EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6422
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Abundant Solitary Solutions for the Fractional Unidirectional Wave Model Using in Oceanography, Coastal engineering, and Meteorology

Wael W. Mohammed[1,*], Monirah W. Alshammary[1], Ahmed E. Matouk[2,3], Naveed Iqbal[1]

[1] *Department of Mathematics, College of Science, University of Ha'il, Ha'il 2440, Saudi Arabia*
[2] *Department of Mathematics, College of Science Al-Zulfi, Majmaah University, Al-Majmaah 11952, Saudi Arabia*
[3] *College of Engineering, Majmaah University, Al-Majmaah 11952, Saudi Arabia*

Abstract. In this paper, we consider the unidirectional wave model (UWM) with beta-derivative operator (BDO). This model simplifies the complexity of wave interactions by providing a one-dimensional approach to understanding wave behavior, particularly under conditions where wave directionality plays a crucial role. Its applications are vital in different area, such as oceanography, coastal engineering, and environmental science, contributing significantly to our understanding of wave dynamics and their impact on coastal and marine environments. Therefore, it is crucial to find the solutions for this model. By applying the $\mathcal{F}$-expansion method, we can obtain abundant solutions including periodic, bright, kink, anti-kink, singular, and dark-bright solitons. Furthermore, the graphs of the solutions are displayed using the MATLAB software to demonstrate how the beta-derivative operator affects the obtained solution.

2020 Mathematics Subject Classifications: 35Q51, 35Q80

Key Words and Phrases: Beta-derivative, $\mathcal{F}$-expansion method, Exact solutions, nonlinear evolution equations

1. Introduction

Nonlinear evolution equations (NLEEs) provide a powerful mathematical framework for studying complex systems with memory effects and non-linear interactions, leading to novel insights and applications in engineering and science [1–5]. The non-linear nature of these equations makes them suitable for modeling a wide range of phenomena in engineering, biology, chemistry, and physics. As a result, it is important to solve these NLEEs. Recently, numerous powerful approaches for discovering exact solutions to

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6422

Email addresses: `w.mohammed@uoh.edu.sa` (W. W. Mohammed), `n.iqbal@uoh.edu.sa` (N. Iqbal)

NLEEs have been introduced, such as, sine-Gordon expansion technique [6], F-expansion method [7], modified extended tanh-function method [8], Jacobi elliptic function expansion [9], exp-function method [10], (G'/G)-expansion [11, 12], sine-cosine method [13], $exp(-\phi(\varsigma))$-expansion method [14], Sumudu perturbation transform method [15], unified Riccati equation expansion technique [16, 17] and etc.

One of these equations is the unidirectional wave model (UWM). UWM is an essential tool in understanding and predicting the behavior of waves in various natural phenomena. This model simplifies the complex interactions of waves by assuming that they propagate in only one direction, making it easier to analyze and interpret wave motion.

One of the major fields where the unidirectional wave model is crucial in oceanography. This model helps scientists and researchers in studying the dynamics of ocean waves, including their formation, propagation, and interaction with other components of the ocean system. By using this model, oceanographers can accurately predict wave height, direction, and speed, which are essential for understanding the impact of waves on coastal areas, marine ecosystems, and human activities at sea.

In coastal engineering, the unidirectional wave model plays a vital role in designing coastal structures that can withstand the forces exerted by waves. Engineers use this model to determine the wave conditions at a given site, which is crucial for designing breakwaters, seawalls, and other coastal defenses that protect shorelines from erosion and damage during storms. By employing the unidirectional wave model, engineers can make informed decisions about the size, shape, and placement of coastal structures, ensuring their effectiveness in minimizing wave impact.

Meteorologists also rely on the unidirectional wave model to forecast and track ocean waves and their effects on weather patterns. By incorporating wave data into their models, meteorologists can better predict the intensity and path of storms, hurricanes, and other extreme weather events that are fueled by ocean waves. This information is essential for issuing timely warnings and alerts to coastal communities, helping to mitigate the potential damage and loss of life caused by these natural disasters.

The UWM with beta-derivative operator (UWM-BDO) takes the following form [18]:

$$\mathcal{D}_t^{\kappa}\mathcal{W} + \frac{1}{6}a_1\mathcal{W}_{xxx} + \frac{3}{2}a_2\mathcal{W}\mathcal{W}_x + \frac{15}{32}a_2^2\mathcal{W}^2\mathcal{W}_x + a_3\mathcal{W}_x + a_4\mathcal{W}_y + a_5\mathcal{W}_z = 0, \tag{1}$$

where $\mathcal{W} = \mathcal{W}(x, y, z, t)$ presents a wave profile; $a_1,\ a_2,\ a_3,\ a_4$ and a_5 are constants. $\mathcal{D}^{\kappa}$ is the beta-derivative operator (BDO) of order κ. The BDO is a novel conformable fractional derivative that was recently proposed by Atangana et al [19]. Now, let us BDO of order $\kappa \in (0, 1]$ for the function $\mathcal{W} : (0, \infty) \to \mathbb{R}$ as follows:

$$\mathcal{D}_t^{\kappa}\mathcal{W}(t) = \lim_{\epsilon \to 0} \frac{\mathcal{W}(t + \epsilon(t + \frac{1}{\Gamma(\beta)})^{1-\beta}) - \mathcal{W}(t)}{\epsilon}.$$

For any constants c_1 and c_2, the BDO meets the following characteristics [19]: (1) $\mathcal{D}_t^{\kappa}[c_1u(t) + c_2v(t)] = c_1\mathcal{D}_t^{\kappa}u(t) + c_2\mathcal{D}_t^{\kappa}v(t)$, (2) $\mathcal{D}_t^{\kappa}[c_1] = 0$, (3) $\mathcal{D}_t^{\kappa}u(t) = (t + \frac{1}{\Gamma(\kappa)})^{1-\kappa}\frac{du}{dt}$, (4) $\mathcal{D}_t^{\kappa}u(v(t)) = (t + \frac{1}{\Gamma(\kappa)})^{1-\kappa}v'(t)u'(v(t))$ (5) If $\theta = \frac{c_1}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^{\kappa}$, then $\mathcal{D}_t^{\kappa}u(t) = c_1\frac{du}{dt}$.

The main contribution of this work is to acquire the exact solutions to the UWM-BDO (1). The $\mathcal{F}$-expansion method is used to create fractional solutions in the form of elliptic, trigonometric and hyperbolic functions. Numerous important scientific phenomena may be investigated using the achieved solutions of the UWM-BDO (1) because this equation has a significant applications in various fields such as oceanography, coastal engineering, and meteorology. To analyze the impact of the BDO on the solutions of UWM-BDO (1), many graphs are created with the MATLAB program.

The following is an outline of the paper: We obtain the wave equation for Eq. (1) in Section 2. The solutions of UWM-BDO (1) are obtained in Section 3. In Section 4, we discuss the influence of BDO on the solutions of (1). Finally, the conclusion of the study is stated.

2. Traveling Wave Equation

To get the wave equation for UWM-BDO (1), we apply

$$\mathcal{W}(x,y,z,t)=\mathcal{V}(\rho),\ \rho=\rho_1 x+\rho_2 y+\rho_3 z+\frac{\lambda}{\kappa}(t+\frac{1}{\Gamma(\kappa)})^{\kappa}, \tag{2}$$

where $\rho_1,\ \rho_2,\ \rho_3$ are non-zero constants and they represent the wave number component in the $x,\ y$ and z directions, respectively; λ is the wave speed. Differentiating Eq. (2) with respect to $t,\ x,\ y$ and z, we get

$$\begin{aligned} \mathcal{D}_t^{\kappa}\mathcal{W} &= \lambda\mathcal{V}',\ \mathcal{W}_x=\rho_1\mathcal{V}',\ \mathcal{W}_y=\rho_2\mathcal{V}', \\ \mathcal{W}_z &= \rho_3\mathcal{V}',\ \mathcal{W}_{xxx}=\rho_1^3\mathcal{V}'''. \end{aligned} \tag{3}$$

Putting Eqs (2) and (3) into Eq. (1), we get

$$\frac{1}{6}a_1\rho_1^3\mathcal{V}'''+(\lambda+a_3\rho_1+a_4\rho_2+a_5\rho_3)\mathcal{V}'+\frac{3}{2}a_2\rho_1\mathcal{V}\mathcal{V}'+\frac{15}{32}a_2^2\rho_1\mathcal{V}^2\mathcal{V}'=0. \tag{4}$$

Integrating Eq. (4) once, we have

$$\mathcal{V}''+A_1\mathcal{V}+A_2\mathcal{V}^2+A_3\mathcal{V}^3=0, \tag{5}$$

where

$$A_1=\frac{6(\lambda+a_3\rho_1+a_4\rho_2+a_5\rho_3)}{a_1\rho_1^3},\ A_2=\frac{9a_2}{2a_1\rho_1^2},\ A_3=\frac{15a_2^2}{16a_1\rho_1^2}.$$

3. Exact solutions of the UWM-BDO

Let us utilize the $\mathcal{F}$-expansion method (for more information see [20–22]). Assuming the solution of Eq. (5) has the form

$$\mathcal{V}(\rho)=\sum_{j=0}^{N}\ell_j\mathcal{F}^j(\rho),\ \ell_N\neq 0, \tag{6}$$

where $\ell_0,\ \ell_1,\ \ell_2,\ \ ,\ \ell_{N-1}$ and ℓ_N are unknown constants to be calculated and $\mathcal{F}$ solves the following auxiliary equation:

$$(\mathcal{F}')^2 = \hbar_1\mathcal{F}^4 + \hbar_2\mathcal{F}^2 + \hbar_3, \tag{7}$$

where $\hbar_1,\ \hbar_2$, and $\hbar_3$ are real numbers. First, let us balance $\mathcal{V}^3$ with $\mathcal{V}''$ in Eq. (5) to calculate the parameters N as

$$3N = N + 2 \ \Rightarrow N = 1.$$

With $N = 1$, Eq. (6) becomes

$$\mathcal{V}(\rho) = \ell_0 + \ell_1\mathcal{F}(\rho), \tag{8}$$

Putting Eq. (8) into Eq. (5), we get

$$\begin{aligned}
&+[2\hbar_1\ell_1 + A_3\ell_1^3]\mathcal{F}^3 + [A_2\ell_1^2 + 3A_3\ell_0\ell_1^2]\mathcal{F}^2\\
&+[\hbar_2\ell_1 + A_1\ell_1 + 2\ell_0\ell_1A_2 + 3\ell_0^2\ell_1A_3]\mathcal{F}\\
&+[A_1\ell_0 + A_2\ell_0^2 + A_3\ell_0^3] = 0.
\end{aligned}$$

For $j = 3, 2, 1, 0$, we balance each coefficient of $\mathcal{F}^j$ with zero to have

$$2\hbar_1\ell_1 + A_3\ell_1^3 = 0,$$

$$A_2\ell_1^2 + 3A_3\ell_0\ell_1^2 = 0,$$

$$\hbar_2\ell_1 + A_1\ell_1 + 2\ell_0\ell_1A_2 + 3\ell_0^2\ell_1A_3 = 0,$$

and

$$A_1\ell_0 + A_2\ell_0^2 + A_3\ell_0^3 = 0.$$

By solving the above equations, we get

$$\ell_0 = \frac{-A_2}{3A_3} = \frac{-8}{5a_2},\ \ell_1 = \pm\sqrt{\frac{-2\hbar_1}{A_3}} = \pm\sqrt{\frac{-32a_1\rho_1^2\hbar_1}{15a_2^2}},\ \text{and}\ \hbar_2 = \frac{A_2^2}{9A_3} = \frac{12}{5a_1\rho_1^2}. \tag{9}$$

Substituting (9) into Eq. (8), we have the solution of the traveling wave Eq. (5) as:

$$\mathcal{V}(\rho) = \frac{-8}{5a_2} \pm \sqrt{\frac{-128\hbar_1}{25a_2^2\hbar_2}}\mathcal{F}(\rho). \tag{10}$$

Consequently, putting Eq. (10) into Eq. (2), we get the next solutions for the UWM-BDO (1):

$$\mathcal{W}(x, y, z, t) = \left\{\frac{-8}{5a_2} \pm \sqrt{\frac{-128\hbar_1}{25a_2^2\hbar_2}}\mathcal{F}(\rho)\right\}, \tag{11}$$

where $\rho = \rho_1x + \rho_2y + \rho_3z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^\kappa$.

There are several cases for the solutions of Eq. (9) that depending on $\hbar_1$, $\hbar_2$ and $\hbar_3$ as follows:

Case 1: If $\hbar_1 = \varpi^2,\ \hbar_2 = -\left(1+\varpi^2\right)$ and $\hbar_3 = 1$, then $\mathcal{F}(\rho) = sn(\rho)$, and Eq. (11) has the form

$$\mathcal{W}(x,y,z,t) = \left\{\frac{-8}{5a_2} \pm \sqrt{\frac{128\varpi^2}{25a_2^2\left(1+\varpi^2\right)}} sn(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^{\kappa})\right\}. \tag{12}$$

If $\varpi \to 1$, then Eq. (12) changes to

$$\mathcal{W}(x,y,z,t) = \left\{\frac{-8}{5a_2} \pm \frac{8}{5a_2}\tanh(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^{\kappa})\right\}. \tag{13}$$

Case 2: If $\hbar_1 = \varpi^2,\ \hbar_2 = -\left(1+\varpi^2\right)$ and $\hbar_3 = 1$, then $\mathcal{F}(\rho) = cd(\rho)$, and Eq. (11) takes the form

$$\mathcal{W}(x,y,z,t) = \left\{\frac{-8}{5a_2} \pm \sqrt{\frac{128\varpi^2}{25a_2^2\left(1+\varpi^2\right)}} cd(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^{\kappa})\right\}. \tag{14}$$

Case 3: If $\hbar_1 = -\varpi^2,\ \hbar_2 = 2\varpi^2 - 1$ and $\hbar_3 = 1 - \varpi^2$, then $\mathcal{F}(\rho) = cn(\rho)$, and Eq. (11) has the form

$$\mathcal{W}(x,y,z,t) = \left\{\frac{-8}{5a_2} \pm \sqrt{\frac{128\varpi^2}{25a_2^2(2\varpi^2-1)}} cn(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^{\kappa})\right\}, \tag{15}$$

for $\frac{1}{\sqrt{2}} < \varpi < 1$. If $\varpi \to 1$, then Eq. (15) becomes

$$\mathcal{W}(x,y,z,t) = \left\{\frac{-8}{5a_2} \pm \sqrt{\frac{128}{25a_2^2}} \operatorname{sech}(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^{\kappa})\right\}. \tag{16}$$

Case 4: If $\hbar_1 = -1,\ \hbar_2 = 2 - \varpi^2$ and $\hbar_3 = \varpi^2 - 1$, then $\mathcal{F}(\rho) = dn(\rho)$, and Eq. (11) has the form

$$\mathcal{W}(x,y,z,t) = \left\{\frac{-8}{5a_2} \pm \sqrt{\frac{128}{25a_2^2(2-\varpi^2)}} dn(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^{\kappa})\right\}. \tag{17}$$

If $\varpi \to 1$, then Eq. (17) becomes Eqs (16).

Case 5: If $\hbar_1 = 1,\ \hbar_2 = -\left(1+\varpi^2\right)$ and $\hbar_3 = \varpi^2$, then $\mathcal{F}(\rho) = ns(\rho)$, and Eq. (11) has the form

$$\mathcal{W}(x,y,z,t) = \left\{\frac{-8}{5a_2} \pm \sqrt{\frac{128}{25a_2^2\left(1+\varpi^2\right)}} ns(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^{\kappa})\right\}. \tag{18}$$

If $\varpi \to 1$, then Eq. (18) tends to

$$\mathcal{W}(x,y,z,t)=\left\{\frac{-8}{5a_2}\pm\frac{8}{5a_2}\coth(\rho_1 x+\rho_2 y+\rho_3 z+\frac{\lambda}{\kappa}(t+\frac{1}{\Gamma(\kappa)})^{\kappa})\right\}. \tag{19}$$

While if $\varpi \to 0$, then Eq. (18) becomes

$$\mathcal{W}(x,y,z,t)=\left\{\frac{-8}{5a_2}\pm\sqrt{\frac{128}{25a_2^2}}\csc(\rho)\right\}. \tag{20}$$

Case 6: If $\hbar_1=1,\ \hbar_2=-\left(1+\varpi^2\right)$ and $\hbar_3=\varpi^2$, then $\mathcal{F}(\rho)=dc(\rho)=\frac{dn(\rho)}{cn(\rho)}$, and Eq. (11) has the form

$$\mathcal{W}(x,y,z,t)=\left\{\frac{-8}{5a_2}\pm\sqrt{\frac{128}{25a_2^2\left(1+\varpi^2\right)}}dc(\rho_1 x+\rho_2 y+\rho_3 z+\frac{\lambda}{\kappa}(t+\frac{1}{\Gamma(\kappa)})^{\kappa})\right\}. \tag{21}$$

If $\varpi \to 0$, then Eq. (21) becomes

$$\mathcal{W}(x,y,z,t)=\left\{\frac{-8}{5a_2}\pm\sqrt{\frac{128}{25a_2^2}}\sec(\rho_1 x+\rho_2 y+\rho_3 z+\frac{\lambda}{\kappa}(t+\frac{1}{\Gamma(\kappa)})^{\kappa})\right\}. \tag{22}$$

Case 7: If $\hbar_1=1-\varpi^2,\ \hbar_2=2\varpi^2-1$ and $\hbar_3=-\varpi^2$, then $\mathcal{F}(\rho)=nc(\rho)$, and Eq. (11) has the form

$$\mathcal{W}(x,y,z,t)=\left\{\frac{-8}{5a_2}\pm\sqrt{\frac{128(1-\varpi^2)}{25a_2^2(2\varpi^2-1)}}nc(\rho_1 x+\rho_2 y+\rho_3 z+\frac{\lambda}{\kappa}(t+\frac{1}{\Gamma(\kappa)})^{\kappa})\right\}, \tag{23}$$

for $\varpi<\frac{1}{\sqrt{2}}$. If $\varpi \to 0$, then Eq. (23) becomes Eqs (22).

Case 8: If $\hbar_1=\varpi^2-1,\ \hbar_2=2-\varpi^2$ and $\hbar_3=-1$, then $\mathcal{F}(\rho)=nd(\rho)$, and Eq. (11) takes the form

$$\mathcal{W}(x,y,z,t)=\left\{\frac{-8}{5a_2}\pm\sqrt{\frac{128(1-\varpi^2)}{25a_2^2(2-\varpi^2)}}nd(\rho_1 x+\rho_2 y+\rho_3 z+\frac{\lambda}{\kappa}(t+\frac{1}{\Gamma(\kappa)})^{\kappa})\right\}. \tag{24}$$

Case 9: If $\hbar_1=-\varpi^2(1-\varpi^2),\ \hbar_2=2\varpi^2-1$ and $\hbar_3=1$, then $\mathcal{F}(\rho)=sd(\rho)$, and Eq. (11) takes the form

$$\mathcal{W}(x,y,z,t)=\left\{\frac{-8}{5a_2}\pm\sqrt{\frac{128\varpi^2(1-\varpi^2)}{25a_2^2(1-2\varpi^2)}}sd(\rho_1 x+\rho_2 y+\rho_3 z+\frac{\lambda}{\kappa}(t+\frac{1}{\Gamma(\kappa)})^{\kappa})\right\}, \tag{25}$$

for $\varpi>\frac{1}{\sqrt{2}}$.

Case 10: If $\hbar_1=1,\ \hbar_2=2\varpi^2-1$ and $\hbar_3=-\varpi^2(1-\varpi^2)$, then $\mathcal{F}(\rho)=ds(\rho)$, and Eq. (11) takes the form

$$\mathcal{W}(x,y,z,t)=\left\{\frac{-8}{5a_2}\pm\sqrt{\frac{-128\hbar_1}{25a_2^2\hbar_2}}ds(\rho_1 x+\rho_2 y+\rho_3 z+\frac{\lambda}{\kappa}(t+\frac{1}{\Gamma(\kappa)})^{\kappa})\right\}, \tag{26}$$

for $\varpi < \frac{1}{\sqrt{2}}$.

Case 12: If $\hbar_1 = \frac{1}{4}$, $\hbar_2 = \frac{\varpi^2-2}{2}$ and $\hbar_3 = \frac{\varpi^2}{4}$, then $\mathcal{F}(\rho) = ns(\rho) \pm ds(\rho)$, and Eq. (11) takes the form

$$\begin{aligned}\mathcal{W}(x,y,z,t) &= \frac{-8}{5a_2} \pm \sqrt{\frac{-128\hbar_1}{25a_2^2\hbar_2}} \Big(ns(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^\kappa) \\ &\quad \pm ds(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^\kappa)\Big). \end{aligned} \tag{27}$$

If $\varpi \to 1$, then Eq. (27) changes to

$$\begin{aligned}\mathcal{W}(x,y,z,t) &= \frac{-8}{5a_2} \pm \sqrt{\frac{1}{9a_3^2}} \Big(\coth(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^\kappa) \\ &\quad \pm \operatorname{csch}(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^\kappa)\Big). \end{aligned} \tag{28}$$

While, if $\varpi \to 0$, then Eq. (27) tends to

$$\begin{aligned}\mathcal{W}(x,y,z,t) &= \frac{-8}{5a_2} \pm \sqrt{\frac{-128\hbar_1}{25a_2^2\hbar_2}} \Big(\csc(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^\kappa) \\ &\quad \pm \cot(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^\kappa)\Big). \end{aligned} \tag{29}$$

Case 13: If $\hbar_1 = \frac{\varpi^2}{4}$, $\hbar_2 = \frac{\varpi^2-2}{2}$ and $\hbar_3 = \frac{\varpi^2}{4}$, then $\mathcal{F}(\rho) = \sqrt{1-\varpi^2}\,(sd(\rho) \pm cd(\rho))$, and Eq. (11) takes the form

$$\begin{aligned}\mathcal{W}(x,y,z,t) &= \frac{-8}{5a_2} \pm \sqrt{\frac{-128\hbar_1}{25a_2^2\hbar_2}} \Big(sd(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^\kappa) \\ &\quad \pm cd(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^\kappa)\Big). \end{aligned} \tag{30}$$

Case 14: If $\hbar_1 = \frac{\varpi^2-1}{4}$, $\hbar_2 = \frac{\varpi^2+1}{2}$ and $\hbar_3 = \frac{\varpi^2-1}{4}$, then $\mathcal{F}(\rho) = \varpi sd(\rho) \pm nd(\rho)$, and Eq. (11) has the form

$$\begin{aligned}\mathcal{W}(x,y,z,t) &= \frac{-8}{5a_2} \pm \sqrt{\frac{-128\hbar_1}{25a_2^2\hbar_2}} \Big(\varpi sd(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^\kappa) \\ &\quad \pm nd(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^\kappa)\Big). \end{aligned} \tag{31}$$

Case 15: If $\hbar_1 = \frac{\varpi^2}{4}$, $\hbar_2 = \frac{\varpi^2-2}{2}$ and $\hbar_3 = \frac{1}{4}$, then $\mathcal{F}(\rho) = \frac{sn(\rho)}{1 \pm dn}$, and Eq. (11) takes the form

$$\mathcal{W}(x,y,z,t) = \frac{-8}{5a_2} \pm \sqrt{\frac{-128\hbar_1}{25a_2^2\hbar_2}} \left(\frac{sn(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^\kappa)}{1 \pm dn(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^\kappa)}\right). \tag{32}$$

If $\varpi \to 1$, then Eq. (32) becomes

$$\mathcal{W}(x,y,z,t) = \frac{-8}{5a_2} \pm \sqrt{\frac{1}{9a_3^2}} \left(\frac{\tanh(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^\kappa)}{1 \pm \operatorname{sech}(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^\kappa)} \right). \tag{33}$$

Case 16: If $\hbar_1 = \frac{-1}{4}$, $\hbar_2 = \frac{\varpi^2+1}{2}$ and $\hbar_3 = \frac{(1-\varpi^2)^2}{4}$, then $\mathcal{F}(\rho) = \varpi cn(\rho) \pm dn(\rho)$, and Eq. (11) takes the form

$$\begin{aligned} \mathcal{W}(x,y,z,t) &= \frac{-8}{5a_2} \pm \sqrt{\frac{64\hbar_1}{25a_2^2(\varpi^2+1)}} \Big(\varpi cn(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^\kappa) \\ &\quad \pm dn(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^\kappa) \Big). \end{aligned} \tag{34}$$

Case 17: If $\hbar_1 = \frac{1}{4}$, $\hbar_2 = \frac{1-2\varpi^2}{2}$ and $\hbar_3 = \frac{1}{4}$, then $\mathcal{F}(\rho) = \frac{sn(\rho)}{1 \pm cn(\rho)}$, and Eq. (11) has the form

$$\mathcal{W}(x,y,z,t) = \frac{-8}{5a_2} \pm \sqrt{\frac{64\hbar_1}{25a_2^2(2\varpi^2-1)}} \frac{sn(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^\kappa)}{1 \pm cn(\rho_1 x + \rho_2 y + \rho_3 z + \frac{\lambda}{\kappa}(t + \frac{1}{\Gamma(\kappa)})^\kappa)}, \tag{35}$$

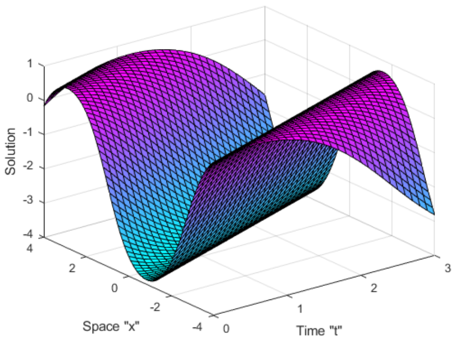
for $\frac{1}{\sqrt{2}} < \varpi < 1$. Where $cn(\rho) = cn(\rho,\varpi)$, $dn(\rho) = dn(\rho,\varpi)$, $sn(\rho) = sn(\rho,\varpi)$, $sc(\rho) = sc(\rho,\varpi)$, $ds(\rho) = ds(\rho,\varpi)$, are the Jacobi elliptic functions (JEFs) for $0 < \varpi < 1$ and ϖ is the elliptic modulus. We note that

JEFs	$\varpi \to 0$	$\varpi \to 1$
$sn(\rho)$	$\sin(\rho)$	$\tanh(\rho)$
$cs(\rho)$	$\cot(\rho)$	$\operatorname{csch}(\rho)$
$cn(\rho)$	$\cos(\rho)$	$\operatorname{sech}(\rho)$
$ds(\rho)$	$\csc(\rho)$	$\operatorname{csch}(\rho)$
$dn(\rho)$	1	$\operatorname{sech}(\rho)$
$sc(\rho)$	$\tan(\rho)$	$\sinh(\rho)$
$ns(\rho)$	$\csc(\rho)$	$\coth(\rho)$

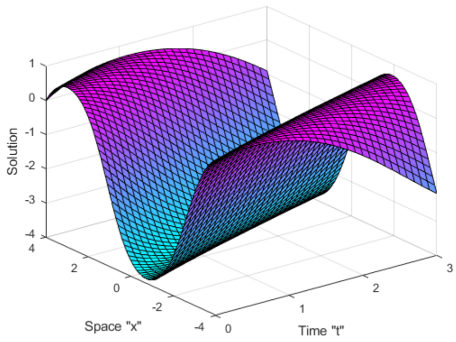
4. Discussion and Effect of BDO

The UWM-BDO provides a powerful tool for modeling wave propagation in a wide range of physical systems. By capturing non-local effects and memory effects, this mathematical model offers a more comprehensive understanding of complex wave phenomena in heterogeneous media. The incorporation of fractional derivatives into the wave equation opens up new avenues for research and applications in areas such as acoustics, electromagnetics, and geophysics, where accurate and efficient wave modeling is essential.

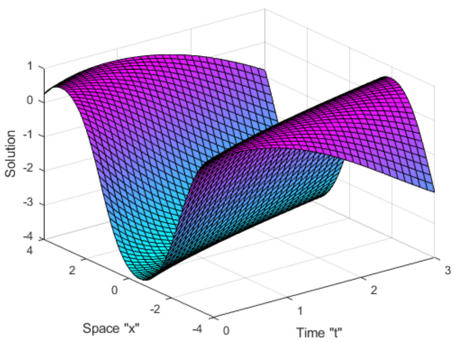
We discuss the impact of the beta-derivative operator on the obtained solutions of UWM-BDO (1). To show how these solutions behave, various graphical representations are offered. For some obtained solutions, such as Eqs. (12), (13) and (16), we simulate the graphical representations for $\rho_1 = \rho_2 = \rho_3 = 1$, $a_2 = 1$ and for varying values of κ. We deduce from these figures that when the fractional order decreases, the surface moves into the right as follows:



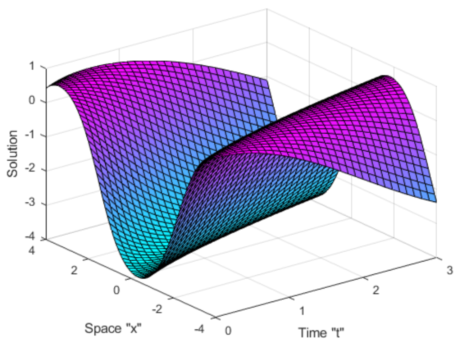
(a) $\kappa = 1$



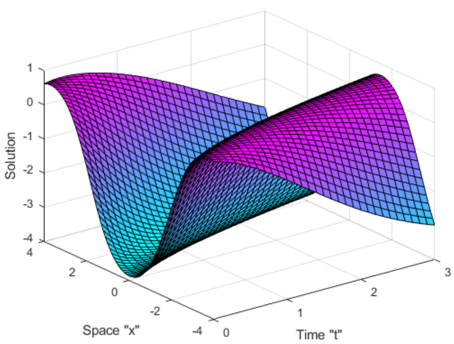
(b) $\kappa = 0.8$



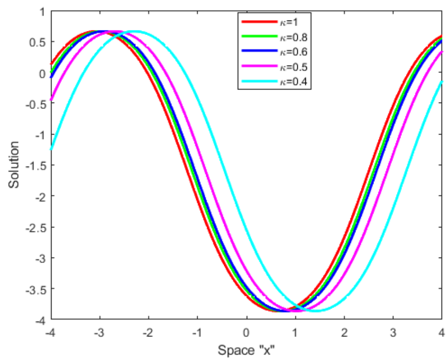
(c) $\kappa = 0.6$



(d) $\kappa = 0.5$

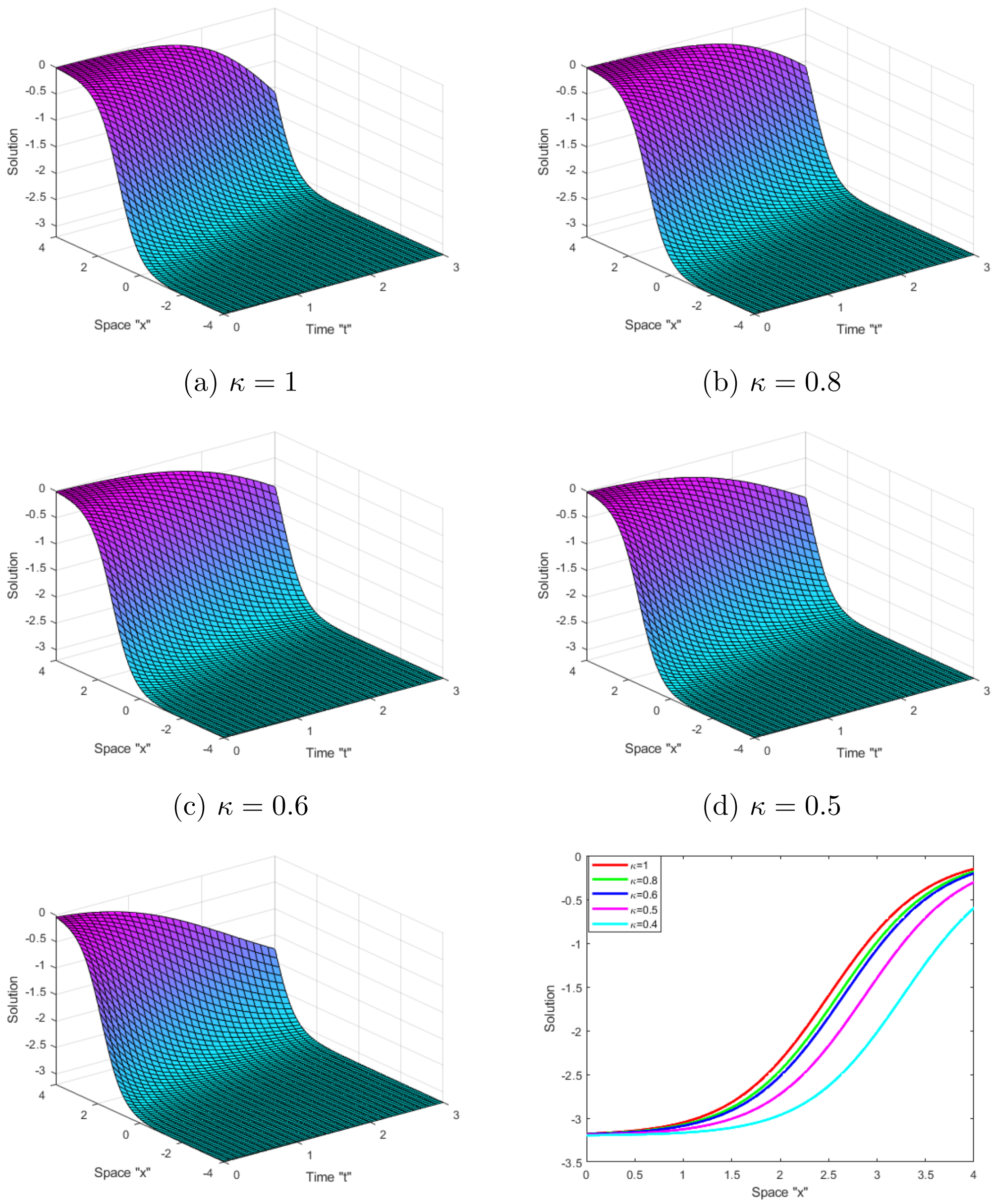


(e) $\kappa = 0.4$



(f) $\kappa = 1,\ 0.8,\ 0.6,\ 0.5,\ 0.4$

Figure 1. (a-e) describe 3D-profile of solution $\mathcal{W}(x, y, z, t)$ in Eq (12) with $\varpi = 0.5$, $a_2 = 1,\ y = z = 0,\ \rho_1 = \rho_2 = \rho_3 = 1,\ x \in [-4, 4]$, and $t \in [0, 3]$ (f) display 2D-profile of Eq. (12) with various κ

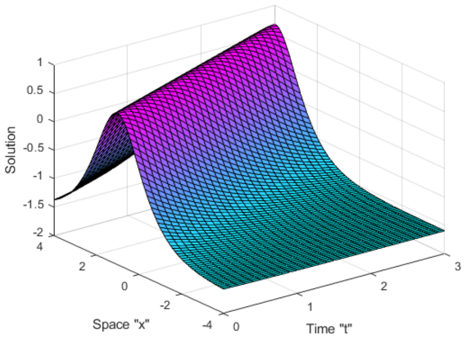


(a) $\kappa = 1$ (b) $\kappa = 0.8$

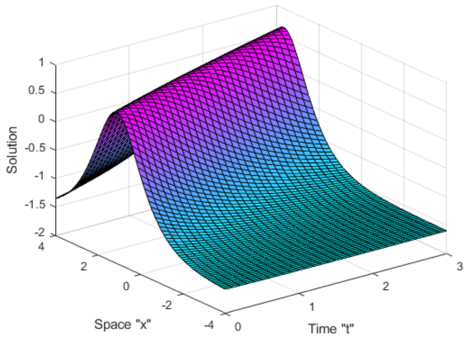
(c) $\kappa = 0.6$ (d) $\kappa = 0.5$

(e) $\kappa = 0.4$ (f) $\kappa = 1,\ 0.8,\ 0.6, 0.5,\ 0.4$

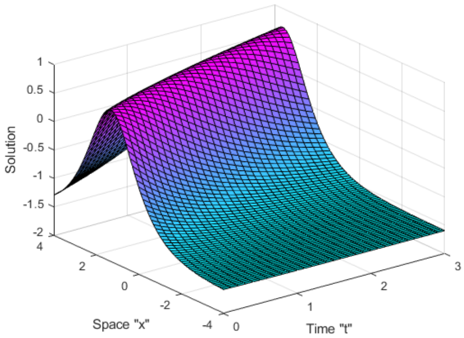
Figure 2. (a-e) describe 3D-profile of solution $\mathcal{W}(x, y, z, t)$ in Eq (13) with $a_2 = 1,\ y = z = 0,\ \rho_1 = \rho_2 = \rho_3 = 1,\ x \in [-4, 4]$, and $t \in [0, 3]$ (f) display 2D-profile of Eq. (13) with various κ



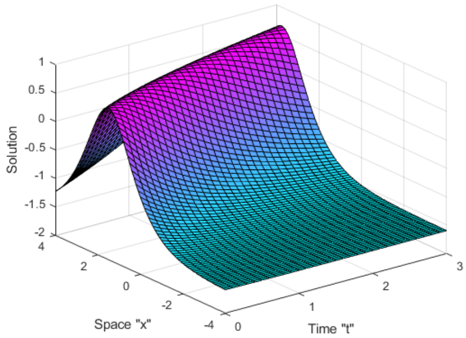
(a) $\kappa = 1$



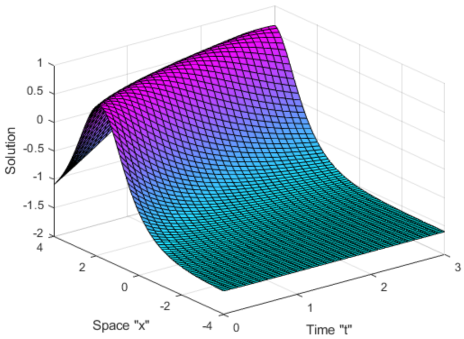
(b) $\kappa = 0.8$



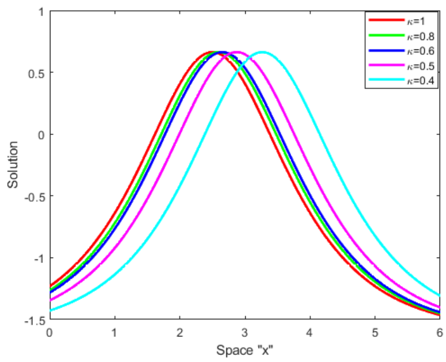
(c) $\kappa = 0.6$



(d) $\kappa = 0.5$



(e) $\kappa = 0.4$



(f) $\kappa = 1,\ 0.8,\ 0.6, 0.5,\ 0.4$

Figure 3. (a-e) describe 3D-profile of solution $\mathcal{W}(x, y, z, t)$ in Eq (33) with $a_2 = 1,\ y = z = 0,\ \rho_1 = \rho_2 = \rho_3 = 1,\ x \in [-4, 4]$, and $t \in [0, 3]$ (f) display 2D-profile of Eq. (33) with various κ

5. Conclusions

In this paper, the unidirectional wave model (UWM) (1) with beta-derivative operator (BDO) was considered. We obtained many various kind of solutions including periodic soliton, bright soliton, kink soliton, anti-kink soliton, singular soliton, and dark-bright soliton by using the $\mathcal{F}$-expansion method. The UWM-BDO (1) has a significant applications in many fields such as oceanography, coastal engineering, and meteorology, allowing for an examination of a variety of scientific phenomena. To study the effect of the beta-derivative operator on the solutions of Eq. (1), many figures (see Figures 1, 2 and 3) are produced by utilizing the MATLAB program. We deduced that when the order of the BDO decreases, the surface of the solutions moves to the right. In the future work, we can get the exact solutions for the UWM with stochastic process.

Acknowledgements

This research has been funded by Scientific Research Deanship at the University of Ha'il-Saudi Arabia through project number RG-25004.

References

[1] K. B. Oldham and J. Spanier. *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*. Academic Press, New York, NY, USA, 1974.

[2] K. S. Miller and B. Ross. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. A Wiley-Interscience Publication, John Wiley & Sons, New York, NY, USA, 1993.

[3] I. Podlubny. *Fractional Differential Equations*, volume 198 of *Mathematics in Science and Engineering*. Academic Press, San Diego, CA, USA, 1999.

[4] F. M. Al-Askar, C. Cesarano, and W. W. Mohammed. Abundant solitary wave solutions for the boiti–leon–manna–pempinelli equation with m-truncated derivative. *Axioms*, 12:466, 2023.

[5] M. Mouy, H. Boulares, S. Alshammari, M. Alshammari, Y. Laskri, and W. W. Mohammed. On averaging principle for caputo–hadamard fractional stochastic differential pantograph equation. *Fractal and Fractional*, 7:31, 2022.

[6] H. M. Baskonus, H. Bulut, and T. A. Sulaiman. New complex hyperbolic structures to the lonngren-wave equation by using sine-gordon expansion method. *Applied Mathematics and Nonlinear Sciences*, 4:129–138, 2019.

[7] B. Lu. The first integral method for some time fractional differential equations. *Journal of Mathematical Analysis and Applications*, 395:684–693, 2012.

[8] A. E. Hamza, M. Alshammari, D. Atta, and W. W. Mohammed. Fractional-stochastic shallow water equations and its analytical solutions. *Results in Physics*, 53:106953, 2023.

[9] Z. L. Yan. Abundant families of jacobi elliptic function solutions of the dimensional integrable davey-stewartson-type equation via a new method. *Chaos, Solitons & Fractals*, 18:299–309, 2003.

[10] J. H. He and X. H. Wu. Exp-function method for nonlinear wave equations. *Chaos, Solitons & Fractals*, 30:700–708, 2006.

[11] M. L. Wang, X. Z. Li, and J. L. Zhang. The (g'/g)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. *Physics Letters A*, 372:417–423, 2008.

[12] H. Zhang. New application of the (g'/g)-expansion method. *Communications in Nonlinear Science and Numerical Simulation*, 14:3220–3225, 2009.

[13] A. M. Wazwaz. The sine-cosine method for obtaining solutions with compact and noncompact structures. *Applied Mathematics and Computation*, 159:559–576, 2004.

[14] K. Khan and M. A. Akbar. The $exp(-\phi(\varsigma))$-expansion method for finding travelling wave solutions of vakhnenko-parkes equation. *International Journal of Dynamical Systems and Differential Equations*, 5:72–83, 2014.

[15] M. Nadeem, O. A. Arqub, M. A. El-Meligy, and A. A. Ibrahim. Fractional analysis of time-fractional emden–fowler model arising in fluid flow of porous media and physical sciences. *International Journal of Geometric Methods in Modern Physics*, 2025.

[16] M. Nadeem, O. A. Arqub, H. M. Hazaimeh, and A. A. Ibrahim. Bifurcation, chaos, and time series analysis of the manakov model utilizing the unified riccati equation expansion technique. *Modern Physics Letters A*, 40:2550074, 2025.

[17] M. Nadeem and O. A. Arqub. Soliton solutions and stability analysis to the hirota-bilinear-like model using the unified riccati equation expansion approach. *Modern Physics Letters A*, 39:2550138, 2025.

[18] A. M. Wazwaz. A hamiltonian equation produces a variety of painlevé integrable equations: Solutions of distinct physical structures. *International Journal of Numerical Methods for Heat & Fluid Flow*, 34:1730–1751, 2024.

[19] A. Atangana and E. F. D. Goufo. Extension of matched asymptotic method to fractional boundary layers problems. *Mathematical Problems in Engineering*, 2014:107535, 2014.

[20] Y. Z. Peng. Exact periodic wave solutions to the coupled schrödinger-kdv equation and ds equations. *Pramana - Journal of Physics*, 62:933–941, 2004.

[21] D. Zhang. Doubly periodic solutions of the modified kawahara equation. *Chaos, Solitons & Fractals*, 25:1155–1160, 2005.

[22] Q. Liu and J. M. Zhu. Exact jacobian elliptic function solutions and hyperbolic function solutions for sawada-kotera equation with variable coefficient. *Physics Letters A*, 352:233–238, 2006.