follows that $2(\boldsymbol{w_2} \odot_2 \boldsymbol{u}) \in \mathcal{H}$.

Write

$$\boldsymbol{w_2} = (w_1, w_2, \ldots, w_n),$$

and let

$$[w_{j,0}, w_{j,1}, \ldots, w_{j,s-1}]_2$$

be the 2-ary expansion of $\boldsymbol{w_j}$, for $j \in \{1, \ldots, n\}$.

By Lemma 6.1,

$$2(\boldsymbol{w_2} \odot_2 \boldsymbol{u}) = 2 \sum_{i=\sigma-1}^{s-2} \left(\boldsymbol{w_2} \odot_2 \mu_i \mathbf{2}^{\boldsymbol{i}} \right) = 2 \sum_{i=\sigma-1}^{s-2} \boldsymbol{R_i} 2^i \in \mathcal{H},$$

where

$$\boldsymbol{R_i} = (r_{1,i}, r_{2,i}, \ldots, r_{n,i}),$$

and

$$r_{j,i} = \begin{cases} 1 & \text{if } w_{j,i} + \mu_i \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\tau = \sum_{i=1}^{s} t_i$. Since $\sigma \leq s-1$, we have $\tau \geq 2$.

If $\tau = 2$, then $\mathcal{H}$ has length $2^{s+1-\sigma}$, and the only vectors in $A^{(t_1,\ldots,t_s)}$ are $\mathbf{1}$ and $w_2 = \boldsymbol{v}$.

If $\tau \geq 3$, for $i \in \{3, \ldots, \tau\}$, the i-th row w_i of $A^{(t_1,\ldots,t_s)}$ has zeros in its first $2^{s+1-\sigma}$ coordinates.

Since $\sigma \leq s-1$ and $\tau \geq 2$, every element of $\mathcal{H}$, when restricted to the first $2^{s+1-\sigma}$ coordinates, takes the form

$$\mu_1 \cdot \mathbf{1} + \mu_2 \cdot \boldsymbol{v}$$

for some $\mu_1, \mu_2 \in \mathbb{Z}_{2^s}[\omega]$.

Now,

$$2 \sum_{i=\sigma-1}^{s-2} \boldsymbol{R_i} 2^i$$

restricted to the first $m = 2^{s+1-\sigma}$ coordinates is

$$2 \sum_{i=\sigma-1}^{s-2} \boldsymbol{R'_i} 2^i,$$

where

$$\boldsymbol{R'_i} = (t_{1,i}, t_{2,i}, \ldots, t_{m,i}).$$

Therefore, we want $\mu_1, \mu_2 \in \mathbb{Z}_{2^s}[\omega]$ such that

$$2 \sum_{i=\sigma-1}^{s-2} \boldsymbol{R'_i} 2^i = \mu_1 \cdot \mathbf{1} + \mu_2 \cdot \boldsymbol{v}.$$

Since the initial entry of $\boldsymbol{v}$ is zero, the first coordinate of $\boldsymbol{v^{(i)}}$ is zero for all $i \in \{0, \ldots, s-1\}$. Thus,

$$\mu_1 = 0,$$

and

$$2 \sum_{i=\sigma-1}^{s-2} \boldsymbol{R'_i} 2^i = \mu_2 \boldsymbol{v}.$$

Note that

$$\boldsymbol{v} = \sum_{i=0}^{s-1} \boldsymbol{v^{(i)}} 2^i = \sum_{i=\sigma-1}^{s-1} \boldsymbol{v^{(i)}} 2^i.$$

Let

$$\boldsymbol{a} = 2 \sum_{i=\sigma-1}^{s-2} \boldsymbol{R'_i} 2^i, \quad \boldsymbol{b} = \mu_2 \boldsymbol{v}.$$

Suppose $\mu_2 \in A = \{0, 2^{s-\sigma+1}\}$. Then $\boldsymbol{b} = \mathbf{0}$. Given the existence of some $\mu_{i_0} \neq 0$, the vector $\boldsymbol{R'_{i_0}}$ has at least one nonzero coordinate, so $\boldsymbol{a} \neq \mathbf{0}$, a contradiction.

On the other hand, if $\mu_2 \in \mathbb{Z}_{2^s}[\omega] \setminus A$, let

$$\boldsymbol{a^{(i)}} = (a_{1,i}, a_{2,i}, \ldots, a_{n,i}), \quad \sigma \leq i \leq s-1,$$

where $a_{j,i} \in \{0,1\}$ for all $j \in \{1,\ldots,m\}$ and $i \in \{\sigma,\ldots,s-1\}$.

Since

$$\boldsymbol{v} = 2^{\sigma-1}(00,\ldots,0,2^{s+1-\sigma}-1,10,\ldots,2^{s+1-\sigma}-1,2^{s+1-\sigma}-1),$$

there exists some $i_1 \in \{\sigma,\ldots,s-1\}$ such that the coordinates of $\boldsymbol{b^{(i_1)}}$ do not belong to $\{0,1\}$, a contradiction.

Therefore, if $\boldsymbol{u} \neq \mathbf{0}$, then

$$2(\boldsymbol{w_2} \odot_2 \boldsymbol{u}) = \mu_1 \cdot \mathbf{1} + \mu_2 \cdot \boldsymbol{v},$$

and hence $\boldsymbol{u} = \mathbf{0}$.

Lemma 6.3: Let $\hat{H} = \hat{H}^{(t_1,\ldots,t_s)}$ be the $\mathbb{Z}_{2^s}[\omega]$-additive GH code of type $(n; t_1, \ldots, t_s)$. Let $\boldsymbol{w_i}$ be the ith row of $A_2^{(t_1,t_2,\ldots,t_s)}$ and $\tau = \sum_{i=1}^{s} t_i$. Define

$$\Xi = \left\{ \boldsymbol{v} = \sum_{i=2}^{\tau-t_s} \mu_i \boldsymbol{w_i} : \mu_i \in \mathbb{Z}_{2^s}[\omega],\ \operatorname{ord}(\boldsymbol{v}) > 2 \right\},$$

$$\mathcal{N} = \left\{ \sum_{i=0}^{s-1} \mu_i \boldsymbol{2^i} : \mu_i \in \mathbb{Z}_2[\omega] \setminus \left\{ \sum_{i=0}^{s-1} \boldsymbol{2^i} \right\} \right\} \quad \text{if } \sigma \leq s-1,$$

and

$$\Xi + \mathcal{N} = \{\boldsymbol{v}_\Xi + \boldsymbol{v}_\mathcal{N} : \boldsymbol{v}_\Xi \in \Xi \cup \{0\}, \boldsymbol{v}_\mathcal{N} \in \mathcal{N}\}.$$

Then

$$\varphi_s(\Xi + \mathcal{N}) \cap K(\varphi_s(\hat{H})) = \{\mathbf{0}\}.$$

Proof: Let $H = \varphi_s(\hat{H})$, which has length $N = 2^{2t} = n \cdot 2^{2(s-1)}$. By Lemma 5, we know that

$$\varphi_s(\mathcal{N}) \cap K(H) = \{\mathbf{0}\},$$

now we prove that

$$\varphi_s(\Xi) \cap K(H) = \emptyset.$$

Let $\boldsymbol{v} = \sum_{i=2}^{\tau-t_s} \mu_i \boldsymbol{w_i} \in \Xi$. Since $\operatorname{ord}(\boldsymbol{v}) > 2$ and $\operatorname{ord}(\boldsymbol{w_i}) \leq 2^{s+1-\sigma}$, we have $\operatorname{ord}(\boldsymbol{v}) = 2^r$ for some $2 \leq r \leq s+1-\sigma$. Through the iterative construction of $A^{(t_1,\ldots,t_s)}$, it is clear that all elements of $\mathbb{Z}_{2^s}[\omega]$ whose order is 2^r or less appear as a coordinate of $\boldsymbol{v}$.

Let

$$[v_{j,0}, v_{j,1}, \ldots, v_{j,s-1}]_2$$

be the 2-ary expansion of v_j, for $j \in \{1, \dots, n\}$. By Corollary 4, we have

$$\varphi_s(\boldsymbol{v}) + \varphi_s(\boldsymbol{2^{s-r}}) = \varphi_s\big(\boldsymbol{v} + \boldsymbol{2^{s-r}} - 2^{s-r+1}\boldsymbol{R_{s-r}}\big),$$

where

$$\boldsymbol{T_{s-r}} = (t_{1,(s-r)}, t_{2,(s-r)}, \dots, t_{n,(s-r)}),$$

and for $j \in \{1, \dots, n\}$,

$$t_{j,(s-r)} = \begin{cases} 1, & \text{if } v_{j,(s-r)} \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

It suffices to show that

$$2^{s-r+1}\boldsymbol{T_{s-r}} \notin \hat{H}$$

to prove that $\varphi_s(\boldsymbol{v}) \notin K(H)$.

Since

$$\boldsymbol{v} = \sum_{i=2}^{\tau - t_s} \mu_i \boldsymbol{w_i} = (v_1, v_2, \dots, v_n)$$

and $\operatorname{ord}(\boldsymbol{v}) = p^r$ for some $2 \leq r \leq s + 1 - \sigma$, according to the construction, v contains each element of $2^{s-1}\mathbb{Z}_{2^s}[\omega]$ exactly α times, $\alpha \geq 0$, and the remaining $n - 4\alpha$ coordinates come from

$$\mathbb{Z}_{2^s}[\omega] \setminus 2^{s-1}\mathbb{Z}_{2^s}[\omega].$$

So,

$$\operatorname{wt}_H\big(\varphi_s(2^{s-r+1}\boldsymbol{R_{s-r}})\big) \leq (n - 4\alpha) \cdot 3 \cdot 2^{2(s-2)} < 3n \cdot 2^{2(s-2)} = \frac{3N}{4} = d(H).$$

Therefore,

$$\varphi_s(\boldsymbol{v}) \notin K(H),$$

and

$$\varphi_s(\Xi) \cap K(H) = \emptyset.$$

We now proceed to show that

$$\varphi_s(\Xi + \mathcal{N}) \cap K(\varphi_s(\hat{H})) = \{\boldsymbol{0}\}.$$

Let

$$\boldsymbol{v} = \boldsymbol{v}_\Xi + \boldsymbol{v}_\mathcal{N} \in \Xi + \mathcal{N} \setminus \{\boldsymbol{0}\},$$

where $\boldsymbol{v}_\Xi \in \Xi$ and $\boldsymbol{v}_\mathcal{N} \in \mathcal{N}$. We previously proved that

$$\varphi_s(\boldsymbol{v}) \notin K(H)$$

if $\boldsymbol{v}_\Xi = \boldsymbol{0}$ or $\boldsymbol{v}_\mathcal{N} = \boldsymbol{0}$. Hence, assume $\boldsymbol{v}_\Xi \neq \boldsymbol{0}$ and $\boldsymbol{0}_\mathcal{N} \neq \boldsymbol{0}$.

We know

$$\boldsymbol{v}_\mathcal{N} = (v, \dots, v).$$

Let

$$[v_0, v_1, \ldots, v_{s-1}]_2$$

be the 2-ary expansion of v.

Consider

$$v_{\mathcal{N}_1} \text{ and } v_{\mathcal{N}_2}$$

as the elements of $\mathbb{Z}_{2^s}[\omega]$ having 2-ary expansions

$$[0, \ldots, 0, v_{s-r}, \ldots, v_{s-1}]_2 \quad \text{and} \quad [v_0, \ldots, v_{s-r-1}, 0, \ldots, 0]_2,$$

respectively.

Then,

$$\boldsymbol{v}_{\mathcal{N}} = \boldsymbol{v}_{\mathcal{N}_1} + \boldsymbol{v}_{\mathcal{N}_2},$$

where

$$\boldsymbol{v}_{\mathcal{N}_i} = (v_{\mathcal{N}_i}, \ldots, v_{\mathcal{N}_i}), \quad i \in \{1, 2\}.$$

Since $\operatorname{ord}(\boldsymbol{v}_\Xi) = 2^r$ with $2 \leq r \leq s + 1 - \sigma$, the 2-ary expansion of each coordinate of v_Ξ takes the form

$$[0, \ldots, 0, v_{\Xi,(s-r)}, \ldots, v_{\Xi,(s-1)}]_2.$$

Note that

$$\operatorname{ord}(\boldsymbol{v}_{\mathcal{N}_1}) \leq \operatorname{ord}(\boldsymbol{v}_\Xi)$$

by construction.

It follows that

$$2\left(\boldsymbol{v}_{\mathcal{N}_2} \odot_2 \boldsymbol{2^{s-r}}\right) = \boldsymbol{0}.$$

Therefore,

$$\operatorname{wt}_H\big(\varphi_s(2(\boldsymbol{v} \odot_2 \boldsymbol{2^{s-r}}))\big) = \operatorname{wt}_H\big(\varphi_s(2((\boldsymbol{v}_\Xi + \boldsymbol{v}_{\mathcal{N}_1}) \odot_2 \boldsymbol{2^{s-r}}))\big).$$

Since $\operatorname{ord}(\boldsymbol{v}_{\mathcal{N}_1}) \leq \operatorname{ord}(\boldsymbol{v}_\Xi)$, there exists a permutation of coordinates π satisfying

$$\pi(\boldsymbol{v}_\Xi + \boldsymbol{v}_{\mathcal{N}_1}) = \boldsymbol{v}_\Xi.$$

Thus,

$$\operatorname{wt}_H\big(\varphi_s(2((\boldsymbol{v}_\Xi + \boldsymbol{v}_{\mathcal{N}_1}) \odot_2 \boldsymbol{2^{s-r}}))\big) = \operatorname{wt}_H\big(\varphi_s(2(\boldsymbol{v}_\Xi \odot_2 \boldsymbol{2^{s-r}}))\big).$$

Since $\operatorname{ord}(\boldsymbol{v}_\Xi) = p^r$ with $2 \leq r \leq s + 1 - \sigma$, as in the previous case, this leads to a contradiction.

Therefore,

$$\varphi_s(\boldsymbol{v}) \notin K(H) \quad \text{and} \quad \varphi_s(\Xi + \mathcal{N}) \cap K(H) = \{\boldsymbol{0}\}.$$

Theorem 6.1: Let $\hat{H} = \hat{H}^{(t_1,\ldots,t_s)}$ be the $\mathbb{Z}_{2^s}[\omega]$-additive Hadamard code of type $(n; t_1, \ldots, t_s)$ such that $\varphi_s(\hat{H})$ is nonlinear. Define $\hat{H}_b$ as the subcode of $\hat{H}$ consisting of all the codewords of order two. Let

$$B = \begin{cases} \{\boldsymbol{2^p}\}_{p=0}^{\sigma-2} & \text{if } \sigma \geq 2, \\ \emptyset & \text{if } \sigma = 1. \end{cases}$$

Then,

$$\langle \varphi_s(\hat{H}_b),\ \varphi_s(B),\ \varphi_s\left(\sum_{i=0}^{s-2} \mathbf{2^i}\right)\rangle \subseteq K(\varphi_s(\hat{H})),$$

and

$$\ker(\varphi_s(\hat{H})) = \sigma + \sum_{i=1}^{s} t_i.$$

Proof: The conclusion directly follows from Proposition 6.1 and Lemma 6.3.

Corollary 6.1: Let $\hat{H} = \hat{H}^{(t_1,\ldots,t_s)}$ be the $\mathbb{Z}_{2^s}[\omega]$-additive Hadamard code of type $(n; t_1, \ldots, t_s)$ such that $\varphi_s(\hat{H})$ is nonlinear. Let $\boldsymbol{w_i}$ be the ith row of $A_2^{(t_1,t_2,\ldots,t_s)}$ and $\tau = \sum_{i=1}^{s} t_i$. Let

$$Q = \{\mathrm{ord}(\boldsymbol{w_j}/2)\boldsymbol{w_j}\}_{j=0}^{r}, \quad B = \begin{cases} \{\mathbf{2^p}\}_{p=0}^{\sigma-2} & \text{if } \sigma \geq 2, \\ \emptyset & \text{if } \sigma = 1. \end{cases}$$

Then $\{\varphi_s(Q), \varphi_s(B), \varphi_s(\sum_{i=0}^{s-2} \mathbf{2^i})\}$ forms a basis for it$K(\varphi(\hat{H}))$.

Example 6.1: Let it $H^{(2,0,0)}$ be the $\mathbb{Z}_8[\omega]$-linear Hadamard code discussed in Example 4. According to Theorem 3.4,

$$\ker(H^{(2,0,0)}) = 3.$$

By Corollary 6.1, $K(H^{(2,0,0)})$ can be constructed from a basis. To begin with, we have that

$$Q = \{\mathbf{40}, (00, 04, \ldots, 00, 04, 40, 44, \ldots, 40, 44, 00, 04, \ldots, 40, 44)\}.$$

Since $\sigma = 1$, in this case $B = \emptyset$. Thus,

$$K(H^{(2,0,0)}) = \langle \varphi_s(\mathbf{40}),\ \varphi_s(00, 04, \ldots, 00, 04, 40, 44, \ldots, 40, 44, 00, 04, \ldots, 40, 44),\ \varphi_s(\mathbf{30})\rangle.$$

7. Classification of $\mathbb{Z}_{2^s}[\omega]$-Linear Hadamard Codes

Our discussion here is aimed at some aspects of classifying $\mathbb{Z}_{2^s}[\omega]$-linear codes for length 2^{2t} for $t \geq 3$ and $s > 2$, but we realize that dimension of the kernel alone is not sufficient for a full classification. Theorem 3 states that for any $t \geq 3$ and $s > 2$, there are at most two $\mathbb{Z}_{2^s}[\omega]$-linear codes of length 2^{2t}, namely; $H^{(1,0,\ldots,0,1,t_s)}$ and $H^{(1,0,\ldots,0,t_s)}$ which are linear. Consequently, our focus can be then placed on $t \geq 5$ and $2 \leq s \leq t-2$ in order to classify the nonlinear codes.

Theorem 7.1 Let $A_{t,s}$ denote the number of inequivalent $\mathbb{Z}_{2^s}[\omega]$-linear Hadamard codes of length 2^{2t}. Then,

$$A_{t,s} = \begin{cases} 0 & \text{if } t \geq 3 \text{ and } s \geq t+2, \\ 1 & \text{if } t \geq 3 \text{ and } s \in \{t-1, t, t+1\}, \\ 1 & \text{if } t = 4 \text{ and } s = 2, \end{cases}$$

and the $\mathbb{Z}_{2^s}[\omega]$-linear Hadamard code is linear when $A_{t,s} = 1$. Moreover, if $t \geq 5$ and $2 \leq s \leq t-2$, then $A_{t,s} \geq 2$, and there exist one linear code and at least one nonlinear code.

Proof: If $t \geq 3$ and $s \geq t+2$, then for the equation

$$t = \left(\sum_{i=1}^{s} (s-i+1)t_i \right) - 1,$$

with $t_1 \geq 1$, there are no nonnegative integer solutions, so $A_{t,s} = 0$.

If $t \geq 3$ and $s = t+1$, there is only one solution $(t_1, \dots, t_s) = (1, 0, \dots, 0)$. If $t \geq 3$ and $s = t$, there is exactly one solution $(1, 0, \dots, 0, 1)$. If $t \geq 3$ and $s = t-1$, there are two solutions $(1, 0, \dots, 0, 2)$ and $(1, 0, \dots, 0, 1, 0)$. Notably, when $t = 3$ and $s = 2$, both solutions are $(1, 2)$ and $(2, 0)$. By Theorem 3.3, for all the above solutions, we obtain a linear code $H^{(t_1, \dots, t_s)}$.

Finally, if $t \geq 5$ and $2 \leq s \leq t-2$, the solutions $(1, 0, \dots, 0, t-s+1)$ and $(1, 0, \dots, 0, 1, t-s-1)$ always exist, yielding a linear code. In these cases, there is at least one additional solution. If $s = 2$,

$$A_{t,s} = \left\lfloor \frac{t-1}{2} \right\rfloor \geq 2 \quad \text{since } t \geq 5.$$

On the other hand, if $s = 3$, $(2, 0, \dots, 0, t-2s+1)$ is a solution because $t \geq 2s-1$ when $t \geq 5$; and if $s \geq 4$, $(1, 0, \dots, 0, 1, 0, t-s-2)$ is a solution. Therefore, for all the cases, $A_{t,s} \geq 2$ by Theorem 5.2.

Example 7.1: The $\mathbb{Z}_8[\omega]$-linear Hadamard codes of length $2^{2t} = 65536$, listed below, are:

$$H^{(1,0,6)}, \quad H^{(1,1,4)}, \quad H^{(1,2,2)}, \quad H^{(1,3,0)}, \quad H^{(2,0,3)}, \quad H^{(2,1,1)}, \quad \text{and } H^{(2,1,1)}.$$

Both of the first two are equivalent because they are linear codes by Theorem 3. According to Theorem 4, the other codes have kernel dimensions of $7, 6, 6, 5,$ and 4. Therefore, from this invariant, we conclude that all these codes are different except $H^{(1,3,0)}$ and $H^{(2,0,3)}$ that share identical kernel dimensions.

We have observed in some instances that codes defined over $\mathbb{Z}_{2^s}$ and codes over $\mathbb{Z}_{2^s}[\omega]$ have the same rank and kernel dimension, meaning that $\text{rank}(H^{(1,3,0)}) = 12$, and $\text{rank}(H^{(2,0,3)}) = 11$, which gives their non-equivalence. As a result, in the case of the $\mathbb{Z}_{2^s}[\omega]$-linear Hadamard codes of length $2^{2t} = 65536$, the rank gives a complete classification that does not require consideration of the kernel.

Example 7.2: Theorem 6.1 verifies that for all $5 \leq t \leq 7$ and $2 \leq s \leq t-2$, the nonlinear $\mathbb{Z}_{2^s}[\omega]$-linear Hadamard codes of length 2^{2t} have different kernel dimensions and we are able to classify these codes according to this invariant. Similar results apply for $t = 8, 9, 10,$ and 11; exceptions occur for certain values of s in each case. In the particular cases of t and s, classification according to kernel analysis provides only partial results.

It was found that for codes over $\mathbb{Z}_{2^s}$ and codes over $\mathbb{Z}_{2^s}[\omega]$, the rank and dimension of kernel are equal. The software Magma can be used to determine the rank and kernel dimension for any $5 \leq t \leq 11$ and $2 \leq s \leq t-2$ [6, 17]. Tables 2 and 5 give the values of $(t_1, \ldots, t_s)$ together with the pair (r, k), where r is the rank and k the dimension of the kernel for all nonlinear $\mathbb{Z}_{2^s}[\omega]$-linear Hadamard codes of length 2^{2t} for $5 \leq t \leq 10$. Such tables show that each length 2^{2t} code has exactly one rank value when $5 \leq t \leq 10$ and $2 \leq s \leq t-2$ is fixed. Thus, all codes in such situations are different, allowing only the rank of the code to be classified, from which we obtain the following assertion.

Table 2: Rank and dimension of kernel for all nonlinear $\mathbb{Z}_{2^s}[\omega]$-linear Hadamard codes of length 2^{2t}

	$t=5$		$t=6$		$t=7$	
	$(t_1,\ldots,t_s)$	(r,k)	$(t_1,\ldots,t_s)$	(r,k)	$(t_1,\ldots,t_s)$	(r,k)
$\mathbb{Z}_4[\omega]$	(3,0)	(7,4)	(3,1)	(8,5)	(3,2)	(9,6)
					(4,0)	(11,5)
$\mathbb{Z}_8[\omega]$	(2,0,0)	(8,3)	(1,2,0)	(8,5)	(1,2,1)	(9,6)
			(2,0,1)	(9,4)	(2,0,2)	(10,5)
					(2,1,0)	(12,4)
$\mathbb{Z}_{16}[\omega]$			(1,1,0,0)	(9,4)	(1,0,2,0)	(9,6)
					(1,1,0,1)	(10,5)
					(2,0,0,0)	(14,3)
$\mathbb{Z}_{32}[\omega]$					(1,0,1,0,0)	(10,5)

Theorem 7.2: Let $A_{t,s}$ represent the number of inequivalent $\mathbb{Z}_{2^s}[\omega]$-linear Hadamard codes of length 2^{2t}. Then, for any $t \geq 3$ and $2 \leq s \leq t-1$,

$$A_{t,s} \leq \left| \left\{ (t_1, \ldots, t_s) \in \mathbb{N}^s : t = \left(\sum_{i=1}^{s} (s-i+1) t_i \right) - 1,\ t_1 \geq 1 \right\} \right| - 1.$$

Moreover, for any values of t within the range $[3, 11]$ and s in the range $[2, t-1]$, including the endpoints, this bound is sharp. Using the outputs of Theorems 5.1 and 5.2, we give the following Table 3 that lists the number of nonequivalent $\mathbb{Z}_{2^s}[\omega]$-linear Hadamard codes of length 2^{2t} where $3 \leq t \leq 11$ and $2 \leq s \leq 9$. Classification is inadequate only when considering the kernel dimension in the highlighted cases in **bold**. However, in all cases, the aforesaid rank turns out to be a good method of classification.

There appear to be $\mathbb{Z}_4[\omega]$-linear Hadamard codes that do not exist as equivalent $\mathbb{Z}_{2^s}[\omega]$-linear Hadamard codes, for $s > 2$.

Example 7.3: Let us consider $H^{(2,0,0)}$ as the $\mathbb{Z}_8[\omega]$-linear Hadamard code of length 1024. From Theorem 3, we know that $\ker(H^{(2,0,0)}) = 3$, i.e., $H^{(2,0,0)}$ cannot be linear.

There are acknowledged to be three $\mathbb{Z}_4[\omega]$-linear Hadamard codes of length 1024 given by $H^{(1,4)}$, $H^{(2,2)}$, and $H^{(3,0)}$. The first two codes have a linear structure, while the last one is nonlinear, and by Theorem 5.2 it is shown that $\ker(H^{(3,0)}) = 4$. Consequently, there does not exist a $\mathbb{Z}_4[\omega]$-linear Hadamard code equivalent to the $\mathbb{Z}_8[\omega]$-linear Hadamard code $H^{(2,0,0)}$.

Example 7.4: It is apparent from Table 2 that for $t = 5$, we have only two nonlinear $\mathbb{Z}_{2^s}[\omega]$-linear Hadamard codes, namely $H^{(3,0)}$ and $H^{(2,0,0)}$. Based on Example 7.3, the

Table 3: Number $A_{t,s}$ of nonequivalent $\mathbb{Z}_{2^s}[\omega]$-linear Hadamard codes of length 2^{2t}

t	3	4	5	6	7	8	9	10	11
$\mathbb{Z}_4[\omega]$	1	1	2	2	3	3	4	4	5
$\mathbb{Z}_8[\omega]$	1	1	2	3	4	6	7	9	11
$\mathbb{Z}_{16}[\omega]$	1	1	1	2	4	5	8	10	14
$\mathbb{Z}_{32}[\omega]$	0	1	1	1	2	4	6	9	12
$\mathbb{Z}_{64}[\omega]$	0	0	1	1	1	2	4	6	10
$\mathbb{Z}_{128}[\omega]$	0	0	0	1	1	1	2	4	6
$\mathbb{Z}_{256}[\omega]$	0	0	0	0	1	1	1	2	4
$\mathbb{Z}_{512}[\omega]$	0	0	0	0	0	1	1	1	2

codes are shown to be distinct because they differ in the kernel's dimension.

More examples can be seen when t is an odd number. For example, based on Tables 2 and 5, when $t = 7$, $t = 9$, and $t = 11$, we observe that the $\mathbb{Z}_4[\omega]$-linear Hadamard codes $H^{(4,0)}$, $H^{(5,0)}$, and $H^{(6,0)}$ do not coincide in equivalence with any $\mathbb{Z}_{2^s}[\omega]$-linear Hadamard codes of the same length and $s > 2$, under both the rank and the dimension of the kernel.

It has been established that for $\mathbb{Z}_2$-linear Hadamard codes, the lower bounds K (the kernel dimension) and RK (the kernel dimension and the rank) are known. In this paper, we have shown that for both $\mathbb{Z}_{2^s}$- and $\mathbb{Z}_{2^s}[\omega]$-linear Hadamard codes, the upper bounds K and RK are the same. The Table 4 contains bounds for the range $3 \leq t \leq 11$.

Table 4: Bounds for the number A_t of nonequivalent $\mathbb{Z}_{2^s}[\omega]$-linear Hadamard codes of length 2^{2t}

t	3	4	5	6	7	8	9	10	11
Lower bound K	1	1	3	3	5	5	7	7	9
Lower bound RK	1	1	3	3	6	7	11	13	20
Upper bound	1	1	3	5	10	16	26	38	57

By studying all nonequivalent $\mathbb{Z}_{2^s}[\omega]$-linear Hadamard codes of length 2^{2t}, it is possible to produce an easy upper bound calculation if t and s are given. Table 4 shows the values for all $3 \leq t \leq 11$.

Theorem 7.3: Let $A_{(t,s)}$ be defined as the number of nonequivalent $\mathbb{Z}_{2^s}[\omega]$-linear Hadamard codes of length 2^{2t}. Let A_t denote the total number of distinct $\mathbb{Z}_{2^s}[\omega]$-linear Hadamard codes of length 2^{2t} for $s \geq 2$. Then,

$$A_t \leq \sum_{s=2}^{t-2} \left(A_{(t,s)} - 1\right) + 1.$$

Theorem 7.4: For lengths 2^{2t}, where $t = 3, 4, 5, 6$ and 7, there exist exactly 1, 1, 3, 3, and 6 distinct nonequivalent $\mathbb{Z}_{2^s}[\omega]$-linear Hadamard codes, respectively.

8. Conclusion and Future Directions

Our research created and studied Generalized Hadamard (GH) codes using Eisenstein local rings $\mathbb{Z}_{2^s}[\omega]$ to develop their complete set of algebraic and combinatorial properties.

Our use of Eisenstein integers established a Gray map to switch from these codes to binary numbers for expert study. Our research revealed specific rules to distinguish $\mathbb{Z}_{2^s}[\omega]$-linear GH codes that are linear from those that are not. Our research into the kernel properties helps reveal how these codes differ from one another. The theory proves that Eisenstein fields offer effective ways to build useful and space-saving error correction codes.

The future of research should study different families of error-correcting codes using Eisenstein algebraic rings and their relatives, such as cyclic, quasi-cyclic, and consta-cyclic codes. Studying automorphism groups and decoding methods for GH codes with $\mathbb{Z}_{2^s}[\omega]$ as field structure will lead to theoretical advancement and practical applications.

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Figure 1: Prof. Dr. Tariq Shah

Data availability

All the data is given in this study.

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Table 5: Rank and kernel for all nonlinear $\mathbb{Z}_{2^s}[\omega]$-linear Hadamard codes of length 2^t

	$t=8$		$t=9$		$t=10$	
	$(t_1,\ldots,t_s)$	(r,k)	$(t_1,\ldots,t_s)$	(r,k)	$(t_1,\ldots,t_s)$	(r,k)
$\mathbb{Z}_4[\omega]$	(3,3)	(10,7)	(3,4)	(11,8)	(3,5)	(12,9)
	(4, 1)	(12, 6)	(4, 2)	(13, 7)	(4,3)	(14,8)
			(5, 0)	(16, 6)	(5,1)	(17,7)
$\mathbb{Z}_8[\omega]$	(1,2,2)	(10,7)	(1,2,3)	(11,8)	(1,2,4)	(12,9)
	(1, 3, 0)	(12, 6)	(1,3,1)	(13,7)	(1,3,2)	(14,8)
	(2, 0, 3)	(11, 6)	(2, 0, 4)	(12, 7)	(1, 4, 0)	(17,7)
	(2,1,1)	(13,5)	(2,1,2)	(14,6)	(2,0,5)	(13,8)
	(3,0,0)	(17,4)	(2,2,0)	(17,5)	(2,1,3)	(15,7)
			(3,0,1)	(18,5)	(2,2,1)	(18,6)
					(3,0,2)	(19,6)
					(3,1,0)	(24,5)
$\mathbb{Z}_{16}[\omega]$	(1, 0, 2, 1)	(10, 7)	(1,0,2,2)	(11,8)	(1,0,2,3)	(12,9)
	(1, 1, 0, 2)	(11, 6)	(1,0,3,0)	(17,7)	(1,0,3,1)	(14,8)
	(1, 1, 1, 0)	(13, 5)	(1,2,0,0)	(18,5)	(1,1,0,4)	(13,8)
	(2, 0, 0, 1)	(15, 4)	(1,1,0,3)	(12,7)	(1,1,1,2)	(15,7)
			(1, 1, 1, 1)	(14,6)	(1,1,2,0)	(18,6)
			(2, 0, 0, 2)	(16,5)	(1,2,0,1)	(19,6)
			(2,0,1,0)	(20,4)	(2,0,0,3)	(17,6)
					(2, 0, 1, 1)	(21,5)
					(2,1,0,0)	(28,4)
$\mathbb{Z}_{32}[\omega]$	(1, 0, 0, 2, 0)	(10, 7)	(1, 0, 0, 2, 1)	(11, 8)	(1, 0, 0, 2, 2)	(12, 9)
	(1, 0, 1, 0, 1)	(11, 6)	(1, 0, 1, 0, 2)	(12, 7)	(1, 0, 0, 3, 0)	(14,8)
	(1, 1, 0, 0, 0)	(15, 4)	(1, 0, 1, 1, 0)	(14, 6)	(1, 0, 1, 0, 3)	(13,8)
			(1, 1, 0, 0, 1)	(16, 5)	(1, 0, 1, 1, 1)	(15,7)
			(2, 0, 0, 0, 0)	(26, 3)	(1, 0, 2, 0, 0)	(19,6)
					(1, 1, 0, 0, 2)	(17,6)
					(1, 1, 0, 1, 0)	(21, 5)
					(2, 0, 0, 0, 1)	(27, 4)
$\mathbb{Z}_{64}[\omega]$	(1, 0, 0, 1, 0, 0)	(11, 6)	(1, 0, 0, 0, 2, 0)	(11, 8)	(1, 0, 0, 0, 2, 1)	(12, 9)
			(1, 0, 0, 1, 0, 1)	(12, 7)	(1, 0, 0, 1, 0, 2)	(13, 8)
			(1, 0, 1, 0, 0, 0)	(16, 5)	(1, 0, 0, 1, 1, 0)	(15, 7)
					(1, 0, 1, 0, 0, 1)	(17, 6)
					(1, 1, 0, 0, 0, 0)	(27, 4)
$\mathbb{Z}_{128}[\omega]$			(1, 0, 0, 0, 1, 0, 0)	(12, 7)	(1, 0, 0, 0, 0, 2, 0)	(19, 2)
					(1, 0, 0, 0, 1, 0, 1)	(13, 8)
					(1, 0, 0, 1, 0, 0, 0)	(17, 6)
$\mathbb{Z}_{256}[\omega]$					(1, 0, 0, 0, 0, 1, 0, 0)	(13, 8)