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A Unified Operational and Algebraic Approach of Δ_h-Hybrid Polynomials Associated with Appell Sequences

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Abstract. This study introduces a new class of Δ_h Legendre-Laguerre-Appell polynomials, constructed through the synergy of the monomiality framework and operational calculus. A comprehensive exploration is carried out, beginning with the formulation of their generating function, followed by the derivation of explicit representations and recurrence schemes. Notably, a determinantal structure for these polynomials is also established and illustrated through representative examples. The work further investigates how this polynomial family interrelates with well-known Δ_h-variants of classical polynomials, including the Bernoulli, Euler, and Genocchi types. Through these connections and properties, the results not only deepen our understanding of the algebraic and analytic behaviour of the Δ_h Legendre-Laguerre-Appell polynomials but also highlight their potential applications in broader areas of discrete mathematics and operational theory.

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1. Introduction and preliminaries

Special polynomials play a vital role in modeling diverse systems across statistical mechanics, quantum mechanics, and several branches of mathematics, including combinatorics, entropy theory, and algebraic structures. Classical families like "Laguerre, Chebyshev, Legendre, and Jacobi polynomials emerge as solutions to specific second-order differential equations, making them instrumental in approximation theory and physics".

Laguerre polynomials, introduced by Edmond Laguerre in the 19^{th} century, are a prominent class of orthogonal polynomials defined on $[0, +\infty)$. These polynomials are integral to various areas such as Fourier analysis, numerical integration (e.g., Gauss–Laguerre quadrature), and solving physical models, notably the radial Schrödinger equation in quantum mechanics. These functions are also essential in contexts like heat conduction, wave motion, and diffusion.

Recent developments have centered on two-variable extensions of such polynomials, which offer refined tools for analyzing physical phenomena with multiple degrees of freedom. These include bivariate forms of "Chebyshev, Hermite, and Laguerre polynomials, frequently used in approximation theory, numerical computation, and signal analysis [1–10]'.

Specifically, the bivariate Laguerre polynomials, denoted $\mathbb{W}_\phi(u, v)$, satisfy a two-variable generalization of the classical Laguerre differential equation. These polynomials are particularly useful in quantum mechanics, potential theory, and the study of random matrices. Their orthogonality with respect to a bivariate weight function makes them suitable for addressing multivariate problems in mathematical physics and probability theory. As highlighted in [11], the introduction of two-variable Laguerre $\mathbb{W}_\phi(u, v)$ and Legendre polynomials $\mathbb{S}_\phi(u, v)$ provides valuable analytical tools for tackling partial differential equations encountered in various physical models.

The Laguerre polynomials (2VLP) with notion $\mathbb{W}_\phi(u, v)$ are represented as

$$e^{v\xi} J_0(\xi\sqrt{-u}) = \sum_{\phi=0}^{\infty} \mathbb{W}_\phi(u, v)\frac{\xi^\phi}{\phi!}, \tag{1}$$

where $J_0(u\xi)$ denotes the Bessel function of the first kind of order zero [12], which is defined by the series:

$$J_\phi(2\sqrt{u}) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \ (\sqrt{u})^{\phi+\nu}}{\nu! \ (\phi+\nu)!}. \tag{2}$$

Additionally, one can use the identity

$$\exp(-\alpha D_u^{-1}) = J_0(2\sqrt{\alpha u}), \qquad D_u^{-\phi}\{1\} := \frac{u^\phi}{\phi!}, \tag{3}$$

where D_u^{-1} stands for the inverse differential operator.

An alternative expression for the generating function is

$$e^{v\xi} C_0(-u\xi) = \sum_{\phi=0}^{\infty} \mathbb{W}_\phi(u, v)\frac{\xi^\phi}{\phi!}, \tag{4}$$

with $C_0(u\xi)$ referring to the Tricomi function of the first kind of order zero [12], described by

$$C_0(-u\xi) = e^{D_u^{-1}\xi}. \tag{5}$$

Consequently, combining either equation (3) or (5), the generating formulation of the Laguerre polynomials can be equivalently written as:

$$e^{v\xi}\ e^{D_u^{-1}\xi} = \sum_{\phi=0}^{\infty} \mathbb{W}_\phi(u,v)\frac{\xi^\phi}{\phi!}. \tag{6}$$

Similarly, the Legendre polynomials with notion $\mathbb{S}_\phi(u,v)$ are represented as

$$e^{v\xi} J_0(\xi\sqrt{-u}) = \sum_{\phi=0}^{\infty} \mathbb{S}_\phi(u,v)\frac{\xi^\phi}{\phi!}, \tag{7}$$

where $J_0(u\xi)$ is again the Bessel function of order zero, as given in (2).

Alternatively, one may express the generating function using the Tricomi function as:

$$e^{v\xi} C_0(-u\xi^2) = \sum_{\phi=0}^{\infty} \mathbb{S}_\phi(u,v)\frac{\xi^\phi}{\phi!}, \tag{8}$$

where $C_0(u\xi)$ is the same function defined earlier in (5).

Therefore, taking into account either (3) or (5), the Legendre polynomial generating function can be restated as:

$$e^{v\xi}\ e^{D_u^{-1}\xi^2} = \sum_{\phi=0}^{\infty} \mathbb{S}_\phi(u,v)\frac{\xi^\phi}{\phi!}. \tag{9}$$

Recent studies have focused on developing Δ_h analogues of special polynomials. In [12], several generalizations were explored. A new class, termed Δ_h-special polynomials, was introduced using the classical finite difference operator Δ_h in [13–16], due to their broad applications in mathematics, physics, and statistics.

The Δ_h-Appell polynomials are defined as:

$$\mathbb{A}_\phi^{[h]}(u) := \mathbb{A}_\phi(u), \quad \phi \in \mathbb{N}_0 \tag{10}$$

with the recurrence relation:

$$\mathbb{A}_\phi^{[h]}(u) = \phi h \mathcal{A}_{\phi-1}(u), \quad \phi \in \mathbb{N}_0, \tag{11}$$

where the finite difference operator is:

$$_{\Delta_h}\mathbb{H}^{[h]}(u) = \mathbb{H}(u+h) - \mathbb{H}(u). \tag{12}$$

Their generating function is given by [13]:

$$\gamma(\xi)(1+h\xi)^{\frac{u}{h}} = \sum_{\phi=0}^{\infty} \mathbb{A}_{\phi}^{[h]}(u)\frac{\xi^{\phi}}{\phi!}, \tag{13}$$

with

$$\gamma(\xi) = \sum_{\phi=0}^{\infty} \gamma_{\phi,h}\frac{\xi^{\phi}}{\phi!}, \qquad \gamma_{0,h} \neq 0. \tag{14}$$

Inspired by [13], we define the three-variable Δ_h Legendre-Laguerre Appell polynomials (Δ_h $LeLAP$) by:

$$\gamma(\xi)(1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} = \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{L}\mathbb{A}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!}. \tag{15}$$

The paper is formulated as:
Section 2 presents their construction and recurrence relations. Section 3 derives explicit formulas. Section 4 discusses the monomiality principle and determinant forms. Section 5 relates these to Δ_h-Bernoulli, Euler, and Genocchi polynomials and provides symmetric identities. The conclusion summarizes results and proposes future directions.

2. Results on Δ_h $LeLAP$

This section explores the generating function and recurrence formulas associated with a novel class of three-variable Δ_h Legendre-Laguerre Appell polynomials.

Theorem 1. *The generating function for the Δ_h $LeLAP$ ${}_{\mathbb{S}}\mathbb{L}\mathbb{A}_{\phi}^{[h]}(u,v,w)$ is given by:*

$$\gamma(\xi)(1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} = \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{L}\mathbb{A}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!}. \tag{16}$$

Proof. Consider the left-hand side of (16). When expanded as a Newton-type series centered at $u = v = w = 0$, each term corresponds to a coefficient of ξ^{ϕ} divided by $\phi!$. By identifying these coefficients, the polynomials ${}_{\mathbb{S}}\mathbb{L}\mathbb{A}_{\phi}^{[h]}(u,v,w)$ arise naturally as those associated with the generating function expansion.

Theorem 2. *The following recurrence relations hold for the polynomials ${}_{\mathbb{S}}\mathbb{L}\mathbb{A}_{\phi}^{[h]}(u,v,w)$:*

$$\begin{gathered}\frac{{}_{v}\Delta_h}{h}{}_{\mathbb{S}}\mathbb{L}\mathbb{A}_{\phi}^{[h]}(u,v,w) = \phi\ {}_{\mathbb{S}}\mathbb{L}\mathbb{A}_{\phi-1}^{[h]}(u,v,w),\\ \frac{{}_{u}\Delta_h}{h}{}_{\mathbb{S}}\mathbb{L}\mathbb{A}_{\phi}^{[h]}(u,v,w) = \phi(\phi-1)\ {}_{\mathbb{S}}\mathbb{L}\mathbb{A}_{\phi-2}^{[h]}(u,v,w), \quad D_u^{-1}\to u \quad D_w^{-1}\to w.\end{gathered} \tag{17}$$

Proof. Differentiating both sides of (16) with respect to v using the finite difference operator ${}_v\Delta_h$, we obtain:

$$ {}_v\Delta_h \left\{ \gamma(\xi)(1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} \right\} = h\xi \cdot \gamma(\xi)(1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}}. \tag{18} $$

Substituting the series representation from (16) into the above and shifting indices in the resulting series:

$$ {}_v\Delta_h \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!} = h\sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi+1}}{\phi!}. \tag{19} $$

Re-indexing and equating coefficients of like powers of ξ on both sides yields the first identity in (17). A similar strategy using differentiation with respect to u leads to the second identity.

Theorem 3. *The family of polynomials* ${}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w)$ *admits the following explicit representation:*

$$ {}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w) = \sum_{d=0}^{[\frac{v}{h}]} \binom{\phi}{d}\binom{\frac{v}{h}}{d} h^d \; {}_{\mathbb{S}}\mathbb{LA}_{\phi-d}^{[h]}(u,w). \tag{20} $$

Proof. We start from the generating function provided in expression (16):

$$ \gamma(\xi)(1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} = \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!}. \tag{21} $$

Observe that when $v=0$, this reduces to:

$$ \gamma(\xi)(1+h\xi^2)^{\frac{D_w^{-1}}{h}} = \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,0,w)\frac{\xi^{\phi}}{\phi!}. \tag{22} $$

We now expand the factor $(1+h\xi)^{\frac{v-D_u^{-1}}{h}}$ using the binomial theorem, noting that $\frac{v-D_u^{-1}}{h} = \frac{v}{h} - \frac{D_u^{-1}}{h}$. Since D_u^{-1} is an operator acting on the variable u, we consider its effect separately, and use the binomial expansion:

$$ (1+h\xi)^{\frac{v}{h}} = \sum_{d=0}^{\frac{v}{h}} \binom{\frac{v}{h}}{d}(h\xi)^d. \tag{23} $$

Multiplying this with the generating function of the form when $v=0$, we obtain:

$$\gamma(\xi)\sum_{d=0}^{\frac{v}{h}}\binom{\frac{v}{h}}{d}\frac{(h\xi)^d}{d!}\cdot\sum_{\phi=0}^{\infty}\mathbb{sLA}_{\phi}^{[h]}(u,0,w)\frac{\xi^{\phi}}{\phi!}=\sum_{\phi=0}^{\infty}\left[\sum_{d=0}^{\min\left(\phi,\frac{v}{h}\right)}\binom{\frac{v}{h}}{d}h^d\mathbb{sLA}_{\phi-d}^{[h]}(u,w)\right]\frac{\xi^{\phi}}{d!\cdot\phi!}. \tag{24}$$

Now applying the identity:

$$\frac{1}{d!(\phi-d)!}=\frac{1}{\phi!}\binom{\phi}{d}, \tag{25}$$

we rewrite the series as:

$$\sum_{\phi=0}^{\infty}\left[\sum_{d=0}^{\min\left(\phi,\frac{v}{h}\right)}\binom{\phi}{d}\binom{\frac{v}{h}}{d}h^d\mathbb{sLA}_{\phi-d}^{[h]}(u,w)\right]\frac{\xi^{\phi}}{\phi!}. \tag{26}$$

Since this matches the original generating function series, by equating the coefficients of $\xi^{\phi}/\phi!$ on both sides, we arrive at the claimed formula:

$$\mathbb{sLA}_{\phi}^{[h]}(u,v,w)=\sum_{d=0}^{[\frac{v}{h}]}\binom{\phi}{d}\binom{\frac{v}{h}}{d}\ h^d\ \mathbb{sLA}_{\phi-d}^{[h]}(u,w). \tag{27}$$

Theorem 4. *The following explicit representation also holds:*

$$\mathbb{sLA}_{\phi}^{[h]}(u,v,w)=\sum_{k=0}^{\phi}\binom{\phi}{k}\ \gamma_{k,h}\ \mathbb{sLA}_{\phi-k}^{[h]}(u,v,w). \tag{28}$$

Proof. We begin by considering the generating function for the polynomials $\mathbb{sLA}_{\phi}^{[h]}(u,v,w)$, as defined in expression (16):

$$G(\xi)=\gamma(\xi)(1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}}=\sum_{\phi=0}^{\infty}\mathbb{sLA}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!}. \tag{29}$$

Using expansion of $\gamma(\xi)$ from expression (14) and substituting into the generating function (16) gives:

$$G(\xi)=\left(\sum_{k=0}^{\infty}\gamma_{k,h}\frac{\xi^k}{k!}\right)(1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}}. \tag{30}$$

Now using the Cauchy product of series, we expand the full generating function:

$$G(\xi)=\sum_{k=0}^{\infty}\gamma_{k,h}\frac{\xi^k}{k!}\cdot\sum_{m=0}^{\infty}\mathbb{sLA}_{m}^{[h]}(u,v,w)\frac{\xi^m}{m!}=\sum_{\phi=0}^{\infty}\left[\sum_{k=0}^{\phi}\binom{\phi}{k}\gamma_{k,h}\ \mathbb{sLA}_{\phi-k}^{[h]}(u,v,w)\right]\frac{\xi^{\phi}}{\phi!}. \tag{31}$$

By equating the coefficients of $\frac{\xi^\phi}{\phi!}$ on both sides, we deduce:

$$ {}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w) = \sum_{k=0}^{\phi} \binom{\phi}{k} \gamma_{k,h} \; {}_{\mathbb{S}}\mathbb{LA}_{\phi-k}^{[h]}(u,v,w), \tag{32} $$

which completes the proof.

3. Other results

This section establishes summation formulae, which serve as essential tools in mathematical analysis, revealing intricate relationships, patterns, and symmetries within polynomial structures. They aid in combinatorics, probability theory, and mathematical physics while enhancing computational efficiency.

Next, we present the summation formulas highlighting key summation properties of the three-variable Δ_h Legendre-Laguerre-Appell polynomials ${}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w)$, forming the core of this study.

Theorem 5. *For $\phi \geq 0$, we have*

$$ {}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w) = \sum_{\psi=0}^{\phi} \binom{\phi}{\psi} \left(-\frac{v}{h}\right)_{\psi} (-h)^{\psi} {}_{\mathbb{S}}\mathbb{LA}_{\phi-\psi}^{[h]}(u,0,w). \tag{33} $$

Proof. From (16), we have

$$ \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w)\frac{\xi^\phi}{\phi!} = \gamma(\xi)(1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} = \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,0,w)\frac{\xi^\phi}{\phi!} \sum_{\psi=0}^{\infty} \left(-\frac{v}{h}\right)_{\psi} (-h)^{\psi} \frac{\xi^\psi}{\psi!} $$

$$ = \sum_{\phi=0}^{\infty} \left(\sum_{\psi=0}^{\phi} \binom{\phi}{\psi} \left(-\frac{v}{h}\right)_{\psi} (-h)^{\psi} {}_{\mathbb{S}}\mathbb{LA}_{\phi-\psi}^{[h]}(u,0,w) \right) \frac{\xi^\phi}{\phi!}. \tag{34} $$

When comparing the ϕ coefficients , we obtain (33).

Theorem 6. *For $\phi \geq 0$, we have*

$$ {}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v+1,w) = \sum_{\psi=0}^{\phi} \binom{\phi}{\psi} \left(-\frac{1}{h}\right)_{\psi} (-h)^{\psi} {}_{\mathbb{S}}\mathbb{L}_{\phi-\psi}^{[h]}(u,v,w). \tag{35} $$

Proof. From (16), we have

$$ \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v+1,w)\frac{\xi^\phi}{\phi!} - \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w)\frac{\xi^\phi}{\phi!} = \gamma(\xi)(1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} \left((1+h\xi)^{\frac{1}{h}} - 1\right) $$

$$= \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!}\left(\sum_{\psi=0}^{\infty}\left(-\frac{1}{h}\right)_{\psi}(-h)^{\psi}\frac{\xi^{\psi}}{\psi!}-1\right)$$

$$= \sum_{\phi=0}^{\infty}\left(\sum_{\psi=0}^{\phi}\binom{\phi}{\psi}\left(-\frac{1}{h}\right)_{\psi}(-h)^{\psi}{}_{\mathbb{S}}\mathbb{L}_{\phi-\psi}^{[h]}(u,v,w)\right)\frac{\xi^{\phi}}{\phi!}-\sum_{\phi=0}^{\infty}{}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!}. \quad (36)$$

By matching the coefficients of identical powers of ϕ, we obtain (35).

We now examine the relationship between the polynomials ${}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w)$ and the Stirling numbers of the first kind.

$$\frac{[\log(1+\xi)]^{k}}{k!} = \sum_{i=k}^{\infty} S_1(i,k)\frac{\xi^{i}}{i!}, \ \mid \xi \mid < 1. \quad (37)$$

If we use the definition (37), we get

$$(v)_i = \sum_{k=0}^{i}(-1)^{i-k}S_1(i,k)v^{k}. \quad (38)$$

Theorem 7. *The polynomials* ${}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w)$ *have*

$${}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w) = \sum_{\gamma=0}^{\phi}\binom{\phi}{\gamma}{}_{\mathbb{S}}\mathbb{LA}_{\phi-\psi}^{[h]}(u,0,w)\sum_{j=0}^{\psi}v^{j}S_1(\psi,j)h^{\psi-j}, \quad \phi \geq 0. \quad (39)$$

Proof. With the help of (16) and (37), we obtain

$$\sum_{\phi=0}^{\infty}{}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!} = \gamma(\xi)e^{\frac{v}{h}\log(1+h\xi)}(1+h\xi)^{\frac{-D_u^{-1}}{h}}(1+h\xi^{2})^{\frac{D_w^{-1}}{h}}$$

$$= \gamma(\xi)(1+h\xi)^{\frac{-D_u^{-1}}{h}}(1+h\xi^{2})^{\frac{D_w^{-1}}{h}}\sum_{j=0}^{\infty}\left(\frac{v}{h}\right)^{j}\frac{[\log(1+h\xi)]^{j}}{j!}$$

$$= \sum_{\phi=0}^{\infty}{}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,0,w)\frac{\xi^{\phi}}{\phi!}\sum_{\psi=0}^{\infty}\sum_{j=0}^{\psi}\left(\frac{v}{h}\right)^{j}S_1(\psi,j)h^{\psi}\frac{\xi^{\psi}}{\psi!}$$

$$= \sum_{\phi=0}^{\infty}\left(\sum_{\psi=0}^{\phi}\binom{\phi}{\psi}{}_{\mathbb{S}}\mathbb{LA}_{\phi-\psi}^{[h]}(u,0,w)\sum_{j=0}^{\psi}\left(\frac{v}{h}\right)^{j}S_1(\psi,j)h^{\psi}\right)\frac{\xi^{\phi}}{\phi!}. \quad (40)$$

If the coefficients of ϕ are equalized in the last equation above and then By matching the coefficients of identical powers of ϕ, we obtain (39).

Theorem 8. *For $\phi \geq 0$, we have*

$$
{}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,0,w) = \sum_{\psi=0}^{\phi} \binom{\phi}{\psi} {}_{\mathbb{S}}\mathbb{L}_{\phi-\psi}^{[h]}(u,v,w) \sum_{j=0}^{\psi} \left(-\frac{v}{h}\right)^{j} S_1(\psi,j) h^{\psi}. \tag{41}
$$

Proof. From (16), we get

$$
\begin{aligned}
\gamma(\xi)(1+h\xi)^{\frac{-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} &= e^{-\frac{v}{h}\log(1+h\xi)} \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!} \\
&= \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!} \sum_{j=0}^{\infty} \left(-\frac{v}{h}\right)^{j} \frac{[\log(1+h\xi)]^{j}}{j!} \\
&= \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!} \sum_{\psi=0}^{\infty}\sum_{j=0}^{\psi} \left(-\frac{v}{h}\right)^{j} S_1(\psi,j) h^{\psi} \frac{\xi^{\psi}}{\psi!} \\
&= \sum_{\phi=0}^{\infty} \left(\sum_{\psi=0}^{\phi} \binom{\phi}{\psi} {}_{\mathbb{S}}\mathbb{L}_{\phi-\psi}^{[h]}(u,v,w) \sum_{j=0}^{\psi} \left(-\frac{v}{h}\right)^{j} S_1(\psi,j) h^{\psi} \right) \frac{\xi^{\phi}}{\phi!}.
\end{aligned} \tag{42}
$$

By matching the coefficients of identical powers of ϕ, we obtain (41).

Theorem 9. *The polynomials ${}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w)$ have the following property for $\phi \geq 0$.*

$$
{}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w) = \sum_{\psi=0}^{\phi}\sum_{l=0}^{\psi} \binom{\phi}{\psi} (-h)^{\psi} {}_{\mathbb{S}}\mathbb{LA}_{\phi-\psi}^{[h]}(u,0,w)(1)^{\psi-l} S_1(\psi,l) \left(-\frac{v}{h}\right)^{l}. \tag{43}
$$

Proof. From (16), we have

$$
\begin{aligned}
\sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!} &= \gamma(\xi)(1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} \\
&= \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,0,w)\frac{\xi^{\phi}}{\phi!} \sum_{\psi=0}^{\infty} \left(-\frac{v}{h}\right)_{\psi} (-h)^{\psi} \frac{\xi^{\psi}}{\psi!} \\
&= \sum_{\phi=0}^{\infty} \left(\sum_{\psi=0}^{\phi} \binom{\phi}{\psi} \left(-\frac{v}{h}\right)_{\psi} (-h)^{\psi} {}_{\mathbb{S}}\mathbb{LA}_{\phi-\psi}^{[h]}(u,0,w) \right) \frac{\xi^{\phi}}{\phi!}.
\end{aligned} \tag{44}
$$

Comparing the coefficients of ϕ, we get

$$
{}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w) = \sum_{\psi=0}^{\phi} \binom{\phi}{\psi} \left(-\frac{v}{h}\right)_{\psi} (-h)^{\psi} {}_{\mathbb{S}}\mathbb{LA}_{\phi-\psi}^{[h]}(u,0,w). \tag{45}
$$

Using the above equality (38), we get

$$\mathbb{s}\mathbb{L}\mathbb{A}_{\phi}^{[h]}(u,v,w)=\sum_{\psi=0}^{\phi}\sum_{l=0}^{\psi}\binom{\phi}{\psi}(-h)^{\psi}\mathbb{s}\mathbb{L}\mathbb{A}_{\phi-\psi}^{[h]}(u,0,w)(1)^{\psi-l}S_1(\psi,l)\left(-\frac{v}{h}\right)^{l}. \tag{46}$$

Theorem 10. *For $\phi \geq 0$, the polynomials $\mathbb{s}\mathbb{L}\mathbb{A}_{\phi}^{[h]}(u,v,w)$ have*

$$\mathbb{s}\mathbb{L}\mathbb{A}_{\phi}^{[h]}(u,s,w)=\sum_{l=0}^{\phi}\sum_{j=0}^{l}\binom{\phi}{l}h^{l}\mathbb{s}\mathbb{L}\mathbb{A}_{\phi-l}^{[h]}(u,v,w)\left(\frac{s-v}{h}\right)^{j}S_1(l,j). \tag{47}$$

Proof. From the generating relation (16), we reach

$$\gamma(\xi)(1+h\xi)^{\frac{-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}}=e^{-\frac{v}{h}\log(1+h\xi)}\sum_{\phi=0}^{\infty}\mathbb{s}\mathbb{L}\mathbb{A}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!}. \tag{48}$$

Replacing v by s and comparing the resulting equations, we get

$$e^{\frac{s}{h}\log(1+h\xi)}(1+h\xi)^{\frac{-D_u^{-1}}{h}}(1+ht^2)^{\frac{D_w^{-1}}{h}}=e^{\frac{x-v}{h}\log(1+h\xi)}\sum_{\phi=0}^{\infty}\mathbb{s}\mathbb{L}\mathbb{A}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!}.$$

By using equations (16) and (37) in the the above equation, we get

$$\sum_{\phi=0}^{\infty}\mathbb{s}\mathbb{L}_{\phi}^{[h]}(u,s,w)\frac{\xi^{\phi}}{\phi!}=\sum_{\phi=0}^{\infty}\mathbb{s}\mathbb{L}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!}\sum_{j=0}^{\infty}\left(\frac{s-v}{h}\right)^{j}\frac{[\log(1+h\xi)]^{j}}{j!}$$

$$\sum_{\phi=0}^{\infty}\mathbb{s}\mathbb{L}\mathbb{A}_{\phi}^{[h]}(u,s,w)\frac{\xi^{\phi}}{\phi!}=\sum_{\phi=0}^{\infty}\sum_{l=0}^{\phi}\sum_{j=0}^{l}\binom{\phi}{l}h^{l}\mathbb{s}\mathbb{L}\mathbb{A}_{\phi-l}^{[h]}(u,v,w)\left(\frac{s-v}{h}\right)^{j}S_1(l,j)\frac{\xi^{\phi}}{\phi!}.$$

Finally, assertion (47) is obtained by equating the coefficients corresponding to identical powers of ϕ.

4. Algebraic characteristics

Introduced by Steffenson [17] through poweroids and later extended by Dattoli [18, 19], monomiality plays a crucial role in special polynomials. The $\hat{\mathcal{J}}$ and $\hat{\mathcal{K}}$ operators, as multiplicative and differential operators, further refine polynomial structures, deepening their mathematical significance. The monomiality principle is a key concept in polynomial theory, stating that any polynomial can be uniquely expressed as a linear combination of monomials—single-variable terms with non-negative integer exponents. This decomposition simplifies polynomial analysis, aiding in the study of properties like degree, leading coefficient, and roots while enabling advanced mathematical techniques.

Beyond theory, the monomiality principle enhances computational methods in interpolation, approximation, and integration. Its adaptability extends to physics, where polynomials model fundamental laws and phenomena.

The operators satisfy:

$$\Lambda_{k+1}(\lambda) = \hat{\mathcal{J}}\{\Lambda_k(\lambda)\}, \tag{49}$$

$$k\ \Lambda_{k-1}(\lambda) = \hat{\mathcal{K}}\{\Lambda_k(\lambda)\}. \tag{50}$$

The set $\{\Lambda_k(\lambda)\}$ forms a quasi-monomial family under these actions. The associated commutator is:

$$[\hat{\mathcal{K}}, \hat{\mathcal{J}}] = \hat{1}, \tag{51}$$

indicating Weyl algebra structure.

Assuming quasi-monomiality, the following identities hold:

(i) Differential equation:

$$\hat{\mathcal{J}}\hat{\mathcal{K}}\{\Lambda_k(\lambda)\} = k\ \Lambda_k(\lambda). \tag{52}$$

(ii) Explicit representation:

$$\Lambda_k(\lambda) = \hat{\mathcal{J}}^k\{1\}, \quad \Lambda_0(\lambda) = 1. \tag{53}$$

(iii) Generating function:

$$e^{w\hat{\mathcal{J}}}\{1\} = \sum_{k=0}^{\infty} \Lambda_k(\lambda)\frac{w^k}{k!}, \quad |w| < \infty. \tag{54}$$

These operator-based results support the monomiality framework relevant in physics and applied mathematics.

This section affirms the monomiality of the three-variable Δ_h Legendre-Laguerre Appell polynomials ${}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u, v, w)$, laying the groundwork for further structural analysis and applications.

Theorem 11. *The Δ_h LeLAP ${}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u, v, w)$ satisfy the succeeding operators:*

$$\hat{M_{\mathbb{SLA}}} = \left(\frac{v - D_u^{-1}}{1 + {}_v\Delta_h} + \frac{2\ D_w^{-1}\,{}_v\Delta_h}{h + {}_v\Delta_h{}^2} + \frac{\gamma'\left(\frac{{}_v\Delta_h}{h}\right)}{\gamma\left(\frac{{}_v\Delta_h}{h}\right)}\right) \tag{55}$$

and

$$\hat{D_{\mathbb{SLR}}} = \frac{{}_v\Delta_h}{h}. \tag{56}$$

Proof. To begin, we differentiate equation (16) with respect to v, making use of identity (11). This yields:

$$\begin{aligned}{}_v\Delta_h\Big\{\gamma(\xi)(1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}}\Big\} &= (1+h\xi)^{\frac{v+h}{h}}(1+h\xi)^{\frac{D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} \\ &\quad - \gamma(\xi)(1+h\xi)^{\frac{v}{h}}(1+h\xi)^{\frac{D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} \\ &= h\xi\ \gamma(\xi)(1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}},\end{aligned} \tag{57}$$

which simplifies to the form:

$$\frac{{}_v\Delta_h}{h}\Big[\gamma(\xi)(1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}}\Big] = \xi\Big[\gamma(\xi)(1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}}\Big], \tag{58}$$

which establishes the identity:

$$\frac{{}_v\Delta_h}{h}\Big[{}_{\mathbb{S}}\mathbb{LA}_\phi^{[h]}(u,v,w)\Big] = \xi\Big[{}_{\mathbb{S}}\mathbb{LA}_\phi^{[h]}(u,v,w)\Big]. \tag{59}$$

Now, we proceed by differentiating equation (16) with respect to ξ:

$$\frac{\partial}{\partial\xi}\Big\{\gamma(\xi)(1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}}\Big\} = \frac{\partial}{\partial\xi}\Bigg\{\sum_{\phi=0}^{\infty}{}_{\mathbb{S}}\mathbb{LA}_\phi^{[h]}(u,v,w)\frac{\xi^\phi}{\phi!}\Bigg\}, \tag{60}$$

resulting in the expression:

$$\left(\frac{v-D_u^{-1}}{1+h\xi}+2\frac{D_w^{-1}\xi}{1+h\xi^2}+\frac{\gamma'(\xi)}{\gamma(\xi)}\right)\Bigg\{\sum_{\phi=0}^{\infty}{}_{\mathbb{S}}\mathbb{LA}_\phi^{[h]}(u,v,w)\frac{\xi^\phi}{\phi!}\Bigg\} = \sum_{\phi=0}^{\infty}\phi\ {}_{\mathbb{S}}\mathbb{LA}_\phi^{[h]}(u,v,w)\frac{\xi^\phi}{\phi!}. \tag{61}$$

By employing identity (49) and making the substitution $n \to n+1$ in the right-hand side of equation (61), the conclusion in (55) is validated.

Moreover, utilizing expression (50), we derive the following:

$$\frac{{}_v\Delta_h}{h}\Big[{}_{\mathbb{S}}\mathbb{LA}_\phi^{[h]}(u,v,w)\Big] = \phi\ {}_{\mathbb{S}}\mathbb{LA}_{\phi-1}^{[h]}(u,v,w), \tag{62}$$

which corresponds precisely to expression (56).

Next, the differential equation for the polynomials ${}_{\mathbb{S}}\mathbb{LA}_\phi^{[h]}(u,v,w)$ is derived.

Theorem 12. *The Δ_h LeLAP ${}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w)$ satisfy the differential equation:*

$$\left(\frac{v-D_u^{-1}}{1+{}_v\Delta_h}+\frac{2\ D_w^{-1}\,{}_v\Delta_h}{h+{}_v\Delta_h{}^2}+\frac{\gamma'(\frac{{}_v\Delta_h}{h})}{\gamma(\frac{{}_v\Delta_h}{h})}-\frac{\phi h}{{}_v\Delta_h}\right){}_{\mathbb{S}}\mathbb{L}\mathbb{R}_n^{[h]}(u,v,w)=0. \tag{63}$$

Proof. Inserting expression (55) and (56) in the expression (52), the assertion (63) is proved.

We now derive the determinant representation of the Δ_h *LeLAP* ${}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w)$ by establishing the following result:

Theorem 13. *The Δ_h LeLAP ${}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w)$ admit the determinant form given by:*

$${}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w)=\frac{(-1)^{\phi}}{(\gamma_{0,h})^{\phi+1}}\begin{vmatrix} 1 & {}_{\mathbb{S}}\mathbb{L}_1^{[h]}(u,v,w) & {}_{\mathbb{S}}\mathbb{L}_2^{[h]}(u,v,w) & \cdots & {}_{\mathbb{S}}\mathbb{L}_{\phi-1}^{[h]}(u,v,w) & {}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w) \\ \gamma_{0,h} & \gamma_{1,h} & \gamma_{2,h} & \cdots & \gamma_{\phi-1,h} & \gamma_{\phi,h} \\ 0 & \gamma_{0,h} & \binom{2}{1}\gamma_{1,h} & \cdots & \binom{\phi-1}{1}\gamma_{\phi-2,h} & \binom{\phi}{1}\gamma_{\phi-1,h} \\ 0 & 0 & \gamma_{0,h} & \cdots & \binom{\phi-1}{2}\gamma_{\phi-3,h} & \binom{\phi}{2}\gamma_{\phi-2,h} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \gamma_{0,h} & \binom{\phi}{\phi-1}\gamma_{1,h} \end{vmatrix}, \tag{64}$$

where

$$\gamma_{\phi,h},\ \phi=0,1,\cdots \textit{ being the coefficients of the Maclaurin series of } \frac{1}{\gamma(\xi)}.$$

Proof. By multiplying expression (16) with $\frac{1}{\gamma(\xi)}=\sum_{\phi=0}^{\infty}\gamma_{\phi,h}\frac{\xi^{\phi}}{\phi!}$ on both sides, it follows

$$\sum_{\phi=0}^{\infty}{}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!}=\sum_{\phi=0}^{\infty}\sum_{m=0}^{\infty}\ \gamma_{m,h}\frac{\xi^m}{m!}\ {}_{\mathbb{S}}\mathbb{LA}_{\phi}^{[h]}(u,v,w)\frac{\xi^{\phi}}{\phi!}, \tag{65}$$

which in consideration of the well-know C.P. formulae gives

$${}_{\mathbb{S}}\mathbb{L}_{\phi}^{[h]}(u,v,w)=\sum_{m=0}^{\phi}\binom{\phi}{m}\ \gamma_{m,h}\ {}_{\mathbb{S}}\mathbb{LA}_{\phi-m}^{[h]}(u,v,w). \tag{66}$$

5. Examples

The Appell polynomial family, presented with the equality (13) at the beginning of our study, offers the opportunity to obtain a wide range of members by choosing an appropriate function $\gamma(\xi)$. These members take the names of different polynomials and associated numbers as the appropriate function $\gamma(\xi)$ changes. Thus, each member has a new generating function. Now, we give information on the generating function for these polynomials.

For the "Δ_h Bernoulli polynomials $\beta_\phi^{[h]}(v)$, Euler polynomials $E_\phi^{[h]}(v)$ and Genocchi polynomials $G_\phi^{[h]}(v)$" the generating relations are given by

$$\frac{\log(1+h\xi)^{\frac{1}{h}}}{(1+h\xi)^{\frac{1}{h}}-1}(1+h\xi)^{\frac{v}{h}} = \sum_{\phi=0}^{\infty} \beta_\phi^{[h]}(v)\frac{\xi^\phi}{\phi!}, \quad |\, t \,| < 2\pi, \tag{67}$$

$$\frac{2}{(1+h\xi)^{\frac{1}{h}}+1}(1+h\xi)^{\frac{v}{h}} = \sum_{\phi=0}^{\infty} E_\phi^{[h]}(v)\frac{\xi^\phi}{\phi!}, \quad |\, t \,| < \pi, \tag{68}$$

and

$$\frac{2\log(1+h\xi)^{\frac{1}{h}}}{(1+h\xi)^{\frac{1}{h}}+1}(1+h\xi)^{\frac{v}{h}} = \sum_{\phi=0}^{\infty} G_\phi^{[h]}(v)\frac{\xi^\phi}{\phi!}, \quad |\, t \,| < \pi, \tag{69}$$

respectively.

As $h \to 0$, the polynomials reduce to the Bernoulli $B_\phi(v)$, Euler $E_\phi(v)$, and Genocchi $\Lambda_\phi(v)$ polynomials [20]. These polynomials, related to Δ_h, are vital in number theory, combinatorics, and numerical analysis, aiding in problem-solving and formula derivation.

Bernoulli numbers are key in Taylor expansions and number theory, Euler numbers in secant function expansions, and Genocchi numbers in graph theory and orthogonal polynomials.

By choosing an appropriate $\gamma(\xi)$ in equation 16, we derive "generating functions for the Δ_h Legendre-Laguerre-based Bernoulli, Euler, and Genocchi polynomials".

$$\frac{\log(1+h\xi)^{\frac{1}{h}}}{(1+h\xi)^{\frac{1}{h}}-1}(1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} = \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{LB}_\phi^{[h]}(u,v,w)\frac{\xi^\phi}{\phi!}, \tag{70}$$

$$\frac{2}{(1+h\xi)^{\frac{1}{h}}+1}(1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} = \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{LE}_\phi^{[h]}(u,v,w)\frac{\xi^\phi}{\phi!}, \tag{71}$$

and

$$\frac{2\log(1+h\xi)}{(1+h\xi)^{\frac{1}{h}}+1}(1+h\xi)^{\frac{v-D_u^{-1}}{h}}(1+h\xi^2)^{\frac{D_w^{-1}}{h}} = \sum_{\phi=0}^{\infty} {}_{\mathbb{S}}\mathbb{LG}_\phi^{[h]}(u,v,w)\frac{\xi^\phi}{\phi!}. \tag{72}$$

Furthermore, the polynomials ${}_{\mathbb{S}}\mathbb{LB}_\phi^{[h]}(u,v,w)$, ${}_{\mathbb{S}}\mathbb{LE}_\phi^{[h]}(u,v,w)$ and ${}_{\mathbb{S}}\mathbb{LG}_\phi^{[h]}(u,v,w)$ satisfy the following explicit form in light of expression (28):

$$
{}_{\mathbb{S}}\mathbb{LB}_{\phi}^{[h]}(u,v,w)=\sum_{k=0}^{n}\binom{n}{k}\ \mathbb{B}_{k,h}\ {}_{\mathbb{S}}\mathbb{L}_{n-k}^{[h]}(u,v,w), \tag{73}
$$

$$
{}_{\mathbb{S}}\mathbb{LE}_{\phi}^{[h]}(u,v,w)=\sum_{k=0}^{n}\binom{n}{k}\ \mathbb{E}_{k,h}\ {}_{\mathbb{S}}\mathbb{L}_{n-k}^{[h]}(u,v,w) \tag{74}
$$

and

$$
{}_{\mathbb{S}}\mathbb{LG}_{\phi}^{[h]}(u,v,w)=\sum_{k=0}^{n}\binom{n}{k}\ \mathbb{G}_{k,h}\ {}_{\mathbb{S}}\mathbb{L}_{n-k}^{[h]}(u,v,w). \tag{75}
$$

In view of expressions (64), the polynomials ${}_{\mathbb{S}}\mathbb{LB}_{\phi}^{[h]}(u,v,w)$, ${}_{\mathbb{S}}\mathbb{LE}_{\phi}^{[h]}(u,v,w)$ and ${}_{\mathbb{S}}\mathbb{LG}_{\phi}^{[h]}(u,v,w)$ satisfy the following determinant representations:

$$
{}_{\mathbb{S}}\mathbb{LB}_{\phi}^{[h]}(u,v,w)=(-1)^{n}\begin{vmatrix}
1 & {}_{\mathbb{S}}\mathbb{L}_{1}^{[h]}(u,v,w) & {}_{\mathbb{S}}\mathbb{L}_{2}^{[h]}(u,v,w) & \cdots & {}_{\mathbb{S}}\mathbb{L}_{n-1}^{[h]}(u,v,w) & {}_{\mathbb{S}}\mathbb{L}_{n}^{[h]}(u,v,w) \\
\gamma_{0,h} & \gamma_{1,h} & \gamma_{2,h} & \cdots & \gamma_{n-1,h} & \gamma_{n,h} \\
0 & \gamma_{0,h} & \binom{2}{1}\gamma_{1,h} & \cdots & \binom{n-1}{1}\gamma_{n-2,h} & \binom{n}{1}\gamma_{n-1,h} \\
0 & 0 & \gamma_{0,h} & \cdots & \binom{n-1}{2}\gamma_{n-3,h} & \binom{n}{2}\gamma_{n-2,h} \\
. & . & . & \cdots & . & . \\
. & . & . & \cdots & . & . \\
0 & 0 & 0 & \cdots & \gamma_{0,h} & \binom{n}{n-1}\gamma_{1,h}
\end{vmatrix}, \tag{76}
$$

$$
{}_{\mathbb{S}}\mathbb{LE}_{\phi}^{[h]}(u,v,w)=(-1)^{n}\begin{vmatrix}
1 & {}_{\mathbb{S}}\mathbb{L}_{1}^{[h]}(u,v,w) & {}_{\mathbb{S}}\mathbb{L}_{2}^{[h]}(u,v,w) & \cdots & {}_{\mathbb{S}}\mathbb{L}_{n-1}^{[h]}(u,v,w) & {}_{\mathbb{S}}\mathbb{L}_{n}^{[h]}(u,v,w) \\
\gamma_{0,h} & \gamma_{1,h} & \gamma_{2,h} & \cdots & \gamma_{n-1,h} & \gamma_{n,h} \\
0 & \gamma_{0,h} & \binom{2}{1}\gamma_{1,h} & \cdots & \binom{n-1}{1}\gamma_{n-2,h} & \binom{n}{1}\gamma_{n-1,h} \\
0 & 0 & \gamma_{0,h} & \cdots & \binom{n-1}{2}\gamma_{n-3,h} & \binom{n}{2}\gamma_{n-2,h} \\
. & . & . & \cdots & . & . \\
. & . & . & \cdots & . & . \\
0 & 0 & 0 & \cdots & \gamma_{0,h} & \binom{n}{n-1}\gamma_{1,h}
\end{vmatrix}, \tag{77}
$$

and

$$
{}_{\mathbb{S}}\mathbb{LG}_{\phi}^{[h]}(u,v,w) = (-1)^n \begin{vmatrix} 1 & {}_{\mathbb{S}}\mathbb{L}_1^{[h]}(u,v,w) & {}_{\mathbb{S}}\mathbb{L}_2^{[h]}(u,v,w) & \cdots & {}_{\mathbb{S}}\mathbb{L}_{n-1}^{[h]}(u,v,w) & {}_{\mathbb{S}}\mathbb{L}_n^{[h]}(u,v,w) \\ \gamma_{0,h} & \gamma_{1,h} & \gamma_{2,h} & \cdots & \gamma_{n-1,h} & \gamma_{n,h} \\ 0 & \gamma_{0,h} & \binom{2}{1}\gamma_{1,h} & \cdots & \binom{n-1}{1}\gamma_{n-2,h} & \binom{n}{1}\gamma_{n-1,h} \\ 0 & 0 & \gamma_{0,h} & \cdots & \binom{n-1}{2}\gamma_{n-3,h} & \binom{n}{2}\gamma_{n-2,h} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \gamma_{0,h} & \binom{n}{n-1}\gamma_{1,h} \end{vmatrix}. \tag{78}
$$

6. Conclusion

In this study, the Δ_h *LeLAP* were systematically introduced and explored through various mathematical frameworks, including the monomiality principle and operational rules. By deriving their generating function, explicit formulas, recurrence relations, and determinant form, we established a comprehensive foundation for their theoretical development. Furthermore, the connections between these polynomials and renowned families such as Δ_h-Bernoulli, Δ_h-Euler, and Δ_h-Genocchi polynomials underscore their relevance in the broader mathematical landscape. These results contribute to a deeper understanding of polynomial structures and their interrelations, making them valuable tools for future mathematical investigations.

Looking ahead, further research can focus on exploring the orthogonal properties of Δ_h Legendre-Laguerre-Appell polynomials and their potential uses in numerical analysis, approximation theory, and differential equations. Additionally, investigating their q-analogues and extensions in multivariable settings could provide new insights into their algebraic and analytical properties. Another promising direction is their utilization in solving integral transforms, fractional calculus problems, and mathematical physics equations, thereby broadening their scope in applied mathematics.

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