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Geometric Properties of a General Subclass of Analytic Functions Involving Multiplier Operator

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Abstract. This paper investigates a general subclass of analytic functions defined in the open unit disk involving a multiplier transformation. Employing the linear approximation theorem, we present sharp coefficient estimates, growth and distortion results, and radii for geometric properties such as close-to-convexity, starlikeness, and convexity. Our approach generalizes several known results.
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1. Introduction

Let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk. Define $\mathcal{A}$ as the class of functions analytic in $\mathbb{U}$ with the normalized form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

We consider the subclass $\mathcal{A}^* \subset \mathcal{A}$, consisting of functions with non-positive Taylor coefficients of the form

$$h(z) = z - \sum_{n=2}^{\infty} c_n z^n, \quad c_n \geq 0. \tag{2}$$

Cho and Srivastava [1] introduced the generalized multiplier transformation given by

$$T_\nu^r f(z) = z + \sum_{n=2}^{\infty} \left(\frac{n+\nu}{1+\nu}\right)^r c_n z^n, \quad r \in \mathbb{N}_0,\ \nu \geq 0. \tag{3}$$

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This operator generalizes previous cases. For instance, when $\nu = 1$, it corresponds to the operator studied by Uralegaddi and Somanatha [2], and for $\nu = 0$, it reduces to the classical Salagean operator [3].

Yousef *et al.* [4] defined a general class of analytic and bi-univalent functions denoted by $\mathcal{B}_{\Sigma}^{\tau}(\lambda, \mu, \delta; \alpha)$, characterized by the following differential inequality

$$\Re\left(1-\delta)\left(\frac{f(z)}{z}\right)^{\tau}+\delta\left(f(z)\right)'\left(\frac{f(z)}{z}\right)^{\tau-1}+\lambda\mu z\left(f(z)\right)''\right)>\alpha, \tag{4}$$

where $\delta \geq 1$, $\tau \geq 0$, $\mu \geq 0$, $0 \leq \alpha < 1$, and $\lambda = \frac{2\delta+\tau}{2\delta+1}$.

This class has been the focus of some researchers in numerous studies addressing some bounding problem such as Fekete-Szegö and the second Hankel determinate (see [5–11]).

In the current work, we adapt the operator $T_{\nu}^{r}h(z)$ within this framework and consider its implications in analytic functions with negative coefficients. Our goal is to define a refined class $\mathcal{B}_{\nu}^{\tau}(\lambda, \mu, \delta; \alpha)$ that encapsulates and extends these concepts, forming the basis for the investigations in the following sections.

Definition 1. *Let $z \in \mathbb{U}$, and let the parameters satisfy $\delta \geq 1$, $\tau \geq 0$, $\mu \geq 0$, and $0 \leq \alpha < 1$. A function $f \in \mathcal{A}$ given by (1) is said to belong to the class $\mathcal{B}_{\nu}(\lambda, \mu, \delta, \tau; \alpha)$ if the following condition holds for all $z \in \mathbb{U}$:*

$$\Re\left((1-\delta)\left(\frac{T_{\nu}^{r}f(z)}{z}\right)^{\tau}+\delta\left(T_{\nu}^{r}f(z)\right)'\left(\frac{T_{\nu}^{r}f(z)}{z}\right)^{\tau-1}+\lambda\mu z\left(T_{\nu}^{r}f(z)\right)''\right)>\alpha, \tag{5}$$

where $\lambda = \frac{2\delta+\tau}{2\delta+1}$.

In this paper, we are generalizing the work that Illafe has done *et al.* in [12, 13] by considering not to eliminate the parameter τ. Furthermore, let $B_{\nu}^{*}(\lambda, \mu, \delta, \tau; \alpha) = \mathcal{B}_{\nu}(\lambda, \mu, \delta, \tau; \alpha) \cap \mathcal{A}^{*}$.

Lemma 1. *[14] Let $f(x)$ be a real-valued function such that $|f(x)| \ll 1$, and let $n \in \mathbb{R}$. Then, the following linear approximation holds*

$$(1 + f(x))^{n} \approx 1 + nf(x)$$

up to first order in $f(x)$.

2. Coefficient Bounds and Characterization

We begin this section by establishing a characterization result that provides the necessary and sufficient conditions for a function to belong to the class $\mathcal{B}_{\nu}^{*}(\lambda, \mu, \delta, \tau; \alpha)$. We begin this section by establishing a characterization result that provides the necessary and sufficient conditions for a function to belong to the class $\mathcal{B}_{\nu}^{*}(\lambda, \mu, \delta, \tau; \alpha)$.

Theorem 1. *Let $h \in \mathcal{A}^*$ be defined by the expansion*

$$h(z) = z - \sum_{n=2}^{\infty} c_n z^n.$$

Then h belongs to the class $\mathcal{B}_\nu^*(\lambda, \mu, \delta, \tau; \alpha)$ if and only if it satisfies the condition:

$$\sum_{n=2}^{\infty} \Big[(\tau - \delta + n\delta + \lambda\mu n(n-1)\Big] \left(\frac{n+\nu}{1+\nu}\right)^r c_n \leq 1 - \alpha. \tag{6}$$

Proof. From the class condition (5) and applying Lemma 1, we can write

$$\Re\left\{(1-\delta)\left(\frac{T_\nu^r h(z)}{z}\right)^\tau + \delta\,(T_\nu^r h(z))'\left(\frac{T_\nu^r h(z)}{z}\right)^{\tau-1} + \lambda\mu z (T_\nu^r h(z))''\right\}$$

$$= \Re\left\{1 + \sum_{n=2}^{\infty} [\tau - \delta + n\delta + \lambda\mu n(n-1)]\left(\frac{n+\nu}{1+\nu}\right)^r c_n z^{n-1}\right\} > \alpha.$$

Taking the limit as $z \to 1^-$ along the real axis yields the validity of inequality (6). On the other hand, assume inequality (6) holds, to prove h belongs to class $\mathcal{B}_\nu^*(\lambda, \mu, \delta, \tau; \alpha)$, we need to show that for every $z \in \mathbb{U}$ that

$$\begin{aligned}
&\left|(1-\delta)\left(\frac{T_\nu^r h(z)}{z}\right)^\tau + \delta\,(T_\nu^r h(z))'\left(\frac{T_\nu^r h(z)}{z}\right)^{\tau-1} + \lambda\mu z\,(T_\nu^r h(z))'' - 1\right| \\
&= \left|\sum_{n=2}^{\infty} (\tau - \delta + n\delta + \lambda\mu n(n-1))\left(\frac{n+\nu}{1+\nu}\right)^r c_n z^{n-1}\right| \\
&\leq \sum_{n=2}^{\infty} (\tau - \delta + n\delta + \lambda\mu n(n-1))\left(\frac{n+\nu}{1+\nu}\right)^r c_n |z|^{n-1} \\
&\leq \sum_{n=2}^{\infty} (\tau - \delta + n\delta + \lambda\mu n(n-1))\left(\frac{n+\nu}{1+\nu}\right)^r c_n \leq 1 - \alpha.
\end{aligned}$$

This completes the proof.

Corollary 1. *Suppose $h(z)$, as defined in equation (2), belongs to the class $\mathcal{B}_\nu^*(\lambda, \mu, \delta, \tau; \alpha)$. Then*

$$c_n \leq \frac{1-\alpha}{[\tau - \delta + n\delta + \lambda\mu n(n-1)]\left(\frac{n+\nu}{1+\nu}\right)^r}, \quad n \geq 2. \tag{7}$$

The bound (7) is sharp and attained by the function

$$h(z) = z - \frac{1-\alpha}{[\tau - \delta + n\delta + \lambda\mu n(n-1)]\left(\frac{n+\nu}{1+\nu}\right)^r} z^n.$$

3. Growth and Distortion Theorems

In this section, we obtain upper and lower bounds for $|h(z)|$ and $|h'(z)|$ when h belongs to the class $\mathcal{B}^*_\nu(\lambda, \mu, \delta, \tau; \alpha)$.

Theorem 2. *Let $h(z) = z - \sum_{n=2}^{\infty} c_n z^n$ be in the class $\mathcal{B}^*_\nu(\lambda, \mu, \delta, \tau; \alpha)$. Then, for $|z| = t < 1$, the following bounds hold*

$$|h(z)| \in \left[t - At^2,\ t + At^2\right], \tag{8}$$

where

$$A = \frac{1-\alpha}{(\tau + \delta + 2\lambda\mu)(\frac{2+\nu}{1+\nu})^r}.$$

The bound given by inequality (8) is sharp, and equality is attained for the function

$$f(z) = z - Az^2.$$

Proof. Let $h(z) \in \mathcal{B}^*_\nu(\lambda, \mu, \delta, \tau; \alpha)$ be defined by (2). Assume $|z| = t < 1$. Then,

$$|h(z)| \le |z| + \sum_{n=2}^{\infty} c_n |z|^n \le t + t^2 \sum_{n=2}^{\infty} c_n.$$

Then,

$$\sum_{n=2}^{\infty} [\tau - \delta + n\delta + \lambda\mu n(n-1)] \left(\frac{n+\nu}{1+\nu}\right)^r c_n \le 1 - \alpha$$

equivalently,

$$\begin{aligned} \sum_{n=2}^{\infty} c_n &\le \frac{1-\alpha}{[\tau - \delta + n\delta + \lambda\mu n(n+1)] \left(\frac{n+\nu}{1+\nu}\right)^r} \\ &\le \frac{1-\alpha}{(\tau + \delta + 2\lambda\mu) \left(\frac{2+\nu}{1+\nu}\right)^r}. \end{aligned}$$

Hence,

$$|h(z)| \le t + \frac{1-\alpha}{(\tau + \delta + 2\lambda\mu) \left(\frac{2+\nu}{1+\nu}\right)^r} t^2$$

Similarly, we can apply the same argument for the lower bound

$$|h(z)| \ge |z| - \sum_{n=2}^{\infty} c_n |z|^n \ge t - \frac{1-\alpha}{(\tau + \delta + 2\lambda\mu) \left(\frac{2+\nu}{1+\nu}\right)^r} t^2.$$

Therefore, the estimate in Theorem 2 follows.

Theorem 3. *Let $h(z)$ be in the class $\mathcal{B}_{\nu}^{*}(\lambda,\mu,\delta,\tau;\alpha)$. Then, for $|z| = t < 1$, the $h'(z)$ satisfies the inequality*

$$1 - Bt \leq |h'(z)| \leq 1 + Bt, \tag{9}$$

where

$$B = \frac{2(1-\alpha)}{(\tau+\delta+2\lambda\mu)\left(\frac{2+\nu}{1+\nu}\right)^{r}}.$$

The bounds are sharp, and equality in (9) is attained by the function

$$h(z) = z - \frac{1-\alpha}{(\tau+\delta+2\lambda\mu)\left(\frac{2+\nu}{1+\nu}\right)^{r}} z^{2}.$$

Proof. A similar argument as in the proof of Theorem 3.1 can be applied here by considering the derivative of f.

4. Closure Properties

In this section, we establish that the class $\mathcal{B}_{\nu}^{*}(\lambda,\mu,\delta,\tau;\alpha)$ is closed under convex combinations and averaging. This follows naturally from the linearity of the operator $T_{\nu}^{r}h(z)$ and the sub-additive behavior of the defining inequality.

Theorem 4. *Let $h_j(z) = z - \sum_{n=2}^{\infty} c_{nj} z^n \in \mathcal{B}_{\nu}^{*}(\lambda,\mu,\delta,\tau;\alpha)$ for $j = 1, 2, \ldots, N$. Then the average function*

$$H(z) := \frac{1}{N}\sum_{j=1}^{N} h_j(z) = z - \sum_{n=2}^{\infty}\left(\frac{1}{N}\sum_{j=1}^{N} c_{nj}\right) z^n$$

also belongs to the class $\mathcal{B}_{\nu}^{*}(\lambda,\mu,\delta,\tau;\alpha)$.

Proof. Let $b_n = \frac{1}{N}\sum_{j=1}^{N} c_{nj}$. Then, using linearity and convexity of the modulus

$$\begin{aligned}
&\sum_{n=2}^{\infty}[\tau-\delta+n\delta+\lambda\mu n(n-1)]\left(\frac{n+\nu}{1+\nu}\right)^{r} b_n \\
&= \sum_{n=2}^{\infty}[\tau-\delta+n\delta+\lambda\mu n(n-1)]\left(\frac{n+\nu}{1+\nu}\right)^{r}\left(\frac{1}{N}\sum_{j=1}^{N} c_{nj}\right) \\
&= \frac{1}{N}\sum_{j=1}^{N}\left(\sum_{n=2}^{\infty}[\tau-\delta+n\delta+\lambda\mu n(n-1)]\left(\frac{n+\nu}{1+\nu}\right)^{r} c_{nj}\right) \\
&\leq \frac{1}{N}\sum_{j=1}^{N}(1-\alpha) = 1-\alpha
\end{aligned}$$

Hence, $H \in \mathcal{B}_{\nu}^{*}(\lambda,\mu,\delta,\tau;\alpha)$.

Theorem 5. *The class $\mathcal{B}_\nu^*(\lambda, \mu, \delta, \tau; \alpha)$ is convex for any $h_1, h_2 \in \mathcal{B}_\nu^*(\lambda, \mu, \delta, \tau; \alpha)$ and any $0 \leq \omega \leq 1$, the function*

$$T(z) := \omega h_1(z) + (1 - \omega)h_2(z)$$

also belongs to $\mathcal{B}_\nu^*(\lambda, \mu, \delta, \tau; \alpha)$.

Proof. Let

$$h_j(z) = z - \sum_{n=2}^{\infty} c_{nj} z^n, \quad j = 1, 2$$

and define

$$T(z) = z - \sum_{n=2}^{\infty} \left[\omega c_{n1} + (1 - \omega)c_{n2}\right] z^n.$$

As before, the inequality becomes

$$\begin{aligned}
&\sum_{n=2}^{\infty} [\tau - \delta + n\delta + \lambda\mu n(n-1)] \left(\frac{n+\nu}{1+\nu}\right)^r [wc_{n1} + (1-w)c_{n2}] \\
&= w \sum_{n=2}^{\infty} [\tau - \delta + n\delta + \lambda\mu n(n-1)] \left(\frac{n+\nu}{1+\nu}\right)^r c_{n1} \\
&+ (1-w) \sum_{n=3}^{\infty} [\tau - \delta + n\delta + \lambda\mu n(n-1)] \left(\frac{n+\nu}{1+\nu}\right)^r c_{n2} \\
&\leq w(1-\alpha) + (1-w)(1-\alpha) = 1 - \alpha
\end{aligned}$$

Thus $T \in \mathcal{B}_\nu^*(\lambda, \mu, \delta, \tau; \alpha)$.

5. Radii of Close-to-Convexity, Starlikeness, and Convexity

Let us now derive the radii within which a function $h \in \mathcal{B}_\nu^*(\lambda, \mu, \delta, \tau; \alpha)$ exhibits the standard geometric behaviors of close-to-convexity, starlikeness, and convexity.

Let $\mathcal{A}$ denote the class of normalized analytic functions in the unit disk $\mathbb{U}$. For $0 \leq \beta < 1$, we define the following important subclasses:

- The class of close-to-convex functions of order β

$$\mathcal{C}(\beta) = \left\{h \in \mathcal{A} : \mathrm{Re}\left\{h'(z)\right\} > \beta\right\}$$

- The class of starlike functions of order β

$$\mathcal{S}^*(\beta) = \left\{h \in \mathcal{A} : \mathrm{Re}\left\{\frac{zh'(z)}{h(z)}\right\} > \beta\right\}$$

- The class of convex functions of order β

$$\mathcal{K}(\beta) = \left\{h \in \mathcal{A} : \mathrm{Re}\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > \beta\right\}$$

In the following, we aim to determine these properties for functions belonging to the class $\mathcal{B}_\nu^*(\lambda, \mu, \delta, \tau; \alpha)$.

Close-to-Convexity Radius

Theorem 6. *Let $h \in \mathcal{B}_\nu^*(\lambda, \mu, \delta, \tau; \alpha)$. Then $h \in \mathcal{C}(\beta)$ in the disk $|z| < r_1$, where*

$$r_1 = \inf_{n\geq 2} \left\{ \frac{(1-\beta)\left[(\tau-\delta)+n\delta+\lambda\mu n(n-1)\right]\left(\frac{n+\nu}{1+\nu}\right)^r}{n(1-\alpha)} \right\}^{1/(n-1)}. \tag{10}$$

Proof.
For $h(z) = z - \sum_{n=2}^{\infty} c_n z^n$, we have

$$h'(z) = 1 - \sum_{n=2}^{\infty} n c_n z^{n-1}.$$

To ensure $\Re\{h'(z)\} > \beta$, it is sufficient that

$$|h'(z) - 1| \leq 1 - \beta \quad \text{for } z \in \mathbb{U}.$$

That is,

$$\sum_{n=2}^{\infty} n c_n r^{n-1} \leq 1 - \beta.$$

Using the coefficient bound property represented by Theorem 1, we get

$$n|z|^{n-1} \leq \frac{(1-\beta)[\tau-\delta+n\delta+\lambda\mu n(n-1)]\left(\frac{n+\nu}{1+\nu}\right)^r}{1-\alpha}$$

$$|z| \leq \inf_{n\geq 2} \left\{ \frac{(1-\beta)\left[(\tau-\delta)+n\delta+\lambda\mu n(n-1)\right]\left(\frac{n+\nu}{1+\nu}\right)^r}{n(1-\alpha)} \right\}^{1/(n-1)}, \tag{11}$$

which yields the radius in (10).

Starlikeness Radius

Theorem 7. *If $h \in \mathcal{B}_\nu^*(\lambda, \mu, \delta, \tau; \alpha)$, then h is starlike of order β in the disk $|z| < r_2$, where*

$$r_2 = \inf_{n\geq 2} \left\{ \frac{(1-\beta)\left[(\tau-\delta)+n\delta+\lambda\mu n(n-1)\right]\left(\frac{n+\nu}{1+\nu}\right)^r}{(n-\beta)(1-\alpha)} \right\}^{1/(n-1)}. \tag{12}$$

Proof. Using similar argument in the previous theorem, we can write

$$\left|\frac{zf'(z)}{f(z)} - 1\right|$$

$$= \left|\frac{\sum_{n=2}^{\infty}(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} a_n z^{n-1}}\right| \leq \frac{\sum_{n=2}^{\infty}(n-1)a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty} a_n |z|^{n-1}} \leq 1-\beta$$

$$(n-\beta)|z|^{n-1} \leq \frac{(1-\beta)\left[\tau-\delta+n\delta+\lambda\mu n(n-1)\right]\left(\frac{n+\nu}{1+\nu}\right)^r}{1-\alpha}$$

$$|z| \leq \inf_{n\geq 2}\left\{\frac{(1-\beta)\left[(\tau-\delta)+n\delta+\lambda\mu n(n-1)\right]\left(\frac{n+\nu}{1+\nu}\right)^r}{(n-\beta)(1-\alpha)}\right\}^{1/(n-1)}. \tag{13}$$

Convexity Radius

Theorem 8. *If $h \in \mathcal{B}_\nu^*(\lambda, \mu, \delta, \tau; \alpha)$, then h is convex of order β in the disk $|z| < r_3$, where*

$$r_3 = \inf_{n\geq 2}\left\{\frac{(1-\beta)\left[(\tau-\delta)+n\delta+\lambda\mu n(n-1)\right]\left(\frac{n+\nu}{1+\nu}\right)^r}{n(n-\beta)(1-\alpha)}\right\}^{1/(n-1)}. \tag{14}$$

Proof. Following arguments analogous to those of Theorems 5.1 and 5.2, we obtain the expression for the radius given in (14).

6. Conclusion

In this paper, we introduce a new subclass of analytic functions $\mathcal{B}_\nu^*(\lambda, \mu, \delta, \tau; \alpha)$, which is defined by a generalized multiplier operator T_ν^r . We derive sharp coefficient estimates, establish growth and distortion theorems, and determine the radii of close-to-convexity, starlikeness, and convexity. The proposed class unifies and generalizes several well-known subclasses in geometric function theory. Potential directions for future work include exploring inclusion relationships, neighborhood properties, and the behavior of partial sums within this class.

Conflict of Interest

The authors declare that there are no conflicts of interest.

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