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Amicable Sets in Paradistributive Latticoids

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Abstract. In this work, we introduce maximal sets and amicable sets in a paradistributive latticoid. We present a few number of examples and counter-examples for them, prove some algebraic properties on them. Also, we define center of a paradistributive latticoid and prove that it is equal to the intersection of all maximal sets. Finally, we obtain some necessary and sufficient conditions for a paradistributive latticoid to become relatively complemented.

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1. Introduction

The exploration of algebraic structures such as lattices and their generalizations constitutes a fundamental area of study in universal algebra and mathematical logic. A recent advancement in this direction is the development of the theory of *paradistributive latticoids* (PDLs) [1], which extend classical distributive lattices by relaxing conventional distributive constraints while maintaining essential structural properties. Introduced by Bandaru and Ajjarapu, PDLs are algebras of type $(2, 2, 1)$, equipped with binary operations $\vee$ and $\wedge$ and a designated greatest element 1, and they satisfy a set of axioms that emulate distributive behavior within a more generalized algebraic setting.

This work builds upon the foundational framework of PDLs to introduce and systematically investigate the concepts of *compatible sets*, *maximal sets*, and *amicable sets*. A

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compatible set is defined as a subset whose elements are mutually commutative under the operations $\vee$ and $\wedge$, and a maximal set is characterized as a compatible set that is maximal with respect to inclusion. The novel notion of *amicability* is introduced to describe maximal sets that exhibit a unifying interaction with the entire lattice, thereby revealing new algebraic insights. Through a rigorous methodology comprising formal definitions, illustrative examples, counterexamples, and theorems, this paper develops a comprehensive theory surrounding these structures. In particular, it establishes that the *center* of a PDL defined as the set of elements compatible with all elements of the lattice is precisely the intersection of all maximal sets, and furthermore, that this center forms a filter. The study of amicable sets serves as a key tool in the structural analysis of PDLs, leading to a new characterization of relatively complemented PDLs a property of considerable interest due to its connections with completeness and decomposability in lattice theory [2, 3]. The contributions of this work enhance the algebraic understanding of paradistributive latticoids by introducing and analyzing new invariants and internal structures. These results extend earlier research on normal PDLs [4] and parapseudo-complementation [5], and are grounded in the classical algebraic foundations laid by Birkhoff [2] and Boole [6]. As such, this paper provides a substantive addition to the theoretical landscape of generalized lattice systems.

2. Preliminaries

In this section, we present necessary definitions and properties of paradistributive latticoids which are taken from [1] for quick reference and to develop the theory.

Definition 2.1. [1] An algebra $(\mathbb{V}, \vee, \wedge, 1)$ of type $(2, 2, 1)$ is called a Paradistributive Latticoid (Abbreviated as PDL), if it assures the following axioms:

(i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(ii) $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$

(iii) $(a \vee b) \wedge b = b$

(iv) $(a \vee b) \wedge a = a$

(v) $a \vee (a \wedge b) = a$

(vi) $a \vee 1 = 1$,

for all $a, b, c \in \mathbb{V}$.

Given a, b in a paradistributive latticoid $\mathbb{V}$, we say that a is less than or equal to b and write $a \leq b$, if $a \wedge b = a$ or equivalently $a \vee b = b$. It is observed that $\leq$ is a partial order on $\mathbb{V}$ and the element 1 is the greatest element. Throughout this paper, we refer $\mathbb{V}$ as a Paradistributive Latticoid with the greatest element 1.

Example 2.2. [1] Let $\mathbb{V}$ be a non-empty set. Fix some element $b_0 \in \mathbb{V}$. For any $a, b \in \mathbb{V}$, define $\vee$ and $\wedge$ on $\mathbb{V}$ by

$$a \vee b = \begin{cases} a, & \text{if } b \neq b_0 \\ b_0, & \text{if } b = b_0 \end{cases} \quad \text{and} \quad a \wedge b = \begin{cases} b, & \text{if } b \neq b_0 \\ a, & \text{if } b = b_0. \end{cases}$$

Then $(\mathbb{V}, \vee, \wedge, b_0)$ is called a disconnected PDL, and b_0 is the greatest element.

Lemma 2.3. *[1] For any $a, b \in \mathbb{V}$,*

(i) $a \wedge a = a$

(ii) $a \vee a = a$

(iii) $(a \wedge b) \vee b = b$

(iv) $a \vee (b \wedge a) = a$

(v) $a \wedge (a \vee b) = a.$

Lemma 2.4. *[1] For any $a, b, c, d \in \mathbb{V}$,*

(i) $a \wedge 1 = a$

(ii) $1 \wedge a = a$

(iii) $1 \vee a = 1$

(iv) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

(v) $a \vee (b \wedge c) = a \vee (c \wedge b)$

(vi) $\vee$ *is associative in* $\mathbb{V}$

(vii) $d \vee [a \wedge (b \wedge c)] = d \vee [(a \wedge b) \wedge c]$

(viii) $a \vee (b \vee c) = a \vee (c \vee b)$

(ix) $a \vee b = 1 \iff b \vee a = 1$

(x) $a \vee b = 1 \implies b \wedge a = a \wedge b.$

Definition 2.5. [1] An element $m \in \mathbb{V}$ is said to be minimal, if for any $a \in \mathbb{V}, m \leq a$ implies $m = a$.

Lemma 2.6. *[1] For any $m \in \mathbb{V}$, the following are equivalent;*

(i) m *is minimal,*

(ii) $a \wedge m = m$, *for all* $a \in \mathbb{V}$

(iii) $a \vee m = a$, *for all* $a \in \mathbb{V}$.

A non-empty subset F of $\mathbb{V}$ is said to be a filter, if for any $a, b \in F, x \in \mathbb{V}$, $a \wedge b \in F$ and $x \vee a \in F$. Let S be a non-empty subset of $\mathbb{V}$. Then $[S) = \{a \vee (\wedge_{i=1}^{n} s_i) \mid s_i \in S, a \in \mathbb{V}, 1 \leq i \leq n$ and n is a positive integer $\}$ is the smallest filter of $\mathbb{V}$ containing S. If $S = \{a\}$, then we write $[S) = [a)$, the principal filter generated by a. The collection $F(\mathbb{V})$ of all filters of $\mathbb{V}$ forms a distributive lattice under set inclusion, in which, the glb and lub of any two filters F and G are given by $F \wedge G = F \cap G$ and $F \vee G = \{f \wedge g \mid f \in F$ and $g \in G\}$ respectively.

Lemma 2.7. *[1] Given a filter F of $\mathbb{V}$ and $a, b \in \mathbb{V}$,*

(i) $[a) = \{x \vee a \mid x \in \mathbb{V}\}$

(ii) $a \in [b) \iff a = a \vee b$

(iii) $a \vee b \in F \iff b \vee a \in F$

(iv) $[a \vee b) = [b \vee a) = [a) \wedge [b)$

(v) $[a \wedge b) = [b \wedge a) = [a) \vee [b)$.

Lemma 2.8. *[1] For any $a, b \in \mathbb{V}$,*

(i) $(a \vee b) \vee b = a \vee b$

(ii) $(a \vee b) \vee a = a \vee b$

(iii) $a \vee (a \vee b) = a \vee b$

(iv) $a \wedge (a \wedge b) = a \wedge b$

(v) $(a \wedge b) \wedge b = a \wedge b$

(vi) $b \wedge (a \wedge b) = a \wedge b$.

Theorem 2.9. *[1] The following are equivalent in $\mathbb{V}$;*

(i) $(\mathbb{V}, \vee, \wedge, 1)$ *is a distributive lattice*

(ii) The poset $(\mathbb{V}, \leq)$ *is directed below*

(iii) $(a \wedge b) \vee a = a$, *for all* $a \in \mathbb{V}$

(iv) $\wedge$ *is commutative*

(v) $\vee$ *is commutative*

(vi) The relation $R = \{(a, b) \in \mathbb{V} \times \mathbb{V} \mid b \vee a = b\}$ *is anti-symmetric.*

3. Amicable Sets in Paradistributive Latticoid

In this section we define compatible sets and then maximal sets in a paradistributive latticoid and extensively study their algebraic properties, and provided a good number of counter-examples in each case. We define the center of a paradistributive latticoid using compatibility, and then we prove that the center is a filter and it is the intersection of all maximal sets in a paradistributive latticoid. Finally, we define amicable sets in a paradistributive latticoid and then we characterize paradistributive latticoids using amicable sets. We obtain some necessary and sufficient conditions for a paradistributive latticoid to become relatively complemented.

Definition 3.1. An element $a \in \mathbb{V}$ is said to be compatible with an element $b \in \mathbb{V}$, if $a \vee b = b \vee a$ or equivalently $a \wedge b = b \wedge a$, and it is denoted by $a \sim b$. A subset S of $\mathbb{V}$ is said to be compatible, if $a \sim b$, for all $a, b \in S$.

Lemma 3.2. *For any $a, b, c \in \mathbb{V}$, we have the following;*

(i) $1 \sim a$

(ii) $\sim$ *is reflexive*

(iii) $\sim$ *is symmetric*

(iv) $a \leq b \implies a \sim b$

(v) $a \sim b \implies c \vee a \sim c \vee b,\ a \wedge c \sim b \wedge c,\ a \vee c \sim b \vee c.$

Proof. Let $a, b, c, \in \mathbb{V}$. Then (i) $a \wedge 1 = a = 1 \wedge a \Rightarrow a \sim 1$ (by Lemma 2.4((i) & (ii)).
(ii) $a \wedge a = a = a \wedge a \Rightarrow a \sim a$ (by Lemma 2.3(i)).
(iii) If $a \sim b$, then $a \wedge b = b \wedge a$. Therefore $b \sim a$.
(iv) If $a \leq b$, then $a \wedge b = a$. Now, $b \wedge a = b \wedge (a \wedge b) = a \wedge b$ (By Lemma 2.8(vi)). Therefore $a \sim b$.
(v) Suppose $a \sim b$. Then,

$$\begin{aligned}(c \vee a) \wedge (c \vee b) &= c \vee (a \wedge b) && \text{(By Definition 2.1(i))}\\ &= c \vee (b \wedge a) && \text{(By Lemma 2.4(v))}\\ &= (c \vee b) \wedge (c \vee a). && \text{(By Definition 2.1(i))}\end{aligned}$$

Therefore $c \vee a \sim c \vee b$. Now,

$$\begin{aligned}(a \wedge c) \vee (b \wedge c) &= (a \vee b) \wedge c && \text{(By Lemma 2.4(iv))}\\ &= (b \vee a) \wedge c && \text{(Since } a \sim b\text{)}\\ &= (b \wedge c) \vee (a \wedge c). && \text{(By Lemma 2.4(iv))}\end{aligned}$$

Therefore $a \wedge c \sim b \wedge c$. Similarly,

$$\begin{aligned}(a \vee c) \wedge (b \vee c) &= (a \wedge b) \vee c && \text{(By Definition 2.1(ii))}\\ &= (b \wedge a) \vee c && \text{(Since } a \sim b\text{)}\\ &= (b \vee c) \wedge (a \vee c). && \text{By Definition 2.1(ii))}\end{aligned}$$

Therefore $a \vee c \sim b \vee c$.

Remark 3.3. Given $a, b, c \in \mathbb{V}$, $a \sim b$ does not imply $(c \wedge a) \sim (c \wedge b)$. For, see the following example;

Example 3.4. Let $\mathbb{V} = \{a, b, c, d, 1\}$ be a set with binary operations $\vee$ and $\wedge$ given in the following;

$\vee$	a	1	b	c	d
a	a	1	a	c	c
1	1	1	1	1	1
b	b	1	b	d	d
c	c	1	c	c	c
d	d	1	d	d	d

$\wedge$	a	1	b	c	d
a	a	a	b	a	b
1	a	1	b	c	d
b	a	b	b	a	b
c	a	c	b	c	d
d	a	d	b	c	d

Then $(\mathbb{V}, \vee, \wedge, 1)$ is a Paradistributive Latticoid, and $1 \sim b$ but $(c \wedge 1) \vee (c \wedge b) \neq (c \wedge b) \vee (c \wedge 1))$.

Theorem 3.5. *$\mathbb{V}$ is distributive if and only if $a \sim b$ implies $c \wedge a \sim c \wedge b$, for all $a, b, c \in \mathbb{V}$.*

Proof. If $\mathbb{V}$ is distributive, then it is easy to verify the result. On the other hand, let $a, b \in L$. By our assumption, $b \wedge 1 \sim b \wedge a$ (that is $(b \wedge 1) \vee (b \wedge a) = (b \wedge a) \vee (b \wedge 1)$). So that $b \vee (b \wedge a) = (b \wedge a) \vee b$. By Definition 2.1(v)., $b = (b \wedge a) \vee b$. Therefore $\mathbb{V}$ is a distributive lattice (By Theorem 2.9.).

Example 3.6. Let I be a non-empty set and $\mathbb{V}$ is a disconnected PDL. Then $(\mathbb{V}^I, \vee, \wedge, 1)$ is a PDL with respect to the point wise operations, where $\bar{1} : I \to \mathbb{V}$ defined by $\bar{1}(i) = 1$ for all $i \in I$, is the greatest element in $\mathbb{V}$. In this $\mathbb{V}$, for any $x, y \in \mathbb{V}$, we have
(i) $x \sim y$ if and only if $x|_{|x| \cap |y|} = y|_{|x| \cap |y|}$ and
(ii) $x \leq y$ if and only if $x_{|x|} = y_{|y|}$, where $|x| = \{i \in I | x(i) \neq 1\}$.

Example 3.7. Let R be a commutative regular ring with unity [1]. For any $a, b \in R$, $a \sim b$ if and only if $ab^2 = a^2 b$. For any $a, b \in \mathbb{V}$,

$$\begin{array}{llcll} a \sim b & \Rightarrow a \vee b & = & b \vee a & \\ & \Rightarrow b_0 a & = & a_0 b & (a \vee b = b_0 a) \\ & \Rightarrow b b_0 a & = & b a_0 b & \\ & \Rightarrow ba & = & b^2 a_0 & (b b_0 = b) \\ & \Rightarrow baa & = & b^2 a_0 a & \\ & \Rightarrow ba^2 & = & b^2 a & (a_0 a = a) \\ & \Rightarrow a^2 b & = & ab^2. & (\text{commutative}) \end{array}$$

Now,

$$\begin{array}{llcll} a^2 b = ab^2 & \Rightarrow ab(a-b) & = & 0 & \\ & \Rightarrow a_0 b_0 (a-b) & = & 0 & (aR = a_0 R) \\ & \Rightarrow a_0 b_0 a - a_0 b_0 b & = & 0 & \\ & \Rightarrow b_0 a_0 a & = & a_0 b & (b b_0 = b) \\ & \Rightarrow b_0 a & = & a_0 b & (a a_0 = a) \\ & \Rightarrow a \wedge b & = & b \wedge a. & \end{array}$$

Therefore $a \sim b$.

Definition 3.8. A maximal compatible set in $\mathbb{V}$ is called a maximal set.

Example 3.9. In the Example 2.2., $\{x, 1\}$ is a maximal set, for any $x \in \mathbb{V}$ and $x \neq 1$.

Example 3.10. Let $\mathbb{V}$ be a disconnected PDL, I an infinite set and $L = \{x \in \mathbb{V}^I \mid |x|$ is finite $\}$, where $|x| = \{i \in I \mid x(i) \neq 1\}$. Then $(L, \vee, \wedge, 1)$ is a PDL with respect to the point wise operations, where $\bar{1} : I \to \mathbb{V}$, defined by $\bar{1}(i) = 1$, for all $i \in I$. In this L, we observe the following;
(i) L has no minimal elements
Let $x \in L$. Then $|x|$ is finite. Choose $i_0 \in I$ such that $i_0 \notin |x|$ and fix $1 \neq d \in \mathbb{V}$. Now, for $i \in I$, define,

$$y(i) = \begin{cases} x(i), & \text{if } i \in |x| \\ d, & \text{if } i = i_0 \\ 1, & \text{otherwise.} \end{cases}$$

Then $y \in L$ and $y \leq x$. Therefore L has no minimal element.
(ii) If $p \in (\mathbb{V} - \{0\})^I$, then the set $M_p = \{p_S \mid S$ is a finite subset of $I\}$ is a maximal set in L, where $p_S : I \to \mathbb{V}$ is defined by

$$p_S(i) = \begin{cases} p(i), & \text{if } i \in S \\ 1, & \text{if } i \notin S, \end{cases}$$

for all $i \in I$. Let S_1, S_2 be two finite subsets of I and $i \in S_1 \cap S_2$. Now,

$$\begin{aligned} (p_{S_1} \wedge p_{S_2})(i) \quad & \Rightarrow p_{S_1}(i) \wedge p_{S_2}(i) \\ & \Rightarrow p(i) \wedge p(i) \\ & \Rightarrow p_{S_2}(i) \wedge p_{S_1}(i) \\ & \Rightarrow (p_{S_2} \wedge p_{S_1})(i). \end{aligned}$$

If $i \notin S_1 \cap S_2$, then $(p_{S_1} \wedge p_{S_2})(i) = p_{S_1}(i) \wedge p_{S_2}(i) = 1$. Therefore $p_{S_1} \wedge p_{S_2} = p_{S_2} \wedge p_{S_1}$, and hence M_p is a compatible set. Let $x \in L$ such that $x \sim p_S$ for all finite subset S of I. Since $|x|$ is a finite subset of I, we have $x \sim p_{|x|}$. Therefore $x_{|x|} = p_{|x|}$ and hence $x = p_{|x|} \in M_p$. Thus M_p is a maximal set.

Example 3.11. The set $M_m = \{a \in \mathbb{V} \mid m \leq a\}$, where m is a minimal element in $\mathbb{V}$, is a maximal set in $\mathbb{V}$.
Let m be a minimal element in $\mathbb{V}$. Then $m \in M_m$. Therefore M_m is non-empty. Let $a, b \in M_m$. Then $m \leq a, b$. Therefore $m \wedge a = m \wedge b = m$. Now, $a \wedge b = (m \vee a) \wedge (m \vee b) = m \vee (a \wedge b) = m \vee (b \wedge a) = (m \vee b) \wedge (m \vee a) = b \wedge a$. Therefore $a \wedge b = b \wedge a$. So that $a \sim b$. Hence M_m is a compatible set. Assume that S is a compatible set such that $M_m \subseteq S$. Let $c \in S$. Then $c \ \sim s$, for all $s \in S$. Since $M_m \subseteq S$, $a \sim m$. Now, $m = a \wedge m = m \wedge a$. Then $m \leq a$. Therefore $a \in M_m$. So that $S \subseteq M_m$. Hence $M_m = S$. Thus M_m is a maximal set in $\mathbb{V}$.

Let $\{\mathbb{V}_i\}_{i\in\Delta}$ be a family of PDLs. Then it is easy to observe that the product $\prod_{i\in\Delta}\mathbb{V}_i$ with point wise operations is a PDL. Now, we have the following;

Theorem 3.12. *For any non-empty subset M of $\prod_{i\in\Delta}\mathbb{V}_i$, M is maximal set in $\prod_{i\in\Delta}\mathbb{V}_i$ if and only if $M = \prod_{i\in\Delta} M_i$, where M_i is a maximal set in $\mathbb{V}_i$.*

Proof. Let $\{\mathbb{V}_i\}_{i\in\Delta}$ be the family of PDLs. If M_i is a maximal set in $\mathbb{V}_i$. Then clearly $\prod_{i\in\Delta} M_i$ is a maximal set. On the other hand, let M be the set such that $M_i = \{a(i) \mid a \in M\}$, then M_i is non-empty and compatible set in $\mathbb{V}_i$. Let $b \in \mathbb{V}_i$ is such that $b \sim a(i)$, for all $a \in M$. Now, define $c \in \prod_{j\in\Delta}\mathbb{V}_j$ by $c(j) = b$, if $i = j$ and 1_j if $i \neq j$. Now, $d \in M$ and $j \in \Delta, (d \wedge c)(j) = d(j) \wedge c(j) = c(j) \wedge d(j) = (c \wedge d)(j)$. Therefore $c \sim d$, for all $d \in M$. So that $c \in M$ and $c(j) = b \in M_i$. Hence M_i is a maximal set in $\mathbb{V}_i$ for all $i \in \Delta$. Since $\prod_{i\in\Delta} M_i$ is a compatible set containing M. Therefore $M = \prod_{i\in\Delta} M_i$.

Lemma 3.13. *Let M be a maximal set in $\mathbb{V}$ and $c \in \mathbb{V}$ such that $c \sim a$, for all $a \in M$. Then $c \in M$.*

Proof. Let $M' = M \cup \{c\}$. Then M' is compatible set and $M \subseteq M'$. By the maximality of M, $M = M'$ and hence $c \in M$.

Theorem 3.14. *Let M be a maximal set in $\mathbb{V}$. Then we have*

(i) M contains 1

(ii) M is distributive lattice with greatest element 1

(iii) M is an initial segment.

Proof. (i) Let $a \in M$. Then $a \wedge 1 = a = 1 \wedge a$. Therefore $1 \sim a$, for all $a \in M$. By Lemma 3.13., $1 \in M$. (ii) Let $a, b \in M$. Then $a \sim b$. Now, $a \wedge (a \wedge b) = a \wedge b$ and $(a \wedge b) \wedge a = (b \wedge a) \wedge a = b \wedge a = a \wedge b$. Therefore $a \sim a \wedge b$. By Lemma 3.13., $a \wedge b \in M$. Now, $a \wedge (a \vee b) = a$ and $(a \vee b) \wedge a = (b \vee a) \wedge a = a$. So that M is closed under $\vee$ and $\wedge$. By Theorem 2.9., M is a distributive lattice with greatest element 1. (iii) Let $a \in M$ and $b \in \mathbb{V}$. If $b \leq a$, then $b \wedge a = b$. Now, $a \wedge b = a \wedge (b \wedge a) = b \wedge a$ (By Lemma 2.8(vi)). Therefore $a \sim b$, for all $a \in M$. By Lemma 3.13., $b \in M$. Hence M is an initial segment.

Definition 3.15. In a PDL $\mathbb{V}$, we define the center of $\mathbb{V}$ as the set $C(\mathbb{V}) = \{a \in V \mid a \sim c, \text{ for all } c \in V\}$.

Theorem 3.16. *$C(\mathbb{V})$ is the intersection of all maximal sets in $\mathbb{V}$.*

Proof. If $a \in C(\mathbb{V})$ and M is a maximal set in $\mathbb{V}$. Then $M \cup \{a\}$ is compatible and by the maximality of M, $a \in M$ (By Lemma 3.13.). Therefore $C(\mathbb{V}) \subseteq M$, for all maximal sets in $\mathbb{V}$. On the other hand, let $a \in \mathbb{V}$ such that $a \notin C(\mathbb{V})$. Then there exists c in $\mathbb{V}$ such that a is not compatible with c. Put $\mathcal{P} = \{\mathbb{C} \subseteq V \mid \mathbb{C} \text{ is compatible}, c \in C \text{ and } a \notin \mathbb{C}\}$. Then $\{c\} \in \mathcal{P}$. Therefore $\mathcal{P}$ is non-empty and $(\mathcal{P}, \subseteq)$ is a partial order set. Let

$\mathbb{C}_1 \subseteq \mathbb{C}_2 \subseteq \mathbb{C}_3 \subseteq ... \subseteq \mathbb{C}_n \subseteq ....$ be a chain in $\mathcal{P}$. If $a \in \cup_{i\in I}\mathbb{C}_i$. Then $a \in \mathbb{C}_i$, for some i which is a contradiction to $a \notin \mathbb{C}_i$, for every i. If $\cup_{i\in I}\mathbb{C}_i$ is not comparable, then there exist $b, c \in \cup_{i\in I}\mathbb{C}_i$ such that b is incomparable with c where $b \in C_i, c \in C_j$, for some $i, j \in I$. But $C_i \subseteq C_j$ or $C_j \subseteq C_i$, $b, c \in C_i$ or $b, c \in C_j$. Then $b \sim c$ which is contradiction. So that $\cup_{i\in I}C_i$ is compatible. Therefore $\cup_{i\in I}C_i$ is an upper bound in $\mathcal{P}$. By Zorn's lemma, $\mathcal{P}$ has a maximal member, say Q. Suppose $\mathbb{N}$ is compatible in V such that $Q \subset \mathbb{N}$. Then $c \in \mathbb{N}$. Since c is incomparable a, $a \notin \mathbb{N}$. Therefore $\mathbb{N} \in \mathcal{P}$. By maximality of Q in $\mathcal{P}$, we get $Q = \mathbb{N}$. Thus Q is maximal set in $\mathbb{V}$ and $a \notin Q$. This proves that $C(\mathbb{V})$ is equal to the intersection of all maximal sets in $\mathbb{V}$.

Theorem 3.17. *$C(\mathbb{V})$ is a filter of $\mathbb{V}$.*

Proof. It is easy to observe that $C(\mathbb{V})$ is the intersection of all maximal sets in $\mathbb{V}$. Since each maximal set is closed under $\wedge$ and $\vee$, we have $C(\mathbb{V})$ is closed under $\wedge$ and $\vee$. Let $a \in C(\mathbb{V}), b, c \in \mathbb{V}$. Then,

$$\begin{aligned} c \vee (b \vee a) &= (c \vee b) \vee a && (\vee \text{ is associative}) \\ &= a \vee (c \vee b) && (\text{since } a \in C(\mathbb{V})) \\ &= a \vee (b \vee c) && (\text{By Lemma 2.4}(viii)) \\ &= (a \vee b) \vee c && (\vee \text{ is associative}) \\ &= (b \vee a) \vee c && (\text{since } a \in C(\mathbb{V})) \\ (b \vee a) \vee c &= c \vee (b \vee a). \end{aligned}$$

Therefore $b \vee a \in C(\mathbb{V})$, for all $b \in \mathbb{V}$. Thus $C(\mathbb{V})$ is a filter of $\mathbb{V}$.

Theorem 3.18. *If $\{\mathbb{V}_\alpha\}_{\alpha\in\Delta}$ is a family of PDLs, then $C(\prod_{\alpha\in\Delta}\mathbb{V}_\alpha) = \prod_{\alpha\in\Delta} C(\mathbb{V}_\alpha)$.*

Proof. Let $a \in C(\prod_{\beta\in\Delta}\mathbb{V}_\beta)$. Then $a \sim x$, for all $x \in \prod_{\beta\in\Delta}\mathbb{V}_\beta$. Let $\alpha \in \Delta$ and $s \in \mathbb{V}_\alpha$. If $x \in \prod_{\beta\in\Delta}\mathbb{V}_\beta)$ such that $x(\alpha) = s$. Since $a \in C(\prod_{\beta\in\Delta}\mathbb{V}_\beta), a \sim x$ and hence $a \wedge x = x \wedge a$. Therefore $(a \wedge x(\alpha)) = a(\alpha \wedge x(\alpha) = x(\alpha) \wedge a(\alpha) = s \wedge a(\alpha)$. This is true for all $s \in \mathbb{V}_\alpha$. So that $a(\alpha) \in C(\mathbb{V}_\alpha)$, for all $\alpha \in \Delta$. Thus $a \in \prod_{\alpha\in\Delta} C(\mathbb{V}_\alpha)$. On the other hand, let $a \in \prod_{\alpha\in\Delta} C(\mathbb{V}_\alpha)$. Then $a(\alpha) \in C(\mathbb{V}_\alpha)$, for all $\alpha \in \Delta$. Let $x \in \prod_{\alpha\in\Delta}\mathbb{V}_\alpha$. Then $x(\alpha) \in \mathbb{V}_\alpha$, for all $\alpha \in \Delta$. Since $a(\alpha) \in C(\mathbb{V}_\alpha)$, for all $\alpha \in \Delta$, $a(\alpha) \sim x(\alpha)$, for all $\alpha \in \Delta$. Therefore $a \sim x$ and hence $a \in C(\prod_{\alpha\in\Delta}\mathbb{V}_\alpha)$. Thus $C(\prod_{\alpha\in\Delta}\mathbb{V}_\alpha) = \prod_{\alpha\in\Delta} C(\mathbb{V}_\alpha)$.

Definition 3.19. A maximal set M in $\mathbb{V}$ is said to be amicable with an element a of $\mathbb{V}$, if there exists $b \in M$ such that $a \vee b = a$. The set M is said to be amicable if it is amicable with every element of $\mathbb{V}$.

Example 3.20. In Example 2.2., for any $x \neq 1$, $\{x, 1\}$ is an amicable set .

Example 3.21. Let $\mathbb{V}$ be a PDL with minimal elements. Then every amicable set is of the form $M_m = \{x \in \mathbb{V} \mid x \leq m\}$, where m is a minimal element in $\mathbb{V}$.

Theorem 3.22. *Let M be a amicable set in $\mathbb{V}$. Then for each $a \in \mathbb{V}$, there exists a unique $a_0 \in M$ such that $[a_0) = [a)$ i.e., $a_0 \vee a = a_0$ and $a \vee a_0 = a$.*

Proof. Let $a \in \vee$. Then there exists $b \in M$ such that $a \vee b = a$. Take $a_0 = b \vee a$. Now, $a_0 \vee b = (b \vee a) \vee b = b \vee (a \vee b = b \vee a = a_0$ (By Lemma 2.4(viii) and $b \vee a_0 = b \vee b \vee a = b \vee a = a_0$. Therefore $a_0 \sim b$. Since $b \in M,\ a_0 \in M$. Now, $a_0 \vee a = b \vee a \vee a = b \vee a = a_0$. Then $[a_0) \subseteq [a)$. Therefore $a \in [a_0)$. Now, $a \vee a = a \vee b \vee a = a \vee a \vee b = a \vee b = a$ (By Lemma 2.8(iii)). Then $[a) \subseteq [a_0)$. Therefore $a \in [a_0)$. Hence $[a_0) = [a)$. Suppose there exists $c \in M$ such that $[c) = [a) = [a_0)$. Since $a_0, c \in M,\ a_0 \sim c$ and hence $c = c \vee a_0 = a_0 \vee c$. Therefore $a_0 \leq c$. Since $a_0 \in [c)$, $a_0 = a_0 \vee c = c \vee a_0$ and hence $c \leq a_0$. Thus $a_0 = c$.

Let us denote $\eta = \{(a, b) \in \mathbb{V} \times \mathbb{V} \mid [a) = [b)\}$. Then we have the following

Theorem 3.23. *η is a congruence relation on $\mathbb{V}$.*

Proof. Let $a, b, c \in \mathbb{V}$. (i) Reflexive: For any $a \in \mathbb{V},\ [a) = [a) \Leftrightarrow (a, a) \in \eta$. (ii) Transitive: For any $(a, b) \in \eta$ and $(b, c) \in \eta,\ [a) = [b) = [c)$. Therefore $[a) = [c)$. Hence $(a, c) \in \eta$. (iii) Symmetric: $(a, b) \in \eta \Leftrightarrow [a) = [b) \Leftrightarrow (b, a) \in \eta$. Therefore η is an equivalence relation on $\mathbb{V}$. Let $(a, b), (c, d) \in \eta$. Then $[a \vee c) = [a) \wedge [c) = [b) \wedge [d) = [b \vee d)$ and $[a \wedge c) = [a) \vee [c) = [b) \vee [d) = [b \wedge d)$. Therefore $(a \vee c,\ b \vee d),\ (a \wedge c,\ b \wedge d) \in \eta$. Hence η is a congruence relation on $\mathbb{V}$.

Lemma 3.24. *For any element $a \in \mathbb{V}$, $a/\eta = \{1\}$ if and only if $a = 1$.*

Proof. Let $a \in \mathbb{V}$. Then $a \in a/\eta$. Suppose that $a/\eta = \{1\}$. Then $[a) = [1)$. Therefore $a = a \vee 1 = 1$. Hence $a = 1$. On the other hand, let $a = 1,\ 1/\eta = \{a \in \mathbb{V} \mid (a, 1) \in \eta\} = \{a \in \mathbb{V} \mid [a) = [1)\} = \{a \in \mathbb{V} \mid a = a \vee 1 = 1\} = \{1\}$. Hence $a/\eta = \{1\}$.

Theorem 3.25. *The quotient lattice $\mathbb{V}/\eta$ forms a distributive lattice with the least element $1/\eta = \{1\}$ and the operations $a/\eta \wedge b/\eta = (a \wedge b)/\eta$ and $a/\eta \vee b/\eta = (a \vee b)/\eta$.*

It is easy to observe that η is the smallest congruence on $\mathbb{V}$ such that $\mathbb{V}/\eta$ is a distributive lattice.

Theorem 3.26. *A compatible set M in $\mathbb{V}$ is amicable if and only if $M \cap (a/\eta)$ is singleton set, for all $a \in \mathbb{V}$.*

Proof. Suppose that M is amicable. Let $a \in \mathbb{V}$. By Theorem 3.22., there exists $a_0 \in M$ such that $[a_0) = [a)$. Then $a_0 \in M \cap (a/\eta)$. Now,

$$\begin{aligned} b \in M \cap (a/\eta) \ &\Rightarrow (a, b) \in \eta \text{ and } b \in M \\ &\Rightarrow [a) = [b) \text{ and } b \in M \\ &\Rightarrow [a_0) = [b) \text{ and } b \in M \quad (\text{since } [a_0) = [a)) \\ &\Rightarrow b = a_0. \quad (\text{since } M \text{ is distributive lattice}) \end{aligned}$$

Therefore $M \cap (a/\eta) = \{a_0\}$. Conversely suppose that $M \cap (a/\eta)$ is a singleton, for all $a \in \mathbb{V}$. Let $b \in \mathbb{V}$ such that $b \sim a$, for all $a \in M$. Then $M \cap (b/\eta) = \{c\}$, for some $c \in M$(By our assumption). Then $[b) = [c)$. Since $b \sim c,\ b = c \in M$. Hence M is maximal set. Now, we show that M is amicable. Let $a \in \mathbb{V}$. By our assumption $M \cap (a/\eta) = \{d\}$, for some $d \in M$. Then $[a) = [d)$. Therefore $a = a \vee d$ and $d \in M$. Hence M is amicable.

Theorem 3.27. *Let M be an amicable set in $\mathbb{V}$ and $x \in \mathbb{V}$. Then $M_x = \{(a \wedge x) \vee a \mid a \in M\}$ is a compatible set.*

Proof. Let $a, b \in M$. Then

$$\begin{aligned}
[(b \wedge x) \vee b] \vee [(a \wedge x) \vee a] &\Rightarrow [(b \wedge x) \vee b] \vee [a \vee (a \wedge x)] && \text{(By Lemma 2.4}(viii))\\
&\Rightarrow [(b \wedge x) \vee b] \vee a && \text{(By Definition 2.1}(v))\\
&\Rightarrow (b \wedge x) \vee (b \vee a) && (\vee \text{ is associative})\\
&\Rightarrow (b \vee (b \vee a)] \wedge (x \vee (b \vee a)] && \text{(By Definition2.1}(ii))\\
&\Rightarrow (a \vee b) \wedge (x \vee (a \vee b)] && (\text{since } a \sim b,\ a, b \in M)\\
&\Rightarrow [a \vee (a \vee b)] \wedge [x \vee (a \vee b)] && \text{(By Lemma2.8}(iii))\\
&\Rightarrow (a \wedge x) \vee (a \vee b) && \\
&\Rightarrow [(a \wedge x) \vee a] \vee b && (\vee \text{ is associative})\\
&\Rightarrow [(a \wedge x) \vee a] \vee [b \vee (b \wedge x)] && \text{(By Definition 2.1}(v))\\
&\Rightarrow [(a \wedge x) \vee a] \vee [(b \wedge x) \vee b] && \text{(By Lemma 2.4}(viii)
\end{aligned}$$

Therefore M_x is compatible set.

Theorem 3.28. *Let M be an amicable set in $\mathbb{V}$. Then M is isomorphic to $\mathbb{V}/\eta$.*

Proof. Define a map $f : M \to \mathbb{V}/\eta$ by a/η, for all $a \in M$. Now, $f(a \wedge b) = (a \wedge b)/\eta = a/\eta \wedge b/\eta = f(a) \wedge f(b)$ and $f(a \vee b) = (a \vee b)/\eta = a/\eta \vee b/\eta = f(a) \vee f(b)$, for all $a, b \in M$. Then f is a homomorphism. Now, $f(a) = f(b) \Rightarrow a/\eta = b/\eta$. Let $x \in a/\eta$. Then $(a, x), (b, x) \in \eta$. Therefore $[a) = [x) = [b)$. Therefore $a = b$. So that f is one-one. By Theorem 3.22., for $x \in \mathbb{V}$, there exists $x_0 \in M$ such that $[x_0) = [x)$. Therefore $x_0/\eta = x/\eta = f(x_0)$. So that f is onto and hence f is bijective.

Theorem 3.29. *Let M be an amicable set in $\mathbb{V}$. Then the lattice of filters of $\mathbb{V}$ is isomorphic with the lattice of filters of M.*

Proof. Let $F(\mathbb{V})$ and $F(M)$ are the lattices of filters of $\mathbb{V}$ and filters of M respectively. Define a map $g : F(\mathbb{V}) \to F(M)$ by $g(F) = F \cap M$, for all $F \in F(\mathbb{V})$. Then clearly g is an order preserving map. Suppose $F, G \in F(\mathbb{V})$ and $F \cap M \subseteq G \cap M$. For $x \in F$, there exists $x_0 \in M$ such that $[x_0) = [x)$. Therefore $x_0 \in F$ and hence $x_0 \in F \cap M \subseteq G \cap M$. For this $x_0 \in G \cap M$, $x \in [x) = [x_0) \subseteq G \cap M$. Therefore $x \in G$. So that $F \subseteq G$. Hence g is one-one. Let $K \in F(M)$. Put $Q = \{x \in \mathbb{V} \mid x_0 \in K \text{ such that } [x) = [x_0)\}$. Let $x, y \in Q$. Then there exist $x_0, y_0 \in K$ such that $[x) = [x_0)$ and $[y) = [y_0)$. Now, $[x \wedge y) = [x) \vee [y) = [x_0) \vee [y_0)) = [x_0 \wedge y_0)$ and $[x \vee y) = [x) \wedge [y) = [x_0) \wedge [y_0)) = [x_0 \vee y_0)$(since K is a filter). Let $a \in \mathbb{V}$. Then there exists $a_0 \in M$ such that $[a) = [a_0)$. Now, $[x \vee a) = [x) \wedge [a) = [x_0) \wedge [a_0) = [x_0 \vee a_0)$. Since $x_0 \vee a_0 \in M$, $x \vee a \in F$ and hence Q is a filter of $\mathbb{V}$ and $g(Q) = Q \cap M = K$. Hence g is bijective and g^{-1} is also order preserving. Thus g is isomorphic.

Definition 3.30. A PDL $(\mathbb{V}, \vee, \wedge, 1)$ is said to be relatively complemented, if every interval $[a, b]$, in $\mathbb{V}$ is a complemented lattice, for all $a, b \in \mathbb{V}$.

Theorem 3.31. $\mathbb{V}$ *is relatively complemented if and only if every maximal set is relatively complemented.*

Proof. Suppose $\mathbb{V}$ is relatively complemented and M is a maximal set in $\mathbb{V}$. For any $a, b \in M$ with $a \leq b$, the interval $[a, b]$ in $\mathbb{V}$ is a subset of M. Let $x \in [a, b]$. Then $a \leq x$. Therefore $a \wedge x = x \wedge a$ and $a \in M$. So that $x \in M$. Hence $[a, b] \subseteq M$. Since $\mathbb{V}$ is relatively complemented, $[a, b]$ is relatively complemented. Hence M is relatively complemented. Conversely suppose that every maximal set is relatively complemented. Let $a, b \in \mathbb{V}$. Then $[a, b]$ is compatible and it contained in a maximal set(since every maximal set is compatible). Therefore $[a, b]$ is relatively complemented. By Definition 3.30., $\mathbb{V}$ is relatively complemented.

Theorem 3.32. $\mathbb{V}$ *is relatively complemented if and only if every amicable is relatively complemented.*

Proof. Let M be an amicable set in $\mathbb{V}$ such that M is relatively complemented. Let $a, b \in \mathbb{V}$ and $a \leq b$. Then there exists $a_0, b_0 \in M$ such that $[a_0) = [a)$ and $[b_0) = [b)$ (By Theorem 3.22.) Now,

$$\begin{aligned}
a_0 \vee b_0 &= b_0 \vee a_0 && (\text{since } a_0, b_0 \in M) \\
&= b_0 \vee a_0 \vee a && (\text{since } a_0 \in [a)) \\
&= b_0 \vee a \vee a_0 && (\text{By Lemma 2.4}(viii)) \\
&= b_0 \vee a && (\text{since } a \in [a_0)) \\
&= b_0 \vee b \vee a && (\text{since } b_0 \in [b)) \\
&= b_0 \vee b && (\text{since } a \leq b) \\
&= b_0. && (\text{since } b_0 \in [b))
\end{aligned}$$

Therefore $a_0 \wedge b_0 = a_0$ and hence $a_0 \leq b_0$. Since M is relatively complemented, there exists $x \in M$ such that $a_0 \vee x = 1$ and $a_0 \wedge x$ is minimal. For $x \vee b \in \mathbb{V}$,

$$\begin{aligned}
a \vee (x \vee b) &= a \vee a_0 \vee x \vee b && (\text{since } a \in [a_0) \\
&= a \vee 1 \vee b && (\text{since } a_0 \vee x = 1) \\
&= 1 \\
a \wedge (x \vee b) &= (a \vee a_0) \wedge (x \vee b) && (\text{since } a \in [a_0) \\
&= \{a \wedge (x \vee b)\} \vee \{a_0 \wedge (x \vee b)\} && (\text{By Lemma 2.4}(iv)) \\
&= \{a \wedge (x \vee b)\} \vee \{(x \vee b) \wedge a_0\} && (\text{By Lemma 2.4}(v)) \\
&= (x \vee b) \wedge \{[a \wedge (x \vee b)] \vee a_0\} && (\text{By Lemma 2.3}(iii)) \\
&= (x \vee b) \wedge \{(a \vee a_0) \wedge [(x \vee b) \vee a_0]\} && (\text{By Definition 2.1}(ii)) \\
&= (x \vee b) \wedge \{(a \vee a_0) \wedge [(x \vee a_0 \vee b]\} && (\text{By Lemma 2.4}(vi)(viii)) \\
&= (x \vee b) \wedge (a \vee a_0) && (\text{since } a_0 \vee x = 1) \\
&= (x \vee b) \wedge a && (\text{since } a \in [a_0) \text{ and } a_0 \in [a) \\
&= (x \wedge a) \vee (b \wedge a) && (\text{By Lemma 2.4}(iv)) \\
&= (x \wedge a) \vee a && (\text{since } a \leq b)) \\
&= a.
\end{aligned}$$

Therefore $a \leq x \vee b \leq 1$. So that $a \vee (x \vee b) = 1$ and $a \wedge (x \vee b) = a$. Hence $\mathbb{V}$ is relatively complemented.

4. Conclusion and Future Work

In this paper, we introduced and analyzed maximal and amicable sets in paradistributive latticoids (PDLs). We established key structural properties, including that the center of a PDL is the intersection of all maximal sets and forms a filter. Additionally, we characterized relatively complemented PDLs using amicable sets. These results contribute to the deeper understanding of compatibility in generalized lattices. Future work may focus on extending these concepts to related algebraic structures, exploring categorical properties, and developing computational approaches for identifying maximal and amicable sets in practical and theoretical applications.

5. Conflict of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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