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A Study of a Coupled System of Fractional Differential Equations with Two Points Integral Boundary Conditions

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Abstract. In this paper, a certain system of fractional differential equations of integral boundary conditions BCs at two points is discussed. The presented coupled fractional system are useful for describing real-world phenomena, such as in physics, biology, and engineering. By utilizing the contraction mapping principle, we demonstrate uniqueness of certain solutions of the given system. Next, we utilize the contraction mapping principle to prove uniqueness of each solution. Further, we address the Hyers-Ulam stability and provide its conditions to show that small changes in the input lead to small changes in the result. Moreover, we provide numerical examples to support and demonstrate our theoretical results.

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Key Words and Phrases: Coupled system, fractional differential equation, fractional derivative, Hyers-Ulam stability

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1. Introduction

Due to their numerous applications in the natural sciences and engineering [1, 2], fractional differential equations (FDEs) have garnered more attention recently [3, 4]. In the last decade, the study of FDEs has garnered significant attention, resulting in a fundamental of articles in many applications across various fields in mathematical modelling of complex systems [5, 6]. Such type of equations have been successfully applied to address real-life problems that often involve integral BCs of blood flow dynamics [7], underground water flow [8], and population dynamics [9, 10]. A coupled system of FDEs with boundary conditions BCs is a complex mathematical model used to describe biological [11], different physical [12, 13], and engineering processes [14, 15]. Unlike regular differential equations, the FDEs use derivatives of non-integer orders to provide a better capture the memory and history of a system. When these equations are coupled, it means they are linked together and must be solved at the same time because the variables affect each other. To ensure that the solutions to real-life problems are realistic, mathematicians used the BCs to show how the solution should behave at the edges of the problem domain [16, 17]. These models are important in fields like materials science since [18] they explain how materials behave over time and in control systems where they manage complex systems. Solving these equations is challenging and requires advanced mathematical methods.

In the last decade, many mathematicians studied a coupled system of FDEs of certain integral BCs. Ntouyas and Obaid [19] discussed a system of FDEs of coupled equations with nonlocal integral boundaries. The Banach's fixed-point theorem and Schauder's alternative are discussed and implemented to investigate both uniqueness and existence of solutions for the proposed coupled FDEs under Riemann-Liouville integral BCs. Ahmed and Ntouyas [20] utilized Schauder's fixed-point and Banach's fixed-point theorems to demonstrate the existence of solutions for coupled fractional DEs. These equations have different cases including those with coupled integral boundary conditions.

The Banach fixed-point theorem and the Leray-Schauder alternative considered important proof techniques of coupled systems of nonlinear FDEs due to their effectiveness in investigating existence and uniqueness of solutions for coupled systems of nonlinear FDEs in those coupled systems that involve complex BCs. The Banach's fixed-point theorem is valuable for proving uniqueness since it ensures a unique fixed point for contraction mappings on a complete metric space. In contrast, the Leray-Schauder is used to prove existence by dealing with compact and continuous mappings that may not be strict contractions. This led to extending the applicability of the fixed-point theory. These theorems provide strong and dependable techniques that have been demonstrated in the literature, making them ideal for nonlinear situations without the need for linearisation. Using fixed-point theorems of Banach and the Leray-Schauder alternative together takes advantage of their strengths which offers a complete approach to handling the complexity of a coupled system of fractional DEs. Such flexibility and dependability make them perfect for the theoretical framework needed to manage the detailed behavior of the coupled system of fractional DEs.

Studying coupled systems of FDEs with integral BCs is crucial for theoretical mathematics and practical applications. These systems arise in various fields such as population dynamics and heat conduction [21, 22]. By considering integral BCs, we capture more realistic behavior and improve accuracy in modeling real-world phenomena [23]. Solutions to these systems provide insights into physical processes and optimal control strategies [24]. This field provides our understanding of complex dynamics and contributes to scientific advancements. Thus, it is crucial to concentrate on studying the coupled systems of FDEs with integral BCs in order to define complete and realistic models, fill in research gaps, and improve our comprehension of stability and control in these systems. In this paper, the coupled system of nonlinear FDEs with two points integral coupled BCs is defined as follows:

$$\begin{cases} {}^{c}D^{\alpha}(t) = u(\tau, \upsilon(\tau), \omega(\tau)), & \tau \in [0, \mathrm{H}], \quad 1 < \alpha \leq 2, \\ {}^{c}D^{\beta}(t) = h(t, \upsilon(\tau), \omega(t)), & \tau \in [0, \mathrm{H}], \quad 1 < \beta \leq 2, \end{cases} \tag{1}$$

where $H > 0$, enhanced with integral BCs in the following form:

$$\begin{cases} \int_0^{\mathrm{H}} \upsilon'^{(t)} dt = \zeta \omega'^{(\mu)}, \quad \int_0^{\mathrm{H}} \omega'(t)\, dt = \eta \upsilon'(\rho), & \mu, \rho \in [0, \mathrm{H}] \\ \upsilon(0) = 0, & \omega(0) = 0, \end{cases} \tag{2}$$

where ${}^{c}D^{i}$ represents the Caputo fractional derivatives of order i, $i{=}\alpha, \beta$, and $u, h{\in}C([0{,}\mathrm{H}]{\times}\mathbb{R}^2{,}\mathbb{R})$ are continuous functions, and ζ, η are real constants.
The model in equations: E. (1) and E. (2) can describe real-life systems where the current state depends on past behavior, like stretchy materials, spreading substances, or controlling machines. It's useful for accurately capturing how these systems behave over time.

2. Preliminaries

In this section, we go over the meanings of fractional derivatives and integrals from [25, 26].
Definition 2.1: The Riemann-Liouville fractional integral of order p for a continuous function h is defined as

$$I^{p} h(t) = \frac{1}{\Gamma(p)} \int_0^t \frac{h(s)}{(t-s)^{1-p}} ds, \qquad p > 0.$$

Definition 2.2: The Caputo fractional derivatives of order p for a continuous function h : $[0, \infty) \to \mathbb{R}$ is given as follows

$${}^{c}D^{p} h(t) = \frac{1}{\Gamma(k-p)} \int_0^t (t-s)^{k-p-1} h^{(k)}(s)\, ds, \qquad k-1 < p < k, \qquad k = [p] + 1.$$

We prove the following auxiliary lemma in order to specify the solution for the problem in E. (1) and E. (2).
Lemma 2.3: Let x, y∈C([0,H],ℝ then the unique solution for the problem for $H > 0$

$$\begin{cases} {}^{c}D^{\alpha}\upsilon(\tau) = x(\tau), & \tau \in [0, \mathrm{H}], \quad 1 < \alpha \leq 2, \\ {}^{c}D^{\beta}\omega(\tau) = y(\tau), & \tau \in [0, \mathrm{H}], \quad 1 < \beta \leq 2, \\ \int_0^{\mathrm{H}} \upsilon'(t)\, dt = \zeta \omega'(\mu), \int_0^{\mathrm{H}} \omega'(t)\, dt = \eta \upsilon'(\rho), \\ \upsilon(0) = 0, \quad \omega(0) = 0, \quad \mu, \rho \in [0, \mathrm{H}] \end{cases} \tag{3}$$

is

$$\upsilon(\tau) = \frac{\tau}{\Lambda}\Bigg(\zeta H \int_0^{\mu} \frac{(\mu - s)^{\beta-2}}{\Gamma(\beta-1)}\, y(s)\, ds - H \int_0^{H}\int_0^{s} \frac{(s-t)^{\alpha-2}}{\Gamma(\alpha-1)}\, x(t)\, dt\, ds$$
$$+ \zeta\eta \int_0^{\rho} \frac{(\rho - s)^{\alpha-2}}{\Gamma(\alpha-1)}\, x(s)\, ds - \zeta \int_0^{H}\int_0^{s} \frac{(s-t)^{\beta-2}}{\Gamma(\beta-1)}\, y(t)\, dt\, ds\Bigg) \tag{4}$$
$$+ \int_0^{\tau} \frac{(\tau - s)^{\alpha-1}}{\Gamma(\alpha)}\, x(s)\, ds.$$

and

$$\omega(\tau) = \frac{\tau}{\Lambda}\Bigg(\zeta\eta \int_0^{\mu} \frac{(\mu - s)^{\beta-2}}{\Gamma(\beta-1)}\, y(s)\, ds - \eta \int_0^{H}\int_0^{s} \frac{(s-t)^{\alpha-2}}{\Gamma(\alpha-1)}\, x(t)\, dt\, ds$$
$$+ H\eta \int_0^{\rho} \frac{(\rho - s)^{\alpha-2}}{\Gamma(\alpha-1)}\, x(s)\, ds - H \int_0^{H}\int_0^{s} \frac{(s-t)^{\beta-2}}{\Gamma(\beta-1)}\, y(t)\, dt\, ds\Bigg) \tag{5}$$
$$+ \int_0^{\tau} \frac{(\tau - s)^{\beta-1}}{\Gamma(\beta)}\, y(s)\, ds.$$

where $\Lambda=\mathrm{H}^2-\zeta\eta\neq 0$.

Proof: The general solution of the coupled system in E. (3) are referred to as follows [26]

$$\upsilon(\tau) = a_0\tau + a_1 + \frac{1}{\Gamma(\alpha)}\int_0^{\tau} (\tau - s)^{\alpha-1} x(s)\, ds, \tag{6}$$

$$\omega(\tau) = b_0\tau + b_1 + \frac{1}{\Gamma(\beta)}\int_0^{\tau} (\tau - s)^{\beta-1} y(s)\, ds, \tag{7}$$

where a_0, a_1, b_0, b_1 are arbitrary constants.
By applying the conditions $\upsilon(0) = 0$ and $\omega(0) = 0$, we obtain a_1=b_1= 0. Here, we have

$$\upsilon'(\tau) = a_0 + \frac{1}{\Gamma(\alpha-1)}\int_0^{\tau} (\tau - s)^{\alpha-2} x(s)\, ds,$$

$$\omega'(\tau) = b_0 + \frac{1}{\Gamma(\beta-1)}\int_0^{\tau} (\tau - s)^{\beta-2} y(s)\, ds.$$

Therefore, in view of the conditions

$$\int_0^{\mathrm{H}} \upsilon'(s) ds = \zeta\omega'(\mu), \int_0^{\mathrm{H}} \omega'(s)\, ds = \eta\upsilon'(\rho),$$

we get

$$a_0\mathrm{H} + \int_0^{\mathrm{H}}\int_0^{s} \frac{(s-t)^{\alpha-2}}{\Gamma(\alpha-1)} x(t) dt ds = \zeta b_0 + \zeta \int_0^{\mu} \frac{(\mu - s)^{\beta-2}}{\Gamma(\beta-1)} y(s)\, ds,$$

and

$$b_0\mathrm{H} + \int_0^{\mathrm{H}}\int_0^{s} \frac{(s-t)^{\beta-2}}{\Gamma(\beta-1)} y(t) dt ds = a_0\eta + \eta \int_0^{\rho} \frac{(\rho - s)^{\alpha-2}}{\Gamma(\alpha-1)} x(s)\, ds,$$

provided that

$$a_0 = \frac{1}{\mathrm{H}}\left(\zeta b_0 + \zeta\int_0^{\mu}\frac{(\mu - s)^{\beta-2}}{\Gamma(\beta-1)}y(s)\,ds - \int_0^{\mathrm{H}}\int_0^{s}\frac{(s-t)^{\alpha-2}}{\Gamma(\alpha-1)}x(t)dtds\right),$$

and

$$b_0 = \frac{1}{\mathrm{H}}\left(a_0\eta + \eta\int_0^{\rho}\frac{(\rho - s)^{\alpha-2}}{\Gamma(\alpha-1)}x(s)\,ds - \int_0^{\mathrm{H}}\int_0^{s}\frac{(s-t)^{\beta-2}}{\Gamma(\beta-1)}y(t)dtds\right).$$

After substituting the value of a_0 into b_0, we derive the final result for such constants in the form

$$\begin{aligned} b_0 = \frac{1}{\Lambda}\Bigg(&\zeta\eta\int_0^{\mu}\frac{(\mu - s)^{\beta-2}}{\Gamma(\beta-1)}\,y(s)\,ds - \eta\int_0^{H}\int_0^{s}\frac{(s-t)^{\alpha-2}}{\Gamma(\alpha-1)}\,x(t)\,dt\,ds \\ &+ H\eta\int_0^{\rho}\frac{(\rho - s)^{\alpha-2}}{\Gamma(\alpha-1)}\,x(s)\,ds - H\int_0^{T}\int_0^{s}\frac{(s-t)^{\beta-2}}{\Gamma(\beta-1)}\,y(t)\,dt\,ds\Bigg). \end{aligned} \tag{8}$$

and

$$\begin{aligned} a_0 = \frac{1}{\Lambda}\Bigg(&\zeta H\int_0^{\mu}\frac{(\mu - s)^{\beta-2}}{\Gamma(\beta-1)}\,y(s)\,ds - H\int_0^{T}\int_0^{s}\frac{(s-t)^{\alpha-2}}{\Gamma(\alpha-1)}\,x(t)\,dt\,ds \\ &+ \zeta\eta\int_0^{\rho}\frac{(\rho - s)^{\alpha-2}}{\Gamma(\alpha-1)}\,x(s)\,ds - \zeta\int_0^{T}\int_0^{s}\frac{(s-t)^{\beta-2}}{\Gamma(\beta-1)}\,y(t)\,dt\,ds\Bigg). \end{aligned} \tag{9}$$

Hence, substituting the values of a_0, a_1, b_0, b_1 in E. (6) and E. (7), gives E. (4) and E. (5). The converse can be established after a straightforward computation. Thus, the proof is therefore completed. ■

In summary, the unique solution for the coupled fractional system defined in E. (3) is obtained by proving Lemma 2.3, where the solution is presented in E. (4) and E. (5).

3. Existence and Uniqueness

This section covers uniqueness and existence of the connected fractional system defined by equations E. (1) and (2). Let's define the space

$$G = \{\upsilon(\tau), \qquad \upsilon(\tau) \in C([0,\mathrm{H}]\},$$

$$Z = \{\omega(\tau), \qquad \omega(\tau) \in C([0,\mathrm{H}]\},$$

with the norm $\|\upsilon\| = \sup_{0\leq\tau\leq\mathrm{H}} |\upsilon(\tau)|$ and $\|\omega\| = \sup_{0\leq\tau\leq\mathrm{H}} |\omega(\tau)|$, respectively. It is obvious that both $(G, \|\ .\|)$ and $(Z, \|\ .\|)$ are considered as Banach spaces. Therefore, the product space $(G\times Z, \|(\upsilon,\omega)\|)$ is also a Banach space as well as $\|(\upsilon,\omega)\| = \|\upsilon\| + \|\omega\|$.
But by considering Lemma (**??**), we establish the operator

$$Q{:}G\times Z\rightarrow G\times Z$$

$$Q\left(\upsilon,\omega\right)\left(\tau\right)=\left(Q_1\left(\upsilon,\omega\right)\left(\tau\right),Q_2\left(\upsilon,\omega\right)\left(\tau\right)\right),$$

where

$$\begin{aligned} Q_1(\upsilon,\omega)(\tau) = \frac{\tau}{\Lambda}\Bigg(& \zeta H \int_0^{\mu} \frac{(\mu-s)^{\beta-2}}{\Gamma(\beta-1)}\, h(s,\upsilon(s),\omega(s))\,ds \\ & - H \int_0^{H}\int_0^{s} \frac{(s-t)^{\alpha-2}}{\Gamma(\alpha-1)}\, u(t,\upsilon(t),\omega(t))\,dt\,ds \\ & + \zeta\eta \int_0^{\rho} \frac{(\rho-s)^{\alpha-2}}{\Gamma(\alpha-1)}\, u(s,\upsilon(s),\omega(s))\,ds \\ & - \zeta \int_0^{H}\int_0^{s} \frac{(s-t)^{\beta-2}}{\Gamma(\beta-1)}\, h(t,\upsilon(t),\omega(t))\,dt\,ds \Bigg) \\ & + \int_0^{\tau} \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}\, u(s,\upsilon(s),\omega(s))\,ds. \end{aligned} \tag{10}$$

and

$$\begin{aligned} Q_2(\upsilon,\omega)(\tau) = \frac{\tau}{\Lambda}\Bigg(& \zeta\eta \int_0^{\mu} \frac{(\mu-s)^{\beta-2}}{\Gamma(\beta-1)}\, h(s,\upsilon(s),\omega(s))\,ds \\ & - \eta \int_0^{H}\int_0^{s} \frac{(s-t)^{\alpha-2}}{\Gamma(\alpha-1)}\, u(t,\upsilon(t),\omega(t))\,dt\,ds \\ & + H\eta \int_0^{\rho} \frac{(\rho-s)^{\alpha-2}}{\Gamma(\alpha-1)}\, u(s,\upsilon(s),\omega(s))\,ds \\ & - H \int_0^{H}\int_0^{s} \frac{(s-t)^{\beta-2}}{\Gamma(\beta-1)}\, h(t,\upsilon(t),\omega(t))\,dt\,ds \Bigg) \\ & + \int_0^{\tau} \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\, h(s,\upsilon(s),\omega(s))\,ds. \end{aligned} \tag{11}$$

Indeed, properties of existence and uniqueness of the solutions of E. (1) and E. (2) may be established by applying the Banach's contraction mapping theory.

Theorem 3.1: Let u, h: $[0,\mathrm{H}]\times\mathbb{R}^2\rightarrow\mathbb{R}$ be jointly continuous functions. Assume that

(i) there exist constants $\Theta,\varpi\in\mathbb{R}$ such that $\forall\ \mathrm{z}_1,\mathrm{z}_2,\mathrm{k}_1,\mathrm{k}_2\in\mathbb{R},\in[0,\mathrm{H}]$, we have

$$\left|\mathrm{u}\left(,\mathrm{z}_1,\mathrm{z}_2\right)-\mathrm{u}\left(,\mathrm{k}_1,\mathrm{k}_2\right)\right|\leq\Theta\left(\left|z_2-z_1\right|+\left|k_2-k_1\right|\right),$$

and

$$\left|h\left(\tau,z_1,z_2\right)-h\left(\tau,k_1,k_2\right)\right|\leq\varpi\left(\left|z_2-z_1\right|+\left|k_2-k_1\right|\right).$$

(i) $\Theta\left(N_1+N_3\right)+\varpi\left(N_2+N_4\right)<1.$

Then the coupled fractional system in the E. (1) and E. (2) has a unique solution on [0,H], where

$$N_1=\frac{\mathrm{H}}{|\Lambda|}\left(\frac{\mathrm{H}^{\alpha+1}}{\Gamma\left(\alpha+1\right)}+\frac{\left|\zeta\eta\right|\rho^{\alpha-1}}{\Gamma\left(\alpha\right)}\right)+\frac{\mathrm{H}^{\alpha}}{\Gamma\left(\alpha+1\right)}\quad,$$

$$N_2 = \frac{\mathrm{H}}{|\Lambda|}\left(\frac{|\zeta|\,\mathrm{H}\mu^{\beta-1}}{\Gamma(\beta)} + \frac{|\zeta|\,\mathrm{H}^{\beta}}{\Gamma(\beta+1)}\right),$$

$$N_3 = \frac{\mathrm{H}}{|\Lambda|}\left(\frac{|\eta|\,\mathrm{H}^{\alpha}}{\Gamma(\alpha+1)} + \frac{\mathrm{H}\,|\eta|\,\rho^{\alpha-1}}{\Gamma(\alpha)}\right),$$

$$N_4 = \frac{\mathrm{H}}{|\Lambda|}\left(\frac{|\zeta\eta|\,\mu^{\beta-1}}{\Gamma(\beta)} + \frac{\mathrm{H}^{\beta+1}}{\Gamma(\beta+1)}\right) + \frac{\mathrm{H}^{\beta}}{\Gamma(\beta+1)}\ .$$

Proof: Let us define
$sup_{0\leq\tau\leq\mathrm{H}}u(\tau,0,0)=u_0<\infty, \sup_{0\leq\tau\leq\mathrm{H}}h(\tau,0,0)=h_0<\infty$ and $\Psi_r=\{(\upsilon,\omega)\in G\times Z\colon \|(\upsilon,\omega)\|\leq r\}$, and $r>0$, such that

$$r \geq \frac{(N_1+N_3)\,u_0 + (N_2+N_4)\,h_0}{1-[\Theta(N_1+N_3)+\varpi(N_2+N_4)]}\ .$$

Hence, we first show that $Q\Psi_r\subseteq\Psi_r$. By our assumption, $(\upsilon,\omega)\in\Psi_r, \tau\in[0,\mathrm{H}]$, we have

$$\begin{aligned}
|u(\tau,\upsilon(\tau),\omega(\tau))| &\leq |u(\tau,\upsilon(\tau),\omega(\tau)) - u(\tau,0,0)| + |u(\tau,0,0)|,\\
&\leq \Theta(|\upsilon(\tau)|+|\omega(\tau)|) + u_0\\
&\leq \Theta(\|\upsilon\|+\|\omega\|) + u\\
&\leq \Theta r + u_0,
\end{aligned}$$

and

$$|h(\tau,\upsilon(\tau),\omega(\tau))| \leq \varpi(|\upsilon(\tau)|+|\omega(\tau)|) + h_0 \leq \varpi(\|\upsilon\|+\|\omega\|) + h_0 \leq \varpi r + h_0,$$

which lead to

$$\begin{aligned}
|Q_1(\upsilon,\omega)(\tau)| \leq{}& \frac{H}{|\Lambda|}\Bigg(|\zeta|H\int_0^{\mu}\frac{(\mu-s)^{\beta-2}}{\Gamma(\beta-1)}ds\,\big(\varpi(\|\upsilon\|+\|\omega\|)+h_0\big)\\
&+ H\int_0^{H}\int_0^{s}\frac{(s-t)^{\alpha-2}}{\Gamma(\alpha-1)}dt\,ds\,\big(\Theta(\|\upsilon\|+\|\omega\|)+u_0\big)\\
&+ |\zeta\eta|\int_0^{\rho}\frac{(\rho-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds\,\big(\Theta(\|\upsilon\|+\|\omega\|)+u_0\big)\\
&+ |\zeta|\int_0^{H}\int_0^{s}\frac{(s-t)^{\beta-2}}{\Gamma(\beta-1)}dt\,ds\,\big(\varpi(\|\upsilon\|+\|\omega\|)+h_0\big)\Bigg)\\
&+ \sup_{0\leq\tau\leq H}\int_0^{\tau}\frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}ds\,\big(\Theta(\|\upsilon\|+\|\omega\|)+u_0\big)\\
\leq{}& \big(\Theta(\|\upsilon\|+\|\omega\|)+u_0\big)\left[\frac{H}{\Lambda}\left(\frac{H^{\alpha+1}}{\Gamma(\alpha+1)}+\frac{|\zeta\eta|\rho^{\alpha-1}}{\Gamma(\alpha)}\right)+\frac{H^{\alpha}}{\Gamma(\alpha+1)}\right]\\
&+ \big(\varpi(\|\upsilon\|+\|\omega\|)+h_0\big)\left[\frac{H}{\Lambda}\left(\frac{|\zeta|H\mu^{\beta-1}}{\Gamma(\beta)}+\frac{|\zeta|H^{\beta}}{\Gamma(\beta+1)}\right)\right]\\
\leq{}& \big(\Theta(\|\upsilon\|+\|\omega\|)+u_0\big)N_1 + \big(\varpi(\|\upsilon\|+\|\omega\|)+h_0\big)N_2\\
\leq{}& (\Theta r+u_0)N_1 + (\varpi r+h_0)N_2.
\end{aligned} \tag{12}$$

In alike manner

$|Q_2(\upsilon,\omega)(\tau)| \leq (\Theta(\|\upsilon\|+\|\omega\|)+u_0)N_3+(\varpi(\|\upsilon\|+\|\omega\|)+h_0)N_4 \leq (\Theta r+u_0)N_3+(\varpi r+h_0)N_4.$

Hence, we derive

$$\|Q_1(\upsilon,\omega)\| \leq (\Theta r+u_0)N_1+(\varpi r+h_0)N_2,$$

and

$$\|Q_2(\upsilon,\omega)\| \leq (\Theta r+u_0)N_3+(\varpi r+h_0)N_4.$$

Consequently, we get

$$\|Q(\upsilon,\omega)\| \leq (\Theta r+u_0)(N_1+N_3)+(\varpi r+h_0)(N_2+N_4) \leq r\ .$$

Therefore, it follows

$\|Q(\upsilon,\omega)\| \leq r$, i.e., $Q\Psi_r \subseteq \Psi_r$.

Now, let $(\upsilon_1,\omega_1),(\upsilon_2,\omega_2) \in G\times Z, \forall\tau\in$ [0,H], $\upsilon,\omega\ \epsilon\mathbb{R}$. Then, we get

$$\begin{aligned}
&|Q_1(\upsilon_1,\omega_1)(\tau)-Q_1(\upsilon_2,\omega_2)(\tau)| \\
&\leq \frac{H}{|\Lambda|}\Bigg(|\zeta|H\int_0^\mu \frac{(\mu-s)^{\beta-2}}{\Gamma(\beta-1)}ds\ \varpi\Big(\|\upsilon_2-\upsilon_1\|+\|\omega_2-\omega_1\|\Big) \\
&\quad + H\int_0^H\int_0^s \frac{(s-t)^{\alpha-2}}{\Gamma(\alpha-1)}dt\,ds\ \Theta\Big(\|\upsilon_2-\upsilon_1\|+\|\omega_2-\omega_1\|\Big) \\
&\quad + |\zeta\eta|\int_0^\rho \frac{(\rho-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds\ \Theta\Big(\|\upsilon_2-\upsilon_1\|+\|\omega_2-\omega_1\|\Big) \\
&\quad + |\zeta|\int_0^H\int_0^s \frac{(s-t)^{\beta-2}}{\Gamma(\beta-1)}dt\,ds\ \varpi\Big(\|\upsilon_2-\upsilon_1\|+\|\omega_2-\omega_1\|\Big)\Bigg) \\
&\quad + \sup_{0\leq\tau\leq H}\int_0^\tau \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}ds\ \Theta\Big(\|\upsilon_2-\upsilon_1\|+\|\omega_2-\omega_1\|\Big).
\end{aligned} \tag{13}$$

That is,

$$\|Q_1(\upsilon_1,\omega_1)-Q_1(\upsilon_2,\omega_2)\| \leq N_1\Theta(\|\upsilon_2-\upsilon_1\|+\|\omega_2-\omega_1\|)+N_2\varpi(\|\upsilon_2-\upsilon_1\|+\|\omega_2-\omega_1\|). \tag{14}$$

Similarly, it follows

$$\|Q_2(\upsilon_1,\omega_1)-Q_2(\upsilon_2,\omega_2)\| \leq N_3\Theta(\|\upsilon_2-\upsilon_1\|+\|\omega_2-\omega_1\|)+N_4\varpi(\|\upsilon_2-\upsilon_1\|+\|\omega_2-\omega_1\|). \tag{15}$$

For $\upsilon,\omega\ \epsilon\mathbb{R}$ we, from (14) and (15), deduced that

$$\|Q(\upsilon_1,\omega_1)-Q(\upsilon_2,\omega_2)\| \leq (\Theta(N_1+N_3)+\varpi(N_2+N_4))(\|\upsilon_2-\upsilon_1\|+\|\omega_2-\omega_1\|),$$

as $\Theta(N_1+N_3)+\varpi(N_2+N_4)<1$. Further, as the operator Q is a contraction operator, the operator Q possesses a unique fixed point according to the Banach's fixed-point theorem. This, indeed, equates to the unique solution of Eqs. (1) and (2). The proof is now complete. ■

The following outcome relies on the Leray-Schauder alternative.

Lemma 3.2: (Leray-Schauder alternative) [27]: Let $F: E\rightarrow E$ be a completely continuous operator and assume $E(F)=\{x\in E : x=\lambda F(x)\ \ for\ some\ 0<\lambda<1\}$. Then, either the set $E(F)$ is unbounded or F has at least one fixed point.

Theorem 3.3: Let $u, h : [0, H] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Assume the following hold true

(i) There exist $_1, _2$,$\phi_1, \phi_2 \geq 0$ where $_1, _2$,ϕ_1, ϕ_2 are real constants and $_0, \phi_0 > 0$ such that $\forall_{\mathrm{i}}, _{\mathrm{i}} \in \mathbb{R}$, $(\mathrm{i} = 1, 2)$,

$$|u(\tau, \upsilon_1, \upsilon_2)| \leq \psi_0 + \psi_1 |\upsilon_1| + \psi_2 |\upsilon_2|,$$

and

$$|h(\tau, \upsilon_1, \upsilon_2)| \leq \phi_0 + \phi_1 |\upsilon_1| + \phi_2 |\upsilon_2|.$$

(i) $(N_1 + N_3)\psi_1 + (N_2 + N_4)\phi_1 < 1$, and $(N_1 + N_3)\psi_2 + (N_2 + N_4)\phi_2 < 1$,

where $N_i, i = 1, 2, 3, 4$ are defined in Theorem 3.1. Then, there exists at least one solution for the coupled fractional system in E. (1) and E. (2).

Proof: The proof of this theorem is divided into two parts:

First Step: Show that $Q: G \times Z \to G \times Z$ is completely continuous, where the continuity of the operator Q holds by the continuity of the functions u, h. Let $R \subseteq G \times Z$ be bounded. Then, there exist positive constants λ_1, λ_2 such that

$$|u(\tau, \upsilon(\tau), \omega(\tau))| \leq \lambda_1, \quad |h(\tau, \upsilon(\tau), \omega(\tau))| \leq \lambda_2, \quad \forall \tau \in [0, \mathrm{H}] \ .$$

Therefore, $\forall (\upsilon, \omega) \in R$, we have

$$|Q_1(\upsilon, \omega)(\tau)| \leq N_1\lambda_1 + N_2\lambda_2.$$

This implies that

$$\|Q_1(\upsilon, \omega)\| \leq N_1\lambda_1 + N_2\lambda_2.$$

Similarly, we get

$$\|Q_2(\upsilon, \omega)\| \leq N_3\lambda_1 + N_4\lambda_2.$$

Therefore, based on the inequalities stated earlier, it can be concluded that the operator Q is uniformly bounded because $\|Q(\upsilon, \omega)\| \leq (N_1 + N_3)\lambda_1 + (N_2 + N_4)\lambda_2$. Furthermore, we prove that the operator Q is equicontinuous. For, assume $\mathrm{k}_1, \mathrm{k}_2 \in [0, \mathrm{H}]$ with $\mathrm{k}_1 < \mathrm{k}_2$. This yields

$$\begin{aligned} |Q_1(\upsilon,\omega)(k_2) - Q_1(\upsilon,\omega)(k_1)| \leq \frac{k_2 - k_1}{|\Lambda|} \Bigg[& |\zeta| H \int_0^{\mu} \frac{(\mu - s)^{\beta-2}}{\Gamma(\beta - 1)} \left| h\big(s, \upsilon(s), \omega(s)\big) \right| ds \\ & + H \int_0^H \int_0^s \frac{(s-t)^{\alpha-2}}{\Gamma(\alpha-1)} \left| u\big(t, \upsilon(t), \omega(t)\big) \right| dt\, ds \\ & + |\zeta\eta| \int_0^{\rho} \frac{(\rho - s)^{\alpha-2}}{\Gamma(\alpha - 1)} \left| u\big(s, \upsilon(s), \omega(s)\big) \right| ds \\ & + |\zeta| \int_0^H \int_0^s \frac{(s-t)^{\beta-2}}{\Gamma(\beta-1)} \left| h\big(t, \upsilon(t), \omega(t)\big) \right| dt\, ds \Bigg]. \end{aligned} \quad (16)$$

$$
\begin{aligned}
|Q_1(\upsilon,\omega)(k_2)-Q_1(\upsilon,\omega)(k_1)| \leq{}& \frac{k_2-k_1}{|\Lambda|}\Bigg(|\zeta|H\lambda_2\int_0^{\mu}\frac{(\mu-s)^{\beta-2}}{\Gamma(\beta-1)}\,ds+H\lambda_1\int_0^{H}\int_0^{s}\frac{(s-t)^{\alpha-2}}{\Gamma(\alpha-1)}\,dt\,ds\\
&+|\zeta\eta|\lambda_1\int_0^{\rho}\frac{(\rho-s)^{\alpha-2}}{\Gamma(\alpha-1)}\,ds+|\zeta|\lambda_2\int_0^{H}\int_0^{s}\frac{(s-t)^{\beta-2}}{\Gamma(\beta-1)}\,dt\,ds\Bigg)\\
&+\lambda_1\left(\int_0^{\tau_2}\frac{|\tau_2-s|^{\alpha-1}}{\Gamma(\alpha)}\,ds+\int_0^{\tau_1}\frac{|\tau_1-s|^{\alpha-1}}{\Gamma(\alpha)}\,ds\right)\\
\leq{}& \frac{k_2-k_1}{|\Delta|}\left(\frac{\lambda_2|\zeta|H\mu^{\beta-1}}{\Gamma(\beta)}+\frac{\lambda_1H^{\alpha+1}}{\Gamma(\alpha+1)}+\frac{\lambda_1|\zeta\eta|\rho^{\alpha-1}}{\Gamma(\alpha)}+\frac{\lambda_2|\zeta|H^{\beta}}{\Gamma(\beta+1)}\right)\\
&+\frac{\lambda_1}{\Gamma(\alpha)}\left(\int_0^{k_1}\left|(k_2-s)^{\alpha-1}-(k_1-s)^{\alpha-1}\right|ds+\int_{k_1}^{k_2}|k_2-s|^{\alpha-1}ds\right).
\end{aligned}
\tag{17}
$$

Now, we can obtain

$$
\begin{aligned}
|Q_1(\upsilon,\omega)(k_2)-Q_1(\upsilon,\omega)(k_1)| \leq{}& \frac{k_2-k_1}{|\Lambda|}\left(\frac{\lambda_2|\zeta|H\,\mu^{\beta-1}}{\Gamma(\beta)}+\frac{\lambda_1H^{\alpha+1}}{\Gamma(\alpha+1)}+\frac{\lambda_1|\zeta\eta|\rho^{\alpha-1}}{\Gamma(\alpha)}+\frac{\lambda_2|\zeta|H^{\beta}}{\Gamma(\beta+1)}\right)\\
&+\frac{\lambda_1}{\Gamma(\alpha+1)}\left(\left|(k_2-k_1)^{\alpha}-k_2^{\alpha}\right|+k_1^{\alpha}+|k_2-k_1|^{\alpha}\right).
\end{aligned}
\tag{18}
$$

Hence, we have $\|Q_1(\upsilon,\omega)(k_2)-Q_1(\upsilon,\omega)(k_1)\|\to 0$ which is independent of υ and ω as $k_2\to k_1$. Also, we can obtain

$$
\begin{aligned}
|Q_2(\upsilon,\omega)(k_2)-Q_2(\upsilon,\omega)(k_1)| \leq{}& \frac{k_2-k_1}{|\Lambda|}\left(\frac{\lambda_2|\zeta\eta|\mu^{\beta-1}}{\Gamma(\beta)}+\frac{\lambda_1|\eta|H^{\alpha}}{\Gamma(\alpha+1)}+\frac{\lambda_1H|\eta|\rho^{\alpha-1}}{\Gamma(\alpha)}+\frac{\lambda_2H^{\beta+1}}{\Gamma(\beta+1)}\right)\\
&+\frac{\lambda_2}{\Gamma(\beta+1)}\left(\left|(k_2-k_1)^{\beta}-k_2^{\beta}\right|+k_1^{\beta}+|k_2-k_1|^{\beta}\right).
\end{aligned}
\tag{19}
$$

which implies that $\|Q_2(\upsilon,\omega)(k_2)-Q_2(\upsilon,\omega)(k_1)\|\to 0$ which is independent of υ and ω as $k_2\to k_1$.

So, the operator $Q(\upsilon,\omega)$ is equicontinuous, and, thus, the operator $Q(\upsilon,\omega)$ is also completely continuous.

Second Step: (Boundedness of operator)

Finally, we show that $B=\{(\upsilon,\omega)\in G\times Z: (\upsilon,\omega)=\lambda Q(\upsilon,\omega), \lambda\in[0,1]\}$ is bounded. Let $(\upsilon,\omega)\in\mathbb{R}$, where $(\upsilon,\omega)=\lambda Q(\upsilon,\omega)$ for any $\tau\in$ [0,H], and

$$\upsilon(\tau)=\lambda Q_1(\upsilon,\omega)(\tau)\ \ and\ \ \omega(\tau)=\lambda Q_2(\upsilon,\omega)(\tau).$$

Therefore,

$$|\upsilon(\tau)|\leq N_1(\psi_0+\psi_1|\upsilon|+\psi_2|\omega|)+N_2(\phi_0+\phi_1|\upsilon|+\phi_2|\omega|),$$

and

$$|\omega(\tau)| \leq N_3(\psi_0 + \psi_1 |\upsilon| + \psi_2 |\omega|) + N_4(\phi_0 + \phi_1 |\upsilon| + \phi_2 |\omega|).$$

Thus, we get

$$\|\upsilon\| \leq N_1(\psi_0 + \psi_1 \|\upsilon\| + \psi_2 \|\omega\|) + N_2(\phi_0 + \phi_1 \|\upsilon\| + \phi_2 \|\omega\|),$$

and

$$\|\omega\| \leq N_3(\psi_0 + \psi_1 \|\upsilon\| + \psi_2 \|\omega\|) + N_4(\phi_0 + \phi_1 \|\upsilon\| + \phi_2 \|\omega\|).$$

This leads to the conclusion that

$$\begin{aligned} &\|\upsilon\| + \|\omega\| \\ &\leq (N_1 + N_3)\psi_0 + (N_2 + N_4)\phi_0 \\ &\quad + ((N_1 + N_3)\psi_1 + (N_2 + N_4)\phi_1)\|\upsilon\| \\ &\quad + ((N_1 + N_3)\psi_2 + (N_2 + N_4)\phi_2)\|\omega\|. \end{aligned}$$

Therefore,

$$\|(\upsilon, \omega)\| \leq \frac{(N_1 + N_3)\psi_0 + (N_2 + N_4)\phi_0}{N_0},$$

where $N_0 = min\{1 - (N_1 + N_3)\psi_1 - (N_2 + N_4)\phi_1, 1 - (N_1 + N_3)\psi_2 - (N_2 + N_4)\phi_2\}$. Thus, this demonstrates that B is bounded and that the operator G has at least one fixed point in accordance to the Leray-Schauder theorem. Therefore, there is at least one solution to both Eqs. (1) and (2) on [0,H]. The proof is complete.

■

In summary, this section discusses and proves the uniqueness of our proposed fractional coupled system in Theorem 3.1. Following this, by applying the Leray-Schauder alternative described in Lemma 3.2 and proving Theorem 3.3, we demonstrate that our proposed fractional coupled system has at least one solution.

4. Hyers-Ulam stability

This section discusses the Hyers-Ulam stability for the boundary value issues in Equations (1) and (2) using an integral form of the general solution, described by

$$\upsilon(\tau) = Q_1(\upsilon, \omega)(\tau), \omega(\tau) = Q_2(\upsilon, \omega)(\tau),$$

where $Q_1$and Q_2 are expressed in Eq.(8) and Eq.(9). Define the following nonlinear operators $M_1, M_2 \in C([0, \mathrm{H}], \mathbb{R}) \times C([0, \mathrm{H}], \mathbb{R}) \to C([0, \mathrm{H}], \mathbb{R})$;

$$\begin{aligned} {}^cD^\alpha \upsilon(\tau) - u(\tau, \upsilon(\tau), \omega(\tau)) &= M_1(\upsilon, \omega)(\tau), \quad \tau \in [0, H], \\ {}^cD^\beta \omega(\tau) - h(\tau, \upsilon(\tau), \omega(\tau)) &= M_2(\upsilon, \omega)(\tau), \quad \tau \in [0, H]. \end{aligned}$$

For some $r_1, r_2 > 0$, we consider the following inequality

$$\|M_1(\upsilon, \omega)\| \leq r_1 \quad \text{and} \quad \|M_2(\upsilon, \omega)\| \leq r_2. \tag{20}$$

Definition 4.1: [27, 28] The coupled fractional system in E.(1) and E.(2) is said to be Hyers-Ulam stable, if there exist $E_{Q_1}, E_{Q_2} > 0$, such that for every solution $(\upsilon^*, \omega^*) \in C([0, H], \mathbb{R}) \times C([0, H], \mathbb{R})$ as given in Eq.(12), there exists a unique solution $(\upsilon, \omega) \in C([0,H],\mathbb{R}) \times C([0,H],\mathbb{R})$ of the system presented in Eq.(1) and Eq.(2) such that

$$\|(\upsilon, \omega) - (\upsilon^*, \omega^*)\| \leq E_{Q_1} r_1 + E_{Q_1} r_2.$$

Theorem 4.2: Let the assumptions of Theorem 3.1 hold. Then, the coupled system in E.(1) and E.(2) is Hyers-Ulam stable.

Proof. Let $(\upsilon, \omega) \in C([0,H],\mathbb{R}) \times C([0,H],\mathbb{R})$ be the solution of the problems in E.(1) and E.(2) satisfying Eq.(8) and Eq.(9) and (υ^*, ω^*) be any solution satisfying E.(20):

$$\begin{aligned} {}^cD^{\alpha} \upsilon^*(\tau) &= u(\tau, \upsilon^*(\tau), \omega^*(\tau)) + M_1(\upsilon^*, \omega^*)(\tau), \quad \tau \in [0, H], \\ {}^cD^{\beta} \omega^*(\tau) &= h(\tau, \upsilon^*(\tau), \omega^*(\tau)) + M_2(\upsilon^*, \omega^*)(\tau), \quad \tau \in [0, H]. \end{aligned}$$

Then, we have

$$\begin{aligned} \upsilon^*(\tau) = Q_1(\upsilon^*, \omega^*)(\tau) + \frac{\tau}{\Lambda}\Bigg[& \zeta H \int_0^{\mu} \frac{(\mu - s)^{\beta-2}}{\Gamma(\beta - 1)} M_2(\upsilon^*, \omega^*)(s)\, ds \\ & - H \int_0^H \int_0^s \frac{(s - t)^{\alpha-2}}{\Gamma(\alpha - 1)} M_1(\upsilon^*, \omega^*)(t)\, dt\, ds \\ & + \zeta\eta \int_0^{\rho} \frac{(\rho - s)^{\alpha-2}}{\Gamma(\alpha - 1)} M_1(\upsilon^*, \omega^*)(s)\, ds \\ & - \zeta \int_0^H \int_0^s \frac{(s - t)^{\beta-2}}{\Gamma(\beta - 1)} M_2(\upsilon^*, \omega^*)(t)\, dt\, ds \Bigg] \\ & + \int_0^{\tau} \frac{(\tau - s)^{\alpha-1}}{\Gamma(\alpha)} M_1(\upsilon^*, \omega^*)(s)\, ds. \end{aligned} \tag{21}$$

It hence follows that

$$\begin{aligned} & |Q_1(\upsilon^*, \omega^*)(\tau) - \upsilon^*(\tau)| \\ & \leq \Bigg| \frac{\tau}{\Lambda} \left(\zeta\eta \int_0^{\mu} \frac{(\mu - s)^{\beta-2}}{\Gamma(\beta - 1)} r_2(s)\, ds - \eta \int_0^{\mathrm{H}} \int_0^s \frac{(s - t)^{\alpha-2}}{\Gamma(\alpha - 1)} r_1(t)\, dt\, ds \right. \\ & \quad \left. + \mathrm{H}\eta \int_0^{\rho} \frac{(\rho - s)^{\alpha-2}}{\Gamma(\alpha - 1)} r_1(s)\, ds - \mathrm{H} \int_0^{\mathrm{H}} \int_0^s \frac{(s - t)^{\beta-2}}{\Gamma(\beta - 1)} r_2(t)\, dt\, ds \right) \\ & \quad + \int_0^{\tau} \frac{(\tau - s)^{\beta-1}}{\Gamma(\beta)} r_2(s)\, ds \Bigg| \\ & \leq \left(\frac{\mathrm{H}}{\Lambda} \left(\frac{\mathrm{H}^{\alpha+1}}{\Gamma(\alpha + 1)} + \frac{|\zeta\eta| \rho^{\alpha-1}}{\Gamma(\alpha)} \right) + \frac{\mathrm{H}^{\alpha}}{\Gamma(\alpha + 1)} \right) r_1 \\ & \quad + \left(\frac{\mathrm{H}}{\Lambda} \left(\frac{|\zeta| \mathrm{H} \mu^{\beta-1}}{\Gamma(\beta)} + \frac{|\zeta| \mathrm{H}^{\beta}}{\Gamma(\beta + 1)} \right) \right) r_2 \\ & \leq N_1 r_1 + N_2 r_2. \end{aligned}$$

Similarly,

$$|Q_2(\upsilon^*,\omega^*)(\tau)-\omega^*(\tau)|\leq \mathrm{N}_3 r_1+\mathrm{N}_4 r_2.$$

We therefore arrive to the following conclusion based on using the operator Q's fixed-point property, given by Eqs. (10) and (11), as follows

$$\begin{aligned}|\upsilon(\tau)-\upsilon^*(\tau)| &= |\upsilon(\tau)-Q_1(\upsilon^*,\omega^*)(\tau)+Q_1(\upsilon^*,\omega^*)(\tau)-\upsilon^*(\tau)|\\ &\leq |Q_1(\upsilon,\omega)(\tau)-Q_1(\upsilon^*,\omega^*)(\tau)|+|M_1(\upsilon^*,\omega^*)(\tau)-\upsilon^*(\tau)|\\ &\leq (N_1\Theta+N_2\varpi)\,\|(\upsilon,\omega)-(\upsilon^*,\omega^*)\|+\mathrm{N}_1 r_1+\mathrm{N}_2 r_2. \end{aligned} \tag{22}$$

Similarly

$$\begin{aligned}|\upsilon(\tau)-\upsilon^*(\tau)| &= |\upsilon(\tau)-Q_2(\upsilon^*,\omega^*)(\tau)+Q_2(\upsilon^*,\omega^*)(\tau)-\upsilon^*(\tau)|\\ &\leq |Q_2(\upsilon,\omega)(\tau)-Q_2(\upsilon^*,\omega^*)(\tau)|+|M_2(\upsilon^*,\omega^*)(\tau)-\upsilon^*(\tau)|\\ &\leq (N_3\Theta+N_4\varpi)\,\|(\upsilon,\omega)-(\upsilon^*,\omega^*)\|+\mathrm{N}_3 r_1+\mathrm{N}_4 r_{2,}. \end{aligned}$$

(**??**)

From E.(22) and E.(**??**) it follows that

$$\|(\upsilon,\omega)-(\upsilon^*,\omega^*)\|\leq(\mathrm{N}_1\Theta+\mathrm{N}_2\varpi+\mathrm{N}_3\Theta+\mathrm{N}_4\varpi)\,\|(\upsilon,\omega)-(\upsilon^*,\omega^*)\|+(\mathrm{N}_1+\mathrm{N}_3)\,r_1+(\mathrm{N}_2+\mathrm{N}_4)\,r_2,$$

$$\|(\upsilon,\omega)-(\upsilon^*,\omega^*)\|\leq\frac{(\mathrm{N}_1+\mathrm{N}_3)\,r_1+(\mathrm{N}_2+\mathrm{N}_4)\,r_2}{1-((\mathrm{N}_1+\mathrm{N}_3)\,\Theta+(\mathrm{N}_2+\mathrm{N}_4)\,\varpi)},$$

where

$E_{Q_1}=\frac{(\mathrm{N}_1+\mathrm{N}_3)}{1-((\mathrm{N}_1+\mathrm{N}_3)\Theta+(\mathrm{N}_2+\mathrm{N}_4)\varpi)}$, and $E_{Q_2}=\frac{(\mathrm{N}_2+\mathrm{N}_4)}{1-((\mathrm{N}_1+\mathrm{N}_3)\Theta+(\mathrm{N}_2+\mathrm{N}_4)\varpi)}$.

Thus we obtain the Hyers-Ulam stability condition.

In summary, the Hyers-Ulam stability condition for the coupled fractional system defined in E.(1) is obtained by proving Lemma 4.2

5. Numerical Examples

Example 1: Examine the following coupled system of FDEs given as

$$\begin{cases} {}^{c}D^{3/2}\upsilon(\tau)=\frac{1}{6\pi\sqrt{81+\tau^2}}\left(\frac{|\upsilon(\tau)|}{3+|\upsilon(\tau)|}+\frac{|\omega(\tau)|}{5+|\upsilon(\tau)|}\right),\\ {}^{c}D^{7/4}\omega(\tau)=\frac{1}{12\pi\sqrt{64+\tau^2}}\left(sin(\upsilon(\tau))+sin(\omega(\tau))\right),\\ \int_0^1\upsilon'(s)\,ds=2\omega'(1)\,,\int_0^1\omega'(s)ds=-\upsilon'(1/2)\,,\\ \upsilon(0)=0,\quad \omega(0)=0, \end{cases} \tag{23}$$

where $\alpha=\frac{3}{2},\beta=\frac{7}{4},\mathrm{H}=1,\zeta=2,\eta=-1,\rho=\frac{1}{2},\mu=1.$

Using the given data, we find that $\Lambda=3, N_1=1.269, N_2=1.1398, N_3=0.5167, N_4=1.554, \Theta=\frac{1}{54\pi}, \varpi=\frac{1}{48\pi}$.

It's clear that u,h are jointly continuous functions and $\Theta(N_1+N_3)+\varpi(N_2+N_4)<1,$ such that

$$\frac{1}{54\pi}(1.269+0.5167)+\frac{1}{48\pi}(1.1398+1.554)=0.0283<1.$$

Therefore, all conditions of Theorem 3.1 are met, and as a result of E.(23) it has a unique solution is in the interval [0,1].

Example 2: Consider the following coupled system of fractional DEs

$$\begin{cases} {}^{c}D^{5/3}\upsilon(\tau) = \frac{1}{80+\tau^4} + \frac{|\upsilon(\tau)|}{120(1+\omega^2(\tau))} + \frac{1}{4\sqrt{2500+\tau^2}}e^{-3\tau}\cos(\omega(\tau)),\ \tau \in [0,1] \\ {}^{c}D^{\frac{6}{5}}\omega(\tau) = \frac{1}{\sqrt{16+\tau^2}}\cos\tau + \frac{1}{150}e^{-3\tau}sin(\omega(\tau)) + \frac{1}{180}\upsilon(\tau),\ \tau \in [0,1] \\ \int_0^1 \upsilon'(s)ds = -3\omega'(1/3), \int_0^1 \omega'(s)ds = \upsilon'(1), \\ \upsilon(0) = 0, \quad \omega(0) = 0, \end{cases} \tag{24}$$

Where $\alpha = \frac{5}{3}, \beta = \frac{6}{5}, \mathrm{H} = 1, \zeta = -3, \eta = 1, \rho = 1, \mu = 1/3$.

Using the given data, we find that $\Lambda = 3, N_1 = 1.269, N_2 = 1.1398, N_3 = 0.5167, N_4 = 1.554, \Theta = \frac{1}{54\pi}, \varpi = \frac{1}{48\pi}$.

It is clear that

$$|u(\tau, \upsilon_1, \upsilon_2)| \leq \frac{1}{80} + \frac{1}{120}\|\upsilon\| + \frac{1}{200}\|\omega\|,$$

and

$$|h(\tau, \upsilon_1, \upsilon_2)| \leq \frac{1}{4} + \frac{1}{180}\|\upsilon\| + \frac{1}{150}\|\omega\|.$$

Thus, $\psi_0 = \frac{1}{80}, \psi_1 = \frac{1}{120}, \psi_2 = \frac{1}{200}, \phi_0 = \frac{1}{4}, \phi_1 = \frac{1}{180}, \phi_2 = \frac{1}{150}$.

We find $(N_1+N_3)\psi_1 + (N_2+N_4)\phi_1 = 0.0298 < 1$ and $(N_1+N_3)\psi_2 + (N_2+N_4)\phi_2 = 0.0269 < 1$. Therefore, according to Theorem 3.3, E.(24) has at least one solution in the interval [0, 1].

6. Conclusion

This article discussed existence, uniqueness, and stability of a coupled system of FDEs with BCs that are two point integral coupled. The fixed-point principle is used to examine the uniqueness of the solutions of the given problem. Then, the existence of solutions for the given coupled system is also verified by using the Leray-Schauder's alternative. Moreover, the Hyers-Ulam stability has been employed to discuss the proposed fractional coupled system's stability. Furthermore, two numerical examples are provided to demonstrate our results. Such an alternative stability theories and availability of solutions for a nonlinear coupled system of three FDEs with nonlocal coupled BCs and non-separated BCs will be our further research.

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