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On Po-injective and Po-surjective Wreath Product of Pomonoids

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Abstract. Let R and S be pomonoids and ${}_RA$ be a left R-poset. The wreath product of the pomonoids R and S by ${}_RA$ is defined as the pomonoid $T = R\times F(A,S)$ while, the wreath product ${}_TC$ of the left R-poset ${}_RA$ with the left S-poset ${}_SB$ over the pomonoid $T = R \times F(A,S)$ is the left T-poset ${}_TC = {}_RA \times {}_SB$ endowed with the monotone action given by $(r,f)(a,b) = (ra, f(a)b)$, where $(r,f) \in R\times F(A,S)$ and $(a,b) \in A\times B$. The po-injectivity and po-cancellative properties on the wreath product ${}_TC$ are studied and the relations between them are established. The relation between po-surjective property and other properties on the wreath product ${}_TC$ are also established. Finally the characterization of some properties of po-flatness such as po-torsion free, properties (P), (E), (P_E), and strongly flat have been examined on the wreath product ${}_TC$ and the relations among them have also been established.

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1. Introduction

In group theory the wreath product is a generalization of the semidirect product. The wreath product is a way to combine two groups, H and K, using the semidirect product. The key feature is that one group, say H acts on K in a specific way, and this action is a crucial part of the construction. The idea of wreath products has been extended to semigroups and posemigroups as well, allowing for a broader application of this construction beyond just groups. The wreath product of semigroups is a generalization of the concept for groups, but with some modifications to accommodate the lack of inverses in semigroups. The construction involves not only the direct product of copies but also an action that reflects the interactions between the two semigroups. As with group wreath

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products, the wreath product of semigroups allows for a structured approach to understanding and constructing certain types of semigroups. It finds applications in areas like automata theory and formal languages, where semigroups play a significant role. Many researchers studied the concept of wreath product of semigroups (monoids) in numerous articles such as [1–5].

Many researchers are interested in generalizing results from the category of semigroups to that of posemigroups, as demonstrated by the work on [6–13]. Knauer and Mikhalev [14], initiated the study of wreath products of ordered semigroups by an ordered action. They considered three different types of ordered wreath products, specifically, order preserving, order reversing and preserving certain zigzag equivalence. Kilp, Knauer and Mikhalev [15] provided a characterization of torsion free wreath products of acts over the wreath product of monoids by describing injective, surjective and cancellative elements of the wreath product. In this paper, we extend the work on the ordered (monotone) wreath product of pomonoids over a poset by generalizing the work of Kilp, Knauer and Mikhalev on monoids in [15], and we will adopt their notations for the sake of simplicity for the reader. For the initial work on ordered wreath product of posets on pomonoids the reader is refered to [14, 16, 17].

A *pomonoid* S is a monoid endowed with a partial order usually denoted by $\leq$ such that it is compatible with the binary operation i.e., for any $s, s', t \in S$, $s \leq s'$ implies $ts \leq ts'$ and $st \leq s't$. Therefore, for any pomonoid S and $r, r', p, p' \in S$, if $r \leq r'$ and $p \leq p'$, then $rp \leq r'p'$. Let A and B be posets, a map $f : A \longrightarrow B$ is called *monotone* if it preserves the order i.e., $a \leq a'$ in A implies $f(a) \leq f(a')$ in B. The set of all monotone mappings from A to B (resp. from A to A) is usually denoted by $F(A, B)$ (resp. $F(A, A)$) and it inherits a point-wise order as follows:

$$f \leq g \Leftrightarrow f(a) \leq g(a), \ \forall\, a \in A.$$

For each $b \in B$, c_b denotes the constant map on A with range $\{b\}$ and such a map is clearly monotone. Let R be a pomoniod and A a poset. We say that A is a left R-poset if there exists a monotone map $R \times A \longrightarrow A$ such that $1a = a$ and $(rp)a = r(pa)$ for all $a \in A$ and $r, p \in R$. We denote a left R-poset by ${}_RA$. The right R-poset is defined dually and it is denoted by A_R.

Let R and S be pomonoids. *The wreath product of the pomonoids R and S by the left R-poset ${}_RA$* is the set $T = R \times F(A, S)$ endowed with the multiplication given by

$$(r,\ f)(p,\ g) = (rp,\ f_pg),$$

where $f_pg(a) = f(pa)g(a)$ for all $a \in A$, $r, p \in R$, $f, g \in F(A, S)$. The map f_p means that $f_p(a) = f(pa)$.

By [14] Proposition 2.1 the wreath product T is a semigroup, and it is a monoid if and only if S is a monoid. Again by [14] Proposition 2.4, T is a posemigroup with componentwise order. Knauer and Mikhalev [14] studied the ordered wreath product where they assumed $F(A, S)$ to contain all montone (isotone), antimonotone (antitone), and zigzag preserving mappings. In this paper, we only consider the monotone (isotone) case and thus under the notations introduced in [14], $F(A,S)$ is precisely the set $I(A,S)$.

The *wreath product* ${}_TC$ of the left R-poset ${}_RA$ with the left S-poset ${}_SB$ over the pomonoid $T = R \times F(A,S)$ is the left T-poset ${}_TC = {}_RA \times {}_SB$ endowed with the monotone action given by

$$(r,f)(a,b) = (ra, f(a)b),$$

for all $(r,f) \in R \times F(A,S)$ and $(a,b) \in A \times B$.

For simplicity, we will refer to the wreath product $T = R \times F(A,S)$ simply as T, and the wreath product T-poset ${}_TC = {}_RA \times {}_SB$ as ${}_TC$ throughout the paper when the context is clear.

In this paper, we first find the necessary conditions for an element of a pomonoid R to act po-injectively and to be po-cancellable respectively on the wreath product ${}_TC$. We also establish necessary conditions on the wreath product T to act po-injectively and to be po-cancellable on the left T-poset ${}_TC$. Finally we examine some of the well known properties of $S-$posets either on ${}_TC$ or on T. This particularly includes the properties such as po-surjectivity, po-torsion free, property (P), property (E), property (P_E), strongly flat, reversible, and solvable.

2. Po-injective action and left po-cancellability

In this section we investigate the po-injective and po-cancellable properties on the wreath product of pomonoids and the relation between these properties.

Let ${}_RA$ be a left R-poset and let r be any element of R. We say that r *acts po-injectively on* A if $ra \leq ra'$ implies $a \leq a'$, where $a, a' \in A$. If this property holds for every $r \in R$, then it can be said that R *acts po-injectively on* A. The strong version of R being acting po-injectively on A is that of R being acting strongly po-injectively on A, which is defined as: R *acts strongly po-injectively on* A if for all $r, r' \in R$, $a, a' \in A$, if $ra \leq r'a'$ and $r \leq r'$ then $a \leq a'$. An element $r \in R$ is called *left po-cancellable* if for all $t, t' \in R$, $rt \leq rt'$ implies $t \leq t'$. A pomoniod R is called *left po-cancellative* if every element in R is left po-cancellable. A pomonoid R is called *strongly left po-cancellative* if for all $r, r', t, t' \in R$, $r \leq r'$ and $rt \leq r't'$ implies $t \leq t'$. The right po-cancellable, right po-cancellative, and strongly right po-cancellative are defined dually.

Let X be a left R-poset and Y a poset. Let ${}_T(X \times Y)$ be a left T-poset where $T = R \times F(A,S)$ is the wreath product defined above, and the action is defined through some monotone mapping $\alpha : F(A,S) \times X \times Y \longrightarrow Y$ such that $(r,f)(x,y) = (rx, \alpha(f,x,y))$. This is equivalent to say that α satisfies the identities:

(i) $\alpha(c_1, x, y) = y$.

(ii) $\alpha(f, px, \alpha(g,x,y)) = \alpha(f_p g, x, y)$, for all $x \in X,\ y \in Y,\ c_1,\ f,\ g \in F(A,S)$ and $p \in R$.

First we give necessary and sufficient condition for an element of T to act po-injectively on ${}_TX \times Y$

Proposition 1. *The element $(r,f) \in T$ acts po-injectively on ${}_T(X \times Y)$ through $\alpha : F(A,S) \times X \times Y \to Y$ if and only if the following conditions are satisfied.*
1) If $x \leq x'$ and $\alpha(f,x,y) \leq \alpha(f,x',y')$ then $y \leq y'$, where $x,x' \in X$ and $y,y' \in Y$.
2) If $rx \leq rx'$ and $x \nleq x'$ then for all $y,y' \in Y$, $\alpha(f,x,y) \nleq \alpha(f,x',y')$ where $x,x' \in X$ and $r \in R$.

Proof. Let $(r,f) \in T$ acts po-injectively on ${}_T(X \times Y)$ through α.
1) First suppose that $x \leq x'$ in X and $\alpha(f,x,y) \leq \alpha(f,x',y')$ where $y,y' \in Y$. Hence, for all $(r,f) \in T$

$$(r,f)(x,y) = (rx, \alpha(f,x,y)) \leq (rx', \alpha(f,x',y')) = (r,f)(x',y').$$

Since (r,f) acts po-injectively on ${}_T(X \times Y)$, it follows that $(x,y) \leq (x',y')$ and therefore, $y \leq y'$ as required.

2) Suppose now that $rx \leq rx'$ and $x \nleq x'$, where $r \in R$ and $x,x' \in X$. Also, suppose that $\alpha(f,x,y) \leq \alpha(f,x',y')$ for some $y,y' \in Y$. Hence,

$$(r,f)(x,y) = (rx, \alpha(f,x,y)) \leq (rx', \alpha(f,x',y')) = (r,f)(x',y').$$

Again since (r,f) acts po-injectively on ${}_T(X \times Y)$, we must have $(x,y) \leq (x',y')$. Thus, $x \leq x'$ and this contradicts the assumption $x \nleq x'$. Therefore, for all $y,y' \in Y$ we have $\alpha(f,x,y) \nleq \alpha(f,x',y')$.

For the other direction, assume the two conditions are satisfied. Suppose that $(r,f)(x,y) \leq (r,f)(x',y')$. This implies $(rx, \alpha(f,x,y)) \leq (rx', \alpha(f,x',y'))$. Therefore, $rx \leq rx'$ and $\alpha(f,x,y) \leq \alpha(f,x',y')$. The condition (2) implies $x \leq x'$ and so from condition (1) we get $y \leq y'$. Therefore, $(x,y) \leq (x',y')$. Hence (r,f) acts po-injectively on T, as required.

In particular, by taking $X = {}_RA,\ Y = {}_SB$ and defining $\alpha : F(A,S) \times A \times S \longrightarrow B$ by $\alpha(f,a,b) = f(a)b$ for all $a \in A, b \in B$, in Proposition 1 we get Theorem 1. However for the sake of clarity we have given its proof.

Theorem 1. *The element $(r,f) \in T$ acts po-injectively on ${}_TC$ if and only if:*
1) if $a \leq a'$ and $f(a)b \leq f(a')b'$ where $a,a' \in A$ and $b,b' \in B$, then $b \leq b'$, and
2) if $ra \leq ra'$ and $a \nleq a'$ where $a,a' \in A$, then $f(a)b \nleq f(a')b'$ for all $b,b' \in B$.

Proof. Let $(r,f) \in T$ acts po-injectively on ${}_TC$.
1) First suppose that $a \leq a'$ in A and $f(a)b \leq f(a')b$ where $a,a' \in A$ and $b,b' \in B$. Then,

$$(r,f)(a,b) = (ra, f(a)b) \leq (ra', f(a')b') = (r,f)(a',b').$$

Since (r,f) acts po-injectively on ${}_TC =_R A \times_S B$, we must have $(a,b) \leq (a',b')$ and so $b \leq b'$, as required.

2) Now suppose that $ra \leq ra'$ and $a \nleq a'$, where $r \in R$ and $a,a' \in A$. Also let $f(a)b \leq f(a')b'$ for some $b,b' \in B$. Then,

$$(r,f)(a,b) = (ra, f(a)b) \leq (ra', f(a')b') = (r,f)(a',b').$$

Since (r,f) acts po-injectively on ${}_TC$ we must have $(a,b) \leq (a',b')$. Thus, $a \leq a'$ and this is a contradiction to the assumption in (2). Therefore, $f(a)b \nleq f(a')b'$ for all $b,b' \in B$.

For the other direction, assume the given conditions (1) and (2) are satisfied. Let $(r,f) \in T$ and suppose that $(r,f)(a,b) \leq (r,f)(a',b')$. So $(ra, f(a)b) \leq (ra', f(a')b')$ which forces $ra \leq ra'$ and $f(a)b \leq f(a')b'$. From condition (2) we have $a \leq a'$ and this together with condition (1) gives $b \leq b'$. Therefore $(a,b) \leq (a',b')$. Then, (r,f) acts po-injectively on ${}_TC$, as required.

In the unordered case, it has been shown in [15] that the analogue of condition (1) in Theorem 1 above is that $f(a)$ acts injectively on B. However, in ordered case condition (1) does not imply that $f(a)$ acts po-injectively on B and conversely it is not enough to have $f(a)$ acting po-injectively or even strongly po-injectively on B to deduce condition (1). Therefore it will be interesting to find that under what conditions the result in ordered case is similar to that of unordered case. However we do have the following.

Corollary 1. *If $(r,f) \in T$ acts po-injectively on ${}_TC$ then*
1) $f(a)$ acts po-injectively on B for any $a \in A$, and
2) if $ra \leq ra'$ and $a \nleq a'$, then for any $b,b' \in B$, $f(a)b \nleq f(a')b'$.

The proof of the following is similar to Theorem 1, however we add the proof for completeness.

Proposition 2. *The element $(r,f) \in T$ is left po-cancellable if and only if the following two conditions hold.*
1) $p \leq p'$ and $f_p g \leq f_{p'} g'$ implies $g \leq g'$, where $p,p' \in R$ and $g,g' \in F(A,S)$.
2) $rp \leq rp'$ and $p \nleq p'$ implies $f_p g \nleq f_{p'} g'$ for any $g,g' \in F(A,S)$.

Proof. Let $(r,f) \in T$ be left po-cancellable.
1) Suppose that $p \leq p'$ in R and $f_p g \leq f_{p'} g'$ where $g,g' \in F(A,S)$. Therefore,

$$(r,f)(p,g) = (rp, f_p g) \leq (rp', f_{p'} g') = (r,f)(p',g').$$

Since (r,f) is left po-cancellable, we must have $(p,g) \leq (p',g')$. Thus $g \leq g'$ as required.

2) Now assume that $r,p,p' \in R$, $rp \leq rp'$ and $p \nleq p'$. Also suppose that $f_p g \leq f_{p'} g'$ for some $g,g' \in F(A,S)$. Therefore,

$$(r,f)(p,g) = (rp, f_p g) \leq (rp', f_{p'} g') = (r,f)(p',g').$$

Since (r,f) is left po-cancellable, we have $(p,g) \leq (p',g')$. Thus, $p \leq p'$ and we arrive at a contradiction. Therefore, $f_p g \nleq f_{p'} g'$ for any $g,g' \in F(A,S)$ as required.

For the other direction, assume the given conditions are satisfied. Suppose that $(r,f)(p,g) \leq (r,f)(p',g')$. So, $(rp, f_p g) \leq (rp', f_{p'} g')$. Thus $rp \leq rp'$ and $f_p g \leq f_{p'} g'$. From condition (2) $p \leq p'$ and condition (1) implies that $g \leq g'$. Therefore $(p,g) \leq (p',g')$. Hence (r,f) is left po-cancellable, as required.

Recall from [16] that a free posemigroup F is a free semigroup F with order defined as:

$$a_1a_2\dots a_n \leq b_1b_2\dots b_t \Leftrightarrow n = t \text{ and } a_i \leq b_i \text{ where } 1 \leq i \leq n.$$

In Example 1.4 of [15] it has been shown that there exist left cancellable elements in the monoid $T = R \times F(A, S)$ for which the first components are not left cancellable in R. In the next example we show that the same example in [15] can be transformed in the ordered case by replacing the free semigroup by the free posemigroup, by defining suitable orders and carefully choosing the required monotone maps. Thus there exists a left po-cancellable element $(r, f) \in T$ but r is not left po-cancellable in R.

Example 1. *Let $A = \{a, b\}$ where $a \leq b$ and $S =< u, v > \cup\{1\}$, where $u \leq v$, be the free pomonoid generated by u, v. Let $P(A) = \{c_a, c_b, 1\}$ be the pomonoid which is also an A-poset under the action given by $c_a.x = a$, $c_b.x = b$ for all $x \in A$. Consider $T = P(A) \times F(A, S)$. Let $f \in F(A, S)$ such that $f(a) = u$ and $f(b) = v$. We show that $(c_a, f) \in T$ is left po-cancellable while c_a is not left po-cancellable, as clearly $c_b \nleq c_a$ as $b \nleq a$ while $c_ac_b \leq c_ac_a$. Now suppose that $(c_a, f)(h_1, g_1) \leq (c_a, f)(h_2, g_2)$ where $(h_1, g_1), (h_2, g_2) \in T$. Then, $(c_ah_1, f_{h_1}g_1) \leq (c_ah_2, f_{h_2}g_2)$. So for any $a \in A$, $f_{h_1}g_1(a) = f(h_1(a))g_1(a) \leq f(h_2(a))g_2(a) = f_{h_2}g_2(a)$. From the definition of free posemigroup and since the image of f is one letter word we get that $h_1 \leq h_2$ and $g_1 \leq g_2$. So $(h_1, g_1) \leq (h_2, g_2)$ and thus (c_a, f) is left po-cancellable.*

Next we discuss the po-injectivity of the wreath product pomonoid T on the wreath product T-poset ${}_TC$ constructed above.

Theorem 2. *The pomonoid T acts po-injectively on ${}_TC$ if and only if the following conditions are satisfied.*
1) $a \leq a'$ and $f(a)b \leq f(a')b'$ implies $b \leq b'$ where $f \in F(A, S)$, $a, a' \in A$, and $b, b' \in B$.
2) R acts po-injectively on A.

Proof. Let T acts po-injectively on ${}_TC$.
1) Suppose that $a \leq a'$ and $f(a)b \leq f(a')b'$, where $a, a' \in A$ and $b, b' \in B$. By condition (1) of Theorem 1, we get that $b \leq b'$.
2) Next suppose that $ra \leq ra'$ where $r \in R$ and $a, a' \in A$. Assume that $a \nleq a'$. Using condition (2) in Theorem 1 we have $c_a(a)b \nleq c_{a'}(a')b'$ for all $b, b' \in B$ and this is a contradiction to T acts po-injectively on ${}_TC$. Therefore, $a \leq a'$ and so R acts po-injectively on A.
Conversely assume that the two conditions are satisfied. Take any $(r, f) \in T$ and (a, b), $(a', b') \in {}_T(A \times B)$ and suppose that $(r, f)(a, b) \leq (r, f)(a', b')$. Therefore $(ra, f(a)b) \leq (ra', f(a')b')$ so $ra \leq ra'$ and $f(a)b \leq f(a')b'$. From condition (2) we have $a \leq a'$. Since $a \leq a'$ and $f(a)b \leq f(a')b'$, from condition (1) we get $b \leq b'$. Hence $(a, b) \leq (a', b')$ and so T acts po-injectively on ${}_TC$, as required.

Corollary 2. *The pomonoid T acts po-injectively on ${}_TC$ if and only if R acts po-injectively on A and S acts po-injectively on B.*

Proof. Suppose that T acts po-injectively on ${}_TC$. By condition (2) of Theorem 2, R acts po-injectively on A. Take any $s \in S$ and $b, b' \in B$ such that $sb \leq sb'$. Clearly

$(ra, sb) \leq (ra, sb')$ and so $(r, c_s)(a, b) \leq (r, c_s)(a, b')$. Since T acts po-injectively on $_TC$, it follows that $(a, b) \leq (a, b')$. Thus $b \leq b'$ and so S acts po-injectively on B as required. Conversely if R acts po-injectively on A and S acts po-injectively on B then conditions (1) and (2) of Theorem 2 are clearly satisfied and hence T acts po-injectively on $_TC$.

Proposition 3. *If $(r, f) \in T$ is left po-cancellable then for all $a \in A$ $s, s' \in S$ and $p, p' \in R$, $rp \leq rp'$ and $p \nleq p'$ imply that $f(pa)s \nleq f(p'a)s'$.*

Proof. Let $(r, f) \in T$ be left po-cancellable. Assume that for all $r, p, p' \in R$, $rp \leq rp'$ and $p \nleq p'$ and there exist some $s, s' \in S$ such that $f(pa)s \leq f(p'a)s'$ for all $a \in A$. Therefore $f_pc_s \leq f_{p'}c_{s'}$ and so,

$$(r, f)(p, c_s) = (rp, f_pc_s) \leq (rp', f_{p'}c_{s'}) = (r, f)(p', c_{s'}).$$

Since (r, f) is left po-cancellable it follows that $(p, c_s) \leq (p', c_{s'})$. Thus $p \leq p'$ and this is a contradiction. Hence $f(pa)s \nleq f(p'a)s'$ for all $s, s' \in S$ and all $a \in A$.

Proposition 4. *The element $(r, f) \in T$ is left po-cancellable if the following two conditions hold:*
1) If $a \leq a'$ and $f(a)s \leq f(a')s'$ then $s \leq s'$.
2) If $rp \leq rp'$ and $p \nleq p'$, then for all $a \in A$, $s, s' \in S$, $f(pa)s \nleq f(p'a)s'$ where $p, p' \in R$.

Proof. Assume the two conditions are satisfied and let

$$(r, f)(p, g_1) \leq (r, f)(p', g_2).$$

Therefore, $(rp, f_pg_1) \leq (rp', f_{p'}g_2)$ and so $rp \leq rp'$ and $f_pg_1 \leq f_{p'}g_2$. If $p \nleq p'$ then from condition (2) it follows that $f(pa)s \nleq f(p'a)s'$ for all $a \in A$ and $s, s' \in S$. Since $g_1(a), g_2(a) \in S$, the condition $f(pa)s \nleq f(p'a)s'$ contradictions the assumption $f_pg_1 \leq f_{p'}g_2$ and so we must have $p \leq p'$. Also, since $pa \leq p'a$ and $f_pg_1(a) = f(pa)g_1(a) \leq f(p'a)g_2(a) = f_{p'}g_2(a)$ for all $a \in A$ so from condition (1) we get that $g_1 \leq g_2$. Hence (r, f) is left po-cancellable, as required.

Theorem 3. *The wreath product T is left po-cancellative if and only if R is left po-cancellable and S is strongly left po-cancellative.*

Proof. Suppose that T is left po-cancellative. Therefore for all (r, f), (p, g) and $(p', g') \in T$ the inequality $(r, f)(p, g) \leq (r, f)(p', g')$ implies that $rp \leq rp'$ and $f_pg \leq f_{p'}g'$. By Proposition 3 we have $p \leq p'$, proving that R is left po-cancellative. Let $s, s', t, t' \in S$ such that $s \leq s'$ and $st \leq s't'$. By taking $f_p = c_s, f_{p'} = c_{s'}, g = c_t$ and $g' = c_{t'}$ and using condition (1) of Propostion 2, we have $t \leq t'$, proving that S is strongly left po-cancellative. Conversely assume that R is left po-cancellative and S is strongly left po-cancellative. Let (r, f), (p, g) and $(p', g') \in T = R \times F(A, S)$ be such that $(r, f)(p, g) \leq (r, f)(p', g')$. This implies $rp \leq rp'$ and $f_pg \leq f_{p'}g'$. Left po-cancellative property of R forces $p \leq p'$. For all $a \in A$, $pa \leq p'a$ and since f is monotone it follows that $f(pa) \leq f(p'a)$. From $f_pg \leq f_{p'}g'$

we have $f(pa)g(a) \leq f(p'a)g'(a)$ for all $a \in A$. Since S is strongly left po-cancellative $g(a) \leq g'(a)$ for all $a \in A$ and so $g \leq g'$. Hence $(p, g) \leq (p', g')$ as required.

Theorem 3 is only true for left po-cancellative. For the right po-cancellative we have the following result whose proof is straightforward and so we omitted it.

Proposition 5. *If the element $(r, f) \in T$ is right po-cancellable then r is right po-cancellable in R.*

Proposition 6. *If $r \in R$ is not left po-cancellable then $(r, c_1) \in T$ is not left po-cancellable.*

Proof. Assume that $r \in R$ is not left po-cancellable and (r, c_1) is left po-cancellable. Then there exist $p, p' \in R$ such that $rp \leq rp'$ and $p \nleq p'$. From Proposition 3, for all $a \in A$, $s, s' \in S$, $c_1(pa)s \nleq c_1(p'a)s'$. Therefore, $s \nleq s'$ for all $s, s' \in S$ which is impossible as $s \leq s$. Therefore, (r, c_1) in T is not left po-cancellable.

3. Po-surjective action and Po-flatness properties of posets

This section will be devoted for studying the po-surjective and po-flatness properties of posets on the wreath product of pomonoids. The relationships among these properties have also been obtained in this section.

Definition 1. *(i) An element $r \in R$ acts po-surjectively on the left R-poset ${}_RA$ if for every $a \in A$ there exists $a' \in A$ such that $ra' \leq a$.*

(ii) $R' \subseteq R$ acts po-surjectively on A, when every $r' \in R'$ acts po-surjectively on ${}_RA$.

(iii) For any fixed $r \in R$ and $a \in {}_RA$ we define the set $a^r := \{x \in A : rx \leq a\}$.

(iv) Let ${}_SB$ be a left $S-$poset, then we say that $SB \leq B$ if for all $b \in B$ there exist $s \in S$ and $b' \in B$ such that $sb' \leq b$.

Proposition 7. *The element $(r, f) \in T$ acts po-surjectively on ${}_TC$ if and only if $f(a^r)B \leq B$ for all $a \in A$.*

Proof. Suppose $(r, f) \in T$ acts po-surjectively on ${}_TC$. Therefore for every $(a, b) \in {}_TC$ it can be found that $(a', b') \in {}_TC$ such that $(r, f)(a', b') \leq (a, b)$. This implies $ra' \leq a$ and $f(a')b' \leq b$. Thus $a' \in a^r$ and we have $f(a^r)B \leq B$.
Conversely, suppose $f(a^r)B \leq B$. Specifically, the assumption implies that $a^r \neq \phi$. Take $(a, b) \in {}_TC$, choose $a' \in a^r$ and $b' \in B$ where $f(a')b' \leq b$. Hence, $(r, f)(a', b') = (ra', f(a')b') \leq (a, b)$. Hence $(r, f) \in T$ acts po-surjectively on ${}_TC$, as required.

Proposition 8. *The element $(r, f) \in T$ acts po-surjectively on ${}_TC$ if and only if s_0 acts po-surjectively on ${}_SB$ for some $s_0 \in f(a^r)$ for all $a \in A$.*

Proof. Let $a \in A$ and $b \in B$. Since $(r, f) \in T$ acts po-surjectively on ${}_TC$, for any $(a, b) \in {}_TC$ it can be found that $(a', b') \in {}_TC$ such that $(r, f)(a', b') \leq (a, b)$. Then, $ra' \leq a$ and $f(a')b' \leq b$ implying $a' \in a^r$. Suppose $f(a') = s_0$. Thus for any $b \in B$ there exists $b' \in B$ such that $s_0 b' \leq b$ for some $s_0 = f(a') \in f(a^r)$. Hence some $s_0 \in f(a^r)$ acts po-surjectively on ${}_SB$.
Now suppose $s_0 \in f(a^r)$ acts po-surjectively on ${}_SB$. Therefore for any $b \in B$ it can be found that $b' \in B$ such that $s_0 b' \leq b$. Clearly, $a^r \neq \emptyset$ as there exists some $a' \in a^r$ such that $f(a') = s_0$. Now for any $(a, b) \in {}_TC = {}_RA \times {}_SB$ there exists some $(a', b') \in {}_Ra^r \times {}_SB \subseteq {}_RA \times {}_SB$ such that $(r, f)(a', b') = (ra', f(a')b') = (ra', s_0 b') \leq (a, b)$. Hence, $(r, f) \in T = R \times F(A, S)$ acts po-surjectively on ${}_TC$.

Theorem 4. *The pomonoid T acts po-surjectively on ${}_TC$ if and only if R acts po-surjectively on ${}_RA$ and S acts po-surjectively on ${}_SB$.*

Proof. Take $(r, c_s) \in T$ for any $r \in R$ and $s \in S$. Since T acts po-surjectively on ${}_TC$ so for every $(a, b) \in_T C$ it can be found that $(a', b') \in {}_TC$ such that $(r, c_s)(a', b') \leq (a, b)$ implying $(ra', sb') \leq (a, b)$ which further implies $ra' \leq a$ and $sb' \leq b$ for arbitrary $r \in R, s \in S, a \in A$ and $b \in B$. Thus, R acts po-surjectively on ${}_RA$ and S acts po-surjectively on ${}_SB$.
Now suppose R acts po-surjectively on ${}_RA$ and S acts po-surjectively on ${}_SB$. We need to show that T acts po-surjectively on ${}_TC$. Assume on contrary that T doesn't act po-surjectively on ${}_TC$, there exists $(r, f) \in T$ such that $(r, f){}_TC \nleq {}_TC$. Therefore there exists $(a, b) \in {}_TC$ such that for every $(a', b') \in {}_TC$ we have $(r, f)(a', b') \nleq (a, b)$ implying that $(ra', f(a')b') \nleq (a, b)$, the following cases arise:
Case 1: If $ra' \nleq a$ and $f(a')b' \nleq b$, a contradiction to both R and S acting po-surjectively on ${}_RA$ and ${}_SB$.
Case 2: If $ra' \nleq a$ and $f(a')b' \leq b$, a contradiction to R acting po-surjectively on ${}_RA$.
Case 3: If $ra' \leq a$ and $f(a')b' \nleq b$, a contradiction to S acting po-surjectively on ${}_SB$.
Hence T acts po-surjectively on ${}_TC$.

Definition 2. *A subset $P \subseteq S$ is referred to as simultaneously right po-cancellable in S if $sp \leq s'p$ for any $p \in P$ and $s, s' \in S$ implies $s \leq s'$.*

Proposition 9. *If the element $(r, f) \in T$ is right po-cancellable, then r acts po-surjectively on ${}_RA$, r is right po-cancellable in R, and for any $a \in A$ the set $f(A)$ is simultaneously right po-cancellable in S.*

Proof. Suppose $(r, f) \in T$ is right po-cancellable. Assuming r is not po-surjectively on ${}_RA$, by way of contadiction. Therefore, we can found that $a \in A$ such that for every $a' \in A, ra' \nleq a$. Clearly, $a \bigcap rA = \emptyset$, for if $x \in a \bigcap rA$ then $x = a$ and $x = ra''$ for some $a'' \in A$ implying $ra'' = a$ for some $a'' \in A$ implying $ra'' \leq a$ for some $a'' \in A$ which is not true. Let $g, g' \in F(A, S)$ where $g(a) \nleq g'(a)$ and $g|_{rA} \leq g'|_{rA}$. Hence, we have $(g_r f)(a') = g(ra')f(a') \leq g'(ra')f(a') = (g'_r f)(a')$ implying $g_r f \leq g'_r f$, where $a' \in A$. Then, $(1, g)(r, f) = (r, g_r f) \leq (r, g'_r f) = (1, g')(r, f)$. Since by our assumption $(r, f) \in T$

is right po-cancellable so $(1, g) \leq (1, g')$ which implies $g \leq g'$ so $g(x) \leq g'(x)$ for all $x \in A$ and $g(a) \leq g'(a)$ which is a contradiction. Thus, r acts po-surjectively on ${}_RA$. The remainder can be proved using a similar argument to Proposition 2.4 in [15], with respect to the order relation.

Theorem 5. *If the pomonoid T is right po-cancellative, then R acts po-surjectively on ${}_RA$ and, R and S are right po-cancellative.*

The proof follows by similar argument as Theorem 2.6 in [15] with respect to the order relation and by using Proposition 9 above.

In [11, 13, 18, 19] many of po-flatness properties such as po-torsion free, property (E), and property (P) in the category of R-posets were considered.

Definition 3. *Let ${}_RA$ be a left R-poset. We say that:*

(i) ${}_RA$ *is* po-torsion free *over R if every left po-cancellable element of R acts po-injectively on ${}_RA$.*

(ii) ${}_RA$ *satisfies* property *(P) if $ra \leq r'a'$ where $r, r' \in R$ and $a, a' \in A$, then $a = ua'', a' = u'a''$ with $ru \leq r'u'$ for some $a'' \in A, u, u' \in R$.*

(iii) ${}_RA$ *satisfies* property *(E) if $ra \leq r'a$ where $r, r' \in R$ and $a \in A$, then $a = ua''$ with $ru \leq r'u$ for some $a'' \in A, u \in R$.*

(iv) ${}_RA$ *satisfies* property *(P_E) if $ra \leq r'a'$ where $r, r' \in R$ and $a, a' \in A$, then $a \leq ua''$ and $u'a'' \leq a'$ with $ru \leq r'u'$ for some $a'' \in A, u, u' \in R$.*

(v) ${}_RA$ *is* strongly flat *if it satisfies properties (P) and (E).*

Theorem 6. *If the left T-poset ${}_TC$ is po-torsion free then ${}_RA$ and ${}_SB$ are po-torsion free.*

Proof. Assume that ${}_TC$ is po-torsion free. Suppose that $r \in R$ is a left po-cancellable. We show that it acts po-injectively on A. Let $ra \leq ra'$ where $a, a' \in A$. We first show that (r, c_1) is left po-cancellable by using Proposition 4. If $a_1 \leq a_2$ and $c_1(a_1)s \leq c_1(a_2)s'$, then $s \leq s'$, and thus the first condition holds. For the second condition, assume that $rp \leq rp'$ and $p \not\leq p'$. However, this is impossible as r is left po-cancellable. Thus (r, c_1) is left po-cancellable. Since ${}_RA \times_S B$ is po-torsion free, (r, c_1) acts po-injectively on ${}_RA \times_S B$. Therefore for any $b \in b$,

$$(r, c_1)(a, b) = (ra, b) \leq (ra', b) = (r, c_1)(a', b).$$

So $(a, b) \leq (a', b)$ and then $a \leq a'$. Thus r acts po-injectively on A and hence ${}_RA$ is po-torsion free.
Now to prove that ${}_SB$ is po-torsion free suppose that $s \in S$ is left po-cancellable and $sb \leq sb'$ for some $b, b' \in B$. We again use Proposition 4 to show that $(1, c_s)$ is left po-cancellable. If $a \leq a'$ and $c_s(a)s \leq c_s(a')s'$, then $ss \leq ss'$. As s is left po-cancellable, $s \leq s'$.

So the first condition holds. Suppose that $p \leq p'$ and $p \nleq p'$, but this is impossible, so the second condition holds. Thus $(1, c_s)$ is left po-cancellable, so $(1, c_s)$ acts po-injectively on ${}_RA \times_S B$. Therefore for any $b \in B$,

$$(1, c_s)(a, b) = (a, sb) \leq (a, sb') = (1, c_s)(a, b').$$

So $(a, b) \leq (a, b')$, and thus $b \leq b'$. Thus s acts po-injectively on B and hence ${}_SB$ is po-torsion free as required.

Recall that in unordered case, when ${}_TC = {}_RA \times {}_SB$ is torsion free, then A and B are both torsion free. Therefore the above theorem generalises this result to the ordered case.

Proposition 10. *For the left T-poset ${}_TC$ the following are true.*

(i) If ${}_TC$ satisfies the property (P) then so does ${}_RA$.

(ii) If ${}_TC$ satisfies the property (E) then so does ${}_RA$.

(iii) If ${}_TC$ is strongly flat then so is ${}_RA$.

(iv) If ${}_TC$ satisfies the property (P_E) then so does ${}_RA$.

Proof. 1) Suppose that T-poset ${}_TC$ satisfies property (P) and let $ra \leq r'a'$, where $r, r' \in R$ and $a, a' \in A$. Now for any $b \in B$,

$$(r, c_1)(a, b) = (ra, b) \leq (r'a', b) = (r', c_1)(a', b).$$

Since ${}_TC$ satisfies property (P) there exists some $(a'', b'') \in {}_TC$ such that

$$(a, b) = (r_1, f_1)(a'', b'') = (r_1a'', f_1(a'')b'')$$

and

$$(a', b) = (r_2, f_2)(a'', b'') = (r_2a'', f_2(a'')b'')$$

with $(r, c_1)(r_1, f_1) \leq (r', c_1)(r_2, f_2)$. Therefore $a = r_1a''$, $a' = r_2a''$ and $(rr_1, (c_1)_{r_1}f_1) \leq (r'r_2, (c_1)_{r_2}f_2)$. It follows that $rr_1 \leq r'r_2$. Hence ${}_RA$ satisfies property (P).
2) It can be proved by an argument similar to case (1).
3) It follows from cases (1) and (2).
4) Suppose that ${}_TC$ satisfies property (P_E) and let $ra \leq r'a'$, where $r, r' \in R$ and $a, a' \in A$. Then for any $b \in B$,

$$(r, c_1)(a, b) = (ra, b) \leq (r'a', b) = (r', c_1)(a', b).$$

Since ${}_TC$ satisfies property (P_E) there exists $(a'', b'') \in {}_TC$ such that

$$(a, b) \leq (r_1, f_1)(a'', b'') = (r_1a'', f_1(a'')b'') \Rightarrow a \leq r_1a''$$

and

$$(r_2a'', f_2(a'')b'') = (r_2, f_2)(a'', b'') \leq (a', b) \Rightarrow r_2a'' \leq a'$$

with $(r, c_1)(r_1, f_1) \leq (r', c_1)(r_2, f_2)$. Thus, $(rr_1, (c_1)_{r_1} f_1) \leq (r'r_2, (c_1)_{r_2} f_2)$ and so $rr_1 \leq r'r_2$. Hence ${}_RA$ satisfies property (P_E) as required.

The concepts of reversible and weakly reversible were considered in the literature, see for example [18, 20]. Next we study these concepts for the case of wreath product and obtain some crucial results.
A pomonoid R is said to be *left reversible* if for every $r, r' \in R$, $rR \cap r'R \neq \emptyset$. If Z is a subset of a poset Y, *the down-set* $(Z]$ of Y is $(Z] = \{y \in Y | y \leq z$ for some $z \in Z\}$. A pomonoid R is called *weakly left reversible* if for every $r, r' \in R$, $rR \cap (r'R] \neq \emptyset$.

Theorem 7. *The wreath product T is left reversible if and only if R and S are left reversible.*

Proof. Let $r, r' \in R$. Suppose that T is left reversible. Then, $(r, c_1)T \cap (r', c_1)T \neq \emptyset$. So there exist (r_1, f_1) and $(r_2, f_2) \in T$ such that $(r, c_1)(r_1, f_1) = (r', c_1)(r_2, f_2)$ and so $(rr_1, (c_1)_{r_1} f_1) = (r'r_2, (c_1)_{r_2} f_2)$. Therefore $rr_1 = r'r_2$ and thus $rR \cap r'R \neq \emptyset$. Hence R is left reversible.

Let $s, s' \in S$. As $(r, c_s)T \cap (r, c_{s'})T \neq \emptyset$, there exist $(r_1, f_1), (r_2, f_2) \in T$ such that $(r, c_s)(r_1, f_1) = (r, c_{s'})(r_2, f_2)$. Thus $(rr_1, (c_s)_{r_1} f_1) = (rr_2, (c_{s'})_{r_2} f_2)$ and so $(c_s)_{r_1} f_1 = (c_{s'})_{r_2} f_2$. Take any $a \in A$, then $c_s(r_1 a) f_1(a) = c_{s'}(r_2 a) f_2(a)$. This implies $sf_1(a) = s' f_2(a)$. As $f_1(a), f_2(a) \in S$, $sS \cap s'S \neq \emptyset$. Hence S is left reversible.

For the other direction, let $(r, f), (p, g) \in T$ be arbitrary. Since R is left reversible, there exist $r', r'' \in R$ such that $rr' = pr''$. Also, as S is left reversible and $f(r'a), g(r''a) \in S$, there exist $s', s'' \in S$ such that $f(r'a)s' = g(r''a)s''$, so $f(r'a)c_{s'}(a) = g(r''a)c_{s''}(a)$ for all $a \in A$, then $f_{r'} c_{s'}(a) = g_{r''} c_{s''}(a)$ for all $a \in A$, which implies $f_{r'} c_{s'} = g_{r''} c_{s''}$.
Now, $(r, f)(r', c_{s'}) = (rr', f_{r'} c_{s'}) = (pr'', g_{r''} c_{s''}) = (p, g)(r'', c_{s''})$, so $(r, f)T \cap (p, g)T \neq \emptyset$. Hence, T is left reversible.

Proposition 11. *If T is weakly left reversible, then R and S are weakly left reversible.*

Proof. Let $r, r' \in R$. As T is weakly left reversible, $(r, c_1)T \cap ((r', c_1)T] \neq \emptyset$. So, we can found $(p, g) \in T$ where $(p, g) = (r, c_1)(r_1, f_1)$ and $(p, g) \leq (r', c_1)(r_2, f_2)$ for some $(r_1, f_1), (r_2, f_2) \in T$. Now

$$(p, g) = (rr_1, (c_1)_{r_1} f_1) \text{ and } (p, g) \leq (r'r_2, (c_1)_{r_2} f_2).$$

Therefore $p = rr_1$ and $p \leq r'r_2$. Thus $p \in rR \cap (r'R]$, and so R is weakly left reversible.
Next take any $s, s' \in S$. Again as T is weakly left reversible, $(1, c_s)T \cap ((1, c_{s'})T] \neq \emptyset$. So there exists $(q, h) \in T$ such that $(q, h) = (1, c_s)(p_1, h_1)$ and $(q, h) \leq (1, c_{s'})(p_2, h_2)$ for some $(p_1, h_1), (p_2, h_2) \in T$. Now

$$(q, h) = (p_1, (c_s)_{p_1} h_1) \text{ and } (q, h) \leq (p_2, (c_{s'})_{p_2} h_2).$$

So $h = (c_s)_{p_1} h_1$ and $h \leq (c_{s'})_{p_2} h_2$. Take any $a \in A$, then $h(a) = c_s(p_1 a) h_1(a)$ and $h(a) \leq c_{s'}(p_2 a) h_2(a)$. So $h(a) = sh_1(a)$ and $h(a) \leq s' h_2(a)$. Thus $h(a) \in sS \cap (s'S]$, and so S is weakly left reversible, as required.

Recall that a posemigroup S is termed *left solvable* if for any $u, v \in S$ there exist $s \in S$ such that $su = v$. However, if for any $u, v \in S$ there exist a unique $s \in S$ such that $su = v$ then S is called *left uniquely solvable.* The *right solvable* and *uniquely solvable* is defined dually. As known that left (resp. right) uniquely solvable semigroup is called *left (resp. right) group.*

Theorem 8. *The posemigroup T is a left solvable if and only if R and S are both left solvable.*

Proof. Suppose that T is a left solvable. For all $p, t \in R$, we know that $(p, g), (t, k) \in T$. Then there exist $(r, f) \in T$ such that $(r, f)(p, g) = (t, k)$. Hence, $(rp, f_p g) = (t, k)$. Then, $rp = t$ and so R is left solvable. Now, for any $s, s' \in S$ we know that $(p, c_s), (t, c_{s'}) \in T$. Hence, there exist $(r, f) \in T$ such that $(r, f)(p, c_s) = (t, c_{s'})$. Hence, $(rp, f_p c_s) = (t, c_{s'})$. So $f_p c_s(a) = f(pa)c_s(a) = f(pa)s = c_{s'}(a) = s'$. Therefore, S is left solvable. Now, suppose that R and S are both left solvable. Suppose that $(p, g), (t, k) \in T$. Since R is left solvable there exist $r \in R$ such that $rp = t$. Since, $g(a), k(a) \in S$ for any $a \in A$ and since S is left solvable then there exist $s \in S$ such that $sg(a) = k(a)$. Hence, $c_s(a)g(a) = k(a)$. Then $(c_s)_p g(a) = c_s(pa)g(a) = c_s(a)g(a) = k(a)$. Therefore, $(c_s)_p g = k$. Thus, $(r, c_s)(p, g) = (rp, (c_s)_p g) = (t, k)$ as required.

Corollary 3. *The left R-poset T is a left uniquely solvable if and only if R and S are both left uniquely solvable.*

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