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Revisiting Best Proximity Results of Relatively Meir-Keeler Condensing Operators in Hyperconvex Spaces

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Abstract. We first prove that if $(\mathcal{G}, \mathcal{H})$ is a nonempty, compact and hyperconvex pair of subsets of a hyperconvex metric space $(\mathcal{M}, d)$, then every cyclic relatively u-continuous mapping T defined on $\mathcal{G} \cup \mathcal{H}$ has a best proximity point. The same result is valid for the case that T is the noncyclic relatively u-continuous map and $(\mathcal{G}, \mathcal{H})$ is a semi-sharp proximinal pair to obtain the existence of best proximity pairs. We then consider the class of relatively **H**-Meir-Keeler condensing operators by applying a concept of measure of noncompactness in the framework of hyperconvex spaces and in a special case in the nonreflexive Banach space ℓ_∞ and revisit the previous best proximity point (pair) results of the paper by M. Gabeleh and C. Vetro [M. Gabeleh, C. Vetro, A new extension of Darbo's fixed point theorem using relatively Meir-Keeler condensing operators, *Bull. Aust. Math. Soc.*, 98 (2018) 286–297]. Examples are given to support our main discussions.

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Key Words and Phrases: Hyperconvex metric space, Best proximity point, Relatively u-continuous map, Meir-Keeler condensing operator

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1. Introduction

Let $(\mathcal{G}, \mathcal{H})$ be a nonempty pair of subsets of a metric space $(\mathcal{M}, d)$. A mapping $T : \mathcal{G} \cup \mathcal{H} \to \mathcal{G} \cup \mathcal{H}$ is said to be *relatively nonexpansive* if $d(Tx, Ty) \le d(x, y)$ for all $(x, y) \in \mathcal{G} \times \mathcal{H}$. In particular case, if $\mathcal{G} = \mathcal{H}$, then T is well-known as a *nonexpansive self-mapping.*
The mapping T is cyclic on $\mathcal{G} \cup \mathcal{H}$ if

$$T(\mathcal{G}) \subseteq \mathcal{H}, \quad T(\mathcal{H}) \subseteq \mathcal{G},$$

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and T is noncyclic provided that

$$T(\mathcal{G}) \subseteq \mathcal{G}, \quad T(\mathcal{H}) \subseteq \mathcal{H}.$$

It is worth mentioning that if the map T is cyclic and $\mathcal{G} \cap \mathcal{H} = \emptyset$, then the fixed point equation $Tx = x$ does not have any solution. Indeed, for any $x \in \mathcal{G} \cup \mathcal{H}$ we have $d(x, Tx) \geq \mathrm{D}(\mathcal{G}, \mathcal{H}) := \inf\{d(u, v) : (u, v) \in \mathcal{G} \times \mathcal{H}\}$. So, in this case, instead of finding a fixed point for the cyclic mapping T we can think about the existence of a point $x^* \in \mathcal{G} \cup \mathcal{H}$ for which

$$d(x^*, Tx^*) = \mathrm{D}(\mathcal{G}, \mathcal{H}).$$

Such points are called *best proximity points* of the cyclic mapping T. On the other hand, if T is noncyclic, then the fixed point equation may have a fixed point, but it is interesting to find a pair of fixed points $(x^*, y^*) \in \mathcal{G} \times \mathcal{H}$ (i.e. $Tx^* = x^*, Ty^* = y^*$) such that $d(x^*, y^*) = \mathrm{D}(\mathcal{G}, \mathcal{H})$. These kinds of points are called *best proximity pairs* for the noncyclic mapping T.

Existence of best proximity points (pairs) for cyclic (noncyclic) relatively nonexpansive mappings was first established by Eldred et al. in [1] and after that R. Espinola ([2]) used a different approach to obtain the existence results of [1] (see also [3, 4] for more information about cyclic maps which satisfy contractive conditions). Here is a main result of [1].

Theorem 1. *Let $(\mathcal{G}, \mathcal{H})$ be a nonempty, compact and convex pair of subsets of a Banach space X. If $T : \mathcal{G} \cup \mathcal{H} \to \mathcal{G} \cup \mathcal{H}$ is a cyclic relatively nonexpansive mapping, then T has a best proximity point.*

We mention that the proof of the above theorem is based on the fact that every nonempty, compact and convex pair of subsets of a Banach space X has the *proximal normal structure* (see Proposition 2.2 and Theorem 2.1 of [1]). Another way to prove Theorem 1 was presented in [5] by applying a concept of proximal diametral sequences.

In order to state the noncyclic version of Theorem 1 we need to recall that a Banach space X is strictly convex if for any two distinct elements $u, v \in \mathcal{S}_X := \{x \in X : \|x\| = 1\}$ we have $\|\frac{u+v}{2}\| < 1$. Hilbert and $L^p(1 < p < +\infty)$ spaces are instances of strictly convex Banach spaces.

Theorem 2. (see Theorem 2.2 of [1]) *Let $(\mathcal{G}, \mathcal{H})$ be a nonempty, compact and convex pair of subsets of a strictly convex Banach space X. If $T : \mathcal{G} \cup \mathcal{H} \to \mathcal{G} \cup \mathcal{H}$ is a noncyclic relatively nonexpansive mapping, then T has a best proximity pair.*

It was announced in [6] that if the Banach space X in Theorem 1 is strictly convex, then Theorem 1 is a special case of Theorem 2.

In what follows we recall the extensions of Theorem 1 and Theorem 2 to a more extensive family of cyclic (noncyclic) relatively nonexpansive mappings.

Definition 1. ([7]) *Let $(\mathcal{G}, \mathcal{H})$ be a nonempty pair in a metric space $(\mathcal{M}, d)$. A mapping $T : \mathcal{G} \cup \mathcal{H} \to \mathcal{G} \cup \mathcal{H}$ is called* relatively u-continuous *if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d(Tx, Ty) < \varepsilon + \mathrm{D}(\mathcal{G}, \mathcal{H})$, whenever $d(x, y) < \delta + \mathrm{D}(\mathcal{G}, \mathcal{H})$, for all $(x, y) \in \mathcal{G} \times \mathcal{H}$. If moreover, T is also cyclic (noncyclic), then T is said to be cyclic (noncyclic) relatively u-continuous.*

Clearly, every relatively nonexpansive map is relatively u-continuous, but the inverse implication may not be hold (see Example 2.1 of [7]).

Theorem 3. (Theorem 3.1 of [7]) *Let $(\mathcal{G}, \mathcal{H})$ be a nonempty, compact and convex pair of subsets of a strictly convex Banach space X. If $T : \mathcal{G} \cup \mathcal{H} \to \mathcal{G} \cup \mathcal{H}$ is a cyclic relatively u-continuous mapping, then T has a best proximity point.*

The noncyclic version of Theorem 3 is as follows.

Theorem 4. (Theorem 4.2 of [8]) *Let $(\mathcal{G}, \mathcal{H})$ be a nonempty, compact and convex pair of subsets of a strictly convex Banach space X. If $T : \mathcal{G} \cup \mathcal{H} \to \mathcal{G} \cup \mathcal{H}$ is a noncyclic relatively u-continuous mapping, then T has a best proximity pair.*

Motivated by Schauder's fixed point problem for compact and continuous self-mappings defined on a bounded, closed and convex subset of a Banach space, the current authors presented the extensions of Theorem 1 and Theorem 2 by shifting the compactness assumption on the pair $(\mathcal{G}, \mathcal{H})$ to the cyclic (noncyclic) mapping T which may not be continuous (see Theorem 3.2 and Theorem 4.1 of [9]). Indeed, the cyclic (noncyclic) mapping $T : \mathcal{G} \cup \mathcal{H} \to \mathcal{G} \cup \mathcal{H}$ is said to be *compact* whenever $\big(T(\mathcal{K}), T(\mathcal{L})\big)$ is relatively compact for any bounded pair $(\mathcal{K}, \mathcal{L}) \subseteq (\mathcal{G}, \mathcal{H})$.

Gabeleh and Vetro ([10]) generalized the aforesaid results by relaxing the compactness assumption of the cyclic (noncyclic) mappings. To state their main corollaries, we remind the reader of some requirements.

From now on, $\mathbf{B}(\mathcal{M})$ denotes the family of all nonempty and bounded subsets of $\mathcal{M}$. Also $\mathbf{K}(\mathcal{M})$ displays the family of all nonempty and compact subsets of $\mathcal{M}$. In the case that X is a Banach space, we use $\mathbf{BCC}(X)$ to denote the class of all nonempty, bounded, closed and convex subsets of X.

Definition 2. *A measure of noncompactness* (MNC for brief) *is a function $\aleph : \mathbf{B}(\mathcal{M}) \to [0, +\infty)$ satisfying the following axioms:*

1. $\aleph(\mathcal{G}) = 0$ *if and only if* $\mathcal{G}$ *is relatively compact,*
2. $\aleph(\mathcal{G}) = \aleph(\overline{\mathcal{G}})$, $\mathcal{G} \in \mathbb{B}(X)$,
3. $\aleph(\mathcal{G} \cup \mathcal{H}) = \max\{\aleph(\mathcal{G}), \aleph(\mathcal{H})\}$, *where* $\mathcal{G}, \mathcal{H} \in \mathbf{B}(\mathcal{M})$.

Some useful properties of the MNC $\aleph$ on $\mathbf{B}(\mathcal{M})$ can be listed as below (see [11] for more details).

(a) $\mathcal{G} \subseteq \mathcal{H}$ implies $\aleph(\mathcal{G}) \leq \aleph(\mathcal{H})$;

(b) $\aleph(\mathcal{G} \cap \mathcal{H}) \leq \min\{\aleph(\mathcal{G}), \aleph(\mathcal{H})\}$, for all $\mathcal{G}, \mathcal{H} \in \mathbf{B}(\mathcal{M})$.

(c) If $\lim_{n \to +\infty} \aleph(\mathcal{G}_n) = 0$ for a nonincreasing sequence $\{\mathcal{G}_n\}$ of nonempty, bounded and closed subsets of $\mathcal{M}$, then

$$\mathcal{G}^{\infty} := \bigcap_{n \geq 1} \mathcal{G}_n \in \mathbf{K}(\mathcal{M}).$$

Here are two well-known examples of MNCs.

Example 1. *Define* $\alpha : \mathbf{B}(\mathcal{M}) \to [0, +\infty)$ *as*

$$\alpha(\mathcal{G}) = \inf \Big\{\varepsilon > 0 : \mathcal{G} \textit{ can be covered by finitely many sets with diameter } \leq \varepsilon\Big\}, \textit{ for all } \mathcal{G} \in \mathbf{B}(\mathcal{M}).$$

Then α *is an* MNC *which was first introduced by Kuratowski* ([12]). *Also a function* $\chi : \mathbf{B}(\mathcal{M}) \to [0, +\infty)$ *which is defined as*

$$\chi(\mathcal{G}) = \inf \Big\{\varepsilon > 0 : \mathcal{G} \textit{ can be covered by finitely many balls with radii } \leq \varepsilon\Big\}, \textit{ for all } \mathcal{G} \in \mathbf{B}(\mathcal{M}),$$

is a generalized version of Kuratowski MNC *which was presented later by Hausdorff.*

For a nonempty pair $(\mathcal{G}, \mathcal{H})$ in a metric space $(\mathcal{M}, d)$ we set

$$\mathcal{G}_0 = \Big\{u \in \mathcal{G} : \text{there exists } v' \in \mathcal{H} \text{ such that } d(u, v') = \mathrm{D}(\mathcal{G}, \mathcal{H})\Big\},$$

$$\mathcal{H}_0 = \Big\{v \in \mathcal{H} : \text{there exists } u' \in \mathcal{G} \text{ such that } d(u', v) = \mathrm{D}(\mathcal{G}, \mathcal{H})\Big\}.$$

The pair $(\mathcal{G}_0, \mathcal{H}_0)$ is said to be a *proximal pair* of $(\mathcal{G}, \mathcal{H})$. There are different conditions to ensure nonemptyness of proximal pairs. For example $(\mathcal{G}_0, \mathcal{H}_0)$ is nonempty if one of the following conditions hold:

(i) $(\mathcal{G}, \mathcal{H})$ is a nonempty and compact pair in a metric space $(\mathcal{M}, d)$;

(ii) $(\mathcal{G}, \mathcal{H})$ is a nonempty pair in a metric space $(\mathcal{M}, d)$ such that $\mathcal{G}$ is compact and $\mathcal{H}$ is approximatively compact w.r.t. $\mathcal{G}$. We recall that the set $\mathcal{H}$ is approximatively compact w.r.t. the set $\mathcal{G}$ whenever for any point $x \in \mathcal{G}$ and any sequence $\{y_n\}$ in the set $\mathcal{H}$ for which $d(x, y_n) \to \mathrm{D}(\{x\}, \mathcal{H})$, then $\{y_n\}$ has a convergent subsequence in $\mathcal{H}$;

(iii) $(\mathcal{G}, \mathcal{H})$ is a nonempty and weakly compact pair in a Banach space X;

(iv) $(\mathcal{G}, \mathcal{H})$ is a nonempty, closed and convex pair in a reflexive Busemann convex metric space $(\mathcal{M}, d)$ such that $\mathcal{H}$ is bounded (see [13]);

(iv) $(\mathcal{G}, \mathcal{H})$ is a nonempty and admissible pair in a hyperconvex metric space $(\mathcal{M}, d)$ (see [14]).

We will say that the nonempty pair $(\mathcal{G}, \mathcal{H})$ is *proximinal* whenever

$$\mathcal{G}_0 = \mathcal{G}, \quad \mathcal{H}_0 = \mathcal{H}.$$

Let $(\mathcal{G}, \mathcal{H})$ be a nonempty pair in a Banach space X and let $T : \mathcal{G} \cup \mathcal{H} \to \mathcal{G} \cup \mathcal{H}$ be a cyclic (noncyclic) relatively nonexpansive mapping. We set

$$\mathbf{M}_{\mathcal{G}\times\mathcal{H}}(T) = \Big\{(\mathcal{L}_1, \mathcal{L}_2) \subseteq (\mathcal{G}, \mathcal{H}) \text{ s.t. } (\mathcal{L}_1, \mathcal{L}_2) \text{ is nonempty, bounded, closed, convex,}$$

$$\text{proximinal and } T-\text{invariant with } \mathrm{D}(\mathcal{L}_1,\mathcal{L}_2)=\mathrm{D}(\mathcal{G},\mathcal{H})\Big\}.$$

It is worth noticing that if for example $(\mathcal{G},\mathcal{H})$ is a nonempty, weakly compact and convex pair in a Banach space X, then $(\mathcal{G}_0,\mathcal{H}_0)\in\mathbf{M}_{\mathcal{G}\times\mathcal{H}}(T)$ (see Lemmas 2.3 and 2.4 of [15]).

We are now ready to recall the concept of *Meir-Keeler condensing operators* which was introduced in [10].

Definition 3. *Let $(\mathcal{G},\mathcal{H})$ be a nonempty and convex pair in a Banach space X and $\aleph$ be an* MNC *on X. A mapping $T:\mathcal{G}\cup\mathcal{H}\to\mathcal{G}\cup\mathcal{H}$ is said to be a Meir-Keeler condensing operator if T is cyclic (noncyclic) and for any $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that for any $(\mathcal{L}_1,\mathcal{L}_2)\in\mathbf{M}_{\mathcal{G}\times\mathcal{H}}(T)$ we have*

$$\varepsilon\le\aleph(\mathcal{L}_1\cup\mathcal{L}_2)<\varepsilon+\delta\Rightarrow\aleph\big(T(\mathcal{L}_1)\cup T(\mathcal{L}_2)\big)<\varepsilon.$$

The next best proximity point (pair) theorems are the main results of [10].

Theorem 5. *Let $(\mathcal{G},\mathcal{H})$ be a nonempty, bounded, closed and convex pair in a Banach space X such that $\mathcal{G}_0$ is nonempty and $\aleph$ is an* MNC *on X. Let $T:\mathcal{G}\cup\mathcal{H}\to\mathcal{G}\cup\mathcal{H}$ be a cyclic relatively nonexpansive mapping which is a Meir-Keeler condensing operator. Then T has a best proximity point.*

Theorem 6. *Let $(\mathcal{G},\mathcal{H})$ be a nonempty, bounded, closed and convex pair in a strictly convex Banach space X such that $\mathcal{G}_0$ is nonempty and $\aleph$ is an* MNC *on X. Let $T:\mathcal{G}\cup\mathcal{H}\to\mathcal{G}\cup\mathcal{H}$ be a noncyclic relatively nonexpansive mapping which is a Meir-Keeler condensing operator. Then T has a best proximity point.*

The main purpose of this article is to present counterpart results of Theorem 5 and 6 in the setting of hyperconvex metric spaces under different conditions.

2. Best proximity version of Schauder's fixed point theorem in Hyperconvex spaces

Let $(\mathcal{M},d)$ be a metric space. Throughout this article, $\mathcal{B}(x;r)$ displays a closed ball centered at $x\in\mathcal{M}$ with radius $r>0$, that is,

$$\mathcal{B}(x;r)=\{y\in\mathcal{M}\ ;\ d(y,x)\le r\}.$$

We mention that a multivalued mapping $F:\mathcal{M}\to 2^{\mathcal{M}}-\{\emptyset\}$ is said to be *almost lower-semicontinuous at a point $x\in\mathcal{M}$* if for each $\varepsilon>0$ there exist an open neighborhood $\mathcal{U}(x)$ of x and a point $z\in Fx$ such that

$$\mathcal{B}(z,\varepsilon)\cap F(y)\neq\emptyset,\ \text{for all } y\in\mathcal{U}(x).$$

For a given $\varepsilon>0$ and $\mathcal{G}\in\mathbf{BC}(\mathcal{M})$, the ε-neighborhood of $\mathcal{G}$ is defined with

$$\mathcal{N}_\varepsilon(\mathcal{G}):=\Big\{u\in\mathcal{M}:\mathrm{D}(u,\mathcal{G})\le\varepsilon\Big\}.$$

A concept of hyperconvexity is based on a property of closed balls in metric spaces which was introduced by Aronszajn and Panitchpakdi ([16]) in 1965 as follows.

Definition 4. *Let $(\mathcal{M}, d)$ be a metric space and $\emptyset \neq \mathcal{G} \subseteq \mathcal{M}$. Then $\mathcal{G}$ is called* hyperconvex *if for any family $\{x_\alpha\}_{\alpha\in I}$ in $\mathcal{G}$ such that $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ for all $\alpha, \beta \in I$, we have*

$$\bigcap_{\alpha\in I} \mathcal{B}(x_\alpha; r_\alpha) \cap \mathcal{G} \neq \emptyset.$$

Particularly, if $\mathcal{G} = \mathcal{M}$, then we say that $\mathcal{M}$ is a hyperconvex metric space.

It is worth mentioning that hyperconvex metric spaces are complete (see [17] for more details). In order to give a suitable characterization of hyperconvexity, we need to recall the following notions.

Definition 5. *A metric space $(\mathcal{M}, d)$ is called* metrically convex *whenever for any two distinct points $x, y \in \mathcal{M}$ there exists an element $z \in \mathcal{M} - \{x, y\}$ for which*

$$d(x, y) = d(x, z) + d(z, y).$$

Definition 6. *A metric space $(\mathcal{M}, d)$ is said to have* binary ball intersection property *provided that any family of closed balls, each two of which intersect must have nonempty intersection.*

We now have the following result regarding hyperconvex spaces.

Proposition 1. (see [17]; p. 77) *A complete metric space $(\mathcal{M}, d)$ is hyperconvex if and only if $\mathcal{M}$ is metrically convex and has the binary ball intersection property.*

Using Proposition 1 it can be shown that the non-reflexive Banach spaces L^∞, l_∞ are hyperconvex. It is interesting to note that hyperconvex subsets of Banach spaces may not be convex.

Example 2. *In the finite dimensional Banach space $(\mathbb{R}^2, \|.\|_\infty)$, suppose*

$$\mathcal{G} = \Big\{(x, x+1) : -1 \leq x \leq 0\Big\} \bigcup \Big\{(x, -x+1) : 0 \leq x \leq 1\Big\}.$$

Then $\mathcal{G}$ is metrically convex and has binary intersection property, which means that $\mathcal{G}$ is a hyperconvex subset of $(\mathbb{R}^2, \|.\|_\infty)$, but clearly $\mathcal{G}$ is not convex.

The next result give us an appropriate condition for convexity of hyperconvex subsets of a Banach space.

Proposition 2. (see Corollary 4.5.18 of [18]) *Let X be a Banach space. Then the following statements are equivalent:*

(i) *X is strictly convex;*

(ii) *Every nonempty and hyperconvex subset of X is convex.*

To state another useful characterization of convexity of hyperconvex subspaces of a Banach space X, we recall the following geometric concept.

Definition 7. ([19]) *Let $(\mathcal{G},\mathcal{H})$ be a nonempty pair in a metric space $(\mathcal{M},d)$ with $\mathcal{G}_0\neq\emptyset$. The pair $(\mathcal{G},\mathcal{H})$ is said to have the* **P***-property if and only if*

$$\begin{cases} d(x_1,y_1)=\mathrm{D}(\mathcal{G},\mathcal{H}) \\ d(x_2,y_2)=\mathrm{D}(\mathcal{G},\mathcal{H}) \end{cases} \Rightarrow d(x_1,x_2)=d(y_1,y_2),$$

where $x_1,x_2\in\mathcal{G}_0$ and $y_1,y_2\in\mathcal{H}_0$.

We now conclude the following result.

Proposition 3. *Let X be a Banach space. Then the following statements are equivalent:*

(i) *Ever pair of nonempty, closed and convex subsets of X has the* **P***-property;*

(ii) *Every nonempty and hyperconvex subset of X is convex.*

Proof. It is sufficient to note that by Theorem 3.1 of [20], every pair of nonempty, closed and convex subsets of X has the **P**-property if and only if X is strictly convex. Now the result follows from Proposition 2.

Definition 8. *A subset $\mathcal{G}$ of a hyperconvex metric space $(\mathcal{M},d)$ is called admissible if $\mathcal{G}$ is the nonempty intersection of a family of closed balls. The set of all admissible subsets of a hyperconvex space $\mathcal{M}$ will be denoted by $\mathfrak{A}(\mathcal{M})$.*

It is well-known that if $(\mathcal{M},d)$ is a hyperconvex space, then every admissible subset of $\mathcal{M}$ is also hyperconvex.
For a nonempty and bounded subset $\mathcal{G}$ of a hyperconvex space $\mathcal{M}$ the cover of $\mathcal{G}$ is defined with

$$\mathrm{cov}(\mathcal{G})=\bigcap\Big\{\mathcal{B}\subseteq\mathcal{M}\ :\ \mathcal{B}\text{ is a closed ball containing }\mathcal{G}\Big\}.$$

Also, we set

$$\delta_x(\mathcal{G}):=\sup_{y\in\mathcal{G}}\Big\{d(x,y):y\in\mathcal{G}\Big\},\ \text{for all } x\in\mathcal{M}.$$

Definition 9. *A set $\mathcal{G}$ in a hyperconvex metric space $(\mathcal{M},d)$ is said to be sub-admissible provided that for any finite subset $\mathcal{E}$ of $\mathcal{G}$ we have $\mathrm{cov}(\mathcal{E})\subseteq\mathcal{G}$.*

The hyperconvex version of Schauder's fixed problem was presented by R. Espinola as follows.

Theorem 7. (Lemma 3 of [2]) *Let $(\mathcal{M},d)$ be a compact and hyperconvex metric space. If $T:\mathcal{M}\to\mathcal{M}$ is continuous, then T has a fixed point.*

An important observation about Theorem 7 is that if $\mathcal{M}$ is a hyperconvex subset of a Banach space X, then $\mathcal{M}$ may not be convex and so the existence of a fixed point for the continuous mapping T cannot be obtained from the Schauder's fixed point theorem.

We are now in position to state the first existence result of this paper which is not only a different version of Theorem 1 in the framework of hyperconvex spaces, but also is a generalization of Theorem 7.

Theorem 8. (Compare to Theorem 1 and Theorem 7) *Let $(\mathcal{M}, d)$ be a hyperconvex metric space and $(\mathcal{G}, \mathcal{H})$ be a nonempty, compact and hyperconvex pair in $\mathcal{M}$. If $T : \mathcal{G}\cup\mathcal{H} \to \mathcal{G}\cup\mathcal{H}$ is a cyclic relatively u-continuous mapping, then T has a best proximity point.*

Proof. By statement (ii) on page 4, the proximal pair $(\mathcal{G}_0, \mathcal{H}_0)$ is nonempty. To see that both sets are compact, assume that $\{x_n\}$ is a sequence in $\mathcal{G}_0$ converging to a point $p \in \mathcal{G}$. Then, there exists a corresponding sequence $\{y_n\}$ in $\mathcal{H}$ such that $d(x_n, y_n) = \mathrm{D}(\mathcal{G}, \mathcal{H})$. By compactness of $\mathcal{H}$, $\{y_n\}$ has a convergent subsequence, also labeled as $\{y_n\}$, with limit q, such that $\lim_{n\to+\infty} d(x_n, y_n) = d(p, q) = \mathrm{D}(\mathcal{G}, \mathcal{H})$. Therefore, $\mathcal{G}_0$ and $\mathcal{H}_0$ are closed subsets of compact sets, hence both are compact. To prove hyperconvexity of the pair $(\mathcal{G}_0, \mathcal{H}_0)$, choose points $\{x_i\}$ in $\mathcal{G}_0$ and $\{y_i\}$ in $\mathcal{H}_0$ such that $d(x_i, y_i) = \mathrm{D}(\mathcal{G}, \mathcal{H})$. Consider the collection of balls $\{\mathcal{B}(x_i; r_i)\}$ and $\{\mathcal{B}(y_i; r_i)\}$ such that $d(x_i, x_j) \le r_i + r_j$ and $d(y_i, y_j) \le r_i + r_j$. By a result of Khamsi et al. [21],

$$d_H\Big(\bigcap_i \mathcal{B}(x_i; r_i), \bigcap_i \mathcal{B}(y_i; r_i)\Big) \le \sup_i d(x_i, y_i) = \mathrm{D}(\mathcal{G}, \mathcal{H}),$$

where d_H is the Hausdorff metric derived from the metric d. This shows that $\mathcal{B}(x_i; r_i) \cap \mathcal{G}_0 \neq \emptyset$, implying that $\mathcal{G}_0$ is hyperconvex, and similarly that $\mathcal{H}_0$ is hyperconvex. Using the fact that T is relatively u-continuous, $(\mathcal{G}_0, \mathcal{H}_0)$ is T-invariant. Define a multivalued mapping $F : \mathcal{G}_0 \to 2^{\mathcal{G}_0}$ by

$$F(v) := \mathcal{B}\big(Tv; \mathrm{D}(\mathcal{G}, \mathcal{H})\big) \bigcap \mathcal{G}_0, \text{ for all } v \in \mathcal{G}_0.$$

Note that the values of F are nonempty. Indeed, for any $v \in G_0$, there exists $y \in H_0$ such that $d(v, y) = \mathrm{D}(\mathcal{G}, \mathcal{H})$ and again using the relatively u-continuity of T, $d(Tv, Ty) = \mathrm{D}(\mathcal{G}, \mathcal{H})$ and so, $Ty \in \mathcal{B}\big(Tv; \mathrm{D}(\mathcal{G}, \mathcal{H})\big) \bigcap \mathcal{G}_0$. We claim that F is almost lower-semicontinuous. Let $x_0 \in \mathcal{G}_0$. Then there is an element $y_0 \in \mathcal{H}_0$ such that $d(x_0, y_0) = \mathrm{D}(\mathcal{G}, \mathcal{H})$. Now for given $\varepsilon > 0$ there exists $\delta > 0$ such that if $z \in \mathcal{G}$ with $d(z, y_0) < \delta + \mathrm{D}(\mathcal{G}, \mathcal{H})$, then $d(Tz, Ty) < \varepsilon + \mathrm{D}(\mathcal{G}, \mathcal{H})$. Set

$$\mathcal{U}(x_0, \delta) := \Big\{u \in \mathcal{G}_0 : d(u, x_0) < \delta\Big\}.$$

Then for any $u \in \mathcal{U}(x_0, \delta)$ we have

$$d(u, y_0) \le d(u, x_0) + d(x_0, y_0) < \delta + \mathrm{D}(\mathcal{G}, \mathcal{H}),$$

and hence,

$$d(Tu, Ty_0) < \varepsilon + \mathrm{D}(\mathcal{G}, \mathcal{H}). \tag{1}$$

We claim that $\mathcal{B}(Ty_0;\varepsilon) \cap Fu \neq \emptyset$ for $u \in \mathcal{U}(x_0, \delta)$. By (1)

$$\mathcal{B}(Ty_0;\varepsilon) \cap \mathcal{B}\big(Tu, \mathrm{D}(\mathcal{G}, \mathcal{H})\big) \neq \emptyset.$$

Consider $\operatorname{cov}(p, q)$, where $p \in \mathcal{B}(Ty_0, \varepsilon) \cap \mathcal{G}_0$, and $q \in Fu$. Since $\mathcal{G}_0$ is hyperconvex, $\operatorname{cov}(p, q) \subseteq \mathcal{G}_0$. Then, as an intersection of admissible sets with pairwise nonempty intersections,

$$\mathcal{B}(Ty_0;\varepsilon) \cap \operatorname{cov}(p, q) \cap \mathcal{B}\big(Tu, \mathrm{D}(\mathcal{G}, \mathcal{H})\big) \neq \emptyset.$$

This shows that $\mathcal{B}(Ty_0;\varepsilon) \cap Fu \neq \emptyset$ and therefore, that F is almost lower-semicontinuous at the point x_0. To see that values of F are sub-admissible sets, let $\{x_i\}$ be a collection of points in $F(v)$ and consider the set $\operatorname{cov}(\{x_i\})$. Hyperconvexity of G_0 implies that $\operatorname{cov}(\{x_i\}) \subseteq \mathcal{G}_0$, and the relation

$$\delta_{\operatorname{cov}(\{x_i\})}(Tv) = \delta_{\{x_i\}}(Tv),$$

deduces that $\operatorname{cov}(\{x_i\}) \subseteq F(v)$, concluding that $F(v)$ is sub-admissible. By Theorem 1 of [22], as an almost lower- semicontinuous mapping with sub-admissible values, the mapping F has a continuous selection $\hbar : \mathcal{G}_0 \to \mathcal{G}_0$ such that $\hbar v \in Fv$ for $v \in \mathcal{G}_0$. By using Theorem 7, the continuous self map $\hbar$ on the compact hyperconvex space $\mathcal{G}_0$ has a fixed point x^*. The definition of the mapping F implies $d(x^*, Tx^*) = \mathrm{D}(\mathcal{G}, \mathcal{H})$ and the proof is completed.

The next result is a consequence of Theorem 8.

Corollary 1. (Theorem 15 of [23]) *Let $(\mathcal{G}, \mathcal{H})$ be a nonempty and admissible pair in a hyperconvex metric space $(\mathcal{M}, d)$ such that $\mathcal{G}_0$ is compact. Let $T : \mathcal{G} \cup \mathcal{H} \to \mathcal{G} \cup \mathcal{H}$ be a cyclic relatively u-continuous mapping. Then T has a best proximity point.*

Proof. By Lemma 2.5 of [14], since $(\mathcal{G}, \mathcal{H})$ is an admissible pair in a hyperconvex space $\mathcal{M}$ the proximal pair $(\mathcal{G}_0, \mathcal{H}_0)$ is nonempty and admissible too, which ensures that $\mathcal{G}_0$ is hyperconvex. Now the result follows from Theorem 8, immediately.

The next example shows the useability of Theorem 8 w.r.t. Theorem 3.

Example 3. *Consider the hyperconvex space ℓ_∞ consists of all real bounded sequences equipped with the supremum norm and let $\{e_n\}_{n\in\mathbb{N}}$ be the canonical basis of ℓ_∞. Set*

$$\mathcal{G} = \big\{te_1 + e_2 \ ; \ t \in [0, 1]\big\}$$
$$\mathcal{H} = \big\{se_1 + e_3 \ ; \ s \in [0, 1]\big\}.$$

Then $(\mathcal{G}, \mathcal{H})$ is a compact, convex, proximinal and hyperconvex pair in ℓ_∞ with $\mathrm{D}(\mathcal{G}, \mathcal{H}) = 1$. Define a mapping $T : \mathcal{G} \cup \mathcal{H} \to \mathcal{G} \cup \mathcal{H}$ with

$$T(te_1 + e_2) = te_1 + e_3, \quad T(se_1 + e_3) = se_1 + e_2, \ \textit{for all } t, s \in [0, 1].$$

Then T is cyclic and for any $\mathbf{x} = te_1 + e_2 \in \mathcal{G}$ and $\mathbf{y} = se_1 + e_3$ we have

$$\|T\mathbf{x} - T\mathbf{y}\|_\infty = 1 = \|\mathbf{x} - \mathbf{y}\|_\infty,$$

which implies that T is relatively u-continuous. It now follows from Theorem 8 that T has a best proximity point, indeed, every point of $\mathcal{G}$ is a best proximity point of T. Note that the Banach space ℓ_∞ is not strictly convex and so the existence of a best proximity point of T cannot be concluded from Theorem 3.

It is worth noticing that the noncyclic version of Theorem 8 holds by adding a geometric property which is inspired from strictly convex Banach spaces.

Definition 10. ([6]) *A nonempty pair $(\mathcal{G},\mathcal{H})$ in a metric space $(\mathcal{M},d)$ is said to be a semi-sharp proximinal pair provided that for any $(x,y)\in\mathcal{G}\times\mathcal{H}$ there exists at most one point $(x',y')\in\mathcal{G}\times\mathcal{H}$ for which*

$$d(x,y')=d(x',y)=\mathrm{D}(\mathcal{G},\mathcal{H}).$$

It is worth noticing that every nonempty and convex pair in a strictly convex Banach space X is a semi-sharp proximinal pair (see [6]). The next example shows that strict convexity assumption of a Banach space X is not an essential condition for semi-sharp proximinality of convex pairs.

Example 4. *In the hyperconvex space ℓ_∞, put*

$$\mathcal{G}=\overline{\mathrm{con}}\Big(\big\{e_{2n-1}+e_{2n}\ ;\ n\in\mathbb{N}\big\}\Big)$$
$$\mathcal{H}=\overline{\mathrm{con}}\Big(\big\{2e_{2n}+e_{2n+1}\ ;\ n\in\mathbb{N}\big\}\Big),$$

where $\overline{\mathrm{con}}(A)$ denotes the closed convex hull of the set $A\subseteq\ell_\infty$. Clearly,

$$\mathrm{D}(\mathcal{G},\mathcal{H})=1,\quad \mathcal{G}_0=\mathcal{G},\quad \mathcal{H}_0=\mathcal{H},$$

and that $(\mathcal{G},\mathcal{H})$ is a semi-sharp proximinal pair.

Theorem 9. (Compare to Theorem 4) *Let $(\mathcal{G},\mathcal{H})$ be a nonempty, compact and hyperconvex pair in a hyperconvex metric space $(\mathcal{M},d)$ which is a semi-sharp proximinal pair. If $T:\mathcal{G}\cup\mathcal{H}\to\mathcal{G}\cup\mathcal{H}$ is a noncyclic relatively u-continuous mapping, then T has a best proximity pair.*

Proof. As in the proof of Theorem 8, $(\mathcal{G}_0,\mathcal{H}_0)$ is nonempty and hyperconvex. T-invariance of these sets follows from relatively u-continuity of T, that is, for any pair $(x,y)\in\mathcal{G}_0\times\mathcal{H}_0$, if $d(x,y)=\mathrm{D}(\mathcal{G},\mathcal{H})$, then $d(Tx,Ty)=\mathrm{D}(\mathcal{G},\mathcal{H})$, implying that $(Tx,Ty)\in(\mathcal{G}_0,\mathcal{H}_0)$.

Choose $x_0\in\mathcal{G}_0$ and $y_0\in\mathcal{H}_0$ such that $d(x_0,y_0)=\mathrm{D}(\mathcal{G},\mathcal{H})$. Then by relatively u-continuity of T we have $d(Tx_0,Ty_0)=\mathrm{D}(\mathcal{G},\mathcal{H})$. Given $\varepsilon>0$, define an open neighborhood of x_0 by $\mathcal{U}(x_0,\delta):=\{u\in\mathcal{G}_0 : d(u,x_0)<\delta\}$. Then $u\in\mathcal{U}(x_0,\delta)$ implies that

$$d(u,y_0)<d(u,x_0)+d(x_0,y_0)<\delta+\mathrm{D}(\mathcal{G},\mathcal{H}),$$

and therefore, by relatively u-continuity of T,

$$d(Tu, Ty_0) < \varepsilon + \mathrm{D}(\mathcal{G}, \mathcal{H}).$$

Now define a multivalued map $F : \mathcal{G}_0 \to 2^{\mathcal{H}_0}$ by

$$F(u) = \mathcal{B}\big(Tu; \mathrm{D}(\mathcal{G}, \mathcal{H})\big) \bigcap \mathcal{H}_0,$$

for $u \in \mathcal{G}_0$. By analogous arguments to those in the proof of Theorem 8, F is an almost lower-semicontinuous mapping with subadmissible values. Then, as above, F has a continuous selection $\mathfrak{h} : \mathcal{G}_0 \to \mathcal{H}_0$. By semi-sharp proximinality of the pair $(\mathcal{G}, \mathcal{H})$, we can define a mapping $\mathcal{P} : \mathcal{H}_0 \to \mathcal{G}_0$ as a projection map that associates to each point p in $\mathcal{H}_0$ the unique point q in $\mathcal{G}_0$ such that $d(p, q) = \mathrm{D}(\mathcal{G}, \mathcal{H})$. We claim that this correspondence is a continuous mapping. Assume that a sequence $\{p_n\}$ in $\mathcal{H}_0$ converges to a point $p \in \mathcal{H}$, and by contrary assume that $\{\mathcal{P}(p_n)\}$ does not converge to $\mathcal{P}(p)$. By compactness of the set $\mathcal{G}_0$, $\{\mathcal{P}(p_n)\}$ has a convergent subsequence $\{q_n\}$ that converges to a point q. Then, $\lim_{n\to+\infty} d(p_n, q_n) = d(p, q) = \mathrm{D}(\mathcal{G}, \mathcal{H})$, where $q \neq \mathcal{P}(p)$, contradicting uniqueness of the proximal point. Therefore, $\mathcal{P}$ is a continuous mapping. As a continuous self mapping on a compact hyperconvex set, the mapping $\mathcal{P}o\mathfrak{h} : \mathcal{G}_0 \to \mathcal{G}_0$ has a fixed point $z \in \mathcal{G}_0$. By definition of this mapping, $z = Tz$, and if w is the unique point in $\mathcal{H}_0$ such that $d(z, w) = \mathrm{D}(\mathcal{G}, \mathcal{H})$, it follows from relatively u-continuity of T that $w = Tw$ and the proof is completed.

The next example guarantees that the semi-sharp proximinality of the pair $(\mathcal{G}, \mathcal{H})$ in Theorem 9 is essential.

Example 5. *Consider the hyperconvex space ℓ_∞ with the canonical basis $\{e_n\}_{n\in\mathbb{N}}$ and let*

$$\mathcal{G} = \Big\{te_1 + 2e_2 : 0 \le t \le 1\Big\},$$
$$\mathcal{H} = \Big\{te_3 : 0 \le t \le 1\Big\}.$$

Then $(\mathcal{G}, \mathcal{H})$ is a compact and hyperconvex pair with $\mathrm{D}(\mathcal{G}, \mathcal{H}) = 2$. Clearly, $(\mathcal{G}, \mathcal{H})$ is not a semi-sharp proximinal pair. Define $T : \mathcal{G} \cup \mathcal{H} \to \mathcal{G} \cup \mathcal{H}$ by

$$T(te_1 + 2e_2) = \sqrt{t}e_1 + 2e_2, \text{ for all } t \in [0, 1], \quad T(te_3) = \begin{cases} 0, & \text{if } t \neq 0, \\ e_3, & \text{if } t = 0. \end{cases}$$

Then

$$d_\infty(Tx, Ty) = 2 = d_\infty(x, y), \text{ for all } (x, y) \in \mathcal{G} \times \mathcal{H},$$

which deduces that T is a noncyclic relatively u-continuous. Note that T does not possess any best proximity pair.

3. Hyperconvex spaces equipped with a suitable MNC

In this section, we extend Theorem 8 and Theorem 9 by relaxing the compactness assumption of the hyperconvex space $\mathcal{M}$ and replacing a suitable MNC on $\mathcal{M}$. For this purpose, we need to recall the following useful tools of hyperconvex spaces which were presented by M.A. Khamsi in [24].

Definition 11. *Let $\mathcal{G}$ be a nonempty subset of a metric space $(\mathcal{M}, d)$. We say that $\mathcal{G}$ is a nonexpansive retract of $\mathcal{M}$ if there exists a nonexpansive mapping $R : \mathcal{M} \to \mathcal{G}$ such that $Rx = x$ for any $x \in \mathcal{G}$.*

Proposition 4. (Proposition 1 of [24]) *A metric space $(\mathcal{M}, d)$ is hyperconvex iff for any metric space $(\mathcal{N}, \rho)$ which contains isometrically $\mathcal{M}$, there exists a nonexpansive retract $R : \mathcal{N} \to \mathcal{M}$.*

Now for a metric space $(\mathcal{M}, d)$ we define

$$\ell_\infty(\mathcal{M}) = \Big\{(x_j)_{j\in\mathcal{M}} \subseteq \mathbb{R} \ : \ \sup_{j\in\mathcal{M}} |x_j| < +\infty\Big\},$$

and let

$$d_\infty\big((x_j), (y_j)\big) := \sup_{j\in\mathcal{M}} |x_j - y_j|, \text{ for all } (x_j), (y_j) \in \ell_\infty(\mathcal{M}).$$

Then $\big(\ell_\infty(\mathcal{M}), d_\infty\big)$ is a hyperconvex metric space. Let $x_0 \in \mathcal{M}$ be a fixed element and define the natural isometric embedding $\eta : \mathcal{M} \to \ell_\infty(\mathcal{M})$ as follows:

$$\eta(x) = \big(d(x, y) - d(x_0, y)\big)_{y\in\mathcal{M}}.$$

Notice that for any $y \in \mathcal{M}$ we have $\sup_{y\in\mathcal{M}} |d(x, y) - d(x_0, y)| \leq d(x, x_0)$ and so, η is well-defined. Also,

$$d_\infty\big(\eta(x), \eta(z)\big) = \sup_{y\in\mathcal{M}} |d(x, y) - d(z, y)| = d(x, z), \text{ for all } x, z \in \mathcal{M},$$

which implies that the metric space $\ell_\infty(\mathcal{M})$ contains an isometric copy $\mathcal{M}$.

If $\mathcal{M}$ is a hyperconvex space, then $\eta(\mathcal{M})$ is an isometric copy of $\mathcal{M}$ in $\ell_\infty(\mathcal{M})$ and is, therefore, a hyperconvex set. Throughout this work we will use $\mathcal{M}$ instead of $\eta(\mathcal{M})$. Similarly, for any hyperconvex subset $\mathcal{G}$ of $\mathcal{M}$ we use $\mathcal{G}$ instead of $\eta(\mathcal{G})$. For $\mathcal{M}$ or any hyperconvex subset $\mathcal{G}$ of $\mathcal{M}$, we define $\mathcal{M}_\infty = \mathrm{cov}(\mathcal{M})$ or $\mathcal{G}_\infty = \mathrm{cov}(\mathcal{G})$ in $\ell_\infty(\mathcal{M})$. Clearly, $\mathcal{M}_\infty$ and $\mathcal{G}_\infty$ are hyperconvex subsets of $\ell_\infty(\mathcal{M})$ and are convex. It is worth noticing that for any hyperconvex space $\mathcal{M}$ there is a nonexpansive retraction R of $\mathcal{M}_\infty$ onto $\mathcal{M}$, and for any hyperconvex subset $\mathcal{G}$ of $\mathcal{M}$, $R(\mathcal{G}_\infty) = \mathcal{G}$ (see [17]).

We are now able to define a suitable MNC on the hyperconvex space $\mathcal{M}_\infty$.

Lemma 1. *Let $(\mathcal{M}, d)$ be a hyperconvex metric space and R be a nonexpansive retract from $\mathcal{M}_\infty$ onto $\mathcal{M}$. Suppose $\aleph$ is an* MNC *on $\mathcal{M}$ and define a function $\aleph_R : \mathbf{B}(\mathcal{M}_\infty) \to [0, \infty)$ as*

$$\aleph_R(\mathcal{G}) := \aleph\big(R(\mathcal{G})\big), \text{ for all } \mathcal{G} \in \mathbf{B}(\mathcal{M}_\infty).$$

Then $\aleph_R$ is an MNC *on $\mathcal{M}_\infty$.*

Proof.
The proof is trivial.

The next lemmas play important roles in our next results.

Lemma 2. (Lemma 2.5 of [14]) *Let $(\mathcal{G},\mathcal{H})$ be a nonempty and admissible pair in a hyperconvex metric space $(\mathcal{M},d)$. Then the proximal pair $(\mathcal{G}_0,\mathcal{H}_0)$ is also nonempty and admissible.*

Lemma 3. *Let $(\mathcal{G},\mathcal{H})$ be a nonempty and admissible pair in a hyperconvex metric space $(\mathcal{M},d)$. Suppose $(\mathcal{E},\mathcal{F}) \subseteq \big(\mathcal{G}_\infty,\mathcal{H}_\infty\big)$ is a nonempty, bounded, closed, convex, and proximinal pair such that $\mathrm{D}(\mathcal{E},\mathcal{F}) = \mathrm{D}(\mathcal{G},\mathcal{H})\big(=\mathrm{D}(\mathcal{G}_\infty,\mathcal{H}_\infty)\big)$. Then $\big(R(\mathcal{E}),R(\mathcal{F})\big)$ is a nonempty, bounded, hyperconvex, and proximinal pair in $\mathcal{M}$ with $\mathrm{D}\big(R(\mathcal{E}),R(\mathcal{F})\big) = \mathrm{D}(\mathcal{G},\mathcal{H})$.*

Proof. As we know, in $\ell_\infty(\mathcal{M})$ each closed, bounded and convex set is also an admissible set. Thus $(\mathcal{E},\mathcal{F})$ is admissible in $\ell_\infty(\mathcal{M})$. Since a nonexpansive retraction preserves hyperconvexity, $\big(R(\mathcal{E}),R(\mathcal{F})\big)$ is a hyperconvex pair. To see the proximinality of the latter pair, choose any point $Rp \in R(\mathcal{E})$, where $p \in \mathcal{E}$. Then there is a point $q \in \mathcal{F}$ such that $d_\infty(p,q) = \mathrm{D}(\mathcal{E},\mathcal{F})\big(=\mathrm{D}(\mathcal{G},\mathcal{H})\big)$, implying that $d(Rp,Rq) \leq \mathrm{D}(\mathcal{E},\mathcal{F})$. Besides, the relation $\big(R(\mathcal{E}),R(\mathcal{F})\big) \subseteq (\mathcal{G},\mathcal{H})$ concludes that $\mathrm{D}(\mathcal{E},\mathcal{F}) = \mathrm{D}\big(R(\mathcal{E}),R(\mathcal{F})\big)$. Therefore, $d(Rp,Rq) = \mathrm{D}(\mathcal{E},\mathcal{F})$ and so, $\big(R(\mathcal{E}),R(\mathcal{F})\big)$ is a proximinal pair.

Motivated by Definition 3 we introduce the class of Meir-Keeler condensing operators in the setting of hyperconvex metric spaces.

Let $(\mathcal{G},\mathcal{H})$ be a nonempty and admissible pair in a hyperconvex space $(\mathcal{M},d)$ and let $T:\mathcal{G}\cup\mathcal{H} \to \mathcal{G}\cup\mathcal{H}$ be a cyclic (noncyclic) relatively u-continuous mapping. Put

$$\mathbf{M}'_{\mathcal{G}\times\mathcal{H}}(T) = \Big\{(\mathcal{L}_1,\mathcal{L}_2) \subseteq (\mathcal{G},\mathcal{H}) \text{ s.t. } (\mathcal{L}_1,\mathcal{L}_2) \text{ is nonempty, bounded, hyperconvex,} \\ \text{proximinal and } T-\text{invariant with } \mathrm{D}(\mathcal{L}_1,\mathcal{L}_2) = \mathrm{D}(\mathcal{G},\mathcal{H})\Big\}.$$

By Lemma 2, under the aforesaid assumptions, the proximal pair $(\mathcal{G}_0,\mathcal{H}_0)$ is also nonempty and admissible. Also, by the fact that T is relatively u-continuous, $(\mathcal{G}_0,\mathcal{H}_0)$ is T-invariant. So,

$$(\mathcal{G}_0,\mathcal{H}_0) \in \mathbf{M}'_{\mathcal{G}\times\mathcal{H}}(T) \neq \emptyset.$$

Definition 12. *Let $(\mathcal{G},\mathcal{H})$ be a nonempty and admissible pair in a hyperconvex space $(\mathcal{M},d)$ and $\aleph$ be an* MNC *on $\mathcal{M}$. A mapping $T:\mathcal{G}\cup\mathcal{H}\to\mathcal{G}\cup\mathcal{H}$ is said to be an* **H**-*Meir-Keeler condensing operator if T is cyclic (noncyclic) and for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for any $(\mathcal{L}_1,\mathcal{L}_2) \in \mathbf{M}'_{\mathcal{G}\times\mathcal{H}}(T)$ we have*

$$\varepsilon \leq \aleph(\mathcal{L}_1\cup\mathcal{L}_2) < \varepsilon+\delta \Rightarrow \aleph\big(T(\mathcal{L}_1)\cup T(\mathcal{L}_2)\big) < \varepsilon.$$

Let us illustrate the concept of **H**-Meir-Keeler condensing operator with the following example.

Example 6. *Consider the hyperconvex space ℓ_∞ equipped with Kuratowski* MNC α *defined in Example 1. Let $(\mathcal{G},\mathcal{H})$ be a nonempty and admissible pair in ℓ_∞ for which $\mathcal{H}$ is compact. Suppose that $T:\mathcal{G}\cup\mathcal{H}\to\mathcal{G}\cup\mathcal{H}$ is a cyclic mapping such that $T\big|_{\mathcal{H}}$ is a contraction, that is, there exists $r\in(0,1)$ such that for any $v,y\in\mathcal{H}$*

$$d_\infty(Tv,Ty)\le rd_\infty(v,y).$$

Then for any $(\mathcal{L}_1,\mathcal{L}_2)\in\mathbf{M}'_{\mathcal{G}\times\mathcal{H}}(T)$ by the compactness of $\mathcal{H}$,

$$\alpha\big(T(\mathcal{L}_1)\cup T(\mathcal{L}_2)\big)=\max\{\underbrace{\alpha\big(T(\mathcal{L}_1)\big)}_{0},\alpha\big(T(\mathcal{L}_2)\big)\}=\alpha\big(T(\mathcal{L}_2)\big)\le r\alpha(\mathcal{L}_2)\le r\alpha(\mathcal{L}_1\cup\mathcal{L}_2),$$

which ensures that T is an **H**-*Meir-Keeler condensing operator.*

In order to prove the main existence results of this section, we need the following auxiliary lemmas.

Lemma 4. *Let $(\mathcal{G},\mathcal{H})$ be a nonempty and admissible pair in a hyperconvex metric space $(\mathcal{M},d)$ and $\aleph$ be an* MNC *on $\mathcal{M}$. Suppose $T:\mathcal{G}\cup\mathcal{H}\to\mathcal{G}\cup\mathcal{H}$ is cyclic (noncyclic) relatively u-continuous mapping which is an* **H**-*Meir-Keeler condensing operator. Then*

$$TR:\mathcal{G}_\infty\cup\mathcal{H}_\infty\to\mathcal{G}_\infty\cup\mathcal{H}_\infty,$$

is a cyclic (noncyclic) relatively u-continuous mapping which a Meir-Keeler condensing operator w.r.t. the MNC $\aleph_R$.

Proof. We assume that T is cyclic. Then

$$TR(\mathcal{G}_\infty)=T(\mathcal{G})\subseteq\mathcal{H}\subseteq\mathcal{H}_\infty,$$
$$TR(\mathcal{H}_\infty)=T(\mathcal{H})\subseteq\mathcal{G}\subseteq\mathcal{G}_\infty,$$

that is, TR is also cyclic. Now let $\varepsilon>0$ be given. Since T is relatively u-continuous, there exists $\delta>0$ such that for any $(x,y)\in\mathcal{G}\times\mathcal{H}$ with $d(x,y)<\delta+\mathrm{D}(\mathcal{G},\mathcal{H})$ we have $d(Tx,Ty)<\varepsilon+\mathrm{D}(\mathcal{G},\mathcal{H})$. Since $R:\mathcal{M}_\infty\to\mathcal{M}$ is a nonexpansive retract, for any $(u,v)\in\mathcal{G}_\infty\times\mathcal{H}_\infty$ with $d_\infty(u,v)<\delta+\mathrm{D}(G,H)$ we have $d_\infty(Ru,Rv)<\delta+\mathrm{D}(G,H)$ and therefore, $d_\infty(TRu,TRv)<\varepsilon+\mathrm{D}(G,H)$, that is, TR is relatively u-continuous. To show that TR is a Meir-Keeler condensing operator consider $\varepsilon>0$ and $(\mathcal{L},\mathcal{K})\in\mathbf{M}_{\mathcal{G}_\infty\times\mathcal{H}_\infty}(TR)$. Then $(\mathcal{L},\mathcal{K})\subseteq(\mathcal{G}_\infty,\mathcal{H}_\infty)$ is a nonempty, bounded, closed, convex, proximinal and TR-invariant with $\mathrm{D}(\mathcal{L},\mathcal{K})=\mathrm{D}(\mathcal{G}_\infty,\mathcal{H}_\infty)$. It follows from Lemma 3 that $\big(R(\mathcal{L}),R(\mathcal{K})\big)$ is bounded, hyperconvex and proximinal with $\mathrm{D}\big(R(\mathcal{L}),R(\mathcal{K})\big)=\mathrm{D}(\mathcal{G},\mathcal{H})$. Since T maps $\mathcal{G}$ into $\mathcal{H}$, $T(R(\mathcal{K}))\subseteq\mathcal{L}\cap\mathcal{H}\subseteq R(\mathcal{L})$. We now have

$$T\big(R(\mathcal{L})\big)=TR(\mathcal{L})\subseteq R(\mathcal{K}),$$

$$T\big(R(\mathcal{K})\big)=TR(\mathcal{K})\subseteq R(\mathcal{L}),$$

which implies that $\big(R(\mathcal{L}), R(\mathcal{K})\big)$ is T-invariant. Thus, $\big(R(\mathcal{L}), R(\mathcal{K})\big) \in \mathbf{M}'_{\mathcal{G}\times\mathcal{H}}(T)$. In view of the fact that T is an $\mathbf{H}$-Meir-Keeler condensing operator, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \underbrace{\aleph\big(R(\mathcal{L}) \cup R(\mathcal{K})\big)}_{=\aleph\big(R(\mathcal{L}\cup\mathcal{K})\big)} < \varepsilon + \delta \Rightarrow \aleph\big(T(R(\mathcal{L})) \cup T((R\mathcal{K}))\big) < \varepsilon.$$

By the definition of $\aleph_R,$ we have that $\aleph_R(B) = \aleph(R(B)) = \aleph(B)$ for any bounded subset of $\mathcal{M}$, and so,

$$\varepsilon \leq \aleph_R\big(\mathcal{L} \cup \mathcal{K}\big) < \varepsilon + \delta \Rightarrow \aleph_R\Big(TR(\mathcal{L}) \cup TR(\mathcal{K})\Big) < \varepsilon.$$

Thereby, TR is an $\aleph_R$-Meir-Keeler condensing operator. In the case that T is noncyclic, the result follows, similarly.

The next theorem is a different version of Theorem 5 in hyperconvex spaces.

Theorem 10. *Let $(\mathcal{G}, \mathcal{H})$ be a nonempty, admissible pair in a hyperconvex metric space $(\mathcal{M}, d)$ and $\aleph$ be an* MNC *on X. Suppose $T : \mathcal{G} \cup \mathcal{H} \to \mathcal{G} \cup \mathcal{H}$ is a cyclic relatively u-continuous mapping which is an* **H***-Meir-Keeler condensing operator. Then T has a best proximity point.*

Proof. It follows from Lemma 4 that the mapping $TR : \mathcal{G}_\infty \cup \mathcal{H}_\infty \to \mathcal{G}_\infty \cup \mathcal{H}_\infty$ is a cyclic relatively u-continuous mapping which is a Meir-Keeler condensing operator w.r.t the MNC $\aleph_R$. It now follows from Theorem 5 that TR has a best proximity point, i.e., there exists a point $p \in \mathcal{G}_\infty \cup \mathcal{H}_\infty$ such that $d_\infty(p, TRp) = \mathrm{D}(\mathcal{G}_\infty, \mathcal{H}_\infty) = \mathrm{D}(\mathcal{G}, \mathcal{H})$. We now have

$$d\big(Rp, TRp\big) \leq d_\infty(p, TRp) = \mathrm{D}(\mathcal{G}, \mathcal{H}).$$

Therefore, Rp is a best proximity point for the mapping T and the proof is completed.

The noncyclic version of Theorem 10 is as follows.

Theorem 11. *Let $(\mathcal{G}, \mathcal{H})$ be a nonempty, admissible and semi-sharp proximinal pair in a hyperconvex metric space $(\mathcal{M}, d)$ and $\aleph$ be an* MNC *on X. Suppose $T : \mathcal{G}\cup\mathcal{H} \to \mathcal{G}\cup\mathcal{H}$ is a noncyclic relatively u-continuous mapping which is an* **H***-Meir-Keeler condensing operator. Then T has a best proximity pair.*

Proof. By Lemma 4, the mapping $TR : \mathcal{G}_\infty \cup \mathcal{H}_\infty \to \mathcal{G}_\infty \cup \mathcal{H}_\infty$ is a noncyclic relatively u-continuous mapping, which is a Meier-Keeler condensing operator w.r.t. the MNC $\aleph_R$. As a mapping defined in the Banach space $\ell_\infty(\mathcal{M})$ that satisfies the conditions of Theorem 6, the mapping TR has a best proximity pair, that is, there exists a pair $(p, q) \in \mathcal{G}_\infty \times \mathcal{H}_\infty$ such that $d_\infty(p, q) = \mathrm{D}(\mathcal{G}_\infty, \mathcal{H}_\infty) = \mathrm{D}(\mathcal{G}, \mathcal{H})$, we obtain $p = TRp$ and $q = TRq$. Since $TRp \in \mathcal{G}$ and $TRq \in \mathcal{H}$, we have $Rp = TRp$ and $Rq = TRq$. Similarly, we can write $d_\infty(Rp, Rq) = \mathrm{D}(G_\infty, H_\infty) = \mathrm{D}(\mathcal{G}, \mathcal{H})$. Thus, Rp and Rq are a best proximity pair for the noncyclic mapping T and we are finished.

4. Conclusion

In this paper, we revisited the main conclusions of [10] related to the existence of best proximity points (pairs) for cyclic (noncyclic) Meir-Keeler condensing operators and obtained similar results in the framework of hyperconvex metric spaces. Examples in the Banach space ℓ_∞ as a well known hyperconvex space, guarantee the useability of our corollaries.

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