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Generous Roman Domination Subdivision Number in Graphs

Jamil J. Hamja[1,5], Seyed Mahmoud Sheikholeslami[2,*], Mina Esmaeili[2], Lutz Volkmann[3], Imelda S. Aniversario[4,5], Lucille M. Bugo[4,5]

[1] *Department of Mathematics, College of Arts and Sciences, MSU-Tawi-Tawi College of Technology and Oceanography, 7500 Tawi-Tawi, Philippines*
[2] *Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, I.R. Iran*
[3] *Lehrstuhl II für Mathematik, RWTH Aachen University, 52056 Aachen, Germany*
[4] *Department of Mathematics and Statistics, College of Science and Mathematics, MSU-Iligan Institute of Technology, 9200 Iligan City, Philippines*
[5] *Center for Mathematical and Theoretical Physical Sciences, Premier Research Institute of Science and Mathematics (PRISM), MSU-Iligan Institute of Technology, 9200 Iligan City, Philippines*

Abstract. Let $G = (V, E)$ be a simple graph, and let $f : V \to \{0, 1, 2, 3\}$ be a function. A vertex u is considered an undefended vertex with respect to f if $f(u) = 0$ and there is no adjacent vertex v satisfying $f(v) \geq 2$. A function f is termed a generous Roman dominating function (GRD-function) if, for every vertex u with $f(u) = 0$, there exists at least one adjacent vertex v such that $f(v) \geq 2$ and the modified function $f' : V \to \{0, 1, 2, 3\}$, defined as

$$f'(u) = \alpha, \quad f'(v) = f(v) - \alpha, \quad \text{where } \alpha \in \{1, 2\},$$

and

$$f'(w) = f(w) \quad \text{for all } w \in V \setminus \{u, v\},$$

ensures that no vertex remains undefended. The weight of a GRD-function f is defined as $f(V) = \sum_{u \in V} f(u)$. The smallest possible weight of a GRD-function on G is known as the generous Roman domination number of G, denoted by $\gamma_{gR}(G)$. The generous Roman domination subdivision number, denoted $\mathrm{sd}_{\gamma_{gR}}(G)$, is the minimum number of edges that must be subdivided (each at most once) to increase $\gamma_{gR}(G)$. In this paper, we establish upper bounds on $\mathrm{sd}_{\gamma_{gR}}(G)$, and determine its exact value for certain families of graphs, including paths, cycles, and ladders. Furthermore, we provide sufficient conditions for a graph G to have a small subdivision number.

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*Corresponding author.
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Email addresses: `jamilhamja@msutawi-tawi.edu.ph` (J. J. Hamja), `s.m.sheikholeslami@azaruniv.ac.ir` (S. M. Sheikholeslami), `minaesmaeeli1999@gmail.com` (M. Esmaeili), `volkm@math2.rwth-aachen.de` (L. Volkmann), `imelda.aniversario@g.msuiit.edu.ph` (I. S. Aniversario), `lucille.bugo@g.msuiit.edu.ph` (L. M. Bugo)

1. Introduction

Motivated by resource–allocation strategies for defending the Roman Empire, where lightly defended regions must be able to call in reinforcements from nearby strongholds, as discussed by ReVelle and Rosing [1] and by Stewart [2], Cockayne et al. introduced *Roman domination* in graphs in 2004 [3]. Since its introduction, Roman domination has attracted wide and sustained interest, spawning a rich family of variants and extensions, e.g., double Roman domination, Roman {2}-domination, independent Roman domination, total Roman domination, bondage and domatic parameters, and directed versions, now documented in well over two hundred publications. For comprehensive accounts of core results, and structural characterizations, we refer to the book chapters and survey papers by Chellali, Jafari Rad, Sheikholeslami, and Volkmann [4, 5]. Focused overviews of specific directions include surveys on varieties of Roman domination [6–8], Roman domatic problems in graphs and digraphs [9], and Roman domination parameters for directed graphs [10]. These sources together chart the evolution of Roman domination from its facility-location and defense-strategy origins [1, 2] to a mature theory with broad connections across domination theory and algorithmic graph problems [4–6, 9, 10].

Among the most recent contributions to Roman domination theory is the notion of *generous Roman domination*, introduced by Benatallah, Blidia, and Ouldrabah in 2024 [11]. This strengthening reflects scenarios where "reinforcements" must be guaranteed from more than one source, thereby increasing robustness compared to classical Roman domination. In their foundational paper, they provided exact values of the generous Roman domination number for paths and cycles, derived an upper bound for general graphs, and characterized cubic graphs of order n with respect to this parameter. Furthermore, they investigated a Nordhaus–Gaddum type inequality for the generous Roman domination number, and they established results concerning its computational complexity, demonstrating that the problem of determining this parameter remains NP-complete for general graphs. Building on this foundation, Sheikholeslami, Chellali, and Kor [12] advanced the study of generous Roman domination in 2025 by determining its exact value for ladder graphs. They also provided an upper bound for trees in terms of the order, the number of leaves, and the number of stems. Their results highlighted structural dependencies of this parameter within tree families, leading to a sharper understanding of its extremal behavior. In particular, they proved that for every tree T on at least three vertices, the generous Roman domination number satisfies

$$\gamma_{gR}(T) \ \geq \ \gamma(T) + 2,$$

where $\gamma(T)$ denotes the domination number of T, and they completely characterized the extremal trees attaining this lower bound. These findings not only extended the applicability of generous Roman domination to broader classes of graphs, but also established meaningful links between classical domination parameters and this new variant. Together, the works of Benatallah et al. [11] and Sheikholeslami et al. [12, 13] demonstrate the rapid development of generous Roman domination as a promising research direction. In particular, they show how the interplay between structural graph properties and domination parameters yields both exact values and tight bounds, while also raising new complexity and extremal problems. This suggests that generous Roman domination may follow a trajectory similar to classical Roman domination, giving rise to a wide array of variants and applications in the years to come.

Given that graphs serve as models for numerous real-world systems, it is natural to explore how structural changes such as vertex or edge deletions, edge additions, or edge subdivisions affect domination-related parameters. In this direction, Velammal [14] introduced the domination subdivision number measuring the minimal number of edge subdivisions needed to increase a graph's domination number, with each edge subdivided at most once. This line of research has since been extended to various domination parameters [15–22].

In this paper, we extend the study of generous Roman domination by examining its behavior under edge subdivision. To this end, we introduce the *generous Roman domination subdivision number*, which is defined as the minimum number of edges in a graph G that must be subdivided (each at most once) in order to increase the generous Roman domination number of G. We establish general upper bounds on generous Roman domination subdivision number and determine its exact value for several families of graphs, including paths, cycles, and ladders. In addition, we provide sufficient conditions under which a graph G admits a small generous Roman domination subdivision number.

2. Terminology and Notation

We consider finite, undirected, and simple graphs G, where the vertex set is denoted by $V = V(G)$ and the edge set by $E = E(G)$. The *order* of G is given by $|V| = n$, while the *size* of G is represented as $|E| = m$. The *open neighborhood* of a vertex $v \in V$ is defined as $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$, whereas its *closed neighborhood* is given by $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v, denoted by $\deg_G(v) = \deg(v)$, refers to the number of neighbors of v, i.e., $\deg_G(v) = |N_G(v)|$. As usual, a *path*, *cycle*, *star*, and *complete graph* with n vertices are denoted by P_n, C_n, $K_{1,n-1}$, and K_n, respectively. Similarly, $K_{n,m}$ denotes the *complete bipartite graph* of order $m + n$, while $DS_{p,q}$ denotes the *double star* of order $p + q + 2$. A vertex with degree one is referred to as a *leaf*, and its adjacent vertex is known as a *support vertex*. A vertex connected to at least two leaves is called a *strong support vertex*. The *Cartesian product* of graphs G and H, denoted by $G\square H$, is the graph with vertex set $V(G\square H) = V(G) \times V(H)$ such that two vertices (v, p) and (u, q) are adjacent in $G\square H$, i.e., $(v, p)(u, q) \in E(G\square H)$, if and only if one of the following holds: $v = u$ and $pq \in E(H)$, or $p = q$ and $vu \in E(G)$.

For any subset $A \subseteq V(G)$ and a function f that maps $V(G)$ to a numerical set, the function sum over A is given by $f(A) = \sum_{x\in A} f(x)$. The total sum over all vertices, $f(V(G))$, is referred to as the *weight* of f, denoted by $\omega(f)$.

Let f be a function from $V(G)$ to $\{0, 1, 2, 3\}$. A vertex u is considered *undefended* with respect to f if $f(u) = 0$ and no adjacent vertex v satisfies $f(v) \geq 2$. The function f is a *generous Roman dominating function* (GRD-function) if, for every vertex u with

$f(u) = 0$, there exists at least one adjacent vertex v with $f(v) \geq 2$ such that the function $g : V \to \{0, 1, 2, 3\}$, given by $g(u) = \alpha, \quad g(v) = f(v) - \alpha, \quad$ where $\alpha \in \{1, 2\}$, and $g(w) = f(w), \quad$ for all $w \in V \setminus \{u, v\}$, ensures that no vertex remains undefended. The *weight* of a GRD-function f is defined as $f(V) = \sum_{u \in V} f(u)$. The total sum over all vertices, $f(V)$, is referred to as the *weight* of f, denoted by $\omega_R^g(f)$, and the smallest possible weight of a GRD-function on G is referred to as the *generous Roman domination number* (GRD-number), denoted by $\gamma_{gR}(G)$, as introduced by Benatallah, Blidia, and Ouldrabah [11]. For any GRD-function f of G, let $V_i = \{v \in V \mid f(v) = i\}$, where $i \in \{0, 1, 2, 3\}$. Since these four sets uniquely define f, we can represent f as (V_0, V_1, V_2, V_3). Furthermore, a $\gamma_{gR}(G)$-function is a GRD-function of G if $\omega_R^g(f) = \gamma_{gR}(G)$. The generous Roman domination is a variant of double Roman domination with less restriction.

Importantly, if $G_1, G_2, \ldots, G_s$ are the components of G, then $\gamma_{gR}(G) = \sum_{i=1}^{s} \gamma_{gR}(G_i)$. Furthermore, if $G_1, G_2, \ldots, G_s$ represent the components of G with order at least 2, then $\mathrm{sd}_{\gamma_{gR}}(G) = \min\{\mathrm{sd}_{\gamma_{gR}}(G_i) \mid 1 \leq i \leq s\}$. Consequently, we restrict our study to connected graphs of order at least two.

3. Preliminary Results

We begin this section with some results that will be utilized later.

Proposition 1. *Let G be a connected graph of order $n \geq 2$, and let $F \subseteq E(G)$ be a subset of edges in G. If G' is obtained by subdividing each edge in F, then $\gamma_{gR}(G') \geq \gamma_{gR}(G)$.*

Proof. The proof is carried out using induction on the size of $|F|$. First, assume that $|F| = 1$, and let $e = uv$ be an element of F. Construct G' from G by introducing a new vertex x to subdivide the edge e. Let f be a $\gamma_{gR}(G')$-function. Since f is a GRD-function on G', it follows that $f(u) + f(v) + f(x) \geq 1$. Let $g : V(G) \to \{0, 1, 2, 3\}$ be a function defined by $g(u) = \min\{3, f(u) + f(x)\}$, and for all $y \in V(G) \setminus \{u\}$, by $g(y) = f(y)$. It is clear that g is a GRD-function on G, and $\omega_R^g(g) \leq \omega_R^g(f)$. Thus, we obtain $\gamma_{gR}(G') \geq \gamma_{gR}(G)$, which establishes the base case. Suppose the statement holds for any subset F of edges with $1 \leq |F| < k$. Let $F \subseteq E(G)$ be a set of edges of size k, where $F = \{e_1, e_2, \ldots, e_k\}, \quad k \geq 2$. Let G' be the graph obtained from G by subdividing all edges in $F \setminus \{e_k\}$. Then, let G'' be the graph formed by further subdividing the edge e_k in G'. By the induction hypothesis, we obtain $\gamma_{gR}(G'') \geq \gamma_{gR}(G') \geq \gamma_{gR}(G)$, which completes the proof. □

Proposition 2. *Let G be a connected graph of order $n \geq 2$, and let $F \subseteq E(G)$ be a set of edges in G. Let G' be the graph obtained from G by subdividing the edges in F. If there exists a $\gamma_{gR}(G')$-function that assigns a value of 1 or 3 to at least one subdivision vertex, then $\gamma_{gR}(G') > \gamma_{gR}(G)$.*

Proof. Let $e = uv \in F$ and suppose that e is subdivided by introducing a new vertex x. Consider a function f that is a $\gamma_{gR}(G')$-function such that $f(x) \in \{1, 3\}$. Construct G'' from G by subdividing all edges in $F - \{e\}$, noting that if $F = \{e\}$, then $G = G''$. If $f(x) = 1$, then the restriction of f to G'' acts as a GRD-function with a weight less than $\omega_R^g(f)$, which leads to the inequality $\gamma_{gR}(G') > \gamma_{gR}(G'') \geq \gamma_{gR}(G)$. Next, assume $f(x) = 3$. Define a function g on G'' by $g(u) = \min\{3, f(u)+1\}$, $g(v) = \min\{3, f(v)+1\}$, and $g(z) = f(z)$, for all $z \in V(G'') - \{u, v\}$. Then, g is a GRD-function with a weight less than $\omega_R^g(f)$, leading to the conclusion that $\gamma_{gR}(G') > \gamma_{gR}(G'') \geq \gamma_{gR}(G)$. Thus, the proof is complete. □

4. Exact Values

The exact values of the GRD-numbers of paths and cycles are determined in [11].

Proposition 3. [11] *For* $n \geq 1$, $\gamma_{gR}(P_n) = \lceil \frac{6n}{7} \rceil$.

Proposition 4. [11] *For* $n \geq 4$, $\gamma_{gR}(C_n) = \lceil \frac{6n}{7} \rceil$.

The following results directly follow from Propositions 3 and 4.

Corollary 1. *For* $n \geq 2$, $\mathrm{sd}_{\gamma_{gR}}(P_n) = \begin{cases} 2 & \text{if } n \equiv 6 \pmod 7 \\ 1 & \text{otherwise.} \end{cases}$

Corollary 2. *For* $n \geq 4$, $\mathrm{sd}_{\gamma_{gR}}(C_n) = \begin{cases} 2 & \text{if } n \equiv 6 \pmod 7 \\ 1 & \text{otherwise.} \end{cases}$

Proposition 5. [11] $\gamma_{gR}(G) = 2$ *if and only if* $G = K_n$ *or* $G = \overline{K_2}$.

Proposition 6. [13] *Let* G *be a connected graph of order* $n \geq 3$ *different from* K_n. *Then* $\gamma_{gR}(G) = 3$ *if and only if* $\Delta(G) = n - 1$.

As a direct consequence of Propositions 5 and 6, we obtain:

Corollary 3. *If* G *is a connected graph of order* $n \geq 2$ *with* $\gamma_{gR}(G) \in \{2, 3\}$, *then* $sd_{\gamma_{gR}}(G) = 1$.

Proof. If $\gamma_{gR}(G) = 2$, then according to Proposition 5, we have $G = K_n$. Furthermore, Proposition 5 implies that subdividing each edge of K_n increases the GRD-number, leading to $sd_{\gamma_{gR}}(G) = 1$. Now, suppose $\gamma_{gR}(G) = 3$. Subdividing any edge e of G results in a new graph G' with order $n+1$ and maximum degree $n-1$. By Proposition 6, it follows that $\gamma_{gR}(G) > 3$, which implies $sd_{\gamma_{gR}}(G) = 1$. □

According to Corollary 3, the following result is obtained.

Corollary 4. *If* G *is a connected graph of order* $n \geq 3$ *with* $sd_{\gamma_{gR}}(G) \geq 2$, *then* $\gamma_{gR}(G) \geq 4$.

Next, we examine the ladder graphs of the form $G = P_2 \square P_n$ and demonstrate that $sd_{\gamma_{gR}}(P_2 \square P_n) = 1$ when n is odd or $n = 4$. We label the vertices of the i-th copy of P_2 in the ladder $P_2 \square P_n$ as u_i and v_i, where $i = 1, 2, \ldots, n$. The GRD-number of ladder graphs is established by Sheikholeslami et al. in [12].

Theorem 1. *For* $n \geq 1$, $\gamma_{gR}(P_2 \square P_n) = \lceil \frac{3n+1}{2} \rceil$.

Proposition 7. $sd_{\gamma_{gR}}(P_2\Box P_4) = 1$.

Proof. Suppose that G' is obtained from $G = P_2\Box P_4$ by introducing a new vertex x through the subdivision of the edge u_2v_2. Let f be a $\gamma_{gR}(G')$-function. It is enough to establish that $\gamma_{gR}(G') > \gamma_{gR}(G)$. If $f(x) \in \{1, 3\}$, then by Proposition 2, we conclude that $\gamma_{gR}(G') > \gamma_{gR}(G)$, as required. We now analyze two cases.

Case 1. $f(x) = 2$.
If $f(u_2) \geq 1$ (a similar argument applies for $f(v_2) \geq 1$), then updating v_2 to the value $\min\{3, f(v_2) + 1\}$ yields a GRD-function for G with a weight smaller than $\omega^g_R(f)$. Now, suppose $f(u_2) = f(v_2) = 0$. In this case, it is evident that $f(u_1) + f(v_1) \geq 2$. Additionally, the function f restricted to $G \setminus \{u_1, v_1, u_2, v_2\}$ serves as a GRD-function of $P_2\Box P_4$ with weight at most $\omega^g_R(f) - 4$. Consequently, we obtain $\omega^g_R(f) \geq 8 > 7 = \gamma_{gR}(G)$ (refer to Theorem 1).

Case 2. $f(x) = 0$.
Without loss of generality, we assume that u_2 is a moving neighbor of x, implying that $f(u_2) \geq 2$. If $f(u_2) = 3$, it is easy to verify that in order to protect the vertices u_4, v_1, v_2, v_3, v_4, we must have $\omega^g_R(f) - f(u_2) \geq 5$, which implies $\omega^g_R(f) \geq 8 > \gamma_{gR}(G)$. Let $f(u_2) = 2$ and define $t = f(u_1) + f(u_2) + f(v_1) + f(v_2)$. Since u_2 is a moving neighbor of x, we must have either $f(u_1) \geq 1$ or $f(v_1) \geq 2$.

First, assume that $f(v_2) \geq 2$. If $f(v_2) = 3$, the result follows as before. If $f(v_2) = 2$, then we have $t \geq 5$. If $t \geq 6$, in order to protect other vertices, we must have $f(u_3) + f(u_4) + f(v_3) + f(v_4) \geq 2$, which leads to $\omega^g_R(f) \geq 8 > \gamma_{gR}(G)$. If $t = 5$, then $f(u_1) = 1$, $f(v_1) = 0$, and v_2 is a moving neighbor only for v_1. To protect the vertices u_4, v_3, v_4, we must have $f(u_3) + f(u_4) + f(v_3) + f(v_4) \geq 3$, so $\omega^g_R(f) \geq 8 > \gamma_{gR}(G)$.

Assume now that $f(v_2) \leq 1$. In this case, u_2 is a moving neighbor only for x. If $f(v_2) = 1$, then to protect the vertices u_1 and v_1, we must have $f(u_1) + f(v_1) \geq 2$. Reassigning u_1 and v_2 the values 0 and 2, respectively, yields a GRD-function of G with a weight less than $\gamma_{gR}(G_1)$, as desired (note that u_2 will be a moving neighbor for u_1 in the new assignment). Thus, we assume that $f(v_2) = 0$. To protect u_1 and v_1, we must have $f(u_1) + f(v_1) \geq 2$. If $f(u_1) + f(v_1) \geq 3$, as before, we observe that $\gamma_{gR}(G') \geq 8 > \gamma_{gR}(G)$. Let $f(u_1) + f(v_1) = 2$. Without loss of generality, assume that $f(v_1) = 2$ and $f(u_1) = 0$. Since u_2 is a moving neighbor only for x, v_1 becomes a moving neighbor of u_1, leading to $f(v_3) \geq 2$. If $f(v_3) = 3$, the result follows as before. Suppose $f(v_3) = 2$. Now, to protect u_4 and v_4, we must have $f(u_4) + f(v_4) + f(u_3) \geq 2$, which again gives $\omega^g_R(f) \geq 8 > \gamma_{gR}(G)$. This concludes the proof. □

Theorem 2. *If $n \geq 1$ is odd, then $sd_{\gamma_{gR}}(P_2\Box P_n) = 1$.*

Proof. Let $G = P_2\Box P_n$. If $n = 1$, the result follows from Corollary 1. Assume $n \geq 3$, and let G' be the graph derived from G by subdividing the edge u_1v_1 with a new vertex x. Let f be a $\gamma_{gR}(G')$-function. If $f(x) \in \{1, 3\}$, then Proposition 2 implies that $\gamma_{gR}(G') > \gamma_{gR}(G)$, so we have $sd_{\gamma_{gR}}(P_2\Box P_n) = 1$. Next, we consider two cases.

Case 1. $f(x) = 2$.
If $f(u_1) \geq 1$ (the case $f(v_1) \geq 1$ is similar), then by reassigning v_1 the value $\min\{3, f(v_1)+1\}$, we obtain a GRD-function for G with a weight smaller than $\omega_R^g(f)$, which leads to $sd_{\gamma_{gR}}(P_2\Box P_n) = 1$. Now, assume that $f(u_1) = f(v_1) = 0$. If x is a moving neighbor of neither u_1 nor v_1, or if x is a moving neighbor of both u_1 and v_1, then we must have $\min\{f(u_2), f(v_2)\} \geq 2$. Reassigning v_1 the value 1, u_2 the value 2, v_2 the value 0 and v_3 the value $\min\{3, f(v_3)+2\}$ gives a GRD-function for G with a weight smaller than $\omega_R^g(f)$, so we have $sd_{\gamma_{gR}}(P_2\Box P_n) = 1$. Next, assume that x is a moving neighbor of u_1 but not of v_1. In this case, v_2 must be a moving neighbor of v_1, so $f(v_2) \geq 2$. It is straightforward to observe that the function f restricted to $G = P_2\Box P_n \setminus \{u_1, v_1\}$ is a GRD-function with weight $\omega_R^g(f) - 2$. By Theorem 1, we can deduce that $\omega_R^g(f) \geq \omega_R^g(f|_{V(G)}) + 2 > \lceil \frac{3n+1}{2} \rceil$, which results in $sd_{\gamma_{gR}}(P_2\Box P_n) = 1$.

Case 2. $f(x) = 0$.
To protect the vertex x, we may assume that $f(u_1) \geq 2$. If $f(u_1) = 3$, then changing the value of u_1 to 2 yields a GRD-function for $P_2\Box P_n$ with a weight of $\omega_R^g(f) - 1$, which implies that $sd_{\gamma_{gR}}(P_2\Box P_n) = 1$. Therefore, we assume that $f(u_1) = 2$. If $f(v_1) \geq 2$, then the function g, defined on $(P_2\Box P_n) \setminus \{u_1, v_1, u_2, v_2\}$ by $g(u_3) = \min\{3, f(u_3) + f(u_2)\}$, $g(v_3) = \min\{3, f(v_3) + f(v_2)\}$, $g(z) = f(z)$ for the remaining vertices z, is a GRD-function with weight at most $\omega_R^g(f) - 4$. By Theorem 1, it follows that $\omega_R^g(f) \geq \omega_R^g(g) + 4 \geq \left\lceil \frac{3(n-2)+1}{2} \right\rceil + 4 > \lceil \frac{3n+1}{2} \rceil$. If $f(v_1) = 1$, then assigning the value 0 to v_1 gives a GRD-function for $P_2\Box P_n$ with weight $\omega_R^g(f) - 1$, leading to $sd_{\gamma_{gR}}(P_2\Box P_n) = 1$. Hence, we assume $f(v_1) = 0$. In this case, v_2 becomes the moving neighbor for v_1, so we must have $f(v_2) \geq 2$. If $f(v_2) = 3$ or if $f(v_2) = 2$ and $f(u_2) \geq 1$, then changing the value of u_1 to 1 results in a GRD-function for $P_2\Box P_n$ with weight $\omega_R^g(f) - 1$, and again $sd_{\gamma_{gR}}(P_2\Box P_n) = 1$. Let $f(v_2) = 2$ and $f(u_2) = 0$. From $f(x) = 0$, it follows that v_2 is only a moving neighbor for v_1 and that u_3 is the only moving neighbor of u_2, meaning $f(u_3) \geq 2$. If $n = 3$, then we have $\omega_R^g(f) \geq 6 > \lceil \frac{3n+1}{2} \rceil$. Therefore, let $n \geq 5$. If $f(u_3) = 3$ or $f(v_3) \geq 1$, then the function f restricted to $(P_2\Box P_n) - \{v_1, v_2, u_1, u_2\}$ is a GRD-function with weight $\omega_R^g(f) - 4$, and as before, we conclude that $\omega_R^g(f) > \lceil \frac{3n+1}{2} \rceil$, leading to $sd_{\gamma_{gR}}(P_2\Box P_n) = 1$. Let $f(u_3) = 2$ and $f(v_3) = 0$. If $f(u_4) \geq 1$ or $f(v_4) \geq 2$, then the function f restricted to $(P_2\Box P_n) \setminus \{v_1, v_2, u_1, u_2\}$ is a GRD-function with weight $\omega_R^g(f) - 4$, and as before, we conclude that $\omega_R^g(f) > \lceil \frac{3n+1}{2} \rceil$, showing that $sd_{\gamma_{gR}}(P_2\Box P_n) = 1$. Hence, we assume $f(u_4) = 0$ and $f(v_4) \leq 1$. If $f(v_4) = 1$, then the function f restricted to $(P_2\Box P_n) \setminus \{v_i, u_i \mid 1 \leq i \leq 4\}$ is a GRD-function with weight $\omega_R^g(f) - 7$, and using Theorem 1, we obtain $\omega_R^g(f) \geq 7 + \left\lceil \frac{3(n-4)+1}{2} \right\rceil > \left\lceil \frac{3n+1}{2} \right\rceil$, Thus $sd_{\gamma_{gR}}(P_2\Box P_n) = 1$. Thus, let $f(v_4) = 0$. To protect v_4, we must have $f(v_5) \geq 2$. On the other hand, since u_3 is the only moving neighbor of u_2, we have $f(u_5) \geq 2$. If $n = 5$, then $\omega_R^g(f) \geq 10 > \lceil \frac{3n+1}{2} \rceil$. Hence, let $n \geq 7$. By reassigning u_7 the value $\min\{3, f(u_7) + f(u_6)\}$ and v_7 the value $\min\{3, f(v_7) + f(v_6)\}$, we obtain a GRD-function on $(P_2\Box P_n) \setminus \{u_i, v_i \mid 1 \leq i \leq 6\}$ with weight at most $\omega_R^g(f) - 10$. By Theorem 1, we have $\omega_R^g(f) \geq \omega_R^g(g) + 10 \geq \left\lceil \frac{3(n-6)}{2} \right\rceil + 10 > \left\lceil \frac{3n+1}{2} \right\rceil$. In conclusion, we obtain $sd_{\gamma_{gR}}(P_2\Box P_n) = 1$. $\square$

5. Sufficient conditions on G having small $sd_{\gamma_{gR}}(G)$

In this section, we present several sufficient conditions for a graph G to have a small value of $sd_{\gamma_{gR}}(G)$.

Proposition 8. *If a connected graph G has a support vertex with at least three leaf neighbors, then $sd_{\gamma_{gR}}(G) = 1$.*

Proof. If G is a star, then $\gamma_{gR}(G) = 3$, and by Corollary 3, we have $sd_{\gamma_{gR}}(G) = 1$. Assume now that G is not a star. Let u, v and y be three leaves adjacent to a support vertex w, and let z be a non-leaf neighbor of w. Let G' be the graph obtained by subdividing the edge uw with a new vertex x, and let f be a $\gamma_{gR}(G')$-function such that $f(w) + f(z)$ is maximized. If $f(w) = 3$, it follows that $f(v) = f(x) = f(y) = 0$, and thus $f(u) = 1$. Reassigning the value 0 to u results in a GRD-function for G with weight smaller than $\omega_R^g(f)$; therefore, $sd_{\gamma_{gR}}(G) = 1$. Now, assume that $f(w) = 2$. If $f(v) \geq 1$ or $f(y) \geq 1$, then by reassigning w the value 3 and v and y the value 0, we obtain a $\gamma_{gR}(G')$-function g such that $g(w) + g(z) > f(w) + f(z)$, contradicting the maximality of $f(w) + f(z)$. Therefore, we must have $f(v) = f(y) = 0$. However, in this case, the vertices v and y cannot be protected by f, which is a contradiction. If $f(w) = 1$, then $f(v) = 1$, and by reassigning w and v the values 2 and 0, respectively, we get a $\gamma_{gR}(G)$-function g such that $g(w) + g(z) > f(w) + f(z)$, contradicting the maximality of $f(w) + f(z)$. Thus, we assume that $f(w) = 0$. In this case, it follows that $f(v) = f(y) = 1$ and $f(x) + f(u) = 2$. By reassigning the values of $w, u, v,$ and y to 3, 1, 0, and 0, respectively, we obtain a $\gamma_{gR}(G)$-function g such that $g(w) + g(z) > f(w) + f(z)$, which contradicts the maximality of $f(w) + f(z)$. Therefore, we conclude that $sd_{\gamma_{gR}}(G) = 1$. □

Proposition 9. *If G has a support vertex w with at least two leaf neighbors, then $sd_{\gamma_{gR}}(G) \leq 2$.*

Proof. If G is a star, the result follows directly from Corollary 3, and if w has at least three leaf neighbors, the conclusion holds by Proposition 8. Thus, assume that G is not a star and that w has exactly two leaf neighbors u and v. Let z be a non-leaf neighbor of w and let G' be the graph derived from G by inserting new vertices x and y to subdivide the edges uw and vw, respectively. Let f be a $\gamma_{gR}(G')$-function that maximizes the sum $f(w) + f(z)$. If $f(w) = 3$, then we have $f(u) = f(v) = 1$, and by reassigning $f(u) = f(v) = 0$, we obtain a GRD-function for G with weight less than $\omega_R^g(f)$, thus implying that $sd_{\gamma_{gR}}(G) \leq 2$. Assume now that $f(w) = 2$. To protect the vertices u, v, x and y, we must have $f(u) + f(v) + f(x) + f(y) \geq 3$. By reassigning $f(w) = 3$, $f(u) = 0$, and $f(v) = 0$, we obtain a GRD-function for G with weight less than $\omega_R^g(f)$, again showing that $sd_{\gamma_{gR}}(G) \leq 2$. Finally, if $f(w) \leq 1$, then to protect the vertices u, v, x and y, we must have $f(u)+f(v)+f(x)+f(y) \geq 4$, and as before, we can conclude that $sd_{\gamma_{gR}}(G) \leq 2$. □

Proposition 10. *For every tree T of order $n \geq 3$, $sd_{\gamma_{gR}}(T) \leq 2$.*

Proof. If T is a star or T contains a strong support vertex, then the result follows directly from Corollary 3 or Proposition 9. Now, assume that T is neither a star nor does it have a strong support vertex. Let $x_1x_2\dots x_k$ represent a diametral path of T, and root T at vertex x_k. According to our previous assumption, we have $\deg(x_2) = 2$. Next, assume that T' is obtained by subdividing the edges x_1x_2 and x_2x_3 with new vertices x and y, respectively, and let f be a $\gamma_{gR}(T')$-function. If $f(x_3) \leq 1$, then we must have $f(x_1) + f(x_2) + f(x) + f(y) \geq 4$. By assigning the values 0 and 2 to x_1 and x_2, respectively, we obtain a GRD-function for T with weight less than $\omega_R^g(f)$. On the other hand, if $f(x_3) \geq 2$, then we have $f(x_1)+f(x_2)+f(x)+f(y) \geq 3$, and assigning the values 0 and 2 to x_1 and x_2, respectively, provides a GRD-function for T with weight less than $\omega_R^g(f)$. Therefore, we conclude that $\mathrm{sd}_{\gamma_{gR}}(G) \leq 2$. □

Proposition 11. *Let G be a connected graph of order $n \geq 3$. If G contains a vertex v that lies in a triangle $uvwu$ such that $N(u) \cup N(w) \subseteq N[v]$, then* $\mathrm{sd}_{\gamma_{gR}}(G) \leq 3$.

Proof. Let G_1 be the graph obtained from G by subdividing the edges vu, vw, and uw with new vertices x_1, x_2 and x_3, respectively. It suffices to prove that $\gamma_{gR}(G_1) > \gamma_{gR}(G)$. Let g represent a minimum GRD-function of G_1. By Proposition 2, we know that for each $i \in \{1, 2, 3\}$, $g(x_i) \notin \{1, 3\}$. If two of the subdivision vertices, say x_1 and x_2, have positive weights under g, then it follows that $g(x_1) + g(x_2) \geq 4$, and by reassigning the value 3 to v, we obtain a GRD-function on G with weight less than $\omega_R^g(g)$. If exactly one of the subdivision vertices, say x_1, has a positive weight under g, then we must have $g(x_2) = g(x_3) = 0$ and $g(u) + g(v) + g(w) \geq 2$. Again, by reassigning the value 3 to v, we get a GRD-function on G with weight less than $\omega_R^g(g)$, as the neighbors of u and z are also neighbors of v. Thus, we assume that $g(x_1) = g(x_2) = g(x_3) = 0$. To protect the subdivision vertices, we must have $g(v) + g(u) + g(w) \geq 4$. The function defined above is a GRD-function on G with weight less than $\omega_R^g(g)$. This completes the proof. □

6. Bounds

In this section, we derive several bounds for the generous Roman domination subdivision number.

Theorem 3. *Let G be a connected graph. If $x \in V(G)$ has degree at least two, then $sd_{\gamma_{gR}}(G) \leq \deg(x)$.*

Proof. Let $s = \deg(x)$ and $N(x) = \{x_1, x_2, \dots, x_s\}$, and let G_1 be the graph obtained from G by subdividing the edges $xx_1, xx_2, \dots, xx_s$ with new vertices $y_1, y_2, \dots, y_s$, respectively. To prove the desired result, it is sufficient to show that $\gamma_{gR}(G_1) > \gamma_{gR}(G)$. Let f be a $\gamma_{gR}(G_1)$-function. By Proposition 2, we may assume that $f(y_i) \notin \{1, 3\}$ for all $1 \leq i \leq s$. If $\sum_{i=1}^{s} f(y_i) + f(x) \geq 4$, then assigning the value 3 to x results in a GRD-function on G with a weight smaller than $\omega_R^g(f)$, as desired. Therefore, we assume that $\sum_{i=1}^{s} f(y_i) + f(x) \leq 3$. If $f(x) \in \{2, 3\}$, then we must have $\sum_{i=1}^{s} f(y_i) = 0$

because $f(y_i) \neq 1$ for each i. In this case, reassigning the value 1 to x results in a GRD-function on G with weight less than $\omega_R^g(f)$, as desired. If $f(x) = 1$, then since $s \geq 2$ and $\sum_{i=1}^{s} f(y_i) + f(x) \leq 3$, there must exist a vertex y_i, say y_1, such that $f(y_1) = 0$. Therefore, x_1 is a moving neighbor of y_1, and thus $f(x_1) \geq 2$. By reassigning the value 0 to x, we obtain a GRD-function on G with weight less than $\omega_R^g(f)$. Finally, assume that $f(x) = 0$. In this case, one of the vertices y_i, say y_1, must be a moving neighbor of x, so $f(y_1) \geq 2$. From $\sum_{i=1}^{s} f(y_i) + f(x) \leq 3$ and the assumption that $f(y_i) \neq 1$ for each i, we conclude that $f(y_i) = 0$ for each $i \in \{2, \ldots, s\}$. Therefore, $f(x_i) \geq 2$, and each x_i is a moving neighbor of y_i for $i \in \{2, \ldots, s\}$. Now, by reassigning the value $\min\{3, f(x_1) + 1\}$ to x_1, we get a GRD-function on G with weight less than $\omega_R^g(f)$, completing the proof. □

As a result of Corollary 1 and Theorem 3, $sd_{\gamma_{gR}}(G)$ is well-defined for any connected graph G of order $n \geq 2$. Moreover, we derive the following result.

Corollary 5. *If G is a connected graph with $\delta \geq 2$, then $sd_{\gamma_{gR}}(G) \leq \delta$.*

Corollary 6. *If G is a connected graph with $\delta = 1$, then $sd_{\gamma_{gR}}(G) \leq 3$.*

Proof. If G is a star, then the result follows directly from Corollary 3. Suppose that G is not a star. Let $v \in V(G)$ be a support vertex, and let v_1 be a leaf adjacent to v. If $\deg(v) = 2$, then, as shown in the proof of Proposition 10, we obtain $sd_{\gamma_{gR}}(G) \leq 3$. Thus, assume that $\deg(v) \geq 3$ and consider two neighbors $v_2, v_3 \in N(v) \setminus \{v_1\}$. Let G_1 be the graph obtained from G by subdividing the edges vv_1, vv_2, vv_3 with new vertices x_1, x_2, x_3, and let f be a $\gamma_{gR}(G_1)$-function such that $f(v)$ is maximized. By Proposition 2, we may assume that $f(x_i) \notin \{1, 3\}$ for all $i \in \{1, 2, 3\}$. If $f(v) + f(v_1) + \sum_{i=1}^{3} f(x_i) \geq 4$, then assigning the value 3 to v produces a GRD-function of G with a weight smaller than $\omega_R^g(f)$. Now, assume that $f(v) + f(v_1) + \sum_{i=1}^{3} f(x_i) \leq 3$. This implies that $f(v) \in \{0, 1, 2\}$. If $f(v) \leq 1$, then we must have $f(x_1) + f(v_1) = 2$ and $f(x_2) = f(x_3) = 0$. Since we previously assumed that $f(y_i) \neq 1$ for all i, the function g defined on G by $g(v_1) = 1, \quad g(v_i) = \min\{3, f(v_i) + f(x_i)\}$ for $i = 2, 3, \quad$ and $\quad g(x) = f(x)$ for all other vertices x, is a GRD-function of G with a weight smaller than $\omega_R^g(f)$. Note that v_2 is a moving neighbor of x_2 under f, and hence, it is a moving neighbor of v under g. Next, assume that $f(v) = 2$. Since $f(v) + f(v_1) + \sum_{i=1}^{3} f(x_i) \leq 3$, it follows that $f(v_1) = 1$ and $f(x_i) = 0$ for all $i \in \{1, 2, 3\}$. In this case, v is the moving neighbor only of x_1, and reassigning v_1 the value 0 produces a GRD-function of G with a weight smaller than $\omega_R^g(f)$. Thus, we conclude that $sd_{\gamma_{gR}}(G) \leq 3$, completing the proof. □

Since every planar graph contains at least one vertex of degree at most five, the following result is an immediate consequence of Corollaries 5 and 6.

Corollary 7. *If G is a planar graph, then $sd_{\gamma_{gR}}(G) \leq 5$.*

In the following, we establish an upper bound on the generous Roman domination number of a graph based on the number of vertices that are at a distance of 2 from a given vertex. For a vertex $x \in V(G)$, let $N_2(x)$ denote the set of vertices in G that are exactly two edges away from x, and define $d_2(x) = |N_2(x)|$. In this setting, we introduce $\delta_2(G) = \min\{d_2(x) \mid x \in V(G) \text{ and } \deg(x) \geq 2\}$.

To establish this result, we begin with a few lemmas. The proof of the following lemmas are essentially similar to the proof of corresponding lemmas in [17].

Lemma 1. *Let G be a connected graph of order $n \geq 3$. If G contains a vertex v_1 that is part of a triangle $v_1v_2v_3v_1$ such that $N(v_2) \subseteq N[v_1]$ and $N(v_3) - N[v_1] \neq \varnothing$, then*

$$\mathrm{sd}_{\gamma_{gR}}(G) \leq 3 + |N(v_3) - N[v_1]|.$$

Proof. Let $w_1, w_2, \ldots, w_k$ be the neighbors of v_3 in $V(G) - N[v_1]$, and let G_1 be obtained from G by subdividing the edges v_1v_2, v_1v_3 and v_2v_3 with vertices x, y and z, respectively, and the edge v_3w_i with vertex x_i for each $1 \leq i \leq k$. Assume that f is a $\gamma_{gR}(G_1)$-function. By Proposition 2, we may assume that $\{1,3\} \cap \{f(x), f(y), f(z), f(x_1), \ldots, f(x_k)\} = \varnothing$. Similar to the proof of Proposition 11, we can observe that $f(x) + f(y) + f(z) + f(v_1) + f(v_2) + f(v_3) \geq 4$. Define a function $g : V(G) \to \{0,1,2,3\}$ by $g(v_1) = 3$, $g(v_2) = g(v_3) = 0$, $g(w_i) = \min\{3, f(w_i) + f(x_i)\}$ for each $1 \leq i \leq k$, and $g(t) = f(t)$ for all $t \in V(G) \setminus \{v_1, v_2, v_3, w_i \mid 1 \leq i \leq k\}$. It is straightforward to verify that g is a GRD-function of G with a weight smaller than $\gamma_{gR}(G_1)$, thereby completing the proof. □

Lemma 2. *Let G be a connected graph of order $n \geq 3$, and let x be a vertex of degree at least 2 in G that satisfies the following conditions:*

(i) *$N(y) \setminus N[x] \neq \varnothing$ for each $y \in N(x)$,*

(ii) *there exist vertices $a, b \in N(x)$ such that $(N(a) \cap N(b)) \setminus N[x] = \varnothing$.*

Then, $sd_{\gamma_{gR}}(G) \leq 3 + |N_2(x)|$.

Proof. Let $\deg(x) = t$ and $N(x) = \{x_1, x_2, \ldots, x_t\}$. We assume, without loss of generality, that $a = x_1$ and $b = x_2$. Additionally, we assume that the pair a, b is selected first among the adjacent vertices in $N(x)$. Therefore, if $ab \in E(G)$, then x must be part of the triangle xx_1x_2x. Moreover, define $S = \{x_1, x_2, \ldots, x_s\}$ as one of the largest subsets of $N(x)$ containing x_1 and x_2, where every pair of vertices a, b in S satisfies Condition (ii). According to item (i), for each $i \in \{1, 2, \ldots, s\}$, we define $N(x_i) \setminus N[x] = \{x_{i_1}, x_{i_2}, \ldots, x_{i_{l_i}}\}$. Next, we define G_1 as the graph obtained from G by subdividing the edges xx_1 and xx_2 with new vertices u_1 and u_2, respectively. For each $i \in \{1, 2, \ldots, s\}$, we subdivide each edge $x_ix_{i_j}$, where $1 \leq j \leq l_i$, by introducing a new vertex x^{i_j}. We define $W_i = \{x^{i_j} \mid 1 \leq j \leq l_i\}$ and $W = \bigcup_{1 \leq i \leq s} W_i$. Furthermore, if x_1 and x_2 are adjacent, we also subdivide the edge x_1x_2 by introducing a new vertex u_3. Finally, let f be a $\gamma_{gR}(G_1)$-function. By Proposition 2, we can assume that no subdivision vertex is assigned the values 1 or 3 under f.

First let $x_1x_2 \in E(G)$. Similar as in the proof of Proposition 11, we can see that $f(x) + f(x_1) + f(x_2) + f(u_1) + f(u_2) + f(u_3) \geq 4$. By reassigning x the value 3, x_1, x_2 the value 0, and x_{i_j} the value $\min\{3, f(x_{i_j}) + f(x^{i_j})\}$ for all i and j, we obtain a GRD-function of G of weight less than $\gamma_{gR}(G_1)$ as desired.

Assume now that $x_1x_2 \notin E(G)$. By the choice of x_1, x_2, we deduce that S is independent. To protect the vertices u_1 and u_2, we must have $f(x)+f(x_1)+f(x_2)+f(u_1)+f(u_2) \geq$

3. If $f(x) + f(u_1) + f(u_2) + \sum_{i=1}^{s} f(x_i) \geq 4$, then reassigning x the value 3, x_i the value 0 for all $i \in \{1, 2, \ldots, s\}$, and x_{i_j} the value $\min\{3, f(x_{i_j}) + f(x^{i_j})\}$ for all i, j, provides a GRD-function of G of weight less than $\gamma_{gR}(G_1)$. Thus, we may assume that $f(x) + f(u_1) + f(u_2) + \sum_{i=1}^{s} f(x_i) \leq 3$. If $f(x) \in \{0, 1, 2\}$, then to protect the vertices u_1, u_2, we must have $f(x)+f(u_1)+f(u_2)+\sum_{i=1}^{s} f(x_i) \geq f(x_1)+f(x_2)+f(u_1)+f(u_2) \geq 4$ contradicting our assumption. Thus, $f(x) = 3$ and so $f(u_1) + f(u_2) + \sum_{i=1}^{s} f(x_i) = 0$.

If $\sum_{j=1}^{l_i} f(x^{i_j}) \geq 4$ for some $1 \leq i \leq s$, say $i = 1$, then reassigning x_1 the value 3 and x_{i_j} the value $\min\{3, f(x_{i_j}) + f(x^{i_j})\}$ for all $i \in \{2, \ldots, s\}$ and all $j \in \{1, 2, \ldots, l_i\}$, provides a GRD-function of G of weight less than $\omega_R^g(f) = \gamma_{gR}(G_1)$. Hence, suppose that $\sum_{j=1}^{l_i} f(x^{i_j}) \leq 3$, for each $i \in \{1, 2, \ldots, s\}$. First, consider the case where there exist some $i \in \{1, 2, \ldots, s\}$ and some $j \in \{1, 2, \ldots, l_i\}$ such that $f(x^{i_j}) = 2$. Assume, without loss of generality, that $i = j = 1$. Then, by updating x_{1_1} to take the value $\min\{3, 1+f(x_{1_1})\}$ and redefining x_{i_j} as $\min\{3, f(x_{i_j}) + f(x^{i_j})\}$, for $i_j \neq 1_1$, we obtain a GRD-function of G with weight strictly less than $\gamma_{gR}(G_1)$. Thus, we may assume that $f(x^{i_j}) = 0$ for all i and j. This directly implies that $f(x_{i_j}) = 2$ for every i and j. Clearly, reassigning x the value 2 results in a GRD-function of G with weight strictly less than $\gamma_{gR}(G_1)$. All in all, we see that the graph G has a GRD-function of weight less than $\gamma_{gR}(G_1)$. Moreover, since G_1 is obtained by inserting at most $3 + |W| \leq 3 + |N_2(x)|$ new vertices, we obtain $sd_{\gamma_{gR}}(G) \leq 3 + |N_2(x)|$. This completes the proof.

Lemma 3. *Let G be a connected graph of order $n \geq 3$ and v be a vertex of degree at least 2 of G satisfying the following conditions:*

(i) *$N(y) \setminus N[v] \neq \emptyset$ for each $y \in N(v)$,*

(ii) *for every pair of vertices a, b in $N(v)$, $(N(a) \cap N(b)) \setminus N[v] \neq \emptyset$.*

Then $sd_{\gamma_{gR}}(G) \leq 3 + |N_2(v)|$.

Proof. If $\deg(v) \leq 3 + |N_2(v)|$, then the result follows from Theorem 3. Henceforth, we assume that $\deg(v) \geq 4 + |N_2(v)|$. Let $N(v) = \{v_1, v_2, \ldots, v_k\}$ and $K = N(v_1) \setminus N[v] = \{w_1, w_2, \ldots, w_p\}$. By (ii) each vertex $y \in N(v) \setminus \{v_1\}$ has a neighbor in K. Let S be one of the largest subsets of $N(v) \setminus \{v_1\}$ such that for every subset $S_1 \subseteq S$, the inequality $|N(S_1) \setminus (N[v] \cup K)| \geq |S_1|$ holds. By the choice of S, we have $|N_2(v)| \geq |K| + |S|$. Furthermore, every vertex u in $U = N(v) \setminus (S \cup \{v_1\})$ has at least one neighbor in K, and the set $N(u) \setminus N[v]$ satisfies $N(u) \setminus N[v] \subseteq K \cup N(S)$. Additionally, the set K dominates $N(v)$ (as stated in item (ii)). From the inequality $4+|K|+|S| \leq 4+|N_2(v)| \leq \deg(v) = |S|+1+|U|$, we conclude that $|U| \geq 4$. If $S \neq \varnothing$, then, without loss of generality, we may assume $S = \{v_2, v_3, \ldots, v_s\}$. Assume that G_1 is obtained from G by subdividing the edges $v_1 w_j$ with new vertices y_j for all $j \in \{1, \ldots, p\}$, and subdividing the edges vv_i with new vertices x_i for $1 \leq i \leq s+2$ when $S \neq \varnothing$, or for $1 \leq i \leq 3$ when $S = \varnothing$. Consequently, the number of subdivided edges is $|K| + |S| + 3$. It suffices to demonstrate that $\gamma_{gR}(G_1) > \gamma_{gR}(G)$. Let f be a $\gamma_{gR}(G_1)$-function. By Proposition 3, we can assume that $f(z) \notin \{1, 3\}$ for all subdivision vertices z.

If $f(v)+f(v_1)+\sum_{i=1}^{s+2} f(x_i) \geq 4$, then by reassigning v the value 3, v_1 the value 0, and w_j the value $\min\{3, f(w_j)+f(y_j)\}$ for all $j \in \{1,\ldots,p\}$, we obtain a GRD-function of G with weight less than $\gamma_{gR}(G_1)$. Thus, we assume that

$$f(v)+f(v_1)+\sum_{i=1}^{s+2} f(x_i) \leq 3. \tag{1}$$

We now distinguish four different situations.

Case 1. $f(v)=3$.
From equation (1), we have $f(v_1)=f(x_1)=\cdots=f(x_{s+2})=0$. If $\sum_{j=1}^{p} f(y_j) \geq 4$, then by assigning the value 3 to v_1, we can obtain a GRD-function for G with weight less than $\gamma_{gR}(G_1)$. Therefore, we assume that $\sum_{j=1}^{p} f(y_j) \leq 3$. Based on our earlier assumption, we also have $\sum_{i=1}^{p} f(y_i) \leq 2$. If there exists some $j \in \{1,2,\ldots,p\}$, say $j=1$, such that $f(y_1)=2$, then it follows that $f(y_j)=0$ for all $j \in \{2,\ldots,p\}$. In this case, reassigning w_1 the value $\min\{3, 1+f(w_1)\}$ provides a GRD-function for G with weight less than $\gamma_{gR}(G_1)$. Hence, we assume that $\sum_{j=1}^{p} f(y_j)=0$. To protect the vertices y_j, the vertex w_j must be the moving neighbor of y_j, implying that $f(w_j)=2$ for each $j \in \{1,2,\ldots,p\}$. Then, by reassigning v the value 2, we obtain a GRD-function for G, where every vertex of U has a neighbor in K, with weight less than $\gamma_{gR}(G_1)$.

Case 2. $f(v)=2$.
From equation (1) and our previous assumption, it follows that $f(x_1)=\cdots=f(x_{s+2})=0$. If $f(v_1)=1$, then by reassigning the value 0 to v_1 and the value $\min\{3, f(w_i)+f(y_i)\}$ to w_i for all $i \in \{1,2,\ldots,p\}$, we obtain a GRD-function for G with weight less than $\gamma_{gR}(G_1)$ (note that v is a moving neighbor of v_1). Now, suppose $f(v_1)=0$. This implies that v is a moving neighbor only for x_1, and thus, v_i is the moving neighbor of x_i in order to protect x_i. Consequently, we have $f(v_i) \geq 2$ for all $i \in \{2,3,\ldots,s+2\}$. If $\sum_{j=1}^{p} f(y_j) \geq 4$, then reassigning the value 3 to v_1 provides a GRD-function for G with weight less than $\gamma_{gR}(G_1)$. Hence, we assume that $\sum_{j=1}^{p} f(y_j) \leq 3$. From our earlier assumption, it follows that either $f(y_j)=0$ for all $j \in \{1,2,\ldots,p\}$ or $f(y_j)=2$ for exactly one j. In the first case, we have $f(w_j) \geq 2$, and w_j is the moving neighbor only for y_j if $f(w_j)=2$. Reassigning the value 1 to v then provides a GRD-function for G with weight less than $\gamma_{gR}(G_1)$. In the second case, reassigning the value 1 to w_j and the value $\min\{3, f(w_i)+1\}$ to w_i for $i \neq j$ provides a GRD-function for G with weight less than $\gamma_{gR}(G_1)$.

Case 3. $f(v)=1$.
To protect x_1, it is required that $f(x_1)+f(v_1) \geq 2$. From our previous assumption and equation (1), it follows that $\sum_{i=1}^{s+2} f(x_i)=0$. If $f(v_1)=2$, then $f(x_1)=0$, which means that v_1 is a moving neighbor only for x_1. By assigning the value 0 to v and the value $\min\{3, f(w_i)+f(y_i)\}$ to w_i for all $i \in \{1,2,\ldots,p\}$, we obtain a GRD-function for G with weight less than $\gamma_{gR}(G_1)$. If $f(x_1)=2$, then by reassigning the value 1 to v_1 and the value $\min\{3, f(w_i)+f(y_i)\}$ to w_i for all $i \in \{1,2,\ldots,p\}$, we obtain a GRD-function for G with weight less than $\gamma_{gR}(G_1)$.

Case 4. $f(v)=0$.
In order to protect x_1, we require that $f(x_1)+f(v_1) \geq 2$, and from equation (1), we know

that $\sum_{i=2}^{s+2} f(x_i) = 0$. This implies that $f(v_i) \geq 2$, and that v_i is a moving neighbor only for x_i for all $i \in \{2, 3, \ldots, s+2\}$. If $\sum_{j=1}^{p} f(y_j) \geq 2$, then by assigning the value 3 to v_1, we can obtain a GRD-function for G with weight less than $\gamma_{gR}(G_1)$. Therefore, we assume that $\sum_{j=1}^{p} f(y_j) = 0$. Since v_1 is a moving neighbor only for x_1, in order to protect y_i, we require that $f(w_i) \geq 2$, and that w_i is a moving neighbor of y_i. Finally, by assigning the value 1 to v_1, we obtain a GRD-function for G with weight less than $\gamma_{gR}(G_1)$. This completes the proof. □

We are now prepared to present our main result.

Theorem 4. *Let G be a connected graph of order $n \geq 3$. Then*

$$\mathrm{sd}_{\gamma_{gR}}(G) \leq 3 + \min\{d_2(x) \mid x \in V \textit{ and } \deg(x) \geq 2\}.$$

Proof. If G is a star graph $K_{1,n-1}$, then $\gamma_{gR}(G) = 3$. Moreover, by Corollary 3, we obtain $sd_{\gamma_{gR}}(G) = 1$, and thus the result holds. Therefore, from this point onward, we assume that $G \neq K_{1,n-1}$. If G contains a leaf, then by Corollary 6, the result also holds. Next, let G be a graph such that $\delta(G) \geq 2$. Using Proposition 11 and Lemmas 1, 2 and 3, we conclude that

$$sd_{\gamma_{gR}}(G) \leq 3 + \min\{d_2(x) \mid x \in V \text{ and } \deg(x) \geq 2\}.$$

□

Let $\delta_2(G) = \min\{d_2(v) \mid v \in V(G) \text{ and } \deg(v) \geq 2\}$, and note that for each vertex v with degree Δ, we have $\delta_2(G) \leq |N_2(v)| \leq n - \Delta - 1$. The next two Corollaries follow directly from Theorem 4.

Corollary 8. *Let G be a connected graph with $\delta(G) \geq 2$, $sd_{\gamma_{gR}}(G) \leq 3 + \delta_2(G)$.*

Corollary 9. *Let G be a connected graph of order $n \geq 3$. Then $sd_{\gamma_{gR}}(G) \leq n - \Delta + 2$.*

Applying Corollaries 5, 6, and 9, we derive the following result.

Proposition 12. *Let G be a connected graph of order $n \geq 3$. Then $sd_{\gamma_{gR}}(G) \leq \frac{n}{2} + 1$.*

Conclusion: This study introduced and explored the concept of the generous Roman domination subdivision number in graphs, focusing on its effect on the generous Roman domination number. By analyzing how edge subdivisions influence this parameter, we established upper bounds and exact values for specific graph families. These findings contributed to a deeper understanding of domination parameters under graph modifications. Future research may focus on characterizing more graph classes where the exact generous Roman domination subdivision number can be determined. Additionally, algorithmic approaches to compute this parameter efficiently in general graphs can be developed to support applications in network defense and resource allocation.

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