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Hermite-Hadamard Inequalities via Riemann-Liouville Fractional Integrals with Generalized Convex Functions

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Abstract. In this study, novel fractional integral inequalities for twice-differentiable geometrically arithmetically (α, m)-convex functions are presented. The classical Riemann-Liouville fractional integrals are used to obtain several new identities. By employing the above convexity, Hermite-Hadamard type inequalities are investigated using these identities. The main findings of this work extend the existing literature and are derived as special cases.

2020 Mathematics Subject Classifications: 26A51, 26A33, 26D15

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1. Introduction and Preliminaries

Fractional calculus explores the integrals and derivatives of arbitrary real or complex orders. It provides a range of tools that can be used to solve differential equations, integral equations, mathematical physics, engineering and machine learning problems. Within the context of Riemann-Liouville fractional calculus, this discussion focuses on the linear operators of fractional integration and differentiation [1, 2]. It also addresses the existence of mild solutions for fractional-order Caputo derivatives in Banach spaces. Moreover, it delves into the existence of mild solutions for nonlocal impulsive differential inclusions [3, 4], considering Neumann boundary conditions in the form where u and v represent

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typical Caputo fractional derivatives [5]. The investigation further encompasses fractional-order delay differential equations [6], while Volterra-Fredholm integral equations involving the Erdlyi-Kober fractional integral operator are discussed in [7]. The Ulam-Hyers type stability for certain nonlinear differential equations are discussed in [8]. This study also derives several tensorial trapezoid-type inequalities in Hilbert spaces by employing classical analytical identities and exploring the convexity properties of functions involving self-adjoint operators [9]. The applications of non-integer order derivatives can be studied in [10] and numerical solutions for various types of fractional diffusion equations in [11]. Additionally, the applications in the dynamics of particles, fields and media can be explored in the book [12].

The theory of convex functions has experienced rapid development. There are several reasons to study this concept. Firstly, modern analysis involve applications of convex functions directly or indirectly. Secondly, this theory plays a significant role in the construction of many existing inequalities, which play a major role in optimization theory. It has been demonstrated that fractional integral inequalities are one of the best tools for the growth of many fields of pure and applied mathematics.

Research on Hermite-Hadamard inequality has been ongoing since its introduction in 1893. The Hermite-Hadamard inequality for fractional integrals was initially formulated by Sarikaya *et al.* in [13]. Such inequalities were further studied by Shuang *et al.* for geometrically arithmetically (GA) s-convex functions in [14]. The Riemann-Liouville fractional Hermite-Hadamard inequalities for twice differentiable geometrically and arithmetically s-convex functions, along with precise error estimates presented in [15, 16] highlighting the significance of Hermite-Hadamard. The inequality provides bounds on the mean function, assisting in error estimation for trapezoid formulas and the construction of generalized means. Fractional Hermite-Hadamard inequalities involving different types of fractional integrals and various classes of convex functions have attracted significant attention of the scientists. Applications to special means, fractional integral inequalities for differentiable convex maps and midpoint formulas were explored in [17], while the integral inequality of the Ostrowski's type and Hermite-Hadamard integral inequality were investigated in [18, 19]. References such as [20–23] provide insights into convex functions, s-convex functions, r-convex functions, (s, m)-convex, and (s, m) logarithmical functions, respectively. Two classes of new Hermite-Hadamard type inequalities, requiring Riemann-Liouville fractional integrals, were constructed for once differentiable and twice differentiable (s, m) and (α, m)-lgorithmically convex functions in [24, 25]. Hermite-Hadamard inequalities for functions satisfying the $s - e$-condition and along with a method for solving nonlinear integral equations through the Riemann-Liouville fractional operator were studied in [26]. Researchers have also explored inequalities using fractional continuities and differences. Hermite-Hadamard type integral inequalities involving the k-Riemann-Liouville fractional operator for twice-differentiable h-convex functions are investigated in [27]. Hermite-Hadamard type inequalities for h-convex function, as well as those involving $(k - p)$-operator with (α,h-m)-p convexity are discussed in [28]. Iqbal *et al.* investigated Grüss inequalities in [29] by considering the notion of generalized fractional derivative and Samraiz *et al.* examined Hermite-Hadamard inequalities for differentiable functions in [30]. Convex functions in the second sense were studied in [31] and the fractional Hermite-Hadamard inequalities in the second sense were investigated in [32].

2. Preliminaries

This section contains basic definitions and introductory information need to explore the main results. One of the early mathematicians to investigate the gamma function was Leonhard Euler in 1729, as documented in [33]. The gamma function, as defined in [34], can be expressed using the following definition.

Definition 1. *The gamma function, for $R(\lambda) > 0$, is defined by the relation:*

$$\Gamma(\lambda) = \int_0^{+\infty} \mathsf{T}^{\lambda-1} e^{-\mathsf{T}} d\,\mathsf{T}\,.$$

The expression below that defines the complete beta function as presented in reference [35].

Definition 2. *The complete beta function defined for positive real numbers $\bar{a}$ and $\bar{b}$, where $\mathbb{R}(\bar{a}) > 0$ and $\mathbb{R}(\bar{b}) > 0$.*

$$B(\bar{a}, \bar{b}) = \int_0^1 \mathsf{T}^{\bar{a}-1}(1-\mathsf{T})^{\bar{b}-1} d\,\mathsf{T}\,.$$

The incomplete beta function, defined in reference [36], given by the the following definition.

Definition 3. *Let $\lambda \in [0, 1]$ and $\bar{a}, \bar{b} > 0$. Then, the incomplete beta function is defined by*

$$B_\lambda(\bar{a}, \bar{b}) = \int_0^\lambda \mathsf{T}^{\bar{a}-1}(1-\mathsf{T})^{\bar{b}-1} d\,\mathsf{T}\,.$$

The beta function is also related to the gamma function through the following relationship:

$$B_\lambda(\bar{a}, \bar{b}) = B_\lambda(\bar{b}, \bar{a}) = \frac{\Gamma(\bar{a})\Gamma(\bar{b})}{\Gamma(\bar{a}+\bar{b})}.$$

The (α, m)-convexity presented in [37] can be considered as a generalization of ordinary convexity and is defined by the following:

Definition 4. *Let a function $\mathcal{F} : [0, \bar{b}] \to \mathbb{R}$, and $(\alpha, m) \in (0, 1]^2$ if,*

$$\mathcal{F}(\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b}) \leq \mathsf{T}^\alpha \mathcal{F}(\bar{a}) + m(1-\mathsf{T}^\alpha)\mathcal{F}(\bar{b}). \tag{1}$$

is valid for all $\bar{a}, \bar{b} \in [0, \bar{b}]$ and $\mathsf{T} \in [0, 1]$, then we say $\mathcal{F}$ is (α, m)-convex on $[0, \bar{b}]$.
This definition generalized the following convexities
(i) If we substitute $m = 1$ in (1), then we get α-convex function.

$$\mathcal{F}(\mathsf{T}^\alpha \bar{a} + (1-\mathsf{T}^\alpha)\bar{b}) \leq \mathsf{T}^\alpha \mathcal{F}(\bar{a}) + (1-\mathsf{T}^\alpha)\mathcal{F}(\bar{b}).$$

(ii) If we substitute $\alpha = 1$ in (1), then we get m-convex function.

$$F(\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b}) \leq \mathsf{T}F(\bar{a}) + m(1-\mathsf{T})F(\bar{b}).$$

(iii) If we substitute $\alpha = 1$ and $m = 1$ in (1), then we get convex function.

$$F(\mathsf{T}\bar{a} + (1-\mathsf{T})\bar{b}) \leq \mathsf{T}F(\bar{a}) + (1-\mathsf{T})F(\bar{b}).$$

The information related to Riemann-Liouville fractional integrals provided in [1] and [2] can be summarized as follows:

Definition 5. *If $F \in L[\bar{a}, \bar{b}]$, then the Riemann-Liouville fractional integrals of order $\theta \in R^+$, denoted by $\chi^{\theta}_{\bar{a}^+}F$ and $\chi^{\theta}_{\bar{a}^-}F$, represent the left and right sided integrals, respectively i.e.,*

$$(\chi^{\theta}_{\bar{a}^+}F)(\lambda) = \frac{1}{\Gamma(\theta)} \int_{\bar{a}}^{\lambda} (\lambda - \mathsf{T})^{\theta-1} F(\mathsf{T}) d\mathsf{T},\ (0 \leq \bar{a} < \lambda)$$

and

$$(\chi^{\theta}_{\bar{b}^-}F)(\lambda) = \frac{1}{\Gamma(\theta)} \int_{\lambda}^{\bar{b}} (\mathsf{T} - \lambda)^{\theta-1} F(\mathsf{T}) d\mathsf{T},\ (0 \leq \lambda < \bar{b}).$$

In [24], the following lemma discussed.

Lemma 1. *For $\mathsf{T} \in [0, 1]$, we obtain the following*

$$(1-\mathsf{T})^{\omega} \leq 2^{1-\omega} - \mathsf{T}^{\omega},\ for\ \omega \in [0,1],$$
$$(1-\mathsf{T})^{\omega} \geq 2^{1-\omega} - \mathsf{T}^{\omega},\ for\ \omega \in [1,+\infty).$$

The following lemma presented in [38] stated as follows:

Lemma 2. *Let $F : [0, \bar{b}] \to \mathbb{R}$ and $(\alpha, m) \in (0, 1]^2$. If*

$$F(\bar{a}^{\mathsf{T}}\bar{b}^{m(1-\mathsf{T})}) \leq \mathsf{T}^{\alpha}F(\bar{a}) + m(1-\mathsf{T}^{\alpha})F(\bar{b}).$$

is valid for all $\bar{a}, \bar{b} \in [0, \bar{b}]$ and $\mathsf{T} \in [0, 1]$, then we say F is GA (α, m)-convex function on $[0, \bar{b}]$.

Mathematicians are exploring fascinating inequalities and extending their reach through various convexities. Generalizations in inequalities through different convexities showcase the versatility and depth of the mathematical approach. Mathematicians not only refine existing inequalities but also explore new insights into the relationships between mathematical entities. These generalizations provide a broader understanding of mathematical structures, enriching the field with powerful tools for analysis and applications across various domains. The extensions and generalizations of inequalities via different convexities reflect the dynamic and evolving nature of mathematics, leading to a deeper comprehension of fundamental mathematical principles. The main objective of the present work is to establish more generalized forms of Hermite-Hadamard inequalities by using GA (α, m)-convex functions. It is important to mention here that it is not easy and not always possible to introduce inequalities by only changing the convexity. Sometimes, it is a complex procedure to investigate predicted results by using a convexity involving new parameters, as in our work. We hope this idea motivates researchers to explore more generalized inequalities by using appropriate convexities.

3. Fractional Integral Inequalities for Geometrically-Arithmetically (α, m)-Convex Functions

In this section, we derive fundamental identities to explore the Hermite-Hadamard inequalities. The first lemma is presented as follows:

Lemma 3. *Let $\digamma : [\bar{a}, \bar{b}] \to \mathbb{R}$ be a differentiable mapping on $(\bar{a}, \bar{b})$ where $\bar{a} < m\bar{b} \leq \bar{b}$, $m\bar{b} = \mu$ and $\mu \in (\bar{a}, \bar{b}]$. If $\digamma' \in L[\bar{a}, \bar{b}]$, then the following equality for fractional integrals holds.*

$$\begin{aligned} &\frac{\digamma(\bar{a}) + \digamma(\mu)}{2} - \frac{\Gamma(\theta+1)}{2(\mu-\bar{a})^{\theta}} \left[\chi^{\theta}_{\bar{a}^+}\digamma(\mu) + \chi^{\theta}_{\mu^-}\digamma(\bar{a})\right] \\ &= \frac{\mu-\bar{a}}{2} \int_0^1 [(1-\mathsf{T})^{\theta} - \mathsf{T}^{\theta}]\digamma'\left(\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b}\right) d\,\mathsf{T}\,. \end{aligned} \tag{2}$$

Proof. Consider

$$\begin{aligned} I &= \int_0^1 [(1-\mathsf{T})^{\theta} - \mathsf{T}^{\theta}]\digamma'\left(\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b}\right) d\mathsf{T} \\ &= \left[\int_0^1 (1-\mathsf{T})^{\theta}\digamma'\left(\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b}\right) d\mathsf{T}\right] + \left[-\int_0^1 \mathsf{T}^{\theta}\digamma'\left(\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b}\right) d\mathsf{T}\right] \\ &= I_1 + I_2. \end{aligned} \tag{3}$$

Applying integration by parts

$$\begin{aligned} I_1 &= \int_0^1 (1-\mathsf{T})^{\theta}\digamma'\left(\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b}\right) d\mathsf{T} \\ &= \frac{\digamma(\mu)}{\mu-\bar{a}} - \frac{\Gamma(\theta+1)}{(\mu-\bar{a})^{\theta+1}}\chi^{\theta}_{\mu^-}\digamma(\bar{a}). \end{aligned} \tag{4}$$

Similarly

$$\begin{aligned} I_2 &= -\left[\int_0^1 \mathsf{T}^{\theta}\digamma'\left(\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b}\right) d\mathsf{T}\right] \\ &= \frac{\digamma(\bar{a})}{\mu-\bar{a}} - \frac{\Gamma(\theta+1)}{(\mu-\bar{a})^{\theta+1}}\chi^{\theta}_{\bar{a}^+}\digamma(\mu). \end{aligned} \tag{5}$$

Using (4) and (5) in (3), it follows that

$$I = \frac{F(\bar{a}) + F(\mu)}{\mu - \bar{a}} - \frac{\Gamma(\theta+1)}{(\mu-\bar{a})^{\theta+1}}[\chi^{\theta}_{\bar{a}^+}F(\mu) + \chi^{\theta}_{\mu^-}F(\bar{a})]. \tag{6}$$

Thus, by multiplying both sides of (6) by $\frac{(\mu-\bar{a})}{2}$, we obtain the required result.

Remark 1. *By substituting $m = 1$ in Lemma 3, we arrive at [20, Lemma 1.5] i.e.,*

$$\begin{aligned}&\frac{F(\bar{a}) + F(\bar{b})}{2} - \frac{\Gamma(\theta+1)}{2(\bar{b}-\bar{a})^{\theta}}\left[\chi^{\theta}_{\bar{a}^+}F(\bar{b}) + \chi^{\theta}_{\bar{b}^-}F(\bar{a})\right]\\ &= \frac{\bar{b}-\bar{a}}{2}\int_0^1 [(1-\mathsf{T})^{\theta} - \mathsf{T}^{\theta}]F'\left(\mathsf{T}\bar{a} + (1-\mathsf{T})\bar{b}\right) d\,\mathsf{T}\,.\end{aligned}$$

Lemma 4. *Let $F : [\bar{a}, \bar{b}] \to \mathbb{R}$ be a twice differentiable mapping on $(\bar{a}, \bar{b})$ with $\bar{a} < \bar{b}$. If $F'' \in L[\bar{a}, \bar{b}]$, then the following fractional integral equality is true.*

$$\begin{aligned}&\frac{F(\bar{a}) + F(\mu)}{2} - \frac{\Gamma(\theta+1)}{2(\mu-\bar{a})^{\theta}}\left[\chi^{\theta}_{\bar{a}^+}F(\mu) + \chi^{\theta}_{\mu^-}F(\bar{a})\right]\\ &= \frac{(\mu-\bar{a})^2}{2}\int_0^1 \frac{1-(1-\mathsf{T})^{\theta+1} - \mathsf{T}^{\theta+1}}{\theta+1}F''\left(\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b}\right) d\,\mathsf{T}\,.\end{aligned} \tag{7}$$

Proof. By comparing Lemma 3 and (7), we can write

$$\begin{aligned}&\frac{(\mu-\bar{a})}{2}\int_0^1 [(1-\mathsf{T})^{\theta} - \mathsf{T}^{\theta}]F'\left(\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b}\right) d\mathsf{T}\\ &= \frac{(\mu-\bar{a})^2}{2}\int_0^1 \frac{1-(1-\mathsf{T})^{\theta+1} - \mathsf{T}^{\theta+1}}{\theta+1}F''\left(\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b}\right) d\,\mathsf{T}\,.\end{aligned} \tag{8}$$

To prove the required result, we are to prove (8). For this purpose, consider

$$\frac{(\mu-\bar{a})}{2}\int_0^1 [(1-\mathsf{T})^{\theta} - \mathsf{T}^{\theta}]F'\left(\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b}\right) d\,\mathsf{T}\,.$$

Integrating by parts the following, we have

$$\begin{aligned}&\int_0^1 [(1-\mathsf{T})^{\theta} - \mathsf{T}^{\theta}]F'\left(\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b}\right) d\mathsf{T}\\ &= \frac{-F'(\bar{a}) + F'(\mu)}{\theta+1} - (\mu-\bar{a})\int_0^1 \frac{(1-\mathsf{T})^{\theta+1} + \mathsf{T}^{\theta+1}}{\theta+1}F''\left(\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b}\right) d\,\mathsf{T}\,.\end{aligned} \tag{9}$$

Note that

$$F'(\mu) - F'(\bar{a}) = \int_{\bar{a}}^{\mu} F''(\lambda)d\lambda. \tag{10}$$

Substituting $\lambda = \mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b}$, we can write

$$F'(\mu) - F'(\bar{a}) = \int_0^1 F''(\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b})(\mu - \bar{a})d\,\mathsf{T}\,. \tag{11}$$

Submitting (11) in (9), we have

$$\begin{aligned}
&\int_0^1 [(1-\mathsf{T})^\theta - \mathsf{T}^\theta] F'\left(\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b}\right) d\mathsf{T} \\
&= \frac{1}{\theta+1}\int_0^1 F''(\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b})(\mu - \bar{a})d\mathsf{T} \\
&- (\mu - \bar{a})\int_0^1 \frac{(1-\mathsf{T})^{\theta+1} + \mathsf{T}^{\theta+1}}{\theta+1} F''\left(\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b}\right) d\,\mathsf{T}\,.
\end{aligned}$$

Multiplying by $\frac{\mu-\bar{a}}{2}$, we obtain

$$\begin{aligned}
&\frac{\mu-\bar{a}}{2}\int_0^1 [(1-\mathsf{T})^\theta - \mathsf{T}^\theta] F'\left(\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b}\right) d\mathsf{T} \\
&= \frac{(\mu-\bar{a})^2}{2}\int_0^1 \frac{1-(1-\mathsf{T})^{\theta+1} - \mathsf{T}^{\theta+1}}{\theta+1} F''\left(\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b}\right) d\,\mathsf{T}\,.
\end{aligned}$$

Hence, the proof is done.

Remark 2. *By substituting $m = 1$ in Lemma 4, we arrive at [**?** , Lemma 2] i.e.,*

$$\begin{aligned}
&\frac{F(\bar{a}) + F(\bar{b})}{2} - \frac{\Gamma(\theta+1)}{2(\bar{b}-\bar{a})^\theta}\left[\chi^\theta_{\bar{a}^+} F(\bar{b}) + \chi^\theta_{\bar{b}^-} F(\bar{a})\right] \\
&= \frac{(\bar{b}-\bar{a})^2}{2}\int_0^1 \frac{1-(1-\mathsf{T})^{\theta+1} - \mathsf{T}^{\theta+1}}{\theta+1} F''\left(\mathsf{T}\bar{a} + (1-\mathsf{T})\bar{b}\right) d\,\mathsf{T}\,.
\end{aligned}$$

Lemma 5. *Let $F : [\bar{a}, \bar{b}] \to \mathbb{R}$ be a twice differentiable mapping on $(\bar{a}, \bar{b})$ with $\bar{a} < \bar{b}$. If $F'' \in L[\bar{a}, \bar{b}]$, then*

$$\begin{aligned}
&\frac{\Gamma(\theta+1)}{2(\mu-\bar{a})^\theta}\left[\chi^\theta_{\bar{a}^+} F(\mu) + \chi^\theta_{\mu^-} F(\bar{a})\right] - F\left(\frac{\bar{a}+\mu}{2}\right) \\
&= \frac{(\mu-\bar{a})^2}{2}\int_0^1 p_0(\mathsf{T}) F''\left(\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b}\right) d\mathsf{T},
\end{aligned}$$

where

$$p_0(\mathsf{T}) = \begin{cases} \mathsf{T} - \frac{1-(1-\mathsf{T})^{\theta+1} - \mathsf{T}^{\theta+1}}{\theta+1}, & \mathsf{T} \in [0, \frac{1}{2}), \\ 1 - \mathsf{T} - \frac{1-(1-\mathsf{T})^{\theta+1} - \mathsf{T}^{\theta+1}}{\theta+1}, & \mathsf{T} \in [\frac{1}{2}, 1). \end{cases}$$

Proof. Consider

$$
\begin{aligned}
&\int_0^1 p_0(\mathsf{T})F''\left(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b}\right)d\mathsf{T}\\
&=\left[\int_0^{\frac{1}{2}}\left(\mathsf{T}-\frac{1-(1-\mathsf{T})^{\theta+1}-\mathsf{T}^{\theta+1}}{\theta+1}\right)F''(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b})d\mathsf{T}\right.\\
&\left.\int_{\frac{1}{2}}^1\left(1-\mathsf{T}-\frac{1-(1-\mathsf{T})^{\theta+1}-\mathsf{T}^{\theta+1}}{\theta+1}\right)F''(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b})d\mathsf{T}\right]\\
&=\left[\left(\int_0^{\frac{1}{2}}\mathsf{T}F''(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b})d\,\mathsf{T}+\int_{\frac{1}{2}}^1(1-\mathsf{T})F''(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b})d\mathsf{T}\right)\right.\\
&\left.-\int_0^1\frac{1-(1-\mathsf{T})^{\theta+1}-\mathsf{T}^{\theta+1}}{\theta+1}F''(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b})d\mathsf{T}\right].
\end{aligned}
\tag{12}
$$

Let

$$
\begin{aligned}
I&=\int_0^{\frac{1}{2}}\mathsf{T}F''\left(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b}\right)d\,\mathsf{T}+\int_{\frac{1}{2}}^1(1-\mathsf{T})F''\left(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b}\right)d\mathsf{T}\\
&=I_1+I_2.
\end{aligned}
\tag{13}
$$

Integrating by parts, we have

$$
\begin{aligned}
I_1&=\int_0^{\frac{1}{2}}\mathsf{T}F''\left(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b}\right)d\mathsf{T}\\
&=\frac{1}{2(\bar{a}-\mu)}F'\left(\frac{\bar{a}+\mu}{2}\right)-\frac{1}{(\bar{a}-\mu)}\left[F\left(\frac{\bar{a}+\mu}{2}\right)-F(\mu)\right],
\end{aligned}
\tag{14}
$$

and

$$
\begin{aligned}
I_2&=\int_{\frac{1}{2}}^1(1-\mathsf{T})F''\left(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b}\right)d\mathsf{T}\\
&=-\frac{1}{2(\bar{a}-\mu)}F'\left(\frac{\bar{a}+\mu}{2}\right)-\frac{1}{(\bar{a}-\mu)}\left[F(\bar{a})-F\left(\frac{\bar{a}+\mu}{2}\right)\right].
\end{aligned}
\tag{15}
$$

Substituting (14) and (15) in (13), it follows that

$$
I=\frac{F(\bar{a})+F(\mu)}{(\mu-\bar{a})^2}-\frac{2}{(\mu-\bar{a})^2}F\left(\frac{\bar{a}+\mu}{2}\right).
\tag{16}
$$

From (12), we obtain

$$
\int_0^1 p_0(\mathsf{T})F''\left(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b}\right)d\mathsf{T}
$$

$$
\begin{aligned}
&= \frac{\digamma(\bar{a}) + \digamma(\mu)}{(\mu - \bar{a})^2} - \frac{2}{(\mu - \bar{a})^2}\digamma\left(\frac{\bar{a} + \mu}{2}\right) \\
&- \int_0^1 \frac{1 - (1 - \mathsf{T})^{\theta+1} - \mathsf{T}^{\theta+1}}{\Upsilon(\theta + 1)} \digamma''(\mathsf{T}\bar{a} + m(1 - \mathsf{T})\bar{b}) d\,\mathsf{T}\,. \qquad (17)
\end{aligned}
$$

Thus, by multiplying both sides of (17) by $\frac{(\mu-\bar{a})^2}{2}$, we have

$$
\begin{aligned}
&\frac{(\mu - \bar{a})^2}{2} \int_0^1 p_0(\mathsf{T})\digamma''\left(\mathsf{T}\bar{a} + m(1 - \mathsf{T})\bar{b}\right) d\mathsf{T} \\
&= \frac{\digamma(\bar{a}) + \digamma(\mu)}{2} - \digamma\left(\frac{\bar{a} + \mu}{2}\right) \\
&- \frac{(\mu - \bar{a})^2}{2} \int_0^1 \frac{1 - (1 - \mathsf{T})^{\theta+1} - \mathsf{T}^{\theta+1}}{\theta + 1} \digamma''\left(\mathsf{T}\bar{a} + m(1 - \mathsf{T})\bar{b}\right) d\,\mathsf{T}\,. \qquad (18)
\end{aligned}
$$

On the other hand, by (7), we obtain

$$
\begin{aligned}
&\frac{\digamma(\bar{a}) + \digamma(\mu)}{2} - \frac{\Gamma(\theta + 1)}{2(\mu - \bar{a})^\theta}\left[\chi^\theta_{\bar{a}+}\digamma(\mu) + \chi^\theta_{\mu-}\digamma(\bar{a})\right] \\
&= \frac{(\mu - \bar{a})^2}{2} \int_0^1 \frac{1 - (1 - \mathsf{T})^{\theta+1} - \mathsf{T}^{\theta+1}}{\theta + 1} \digamma''\left(\mathsf{T}\bar{a} + m(1 - \mathsf{T})\bar{b}\right) d\,\mathsf{T}\,. \qquad (19)
\end{aligned}
$$

Combining (18) and (19), we have obtained the conclusion of the proof.

Remark 3. *By substituting $m = 1$ in Lemma 5, we arrive at [20, Lemma 2.1] i.e.,*

$$
\begin{aligned}
&\frac{\Gamma(\theta + 1)}{2(\bar{b} - \bar{a})^\theta}\left[\chi^\theta_{\bar{a}+}\digamma(\bar{b}) + \chi^\theta_{\bar{b}-}\digamma(\bar{a})\right] - \digamma\left(\frac{\bar{a} + \bar{b}}{2}\right) \\
&= \frac{(\bar{b} - \bar{a})^2}{2} \int_0^1 p_0(\mathsf{T})\digamma''\left(\mathsf{T}\bar{a} + (1 - \mathsf{T})\bar{b}\right) d\,\mathsf{T}\,.
\end{aligned}
$$

Lemma 6. *Let $\digamma : [\bar{a}, \bar{b}] \to \mathbb{R}$ be a twice differentiable mapping on $(\bar{a}, \bar{b})$ with $\bar{a} < \bar{b}$. If $\Upsilon > 0$, $\digamma'' \in L[\bar{a}, \bar{b}]$, then*

$$
\begin{aligned}
&\frac{\digamma(\bar{a}) + \digamma(\mu)}{\Upsilon(\Upsilon + 1)} + \frac{2}{\Upsilon + 1}\digamma\left(\frac{\bar{a} + \mu}{2}\right) - \frac{\Gamma(\theta + 1)}{\Upsilon(\mu - \bar{a})^\theta}\left[\chi^\theta_{\bar{a}+}\digamma(\mu) + \chi^\theta_{\mu-}\digamma(\bar{a})\right] \\
&= (\mu - \bar{a})^2 \int_0^1 q_0(\mathsf{T})\digamma''\left(\mathsf{T}\bar{a} + m(1 - \mathsf{T})\bar{b}\right) d\mathsf{T},
\end{aligned}
$$

where

$$
q_0(\mathsf{T}) = \begin{cases} \frac{1-(1-\mathsf{T})^{\theta+1}-\mathsf{T}^{\theta+1}}{\Upsilon(\theta+1)} - \frac{\mathsf{T}}{\Upsilon+1}, & \mathsf{T} \in [0, \frac{1}{2}), \\ \frac{1-(1-\mathsf{T})^{\theta+1}-\mathsf{T}^{\theta+1}}{\Upsilon(\theta+1)} - \frac{1-\mathsf{T}}{\Upsilon+1}, & \mathsf{T} \in [\frac{1}{2}, 1). \end{cases}
$$

Proof. Consider

$$\begin{aligned}
&\int_0^1 q_0(\mathsf{T})F''\left(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b}\right)d\mathsf{T}\\
&=\left[\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mathsf{T})^{\theta+1}-\mathsf{T}^{\theta+1}}{\Upsilon(\theta+1)}-\frac{\mathsf{T}}{\Upsilon+1}\right)F''(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b})d\mathsf{T}\right.\\
&\left.\int_{\frac{1}{2}}^1\left(\frac{1-(1-\mathsf{T})^{\theta+1}-\mathsf{T}^{\theta+1}}{\Upsilon(\theta+1)}-\frac{1-\mathsf{T}}{\Upsilon+1}\right)F''(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b})d\mathsf{T}\right]\\
&=\left[\int_0^1\frac{1-(1-\mathsf{T})^{\theta+1}-\mathsf{T}^{\theta+1}}{\Upsilon(\theta+1)}F''(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b})d\mathsf{T}\right.\\
&\left.-\frac{1}{\Upsilon+1}\left(\int_0^{\frac{1}{2}}\mathsf{T}F''(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b})d\,\mathsf{T}+\int_{\frac{1}{2}}^1(1-\mathsf{T})F''(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b})d\mathsf{T}\right)\right].
\end{aligned}\tag{20}$$

By using (16), we obtain

$$\begin{aligned}
&\int_0^1 q_0(\mathsf{T})F''\left(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b}\right)d\mathsf{T}\\
&=\int_0^1\frac{1-(1-\mathsf{T})^{\theta+1}-\mathsf{T}^{\theta+1}}{\Upsilon(\theta+1)}F''(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b})d\mathsf{T}\\
&-\frac{1}{\Upsilon+1}\left(\frac{F(\bar{a})+F(\mu)}{(\mu-\bar{a})^2}-\frac{2}{(\mu-\bar{a})^2}F\left(\frac{\bar{a}+\mu}{2}\right)\right).
\end{aligned}\tag{21}$$

Multiplying both sides of (21) by $(\mu-\bar{a})^2$, we get

$$\begin{aligned}
&(\mu-\bar{a})^2\int_0^1 q_0(\mathsf{T})F''\left(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b}\right)d\mathsf{T}\\
&=(\mu-\bar{a})^2\int_0^1\frac{1-(1-\mathsf{T})^{\theta+1}-\mathsf{T}^{\theta+1}}{\Upsilon(\theta+1)}F''(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b})d\mathsf{T}\\
&-\frac{F(\bar{a})+F(\mu)}{(\Upsilon+1)}+\frac{2}{\Upsilon+1}F\left(\frac{\bar{a}+\mu}{2}\right).
\end{aligned}\tag{22}$$

By multiplying $\frac{1}{\Upsilon}$ on both sides of Lemma 4, we can write

$$\begin{aligned}
&\frac{F(\bar{a})+F(\mu)}{\Upsilon}-\frac{\Gamma(\theta+1)}{\Upsilon(\mu-\bar{a})^{\theta}}\left[\chi^{\theta}_{\bar{a}^+}F(\mu)+\chi^{\theta}_{\mu^-}F(\bar{a})\right]\\
&=(\mu-\bar{a})^2\int_0^1\frac{1-(1-\mathsf{T})^{\theta}-\mathsf{T}^{\theta}}{\Upsilon(\theta+1)}F'\left(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b}\right)d\,\mathsf{T}\,.
\end{aligned}\tag{23}$$

Combining (22) and (23), we have obtained the conclusion of the proof.

Remark 4. *By substituting $m = 1$ in Lemma 6, we arrive at [20, Lemma 2.2] i.e.,*

$$\frac{F(\bar{a}) + F(\bar{b})}{\Upsilon(\Upsilon+1)} + \frac{2}{\Upsilon+1} F\left(\frac{\bar{a}+\bar{b}}{2}\right) - \frac{\Gamma(\theta+1)}{\Upsilon(\bar{b}-\bar{a})^{\theta}} \left[\chi^{\theta}_{\bar{a}^+} F(\bar{a}) + \chi^{\theta}_{\bar{b}^-} F(\bar{b})\right]$$
$$= (\bar{b}-\bar{a})^2 \int_0^1 q_0(\mathsf{T}) F''\left(\mathsf{T}\bar{a} + (1-\mathsf{T})\bar{b}\right) d\,\mathsf{T}\,.$$

4. Hermite-Hadamard Inequalities for GA (α, m)-Convex Functions

This section is dedicated to explore Hermite-Hadamard type inequalities by using the results proved in the previous section. To establish the inequalities, we need the following lemma.

Lemma 7. *For $\mathsf{T} \in [0,1]$, $\bar{a}, \bar{b} > 0$, we have*

$$\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b} \leq (\bar{a}^{\mathsf{T}} \bar{b}^{m(1-\mathsf{T})}).$$

Theorem 1. *Let $F : [\bar{a}, \bar{b}] \longrightarrow \mathbb{R}$ be a twice differentiable function such that $|F''|$ is Lebesgue integerable, increasing and GA (α, m)-convex function on $[\bar{a}, \bar{b}]$. Then for given parameters $\theta \in (0, +\infty)$ and $(\alpha, m) \in (0,1]^2$, where $0 \leq \bar{a} < \bar{b}$ the following inequality*

$$\left|\frac{F(\bar{a}) + F(\mu)}{2} - \frac{\Gamma(\theta+1)}{2(\mu-\bar{a})^{\theta}} \left[\chi^{\theta}_{\bar{a}^+} F(\mu) + \chi^{\theta}_{\mu^-} F(\bar{a})\right]\right|$$
$$\leq \frac{(\mu-\bar{a})^2}{2(\theta+1)} \left[|F''(\bar{a})| \left(\frac{1}{\alpha+1} - \frac{1}{\theta+\alpha+2}\right) + m|F''(\bar{b})| \left(\frac{\alpha}{\alpha+1} + \frac{1}{\theta+\alpha+2}\right)\right] \tag{24}$$

holds true.

Proof. By using Lemma 2, Lemma 4 and Lemma 7, we have

$$\left|\frac{F(\bar{a}) + F(\mu)}{2} - \frac{\Gamma(\theta+1)}{2(\mu-\bar{a})^{\theta}} \left[\chi^{\theta}_{\bar{a}^+} F(\mu) + \chi^{\theta}_{\mu^-} F(\bar{a})\right]\right|$$
$$\leq \frac{(\mu-\bar{a})^2}{2} \int_0^1 \left|\frac{1-(1-\mathsf{T})^{\theta+1} - \mathsf{T}^{\theta+1}}{\theta+1}\right| \left|F''\left(\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b}\right)\right| d\mathsf{T}$$
$$\leq \frac{(\mu-\bar{a})^2}{2} \int_0^1 \left|\frac{1-(1-\mathsf{T})^{\theta+1} - \mathsf{T}^{\theta+1}}{\theta+1}\right| \left|F''\left(\bar{a}^{\mathsf{T}} \bar{b}^{m(1-\mathsf{T})}\right)\right| d\mathsf{T}$$
$$\leq \frac{(\mu-\bar{a})^2}{2(\theta+1)} \int_0^1 \left|1-(1-\mathsf{T})^{\theta+1} - \mathsf{T}^{\theta+1}\right| \left[\mathsf{T}^{\alpha} \left|F''(\bar{a})\right| + m(1-\mathsf{T}^{\alpha}) \left|F''(\bar{b})\right|\right] d\mathsf{T}$$
$$\leq \frac{(\mu-\bar{a})^2}{2(\theta+1)} \left[|F''(\bar{a})| \left(\frac{1}{\alpha+1} - \frac{1}{\theta+\alpha+2}\right) + m|F''(\bar{b})| \left(\frac{\alpha}{\alpha+1} + \frac{1}{\theta+\alpha+2}\right)\right].$$

Hence, the proof is done.

Example 1. *Let $F(\mathsf{T}) = \mathsf{T}^3$ and by taking the values of the parameters*

- $\bar{a} = 0, \mu = 1$ *and* $\bar{b} = 2$
- $\alpha = 0.5, m = 0.5$ *and* $\theta = 1$

The LHS of the inequality

$$\left|\frac{F(\bar{a}) + F(\mu)}{2} - \frac{\Gamma(\theta+1)}{2(\mu-\bar{a})^{\theta}}\left[\chi^{\theta}_{\bar{a}^+}F(\mu) + \chi^{\theta}_{\mu^-}F(\bar{a})\right]\right|$$

$$\left|\frac{(0)^3 + (1)^3}{2} - \frac{\Gamma(1+1)}{2(1-0)^1}\left[\chi^{1}_{0^+}F(1) + \chi^{1}_{1^-}F(0)\right]\right|$$

$$\left|\frac{(0)^3 + (1)^3}{2} - \frac{\Gamma(1+1)}{2(1-0)^1}\left[\frac{1}{\Gamma(1)}\int_0^1 (1-\mathsf{T})^{1-1}(\mathsf{T})^3 d\,\mathsf{T} + \frac{1}{\Gamma(1)}\int_1^2 (\mathsf{T}-0)^{1-1}(0)^3 d\mathsf{T}\right]\right|$$

$$\left|\frac{1}{2} - \frac{\Gamma(2)}{2}\left[\frac{1}{\Gamma(1)}\left[\frac{\mathsf{T}^4}{4}\right]_0^1 + 0\right]\right|$$

$$\left|\frac{1}{2} - \frac{1}{2} \times \frac{1}{4}\right|$$

$$|0.5 - 0.125| = 0.375.$$

The RHS of the inequality

$$\frac{(\mu-\bar{a})^2}{2(\theta+1)}\left[|F''(\bar{a})|\left(\frac{1}{\alpha+1} - \frac{1}{\theta+\alpha+2}\right) + m|F''(\bar{b})|\left(\frac{\alpha}{\alpha+1} + \frac{1}{\theta+\alpha+2}\right)\right]$$

$$\frac{(1-0)^2}{2(1+1)}\left[|6(0)|\left(\frac{1}{0.5+1} - \frac{1}{1+0.5+2}\right) + 0.5|6(2)|\left(\frac{0.5}{0.5+1} + \frac{1}{1+0.5+2}\right)\right].$$

To simplify, $\frac{0.5}{1.5} = \frac{1}{3}$, $\frac{1}{0.5} = \frac{1}{2}$ *and* $\frac{1}{3.5} = \frac{2}{7}$

$$\frac{1}{4}\left[0.5(12)\left(\frac{0.5}{1.5} + \frac{1}{3.5}\right)\right] \approx 0.9286$$

Hence, the chosen values satisfy the inequality $0.375 \leq 0.9286$*. This example confirms the validity of the fractional Hermite-Hadamard-type inequality and demonstrates its usefulness in numerical analysis for estimating error bounds in numerical integration, as well as in inequality theory, fractional differential equations and mathematical modeling.*

Theorem 2. *Let* $F : [\bar{a}, \bar{b}] \longrightarrow \mathbb{R}$ *be a twice differentiable function on* $[\bar{a}, \bar{b}]$ *and* $1 < \lambda_2 < +\infty$*. If* $|F''|^{\lambda_2}$ *is Lebesgue integerable, increasing and GA* (α, m)*-convex on* $[\bar{a}, \bar{b}]$*. Then for given parameters* $\theta \in (0, +\infty)$ *and* $(\alpha, m) \in (0, 1]^2$*, where* $0 \leq \bar{a} < \bar{b}$ *the following inequality*

$$\left|\frac{F(\bar{a}) + F(\mu)}{2} - \frac{\Gamma(\theta+1)}{2(\mu-\bar{a})^{\theta}}\left[\chi^{\theta}_{\bar{a}^+}F(\mu) + \chi^{\theta}_{\mu^-}F(\bar{a})\right]\right|$$

$$\leq \frac{(\mu-\bar{a})^2 \max(1-2^{1-\theta}, 2^{1-\theta}-1)}{2(\theta+1)} \left(\frac{|F''(\bar{a})|^{\lambda_2} + m\alpha |F''(\bar{b})|^{\lambda_2}}{\alpha+1} \right)^{\frac{1}{\lambda_2}} \tag{25}$$

holds true.

Proof. The proof of this result is split into two situations presented as follows.
Case(i) Let $\theta \in (0,1)$. By utilizing Lemma 1, 2, 4, 7 and applying Hölder's inequality, we have

$$\begin{aligned}
&\left| \frac{F(\bar{a}) + F(\mu)}{2} - \frac{\Gamma(\theta+1)}{2(\mu-\bar{a})^\theta} \left[\chi^\theta_{\bar{a}+} F(\bar{b}) + \chi^\theta_{\bar{b}-} F(\bar{a}) \right] \right| \\
&\leq \frac{(\mu-\bar{a})^2}{2} \int_0^1 \left| \frac{1-(1-\mathsf{T})^{\theta+1} - \mathsf{T}^{\theta+1}}{\theta+1} \right| |F''(\mathsf{T}\bar{a} + (1-\mathsf{T})\bar{b})| d\mathsf{T} \\
&\leq \frac{(\mu-\bar{a})^2}{2} \left(\int_0^1 \left| \frac{1-(1-\mathsf{T})^{\theta+1} - \mathsf{T}^{\theta+1}}{\theta+1} \right|^{\lambda_1} d\mathsf{T} \right)^{\frac{1}{\lambda_1}} \\
&\times \left(\int_0^1 |F''(\mathsf{T}\bar{a} + m(1-\mathsf{T})\bar{b})|^{\lambda_2} d\mathsf{T} \right)^{\frac{1}{\lambda_2}} \\
&\leq \frac{(\mu-\bar{a})^2}{2(\theta+1)} \left(\int_0^1 \left| 1-(1-\mathsf{T})^{\theta} - \mathsf{T}^{\theta} \right|^{\lambda_1} d\mathsf{T} \right)^{\frac{1}{\lambda_1}} \left(\int_0^1 |F''(\bar{a}^{\mathsf{T}} \bar{b}^{m(1-\mathsf{T})})|^{\lambda_2} d\mathsf{T} \right)^{\frac{1}{\lambda_2}} \\
&\leq \frac{(\mu-\bar{a})^2}{2(\theta+1)} \left(\int_0^1 |1-(1-\mathsf{T})^{\theta} - \mathsf{T}^{\theta}|^{\lambda_1} d\mathsf{T} \right)^{\frac{1}{\lambda_1}} \\
&\times \left(\int_0^1 (\mathsf{T}^\alpha |F''(\bar{a})|^{\lambda_2} + m(1-\mathsf{T}^\alpha)|F''(\bar{b})|^{\lambda_2}) d\mathsf{T} \right)^{\frac{1}{\lambda_2}} \\
&\leq \frac{(\mu-\bar{a})^2}{2(\theta+1)} \left(\frac{|F''(\bar{a})|^{\lambda_2} + m\alpha |F''(\bar{b})|^{\lambda_2}}{\alpha+1} \right)^{\frac{1}{\lambda_2}} \left(\int_0^1 \left| 1-(1-\mathsf{T})^{\theta} - \mathsf{T}^{\theta} \right|^{\lambda_1} \right)^{\frac{1}{\lambda_1}} \\
&\leq \frac{(\mu-\bar{a})^2}{2(\theta+1)} \left(\frac{|F''(\bar{a})|^{\lambda_2} + m\alpha |F''(\bar{b})|^{\lambda_2}}{\alpha+1} \right)^{\frac{1}{\lambda_2}} \left(\int_0^1 [(1-\mathsf{T})^{\theta} + \mathsf{T}^{\theta} - 1]^{\lambda_1} \right)^{\frac{1}{\lambda_1}} \\
&\leq \frac{(\mu-\bar{a})^2 (2^{1-\theta}-1)}{2(\theta+1)} \left(\frac{|F''(\bar{a})|^{\lambda_2} + m\alpha |F''(\bar{b})|^{\lambda_2}}{\alpha+1} \right)^{\frac{1}{\lambda_2}},
\end{aligned} \tag{26}$$

where $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1$.
Case(ii): Let $\theta \in [1, +\infty)$. By utilizing Lemma 1, 2, 3, 7 and applying Hölder's inequality, we have

$$\begin{aligned}
&\left| \frac{F(\bar{a}) + F(\mu)}{2} - \frac{\Gamma(\theta+1)}{2(\mu-\bar{a})^\theta} \left[\chi^\theta_{\bar{a}+} F(\mu) + \chi^\theta_{\bar{b}-} F(\bar{a}) \right] \right| \\
&\leq \frac{(\mu-\bar{a})^2}{2(\theta+1)} \left(\frac{|F''(\bar{a})|^{\lambda_2} + m\alpha |F''(\bar{b})|^{\lambda_2}}{\alpha+1} \right)^{\frac{1}{\lambda_2}} \left(\int_0^1 [1-(1-\mathsf{T})^{\theta} - \mathsf{T}^{\theta}]^{\lambda_1} d\mathsf{T} \right)^{\frac{1}{\lambda_1}}
\end{aligned}$$

$$\leq \frac{(\mu-\bar{a})^2(1-2^{1-\theta})}{2(\theta+1)}\left(\frac{|F''(\bar{a})|^{\lambda_2}+m\alpha|F''(\bar{b})|^{\lambda_2}}{\alpha+1}\right)^{\frac{1}{\lambda_2}}. \tag{27}$$

From (26) and (27), we have obtained the required result.

Remark 5. *In accordance with the selection of parameters $\alpha = 1$ and $m = 1$ in Theorem 2 and $s = 1$ in [2, Theorem 3.2], we get*

$$\left|\frac{F(\bar{a})+F(\bar{b})}{2}-\frac{\Gamma(\theta+1)}{2(\bar{b}-\bar{a})^\theta}\left[\chi^\theta_{\bar{a}+}F(\bar{b})+\chi^\theta_{\bar{b}-}F(\bar{a})\right]\right|$$
$$\leq \frac{(\bar{b}-\bar{a})^2\max(1-2^{1-\theta},2^{1-\theta}-1)}{2(\theta+1)}\left(\frac{|F''(\bar{a})|^{\lambda_2}+|F''(\bar{b})|^{\lambda_2}}{2}\right)^{\frac{1}{\lambda_2}}.$$

The following main result is based on utilization of Lemma 5.

Theorem 3. *Let $F : [\bar{a},\bar{b}] \longrightarrow \mathbb{R}$ be a twice differentiable function such that $|F''|$ is Lebesgue integerable, increasing and GA (α,m)-convex function on $[\bar{a},\bar{b}]$. Then for given parameters $\theta\in(0,+\infty)$ and $(\alpha,m)\in(0,1]^2$, where $0\leq\bar{a}<\bar{b}$ the following inequality*

$$\left|\frac{\Gamma(\theta+1)}{2(\bar{b}-\bar{a})^\theta}\left[\chi^\theta_{\bar{a}+}F(\mu)+\chi^\theta_{\mu-}F(\bar{a})\right]-F\left(\frac{\bar{a}+\mu}{2}\right)\right|$$
$$\leq \frac{(\mu-\bar{a})^2}{2(\theta+1)}\left[|F''(\bar{a})|\left(\frac{\theta-\theta 2^{-\alpha-1}-2^{-\alpha-1}}{\alpha+1}-\frac{\theta+1}{\alpha+2}\right.\right.$$
$$\left.+\frac{1}{\theta+\alpha+2}+2B_{0.5}(\theta+2,\alpha+1)\right)$$
$$+m|F''(\bar{b})|\left(\frac{\theta-3}{4}+\frac{1}{\theta+2}-\frac{\theta-\theta 2^{-\alpha-1}-2^{-\alpha-1}}{\alpha+1}-\frac{\theta+1(1-2^{-\alpha})}{\alpha+2}\right.$$
$$\left.\left.-\frac{1}{\theta+\alpha+2}-2B_{0.5}(\alpha+1,\theta+2)\right)\right]$$

holds true.

Proof. By using Lemma 2, 5 and 7 we have

$$\left|\frac{\Gamma(\theta+1)}{2(\mu-\bar{a})^\theta}[\chi^\theta_{\bar{a}+}F(\mu)+\chi^\theta_{\mu-}F(\bar{a})]-F\left(\frac{\bar{a}+\mu}{2}\right)\right|$$
$$\leq \frac{(\mu-\bar{a})^2}{2}\int_0^1|p_0(\mathsf{T})||F''(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b})|d\mathsf{T}$$
$$\leq \frac{(\mu-\bar{a})^2}{2}\int_0^1|p_0(\mathsf{T})||F''(\bar{a}^{\mathsf{T}}\bar{b}^{m(1-\mathsf{T})}|d\mathsf{T}$$
$$\leq \frac{(\mu-\bar{a})^2}{2}\int_0^1|p_0(\mathsf{T})|(\mathsf{T}^\alpha|F''(\bar{a})|+m(1-\mathsf{T}^\alpha)|F''(\bar{b})|)d\mathsf{T}$$

$$\leq \frac{(\mu-\bar{a})^2}{2}\left[\int_0^{\frac{1}{2}}\left|\mathsf{T}-\frac{1-(1-\mathsf{T})^{\theta+1}-\mathsf{T}^{\theta+1}}{\theta+1}\right|\right.$$

$$\times\left(\mathsf{T}^{\alpha}|F''(\bar{a})|+m(1-\mathsf{T}^{\alpha})|F''(\bar{b})|\right)d\mathsf{T}$$
$$\left.+\int_{\frac{1}{2}}^{1}\left|1-\mathsf{T}-\frac{1-(1-\mathsf{T})^{\theta+1}-\mathsf{T}^{\theta+1}}{\theta+1}\right|(\mathsf{T}^{\alpha}|F''(\bar{a})|+m(1-\mathsf{T}^{\alpha})|F''(\bar{b})|)d\mathsf{T}\right]$$
$$\leq \frac{(\mu-\bar{a})^2}{2(\theta+1)}\left[\int_0^{\frac{1}{2}}\left|\mathsf{T}(\theta+1)-1+(1-\mathsf{T})^{\theta+1}+\mathsf{T}^{\theta+1}\right|\right.$$
$$\times(\mathsf{T}^{\alpha}|F''(\bar{a})|+m(1-\mathsf{T}^{\alpha})|F''(\bar{b})|)d\mathsf{T}$$
$$\left.+\int_{\frac{1}{2}}^{1}\left|\theta-\mathsf{T}(\theta+1)+(1-\mathsf{T})^{\theta+1}+\mathsf{T}^{\theta+1}\right|(\mathsf{T}^{\alpha}|F''(\bar{a})|+m(1-\mathsf{T}^{\alpha})|F''(\bar{b})|)d\mathsf{T}\right]. \tag{28}$$

Consider

$$\int_0^{\frac{1}{2}}\left|\mathsf{T}(\theta+1)-1+(1-\mathsf{T})^{\theta+1}+\mathsf{T}^{\theta+1}\right|(\mathsf{T}^{\alpha}|F''(\bar{a})|+m(1-\mathsf{T}^{\alpha})|F''(\mu)|)d\mathsf{T}$$
$$\leq |F''(\bar{a})|\left((\theta+1)\frac{(\frac{1}{2})^{\alpha+2}}{\alpha+2}-\frac{(\frac{1}{2})^{\alpha+1}}{\alpha+1}+B_{0.5}(\alpha+1,\theta+2)+\frac{(\frac{1}{2})^{\theta+\alpha+2}}{\theta+\alpha+2}\right)$$
$$+m|F''(\bar{b})\left(\frac{\theta-3}{8}+\frac{2^{-\alpha-1}}{\alpha+1}-\frac{(\theta+1)2^{-\alpha-2}}{\alpha+2}\right.$$
$$\left.-\frac{2^{-\theta-\alpha-2}}{\theta+\alpha+2}-B_{0.5}(\alpha+1,\theta+2)\right). \tag{29}$$

Also, consider

$$\int_{\frac{1}{2}}^{1}\left|\theta-\mathsf{T}(\theta+1)+(1-\mathsf{T})^{\theta+1}+\mathsf{T}^{\theta+1}\right|(\mathsf{T}^{\alpha}|F''(\bar{a})|+m(1-\mathsf{T}^{\alpha})|F''(\bar{b})|)d\mathsf{T}$$
$$\leq |F''(\bar{a})|\left(\theta\frac{1-(\frac{1}{2})^{\alpha+1}}{\alpha+1}-(\theta+1)\frac{1-(\frac{1}{2})^{\alpha+2}}{\alpha+2}\right.$$
$$\left.+B_{0.5}(\alpha+1,\theta+2)+\frac{1-(\frac{1}{2})^{\theta+\alpha+2}}{\theta+\alpha+2}\right)$$
$$+m|F''(\bar{b})|\left(\frac{\theta-3}{8}+\frac{1}{\theta+2}-\frac{\theta-\theta 2^{-\alpha-1}}{\alpha+1}\right.$$
$$\left.+\frac{(\theta+1)(1-2^{-\alpha-2})}{\alpha+2}-\frac{1-2^{-\theta-\alpha-2}}{\theta+\alpha+2}-B_{0.5}(\alpha+1,\theta+2)\right). \tag{30}$$

Substituting values of (29) and (30) in (28), we obtain

$$
\begin{aligned}
&\left|\frac{\Gamma(\theta+1)}{2(\mu-\bar{a})^{\theta}}[\chi^{\theta}_{\bar{a}^{+}}\digamma(\mu)+\chi^{\theta}_{\mu^{-}}\digamma(\bar{a})]-\digamma\left(\frac{\bar{a}+\mu}{2}\right)\right| \\
&\leq \frac{(\mu-\bar{a})^2}{2(\theta+1)}|\digamma''(\bar{a})|\left((\theta+1)\frac{(\frac{1}{2})^{\alpha+2}}{\alpha+2}-\frac{(\frac{1}{2})^{\alpha+1}}{\alpha+1}+B_{0.5}(\alpha+1,\theta+2)+\frac{(\frac{1}{2})^{\theta+\alpha+2}}{\theta+\alpha+2}\right) \\
&+\frac{(\mu-\bar{a})^2}{2(\theta+1)}m|\digamma''(\bar{b})|\left(\frac{\theta-3}{8}+\frac{(\frac{1}{2})^{\alpha+1}}{\alpha+1}-\frac{(\theta+1)(\frac{1}{2})^{\alpha+2}}{\alpha+2}\right. \\
&\left.-\frac{(\frac{1}{2})^{\theta+\alpha+2}}{\theta+\alpha+2}-B_{0.5}(\alpha+1,\theta+2)\right) \\
&+\frac{(\mu-\bar{a})^2}{2(\theta+1)}|\digamma''(\bar{a})|\left(\theta\frac{1-(\frac{1}{2})^{\alpha+1}}{\alpha+1}-(\theta+1)\frac{1-(\frac{1}{2})^{\alpha+2}}{\alpha+2}\right.
\end{aligned}
$$

$$
\begin{aligned}
&\left.+B(\alpha+1,\theta+2)+\frac{1-(\frac{1}{2})^{\theta+\alpha+2}}{\theta+\alpha+2}\right) \\
&+\frac{(\mu-\bar{a})^2}{2(\theta+1)}m|\digamma''(\bar{b})|\left(\frac{\theta-3}{8}+\frac{1}{\theta+2}-\frac{\theta-\theta 2^{-\alpha-1}}{\alpha+1}\right. \\
&\left.+\frac{(\theta+1)(1-2^{-\alpha-2})}{\alpha+2}-\frac{1-2^{-\theta-\alpha-2}}{\theta+\alpha+2}-B_{0.5}(\alpha+1,\theta+2)\right).
\end{aligned}
$$

Hence, the proof is done.

Theorem 4. *Let $\digamma:[\bar{a},\bar{b}]\longrightarrow\mathbb{R}$ be a twice differentiable function such that $|\digamma''|$ is Lebesgue integerable, increasing and GA (α,m)-convex function on $[\bar{a},\bar{b}]$. Then for given parameters $\theta\in(0,+\infty)$ and $(\alpha,m)\in(0,1]^2$, where $0\leq\bar{a}<\bar{b}$. Thus, let $1<\curlywedge_2<+\infty$ the following inequality*

$$
\begin{aligned}
&\left|\frac{\Gamma(\theta+1)}{2(\mu-\bar{a})^{\theta}}\left[\chi^{\theta}_{\bar{a}^{+}}\digamma(\mu)+\chi^{\theta}_{\mu^{-}}\digamma(\bar{a})\right]-\digamma\left(\frac{\bar{a}+\mu}{2}\right)\right| \\
&\leq\frac{(\mu-\bar{a})^2}{2(\theta+1)}\left(\frac{|\digamma''(\bar{a})|^{\curlywedge_2}+m\alpha|\digamma''(\bar{b})|^{\curlywedge_2}}{\alpha+1}\right)^{\frac{1}{\curlywedge_2}} \\
&\times\left(\frac{(\theta+1)2^{-\curlywedge_1-1}+(\theta+0.5)^{\curlywedge_1+1}-\theta^{\curlywedge_1+1}}{\curlywedge_1+1}\right)^{\frac{1}{\curlywedge_1}}
\end{aligned}
$$

holds true, where $\frac{1}{\curlywedge_1}+\frac{1}{\curlywedge_2}=1$.

Proof. By utilizing Lemma 5, 7 and applying Hölder's inequality, we have

$$
\begin{aligned}
&\left|\frac{\Gamma(\theta+1)}{2(\mu-\bar{a})^{\theta}}[\chi^{\theta}_{\bar{a}^+}\mathsf{F}(\mu)+\chi^{\theta}_{\mu^-}\mathsf{F}(\bar{a})]-\mathsf{F}\left(\frac{\bar{a}+\mu}{2}\right)\right|\\
&\leq \frac{(\mu-\bar{a})^2}{2}\int_0^1 |p_0(\mathsf{T})||\mathsf{F}''(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b})|d\mathsf{T}\\
&\leq \frac{(\mu-\bar{a})^2}{2}\int_0^1 |p_0(\mathsf{T})||\mathsf{F}''(\bar{a}^{\mathsf{T}}\bar{b}^{m(1-\mathsf{T})}|d\mathsf{T}\\
&\leq \frac{(\mu-\bar{a})^2}{2}\int_0^1 |p_0(\mathsf{T})|\mathsf{T}^{\alpha}|\mathsf{F}''(\bar{a})|+m(1-\mathsf{T}^{\alpha})|\mathsf{F}''(\bar{b})|d\mathsf{T}\\
&\leq \frac{(\mu-\bar{a})^2}{2}\int_0^1 \left(|(p_0(\mathsf{T}))^{\lambda_1}|\right)^{\frac{1}{\lambda_1}}\left(|\mathsf{F}''(\bar{a})|^{\lambda_2}\int_0^1 \mathsf{T}^{\alpha}d\,\mathsf{T}+m|\mathsf{F}''(\bar{b})|^{\lambda_2}\int_0^1(1-\mathsf{T}^{\alpha})d\mathsf{T}\right)^{\frac{1}{\lambda_2}}\\
&\leq \frac{(\mu-\bar{a})^2}{2}\int_0^1 \left(|(p_0(\mathsf{T}))^{\lambda_1}|\right)^{\frac{1}{\lambda_1}}\left(|\mathsf{F}''(\bar{a})|^{\lambda_2}\frac{1}{\alpha+1}+m|\mathsf{F}''(\bar{b})|^{\lambda_2}\frac{\alpha}{\alpha+1}\right)^{\frac{1}{\lambda_2}}
\end{aligned}
$$

Substituting the value $p_0(\mathsf{T})$ of Lemma 4.

$$
\begin{aligned}
&\leq \frac{(\mu-\bar{a})^2}{2}\left(\frac{|\mathsf{F}''(\bar{a})|^{\lambda_2}+m\alpha|\mathsf{F}''(\bar{b})|^{\lambda_2}}{\alpha+1}\right)^{\frac{1}{\lambda_2}}\left(\int_0^{\frac{1}{2}}\left|\mathsf{T}-\frac{1-(1-\mathsf{T})^{\theta+1}-\mathsf{T}^{\theta+1}}{\theta+1}\right|d\mathsf{T}\right.\\
&\left.+\int_{\frac{1}{2}}^1\left|1-\mathsf{T}-\frac{1-(1-\mathsf{T})^{\theta+1}-\mathsf{T}^{\theta+1}}{\theta+1}\right|d\mathsf{T}\right)^{\frac{1}{\lambda_1}}\\
&\leq \frac{(\mu-\bar{a})^2}{2(\theta+1)}\left(\frac{|\mathsf{F}''(\bar{a})|^{\lambda_2}+m\alpha|\mathsf{F}''(\bar{b})|^{\lambda_2}}{\alpha+1}\right)^{\frac{1}{\lambda_2}}\\
&\times\left(\int_0^{\frac{1}{2}}\left|\theta(\mathsf{T})+\mathsf{T}-1+(1-\mathsf{T})^{\theta+1}+\mathsf{T}^{\theta+1}\right|^{\lambda_1}d\mathsf{T}\right.\\
&\left.+\int_{\frac{1}{2}}^1\left|\theta-\mathsf{T}+(1-\mathsf{T})^{\theta+1}+\mathsf{T}^{\theta+1}\right|^{\lambda_1}d\mathsf{T}\right)^{\frac{1}{\lambda_1}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{(\mu-\bar{a})^2}{2(\theta+1)}\left(\frac{|\mathsf{F}''(\bar{a})|^{\lambda_2}+m\alpha|\mathsf{F}''(\bar{b})|^{\lambda_2}}{\alpha+1}\right)^{\frac{1}{\lambda_2}}\\
&\times\left((\theta+1)\int_0^{\frac{1}{2}}\mathsf{T}^{\lambda_1}d\,\mathsf{T}+\int_{\frac{1}{2}}^1(\theta-\mathsf{T}+1)^{\lambda_1}d\mathsf{T}\right)^{\frac{1}{\lambda_1}}\\
&\leq \frac{(\mu-\bar{a})^2}{2(\theta+1)}\left(\frac{|\mathsf{F}''(\bar{a})|^{\lambda_2}+m\alpha|\mathsf{F}''(\bar{b})|^{\lambda_2}}{\alpha+1}\right)^{\frac{1}{\lambda_2}}\left((\theta+1)\frac{(\frac{1}{2})^{\lambda_1+1}}{\lambda_1+1}-\frac{(\theta)^{\lambda_1+1}}{\lambda_1+1}+\frac{(\theta+\frac{1}{2})^{\lambda_1+1}}{\lambda_1+1}\right)^{\frac{1}{\lambda_1}}
\end{aligned}
$$

$$
\leq \frac{(\mu-\bar{a})^2}{2(\theta+1)}\left(\frac{|\digamma''(\bar{a})|^{\leftthreetimes_2}+m\alpha|\digamma''(\bar{b})|^{\leftthreetimes_2}}{\alpha+1}\right)^{\frac{1}{\leftthreetimes_2}}
$$

$$
\times\left(\frac{(\theta+1)2^{-\leftthreetimes_1-1}-\theta^{\leftthreetimes_1+1}+(\theta+0.5)^{\leftthreetimes_1+1}}{\leftthreetimes_1+1}\right)^{\frac{1}{\leftthreetimes_1}}.
$$

Hence, the proof is done.

Remark 6. *In accordance with the selection of parameters $\alpha=1$ and $m=1$ in Theorem 4 and $s=1$ in [2, Theorem 4.2], we get*

$$
\left|\frac{\Gamma(\theta+1)}{2(\bar{b}-\bar{a})^{\theta}}[\chi^{\theta}_{\bar{a}^+}\digamma(\bar{b})+\chi^{\theta}_{\bar{b}^-}\digamma(\bar{a})]-\digamma\left(\frac{\bar{a}+\bar{b}}{2}\right)\right|
$$

$$
\leq\frac{(\bar{b}-\bar{a})^2}{2(\theta+1)}\left(\frac{|\digamma''(\bar{a})|^{\leftthreetimes_2}+|\digamma''(\bar{b})|^{\leftthreetimes_2}}{2}\right)^{\frac{1}{\leftthreetimes_2}}
$$

$$
\times\left(\frac{(\theta+1)2^{-\leftthreetimes_1-1}+(\theta+0.5)^{\leftthreetimes_1+1}-\theta^{\leftthreetimes_1+1}}{\leftthreetimes_1+1}\right)^{\frac{1}{\leftthreetimes_1}}.
$$

In the following result, we utilize Lemma 6.

Theorem 5. *Let $\digamma:[\bar{a},\bar{b}]\longrightarrow\mathbb{R}$ be a twice differentiable function such that $|\digamma''|$ is Lebesgue integerable, increasing and GA (α,m)-convex function on $[\bar{a},\bar{b}]$. Then for some given parameters $\theta\in(0,+\infty)$ and $(\alpha,m)\in(0,1]^2$, where $0\leq\bar{a}<\bar{b}$ the following inequality*

$$
\left|\frac{\digamma(\bar{a})+\digamma(\mu)}{\Upsilon(\Upsilon+1)}+\frac{2}{\Upsilon+1}\digamma\left(\frac{\bar{a}+\mu}{2}\right)-\frac{\Gamma(\theta+1)}{\Upsilon(\mu-\bar{a})^{\theta}}[\chi^{\theta}_{\bar{a}^+}\digamma(\mu)+\chi^{\theta}_{\mu^-}\digamma(\bar{a})]\right|
$$

$$
\leq\frac{(\mu-\bar{a})^2}{\Upsilon(\theta+1)(\Upsilon+1)}\max\Bigg[[(\Upsilon+1)-(\Upsilon+1)2^{-\theta}]
$$

$$
\times\left(\frac{2^{-\alpha}|\digamma''(\bar{a})+(\alpha+1-2^{-\alpha})m|\digamma''(\bar{b})}{2(\alpha+1)}\right)
$$

$$
-\Upsilon(\theta+1)\left(\frac{2^{-\alpha+1}|\digamma''(\bar{a})|+(\alpha+2-2^{-\alpha+1})m|\digamma''(\bar{b})|}{8(\alpha+2)}\right)
$$

$$
,\Upsilon(\theta+1)\left(\frac{2^{-\alpha+1}|\digamma''(\bar{a})|+(\alpha+2-2^{-\alpha+1})m|\digamma''(\bar{b})|}{8(\alpha+2)}\right)\Bigg]
$$

$$
+\frac{(\mu-\bar{a})^2}{\Upsilon(\theta+1)(\Upsilon+1)}\max\Bigg[[(\Upsilon+1)-(\Upsilon+1)2^{-\theta}-\Upsilon(\theta+1)]
$$

$$
\times\left(\frac{(2-2^{-\alpha})|\digamma''(\bar{a})|+(\alpha-1+2^{-\alpha})m|\digamma''(\bar{b})|}{2(\alpha+1)}\right)
$$

$$
+\Upsilon(\theta+1)\left(\frac{(8-2^{-\alpha+1})|\digamma''(\bar{a})|+(3\alpha-2+2^{-\alpha+1})m|\digamma''(\bar{b})|}{8(\alpha+2)}\right)
$$

$$
,\Upsilon(\theta+1)\left(\frac{(2-2^{-\alpha})|\digamma''(\bar{a})|+(\alpha-1+2^{-\alpha})m|\digamma''(\bar{b})|}{2(\alpha+1)}\right.
$$

$$-\frac{(8-2^{-\alpha+1})|F''(\bar{a})|+(3\alpha-2+2^{-\alpha+1})m|F''(\bar{b})}{8(\alpha+2)}\Bigg)\Bigg]$$

holds true.

Proof. By using Lemma 2, 6 and 7, we have

$$\begin{aligned}
&\left|\frac{F(\bar{a})+F(\mu)}{\Upsilon(\Upsilon+1)}+\frac{2}{\Upsilon+1}F\left(\frac{\bar{a}+\mu}{2}\right)-\frac{\Gamma(\theta+1)}{\Upsilon(\mu-\bar{a})^{\theta}}[\chi^{\theta}_{\bar{a}^{+}}F(\mu)+\chi^{\theta}_{\mu^{-}}F(\bar{a})]\right|\\
&\leq(\mu-\bar{a})^2\int_0^1|q_0(\mathsf{T})||F''(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b})|d\mathsf{T}\\
&\leq(\mu-\bar{a})^2\int_0^1|q_0(\mathsf{T})||F''(\bar{a}^{\mathsf{T}}\bar{b}^{m(1-\mathsf{T})}|d\mathsf{T}\\
&\leq\frac{(\mu-\bar{a})^2}{2}\int_0^1|q_0(\mathsf{T})|[\mathsf{T}^{\alpha}|F''(\bar{a})|+m(1-\mathsf{T}^{\alpha})|F''(\bar{b})|]d\mathsf{T}\\
&\leq\frac{(\mu-\bar{a})^2}{2}\left(\int_0^{\frac{1}{2}}\left|\frac{1-(1-\mathsf{T})^{\theta+1}-\mathsf{T}^{\theta+1}}{\Upsilon(\theta+1)}-\frac{\mathsf{T}}{\Upsilon+1}\right|\right.\\
&\times[\mathsf{T}^{\alpha}|F''(\bar{a})|+m(1-\mathsf{T}^{\alpha})|F''(\bar{b})|]d\mathsf{T}\\
&\left.+\int_{\frac{1}{2}}^1\left|\frac{1-(1-\mathsf{T})^{\theta+1}-\mathsf{T}^{\theta+1}}{\Upsilon(\theta+1)}-\frac{1-\mathsf{T}}{\Upsilon+1}\right|[\mathsf{T}^{\alpha}|F''(\bar{a})|+m(1-\mathsf{T}^{\alpha})|F''(\bar{b})|]d\mathsf{T}\right)\\
&\leq\frac{(\mu-\bar{a})^2}{2(\theta+1)(\Upsilon+1)}\\
&\times\left(\int_0^{\frac{1}{2}}[(\Upsilon+1)-(\Upsilon+1)(1-\mathsf{T})^{\theta+1}-(\Upsilon+1)\,\mathsf{T}^{\theta+1}-\mathsf{T}\,\Upsilon(\theta+1)]\right.\\
&\times(\mathsf{T}^{\alpha}|F''(\bar{a})|+m(1-\mathsf{T}^{\alpha})|F''(\bar{b})|)d\mathsf{T}\\
&+\int_{\frac{1}{2}}^1[(\Upsilon+1)-(\Upsilon+1)(1-\mathsf{T})^{\theta+1}-(\Upsilon+1)\,\mathsf{T}^{\theta+1}-\Upsilon\,(\theta+1)(1-\mathsf{T})]\\
&\left.\times[\mathsf{T}^{\alpha}|F''(\bar{a})|+m(1-\mathsf{T}^{\alpha})|F''(\bar{b})|d\mathsf{T}]\right).
\end{aligned}\tag{31}$$

Let us consider

$$\begin{aligned}
&\int_0^{\frac{1}{2}}[\Upsilon+1-(\Upsilon+1)[(1-\mathsf{T})^{\theta+1}+\mathsf{T}^{\theta+1}]-\mathsf{T}\,\Upsilon\,(\theta+1)]\\
&\times[\mathsf{T}^{\alpha}|F''(\bar{a})|+m(1-\mathsf{T}^{\alpha})|F''(\bar{b})|]d\mathsf{T}\\
&\leq[(\Upsilon+1)-(\Upsilon+1)2^{-\theta}]\left(|F''(\bar{a})|\int_0^{\frac{1}{2}}\mathsf{T}^{\alpha}d\,\mathsf{T}+m|F''(\bar{b})|\int_0^{\frac{1}{2}}(1-\mathsf{T}^{\alpha})d\mathsf{T}\right)\\
&-\mathsf{T}\,\Upsilon\,(\theta+1)\left(|F''(\bar{a})|\int_0^{\frac{1}{2}}\mathsf{T}^{\alpha}d\,\mathsf{T}+m|F''(\bar{b})|\int_0^{\frac{1}{2}}(1-\mathsf{T}^{\alpha})d\mathsf{T}\right)
\end{aligned}$$

$$\leq [(\Upsilon+1)-(\Upsilon+1)2^{-\theta}]\left(\frac{|F''(\bar{a})2^{-\alpha}+m|F''(\bar{b})(\alpha+1-2^{-\alpha})}{2(\alpha+1)}\right)$$
$$-\Upsilon(\theta+1)\left(\frac{|F''(\bar{a})|2^{-\alpha+1}+m|F''(\bar{b})(\alpha+2-2^{-\alpha+1})}{8(\alpha+2)}\right), \tag{32}$$

and

$$\int_0^{\frac{1}{2}}[-\Upsilon-1+(\Upsilon+1)2^{-\theta}+\mathsf{T}\,\Upsilon\,(\theta+1)]\left[|F''(\bar{a})|\,\mathsf{T}^{\alpha}+m|F''(\bar{b})|(1-\mathsf{T}^{\alpha})\right]d\mathsf{T}$$
$$\leq [-\Upsilon-1+(\Upsilon+1)2^{-\theta}]\int_0^{\frac{1}{2}}\left[|F''(\bar{a})|\,\mathsf{T}^{\alpha}+m|F''(\bar{b})|(1-\mathsf{T}^{\alpha})\right]d\mathsf{T}$$
$$+\Upsilon(\theta+1)\int_0^{\frac{1}{2}}\left[|F''(\bar{a})|\,\mathsf{T}^{\alpha+1}+m|F''(\bar{b})|\,\mathsf{T}\,(1-\mathsf{T})^{\alpha}\right]d\mathsf{T}$$
$$\leq \Upsilon(\theta+1)\left(\frac{|F''(\bar{a})|2^{-\alpha+1}+m|F''(\bar{b})(\alpha+2-2^{-\alpha+1})}{8(\alpha+2)}\right), \tag{33}$$

and

$$\int_{\frac{1}{2}}^1\left[(\Upsilon+1)-(\Upsilon+1)2^{-\theta}+\mathsf{T}\,\Upsilon\,(\theta+1)-\Upsilon(\theta+1)\right]$$
$$\times\left(|F''(\bar{a})|\,\mathsf{T}^{\alpha}\,d\,\mathsf{T}+m|F''(\bar{b})|(1-\mathsf{T}^{\alpha})d\mathsf{T}\right)$$
$$\leq [(\Upsilon+1)-(\Upsilon+1)2^{-\theta}-\Upsilon(\theta+1)]\left(|F''(\bar{a})|\int_{\frac{1}{2}}^1\mathsf{T}^{\alpha}d\,\mathsf{T}+m|F''(\bar{b})|\int_{\frac{1}{2}}^1(1-\mathsf{T}^{\alpha})d\mathsf{T}\right)$$
$$+\Upsilon(\theta+1)\left(|F''(\bar{a})|\int_{\frac{1}{2}}^1\mathsf{T}^{\alpha+1}d\,\mathsf{T}+m|F''(\bar{b})|\int_{\frac{1}{2}}^1\mathsf{T}(1-\mathsf{T}^{\alpha})d\mathsf{T}\right)$$
$$\leq [(\Upsilon+1)-(\Upsilon+1)2^{-\theta}-\Upsilon(\theta+1)]\left(\frac{(2-2^{-\alpha})|F''(\bar{a})|+(\alpha-1+2^{-\alpha})m|F''(\bar{b})|}{2(\alpha+1)}\right)$$
$$+\Upsilon(\theta+1)\left(\frac{(8-2^{-\alpha+1})|F''(\bar{a})|+(3\alpha-2+2^{-\alpha+1})m|F''(\bar{b})|)}{8(\alpha+2)}\right), \tag{34}$$

and

$$\int_{\frac{1}{2}}^1[-\Upsilon-1+(\Upsilon+1)2^{-\theta}-\mathsf{T}\,\Upsilon\,(\theta+1)+\Upsilon(\theta+1)]$$
$$\left(|F''(\bar{a})|\,\mathsf{T}^{\alpha}\,d\,\mathsf{T}+m|F''(\bar{b})|(1-\mathsf{T}^{\alpha})d\mathsf{T}\right)$$
$$\leq [-\Upsilon-1+(\Upsilon+1)+\Upsilon(\theta+1)]\left(|F''(\bar{a})|\int_{\frac{1}{2}}^1\mathsf{T}^{\alpha}d\,\mathsf{T}-m|F''(\bar{b})|\int_{\frac{1}{2}}^1(1-\mathsf{T}^{\alpha})d\mathsf{T}\right)$$
$$+\Upsilon(\theta+1)\left(|F''(\bar{a})|\int_{\frac{1}{2}}^1\mathsf{T}^{\alpha+1}d\,\mathsf{T}+m|F''(\bar{b})|\int_{\frac{1}{2}}^1\mathsf{T}(1-\mathsf{T}^{\alpha})d\mathsf{T}\right)$$

$$\leq \Upsilon(\theta+1)\left(\frac{(2-2^{-\alpha})|F''(\bar{a})|(\alpha-1+2^{-\alpha})m|F''(\bar{b})|)}{2(\alpha+1)}\right.$$
$$\left.-\frac{(8-2^{-\alpha+1})|F''(\bar{a})|+(3\alpha-2+2^{-\alpha+1})m|F''(\bar{b})}{8(\alpha+2)}\right). \tag{35}$$

Substituting the values of (32), (33), (34) and (35) in (31), we obtained the required result.

Theorem 6. *Let $F : [\bar{a}, \bar{b}] \longrightarrow \mathbb{R}$ be a twice differentiable function such that $|F''|$ is Lebesgue integrable, increasing and GA (α, m)-convex function on $[\bar{a}, \bar{b}]$. Then for given parameters $\theta \in (0, +\infty]$ and $(\alpha, m) \in (0, 1]^2$, where $0 \leq \bar{a} < \bar{b}$. Thus, let $1 < \lambda_2 < +\infty$ then, the following inequality*

$$\left|\frac{F(\bar{a})+F(\mu)}{\Upsilon(\Upsilon+1)}+\frac{2}{\Upsilon+1}F\left(\frac{\bar{a}+\mu}{2}\right)-\frac{\Gamma(\theta+1)}{\Upsilon(\mu-\bar{a})^{\theta}}[\chi^{\theta}_{\bar{a}^+}F(\mu)+\chi^{\theta}_{\mu^-}F(\bar{a})]\right|$$
$$\leq \frac{(\mu-\bar{a})^2}{[\Upsilon(\theta+1)(\Upsilon+1)]^{1+\lambda_1^{-1}}}\left(\frac{|F''(\bar{a})|^{\lambda_2}+m\alpha|F''(\bar{b})|^{\lambda_2}}{\alpha+1}\right)^{\frac{1}{\lambda_2}}$$
$$\times\left[\max\left(\left(\Upsilon+1-(\Upsilon+1)2^{-\theta}\right)^{\lambda_1+1}-\left(1+0.5\,\Upsilon\,(1-\theta)-(1+\Upsilon)2^{-\theta}\right)^{\lambda_1+1}\right.\right.$$
$$\left.,(\Upsilon(\theta+1))^{\lambda_1+1}2^{-\lambda_1-1}\right)$$
$$+\max\left(\left(\Upsilon+1-(\Upsilon+1)2^{-\theta}\right)^{\lambda_1+1}-\left(1+0.5\,\Upsilon\,(1-\theta)-(1+\Upsilon)2^{-\theta}\right)^{\lambda_1+1}\right.$$
$$\left.\left.,0.5\,\Upsilon\,(\theta+1)^{\lambda_1+1}\right)\right)^{\frac{1}{\lambda_1}}\Big)\Big]$$

holds true, where $\frac{1}{\lambda_1}+\frac{1}{\lambda_2}=1$.

Proof. By utilizing Definitions 2, 4, Lemma 2, 6, 7 and applying Hölder's inequality, we have

$$\left|\frac{F(\bar{a})+F(\mu)}{\Upsilon(\Upsilon+1)}+\frac{2}{\Upsilon+1}F\left(\frac{\bar{a}+\mu}{2}\right)-\frac{\Gamma(\theta+1)}{\Upsilon(\mu-\bar{a})^{\theta}}[\chi^{\theta}_{\bar{a}^+}F(\mu)+\chi^{\theta}_{\mu^-}F(\bar{a})]\right|$$
$$\leq(\mu-\bar{a})^2\int_0^1|q_0(\mathsf{T})||F''(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b})|d\mathsf{T}$$
$$\leq(\mu-\bar{a})^2\left(\int_0^1|q_0(\mathsf{T})|^{\lambda_1}\right)^{\frac{1}{\lambda_1}}\left(\int_0^1|F''(\mathsf{T}\bar{a}+m(1-\mathsf{T})\bar{b})|^{\lambda_2}d\mathsf{T}\right)^{\frac{1}{\lambda_2}}$$
$$\leq(\mu-\bar{a})^2\left(\int_0^1|q_0(\mathsf{T})|^{\lambda_1}\right)^{\frac{1}{\lambda_1}}\left(\int_0^1|F''(\bar{a}^{\mathsf{T}}\bar{b}^{m(1-\mathsf{T})}|^{\lambda_2}d\mathsf{T}\right)^{\frac{1}{\lambda_2}}$$
$$\leq(\mu-\bar{a})^2\left(\int_0^1|q_0(\mathsf{T})|^{\lambda_1}\right)^{\frac{1}{\lambda_1}}\left(\int_0^1\mathsf{T}^{\alpha}|F''(\bar{a})|^{\lambda_2}+m(1-\mathsf{T}^{\alpha})|F''(\bar{b})|^{\lambda_2}\right)^{\frac{1}{\lambda_2}}$$

$$
\begin{aligned}
&\leq (\mu-\bar{a})^2\left(\frac{|\mathsf{F}''(\bar{a})|^{\leftthreetimes_2}+m\alpha|\mathsf{F}''(\bar{b})|^{\leftthreetimes_2}}{\alpha+1}\right)^{\frac{1}{\leftthreetimes_2}}\left(\int_0^1 |q_0(\mathsf{T})|^{\leftthreetimes_1}\right)^{\frac{1}{\leftthreetimes_1}}\\
&\leq (\mu-\bar{a})^2\left(\frac{|\mathsf{F}''(\bar{a})|^{\leftthreetimes_2}+m\alpha|\mathsf{F}''(\bar{b})|^{\leftthreetimes_2}}{\alpha+1}\right)^{\frac{1}{\leftthreetimes_2}}\\
&\times\left(\int_0^{\frac{1}{2}}\left|\frac{1-(1-\mathsf{T})^{\theta+1}-\mathsf{T}^{\theta+1}}{\Upsilon(\theta+1)}-\frac{\mathsf{T}}{\Upsilon+1}\right|^{\leftthreetimes_1}d\mathsf{T}\right.\\
&\left.+\int_{\frac{1}{2}}^1\left|\frac{1-(1-\mathsf{T})^{\theta+1}-\mathsf{T}^{\theta+1}}{\Upsilon(\theta+1)}-\frac{1-\mathsf{T}}{\Upsilon+1}\right|^{\leftthreetimes_1}d\mathsf{T}\right)^{\frac{1}{\leftthreetimes_1}}\\
&\leq \frac{(\mu-\bar{a})^2}{\Upsilon(\Upsilon+1)(\theta+1)}\left(\frac{|\mathsf{F}''(\bar{a})|^{\leftthreetimes_2}+m\alpha|\mathsf{F}''(\bar{b})|^{\leftthreetimes_2}}{\alpha+1}\right)^{\frac{1}{\leftthreetimes_2}}\\
&\times\left(\int_0^{\frac{1}{2}}\Big|(\Upsilon+1)-(\Upsilon+1)[(1+\mathsf{T})^{\theta+1}+\mathsf{T}^{\theta+1}]-\mathsf{T}\,\Upsilon\,(\theta+1)\Big|^{\leftthreetimes_1}d\mathsf{T}\right.\\
&\left.+\int_{\frac{1}{2}}^1\Big|(\Upsilon+1)-(\Upsilon+1)[(1+\mathsf{T})^{\theta+1}+\mathsf{T}^{\theta+1}]-\Upsilon(\theta+1)(1-\mathsf{T})\Big|^{\leftthreetimes_1}d\mathsf{T}\right)^{\frac{1}{\leftthreetimes_1}}. \qquad (36)
\end{aligned}
$$

Let us consider

$$
\int_0^{\frac{1}{2}}\Big[(\Upsilon+1)-(\Upsilon+1)[(1+\mathsf{T})^{\theta+1}+\mathsf{T}^{\theta+1}]-\mathsf{T}\,\Upsilon\,(\theta+1)\Big]^{\leftthreetimes_1}d\mathsf{T}
$$

$$
\begin{aligned}
&\leq -\frac{1}{\Upsilon(\theta+1)}\left[\frac{[(\Upsilon+1)-(\Upsilon+1)2^{-\theta}-\mathsf{T}\,\Upsilon\,(\theta+1)]^{\leftthreetimes_1+1}}{\leftthreetimes_1+1}\Bigg|_0^{\frac{1}{2}}\right]\\
&\leq \frac{[(\Upsilon+1)-(\Upsilon+1)2^{-\theta}]^{\leftthreetimes_1+1}-[1+0.5\,\Upsilon\,(1-\theta)-(1+\Upsilon)2^{-\theta}]^{\leftthreetimes_1+1}}{\Upsilon(\theta+1)(\leftthreetimes_1+1)}, \qquad (37)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^{\frac{1}{2}}\Big[-(\Upsilon+1)+(\Upsilon+1)[(1+\mathsf{T})^{\theta+1}+\mathsf{T}^{\theta+1}]+\mathsf{T}\,\Upsilon\,(\theta+1)\Big]^{\leftthreetimes_1}d\mathsf{T}\\
&\leq \frac{1}{\Upsilon(\theta+1)}\left[\frac{[-\Upsilon-1+(\Upsilon+1)2^{-\theta}+\Upsilon(\theta+1)\mathsf{T}]^{\leftthreetimes_1+1}}{\leftthreetimes_1+1}\Bigg|_0^{\frac{1}{2}}\right]\\
&\leq \frac{[\Upsilon(\theta+1)]^{\leftthreetimes_1+1}2^{-\leftthreetimes_1-1}}{\Upsilon(\theta+1)(\leftthreetimes_1+1)}, \qquad (38)
\end{aligned}
$$

and

$$
\int_{\frac{1}{2}}^1\Big[(\Upsilon+1)-(\Upsilon+1)[(1+\mathsf{T})^{\theta+1}+\mathsf{T}^{\theta+1}]-\Upsilon(\theta+1)(1-\mathsf{T})\Big]^{\leftthreetimes_1}d\mathsf{T}
$$

$$\leq -\frac{1}{\Upsilon(\theta+1)}\left[\frac{[(\Upsilon+1)-(\Upsilon+1)2^{-\theta}-\Upsilon(\theta+1)(1-\mathsf{T})]^{\leftthreetimes_1+1}}{\leftthreetimes_1+1}\Bigg|_{\frac{1}{2}}^{1}\right]$$

$$\leq \frac{[((\Upsilon+1)-(\Upsilon+1)2^{-\theta})-(1+0.5\,\Upsilon\,(1-\theta)+(\Upsilon+1)2^{-\theta})]^{\leftthreetimes_1+1}}{\Upsilon(\theta+1)(\leftthreetimes_1+1)}, \tag{39}$$

and

$$\int_{\frac{1}{2}}^{1}\Big[(-\,\Upsilon\,-1+(\Upsilon+1)2^{-\theta}+\Upsilon(\theta+1)(1-\mathsf{T}))d\mathsf{T}\Big]^{\leftthreetimes_1}$$

$$\leq \frac{[\Upsilon(\theta+1)]^{\leftthreetimes_1+1}2^{-\leftthreetimes_1-1}}{\Upsilon(\theta+1)(\leftthreetimes_1+1)}. \tag{40}$$

Substituting (37), (38), (39) and (40) in (36), we obtained the required result.

Remark 7. *In accordance with the selection of parameters $\alpha = 1$ and $m = 1$ in Theorem 6 and $s = 1$ in [2, Theorem 5.2], we obtained the same result i.e.,*

$$\left|\frac{F(\bar{a})+F(\bar{b})}{\Upsilon(\Upsilon+1)}+\frac{2}{\Upsilon+1}F\left(\frac{\bar{a}+\bar{b}}{2}\right)-\frac{\Gamma(\theta+1)}{\Upsilon(\bar{b}-\bar{a})^{\theta}}[\chi^{\theta}_{\bar{a}^+}F(\bar{b})+\chi^{\theta}_{\bar{b}^-}F(\bar{a})]\right|$$

$$\leq \frac{(\bar{b}-\bar{a})^2}{[\Upsilon(\theta+1)(\Upsilon+1)]^{1+\leftthreetimes_1^{-1}}}\left(\frac{|F''(\bar{a})|^{\leftthreetimes_2}+|F''(\bar{b})|^{\leftthreetimes_2}}{2}\right)^{\frac{1}{\leftthreetimes_2}}$$

$$\times\Bigg[\max\Bigg(\Big(\Upsilon+1-(\Upsilon+1)2^{-\theta}\Big)^{\leftthreetimes_1+1}-\Big(1+0.5\,\Upsilon\,(1-\theta)-(1+\Upsilon)2^{-\theta}\Big)^{\leftthreetimes_1+1}$$

$$,(\Upsilon(\theta+1))^{\leftthreetimes_1+1}\,2^{-\leftthreetimes_1-1}\Bigg)$$

$$+\max\Bigg(\Big(\Upsilon+1-(\Upsilon+1)2^{-\theta}\Big)^{\leftthreetimes_1+1}-\Big(1+0.5\,\Upsilon\,(1-\theta)-(1+\Upsilon)2^{-\theta}\Big)^{\leftthreetimes_1+1}$$

$$,0.5\,\Upsilon\,(\theta+1)^{\leftthreetimes_1+1}\Bigg)\Bigg)^{\frac{1}{\leftthreetimes_1}}\Bigg)\Bigg].$$

5. Conclusions

The importance of convexity and fractional calculus in real-life implications is unavoidable. Both concepts represent nature well. In the presented work, we utilized both concepts and obtained some good results. This paper introduces new fractional integral inequalities that are applicable to twice-differentiable geometrically arithmetically (α, m)-convex functions. The utilization of classical Riemann-Liouville fractional integrals leads to the derivation of new identities. Consequently, we are able to explore the Hermite-Hadamard type inequalities based on the mentioned convexity. Hölder's inequality is employed to investigate the mean inequalities, which have strong applicability in optimization theory. These findings significantly extend the existing literature, presenting them as special cases of our newly established consequences. This work contributes to a deeper understanding of the properties and applications of GA (α, m)-convex functions in the context of fractional integral inequalities. The presented work opens avenues for further exploration and expansion of inequalities applicable in optimization theory. The mathematicians can compare the proposed approach with alternative methods or convexities to evaluate its efficiency and uniqueness in handling similar mathematical problems.

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Declarations

Availability of data and material
No data were used to support this study.
Ethical Approval
Not Applicable. **Competing interests**
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Authors' contributions
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