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Computing Metric and Connected Metric Dimension of Some Graphs

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Abstract. For a connected graph $G = (V, E)$, a set $B \subseteq V(G)$ is a resolving set if every vertex in G is uniquely identified by its distances to the vertices in B. A resolving set that induces a connected subgraph is called a connected resolving set. The minimum cardinality of such a set is the connected metric dimension, denoted $cdim(G)$. In this paper, we compute the metric and connected metric dimensions for several classes of corona product graphs and propose an approximate algorithm to determine the connected metric dimension of arbitrary graphs.

2020 Mathematics Subject Classifications: 05C78, 05C15

Key Words and Phrases: Algebraic Graph, metric basis, resolving set, metric dimensions, Connected metric dimensions, Algorithm, Network security, Edge Computing

1. Introduction

Let G be a connected graph and $d\ (x, y)$ be the distance between the vertices x and y. A subset of vertices $W = (w_1, \ldots, w_k)$ is called a resolving set for G if for every two distinct vertices $x, y \in V(G)$, there is a vertex $w_i \in W$ such that $d(x, [w_i]) \neq d(y, w_i,)$. The metric dimension $dim(G)$ of G is the minimum cardinality of a resolving set for G. The concept of metric dimension was put forward by Slater [1], where it was expressed as locating sets, and later by Harary and Melter [2] called it as a metric dimension where the associate editor coordinating the review of this manuscript and approving it for publication was Yilun Shang. the metric generators were termed as resolving sets. There are numerous metric dimension applications, such as identifying an intruder in a network, robotics navigation, chemistry, and pattern recognition or image processing; for further studies related to this invariant, some of the references are, see, for instance, [3–5]. Some

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of the recent studies on the metric dimension are in [6–8]. The metric dimension $dim(G)$ of G is the minimum cardinality of a resolving set for G. Slater [9, 10] introduced the concept of a resolving set for a connected graph under the term locating set. He referred to a minimum resolving set as a reference set, and the cardinality of a minimum resolving set as the location number of a graph. Independently, Harary and Melter [11] studied these concepts under the term metric dimension. A resolving set S of G is called a connected resolving set of G if $G[S]$ is connected, and the connected metric dimension of G, denoted by $cdim(G)$, is the minimum cardinality of S over all connected resolving sets of G. For $v \in V(G)$, we define the connected metric dimension at v of G, denoted by $cdim(v)$, to be the minimum cardinality of a resolving set of G which contains v and induces a connected sub graph of G; then $cdim(G) = minv \in V(G)\{cdimG(v)\}$ for any vertex $v \in V(G)$. Moreover, connected metric dimension theory is used by wireless communication networks, electrical networks, chemical structures, and commercial networks [12–28]. This paper is organized as follows: the metric and connected metric dimension introduced in Sect. 2 the theorems that determine the metric and connected metric dimension of some graphs along with examples in Section 3. In Sect. 4 an approximate algorithm which finds the minimum connected metric dimension Finally, conclusions are drawn in Sect. 5.

Definition 1 [21–23, 29]**:** The corona $G_1 \bigodot G_2$ of two graphs G_1 (with n_1 vertices and m_1 edges)and G_2 (with n_2 vertices and m_2 edges) is defined as the graph obtained by taking one copy of G_1 and n_1 copies of G_2, and then joining the i^{th}vertex of $G_1$with an edge to every vertex in the i^{th} copy of G_2. It follows from the definition of the corona that $G_1 \bigodot G_2$ has $n_1 + n_1n_2$ vertices and $m_1n_1m_2 + n_1n_2$ edges. As show in Figure 1.

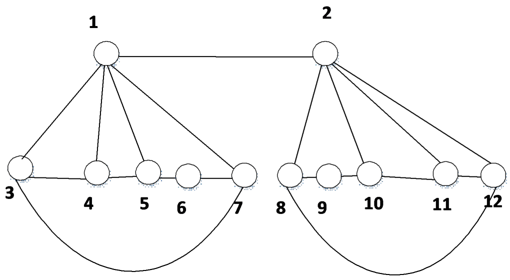

Figure 1. The corona product between two graphs G_1 and G_2.

Definition 2 [12]: For a connected graph $G = (V, E)$, a set of vertices $B \subseteq V(G)$ resolves G if every vertex of G is uniquely determined by its vector of distances to the vertices in B. Mathematically: $r(v|\ B) = (d(v,\ x_1),\ d(v,\ x_2), ..., d(v,\ x_k))$ is unique for every $v \in V(G)$.

Definition 3 (Metric basis): The minimum resolving set is called the metric basis.

Definition 4 (Metric dimension) [13]: The cardinality of the basis is called the metric dimension of G denoted by $dim(G)$. As show in Figures 2, 3.

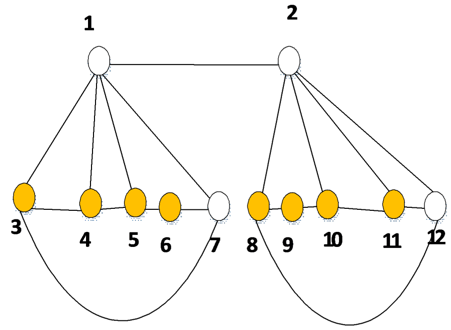

Figure 2. Metric dimension of $P_2 \odot C_5$.

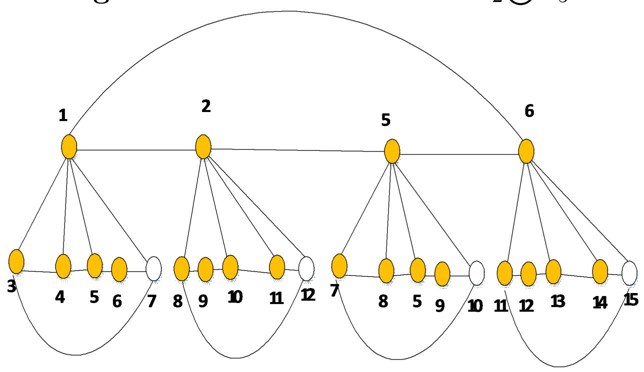

Figure 3. Metric dimension of $C_4 \odot C_5$.

Definition 5 [11]: A metric basis B of G is connected if the sub graph induced by B is a nontrivial connected sub graph of G. The cardinality number of the connected metric basis is the connected metric dimension of G and is denoted $cdim(G)$. As show in Figure 4.

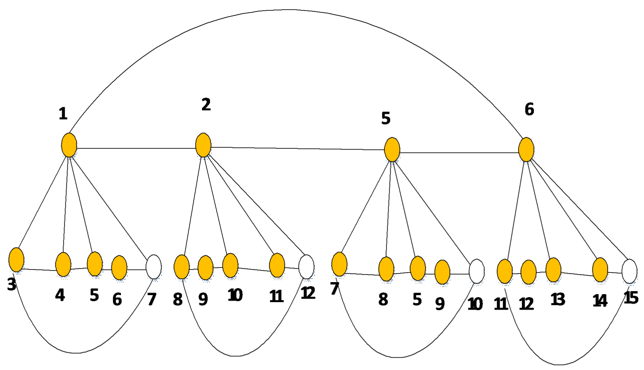

Figure 4. Connected metric dimension of $C_4 \odot C_5$.

The remaining of this paper is organized as follows: the metric and connected metric dimension introduced in Sect. 2 the theorems that determine the metric and connected metric dimension of some graphs. with an example in Section 3. In Sect. 4 an approximate algorithm which finds the minimum connected metric dimension Finally, conclusions are drawn in Sect. 5.

2. Main result

In this section, we prove the theorems that determine the metric and connected metric dimension of some graphs.

Theorem 1: Let G be $P_n \bigodot P_3$ graphs, then $C.M.D\,(P_n \bigodot P_3)$ equal $2n$, $n \geq 1$.

Proof. Let P_n be the path graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edges $v_i \sim v_{i+1}$ for $1 \leq i < n$. In the corona product $P_n \odot P_3$, for each vertex v_i of P_n, we attach a copy of P_3 denoted by u_{i1}, u_{i2}, u_{i3} and connect each u_{ij} to v_i , as show in Figure 5.

Define the set

$$R = \{v_1, v_2, \ldots, v_n\} \cup \{u_{i1} \mid 1 \leq i \leq n\}.$$

Clearly, $|R| = 2n$. We will show that R is a connected resolving set and that no smaller such set exists.

(i) R is a resolving set:

We must show that for any two distinct vertices $x, y \in V(G)$, the distance vectors $r(x|R)$ and $r(y|R)$ are distinct.

- For vertices in different P_3 copies, the distances to the corresponding v_i and u_{i1} in R distinguish them.
- Within each P_3 copy, the inclusion of u_{i1} and v_i in R ensures that the other two vertices u_{i2}, u_{i3} are distinguishable by their distances to u_{i1} and v_i.
- Vertices v_i are all in R, so they are trivially distinguished.
- Any vertex in P_n and any vertex in a P_3 copy are distinguished by their distances to R.

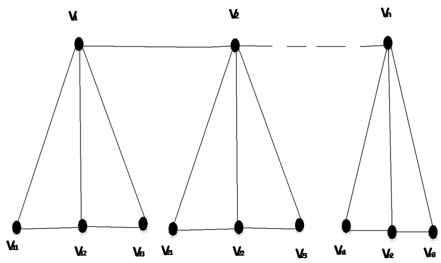

Figure 5. $P_n \bigodot P_3$ graph.

$$\boldsymbol{w}\,(\boldsymbol{v_i}, \boldsymbol{R}) = \begin{cases} \{0, 1, 2, \ldots, n-1, 1, 2, 3, \ldots, n\} \quad , i = 1 \\ \{1, 0, 1, \ldots, n-2, 2, 1, 2, \ldots, n-1\}\ , i = 2 \\ \{2, 1, 0, 1, \ldots, n-3, 3, 2, 1, 2, \ldots, n-2\}\,, i = 3 \\ \vdots \\ \vdots \\ \vdots \\ \{n-2, n-3, n-4, \ldots, 1, 2, , 3, n-2, \ldots, 2, 1, 2, 3\}\,, i = n-2 \\ \{n-1, n-2, n-3, \ldots, 2, 1, 2, 3, 2, 1, 2, \ldots, 3, 2, 1, 2\}\,, i = n-1 \\ \{n, n-1, n-2, n-3, \ldots, 3, 2, 1, n, \ldots, 3, 2, 1\}\,.i = n \end{cases}$$

$$\boldsymbol{w}\left(\boldsymbol{u_{ij}},\boldsymbol{R}\right) = \begin{cases} \{1,2,3,\ldots,n,j-1,3,4,\ldots,n,n+1\} & ,i=1 \\ \{2,1,2,\ldots,n-1,3,j-1,3,4,\ldots,n-1,n\} & ,i=2 \\ \{3,2,1,2,\ldots,n-2,4,3,j-1,3,4,\ldots,n-2,n-1\} & ,i=3 \\ \vdots \\ \vdots \\ \vdots \\ \{n-2,n-3,n-4,\ldots,2,3,\ldots,n-1,n-2,4,3,j-1,3,4\} & ,i=n-2 \\ \{n-1,n-2,n-3,\ldots,n-1,n,n-1,\ldots,4,3,j-1,3\} & ,i=n-1 \\ \{n,n-1,n-2,\ldots,1,n+1,n,\ldots,4,3,j-1\} & .i=n \end{cases}$$

(ii) R is connected:
The subgraph induced by $\{v_1,\ldots,v_n\}$ is a path and hence connected. Each u_{i1} is adjacent to v_i, so the full induced subgraph on R is connected.

(iii) Minimality:
Suppose there exists a connected resolving set R' with $|R'| < 2n$. Then at least one v_i or u_{i1} is missing from R'. If $v_i \notin R'$, then u_{i1} may be disconnected from the rest of R', violating connectivity. If $u_{i1} \notin R'$, then u_{i2} and u_{i3} may not be distinguishable. Thus, removing any vertex from R breaks either the resolving property or connectivity.

Therefore, R is a minimum connected resolving set, and $\text{cdim}(P_n \odot P_3) = 2n$.

Theorem 2: Let G be $P_n \bigodot C_3$ graphs, then the connected metric dimension of $P_n \bigodot C_3$ equal to $3n$, $n \geq 1$.

Proof. Let P_n be a path graph with vertices $v_1, v_2, \ldots, v_n$. For each vertex v_i in P_n, attach a copy of the cycle C_3 with vertices u_{i1}, u_{i2}, u_{i3}, and connect each u_{ij} to v_i, as show in Figure 6. The resulting graph $G = P_n \odot C_3$ has n central vertices and $3n$ peripheral vertices, totaling $4n$ vertices.

We define the set

$$R = \{v_1, v_2, \ldots, v_n\} \cup \{u_{i1}, u_{i2} \mid 1 \leq i \leq n\},$$

which includes all n central vertices and two vertices from each C_3 copy. Thus, $|R| = n + 2n = 3n$.

(i) R is a resolving set.

Consider any two distinct vertices $x, y \in V(G)$. We analyze the following cases:

- **Case 1:** x and y belong to different C_3 copies. Since each v_i is in R, and each u_{ij} is connected to v_i, the distances to v_i and the included u_{i1}, u_{i2} uniquely identify vertices in each C_3 copy.

- **Case 2:** x and y belong to the same C_3 copy. The inclusion of two vertices from each C_3 ensures that all three vertices in the cycle are distinguishable by their distances to the two included vertices.

- **Case 3:** One of x or y is a central vertex v_i, and the other is a peripheral vertex u_{ij}. Since v_i is in R and u_{i1}, u_{i2} are also in R, the distance vectors to R will differ.

- **Case 4:** x and y are distinct central vertices. Since all v_i are in R, they are trivially resolved.

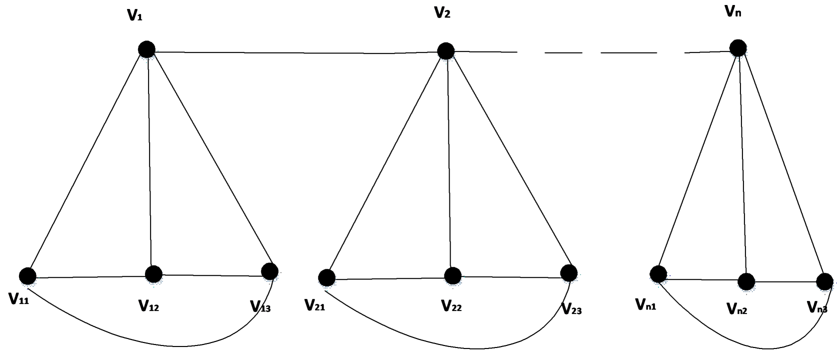

Figure 6. $P_n \bigodot C_3$ graph.

$$\boldsymbol{w}\left(\boldsymbol{v_i}, \boldsymbol{R}\right) = \begin{cases} \{0,1,2,\ldots,n-1,1,1,2,2,3,3,\ldots,n,n\} & ,i=1 \\ \{1,0,1,\ldots,n-2,2,2,1,1,2,2\ldots,n-1,n-1\} & ,i=2 \\ \{2,1,0,1,\ldots,n-3,3,3,2,2,1,1,\ldots,n-2,n-2\} & ,i=3 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \{n-1,n-2,n-3,\ldots,0,n,n.n-1,n-1,\ldots,1,1\} & .i=n \end{cases}$$

$$\boldsymbol{w}\left(\boldsymbol{u_{ij}}, \boldsymbol{R}\right) = \begin{cases} \{1,2,3,\ldots,n,n,j-1,j,3,3,4,4,\ldots,n,n,n+1,n+1\} & ,i=1,j=1 \\ \{1,2,3,\ldots,n,n,j-1,j-2,3,3,4,4,\ldots,n,n,n+1,n+1\} & ,i=1,j=2 \\ \{1,2,3,\ldots,n,n,j-2,j-2,3,3,4,4,\ldots,n,n,n+1,n+1\} & ,i=1,j=3 \\ \{2,1,2,3,4\ldots,n-1,3,3,j-1,j,3,3,4,4,\ldots,n,n\} & ,i=2,j=1 \\ \{2,1,2,3,4\ldots,n-1,3,3,j-1,j-2,3,3,4,4,\ldots,n,n\} & ,i=2,j=2 \\ \{2,1,2,3,4\ldots,n-1,3,3,j-2,j-2,3,3,4,4,\ldots,n,n\} & ,i=2,j=3 \\ \vdots \\ \vdots \\ \vdots \\ \{n-1,n-2,\ldots,1,n+1,n+1,n,n,n-1,n-1,j-1,j\} & ,i=n\ ,j=1 \\ \{n-1,n-2,\ldots,1,n+1,n+1,n,n,n-1,n-1,j-1,j-2\} & ,i=n\ ,j=2 \\ \{n-1,n-2,\ldots,1,n+1,n+1,n,n,n-1,n-1,j-2,j-2\} & ,i=n\ ,j=3 \end{cases}$$

Hence, R is a resolving set.

(ii) R induces a connected subgraph.

The subgraph induced by $\{v_1,\ldots,v_n\}$ is a path and hence connected. Each u_{i1} and u_{i2} is adjacent to v_i, so the full induced subgraph is connected.

(iii) R is minimal.

Suppose there exists a connected resolving set R' with $|R'| < 3n$. Then for some i, at most one of u_{i1}, u_{i2} is in R'. But in a C_3 cycle, at least two vertices are needed to resolve all three vertices. Removing any v_i would disconnect the corresponding C_3 copy from the rest of the graph. Thus, R' cannot be both resolving and connected.

Therefore, R is a minimal connected resolving set of size $3n$.

Theorem 3: The connected metric dimension of $C_n \bigodot P_3$ equal to $2n$, if $n \geq 3$.

Proof. The corona product $C_n \odot P_3$ is formed by taking a cycle C_n with vertices $v_1, v_2, \ldots, v_n$ and attaching to each v_i a copy of P_3 with vertices u_{i1}, u_{i2}, u_{i3}, where each u_{ij} is connected to v_i , as show in Figure 7.

We construct a set R consisting of:

$$R = \{v_1, v_2, \ldots, v_n\} \cup \{u_{i1} \mid 1 \leq i \leq n\}$$

Thus, $|R| = 2n$.

Resolving Property:

- Each v_i is in R, so any pair of v_i, v_j is trivially resolved.
- For each P_3 copy, the inclusion of u_{i1} and v_i ensures that the other two vertices u_{i2}, u_{i3} are uniquely identified by their distances to u_{i1} and v_i.
- Vertices from different P_3 copies are distinguished by their distances to the corresponding v_i and u_{i1}.

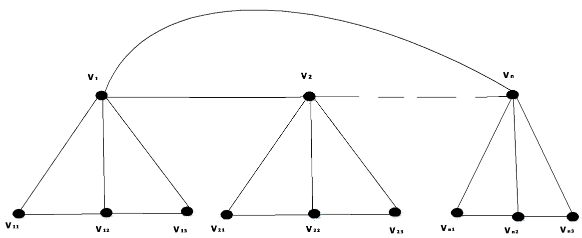

Figure 7. The $C_n \bigodot P_3$ graph.

$$\boldsymbol{w}\left(\boldsymbol{v_i}, \boldsymbol{R}\right) = \begin{cases} \{0, 1, 2, \ldots, 1, 1, 2, \ldots, 2\} & , i = 1 \\ \{1, 0, 1, \ldots, 2, 2, 1, 2, \ldots, 2\} & , i = 2 \\ \{2, 1, 0, 1, \ldots, 2, 3, 2, 1, \ldots, 3\} & , i = 3 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \{1, 2, 3, \ldots, 0, 2, 3, n-2, \ldots, 2, 1\} & . i = n \end{cases}$$

$$\boldsymbol{w}\left(\boldsymbol{u_{ij}},\boldsymbol{R}\right)=\begin{cases}\{1,2,3,\ldots,2,j-1,3,4,\ldots,3\} & ,i=1\\ \{2,1,2,\ldots,3,3,j-1,3,\ldots,n-2,n-1,4\} & ,i=2\\ \{3,2,1,2,\ldots,n-2,4,3,j-1,3,4,\ldots,n-2,n-1\} & ,i=3\\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \{1,2,3,4\ldots,1,3,4,\ldots,4,3,j-1\} & .i=n\end{cases}$$

Connectivity: The subgraph induced by R includes the cycle C_n (which is connected) and each u_{i1} is connected to v_i, forming a connected tree-like structure rooted at the cycle.

Minimality: Suppose a smaller connected resolving set R' exists with $|R'| < 2n$. Then at least one v_i or u_{i1} is missing. If $v_i \notin R'$, then u_{i1} may become disconnected or fail to resolve its P_3 copy. If $u_{i1} \notin R'$, then u_{i2}, u_{i3} may not be distinguishable. Hence, R is minimal.

Therefore, $\text{cdim}(C_n \odot P_3) = 2n$.

Theorem 4: If $n \geq 3$, then connected metric dimension of $C_n \bigodot C_3$ graphs are equal $3n$.

Proof. The corona product $C_n \odot C_3$ is formed by taking a cycle C_n with vertices $v_1, v_2, \ldots, v_n$ and attaching to each v_i a copy of C_3 with vertices u_{i1}, u_{i2}, u_{i3}, where each u_{ij} is connected to v_i , as show in Figure 8.

We construct a set R consisting of:

$$R = \{v_1, v_2, \ldots, v_n\} \cup \{u_{i1}, u_{i2} \mid 1 \leq i \leq n\}$$

Thus, $|R| = 3n$.

Resolving Property:

- Each v_i is in R, so all cycle vertices are resolved.
- Including two vertices from each C_3 copy ensures that all three vertices in the triangle are distinguishable.
- Vertices from different C_3 copies are resolved by their distances to the corresponding v_i, u_{i1}, and u_{i2}.

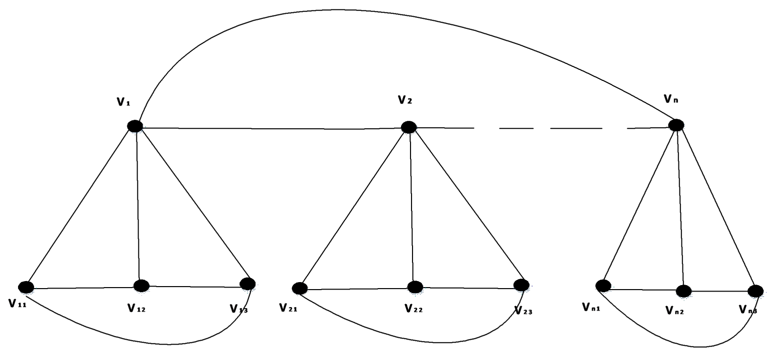

Figure 8. The $C_n \bigodot C_3$ graph.

$$\boldsymbol{w}\left(\boldsymbol{v_i}, \boldsymbol{R}\right) = \begin{cases} \{0,1,2,\ldots,n-1,1,1,2,2,3,3,\ldots,n,n\} & ,i=1 \\ \{1,0,1,\ldots,n-2,2,2,1,1,2,2\ldots,n-1,n-1\} & ,i=2 \\ \{2,1,0,1,\ldots,n-3,3,3,2,2,1,1,\ldots,n-2,n-2\} & ,i=3 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \{n-1,n-2,n-3,\ldots,0,n,n.n-1,n-1,\ldots,1,1\} & .i=n \end{cases}$$

$$\boldsymbol{w}\left(\boldsymbol{u_{ij}}, \boldsymbol{R}\right) = \begin{cases} \{1,2,3,\ldots,n,0,1,\ldots,n,n,n+1,n+1\} & ,i=1,j=1 \\ \{1,2,3,\ldots,n,1,0,\ldots,n,n,n+1,n+1\} & ,i=1,j=2 \\ \{1,2,3,\ldots,n,1,1,\ldots,n,n,n+1,n+1\} & ,i=1,j=3 \\ \{2,1,2,3\ldots,n-1,3,3,j-1,j,3,3,4,4,\ldots,n,n\} & ,i=2,j=1 \\ \{2,1,2,3\ldots,n-1,3,3,j-1,j-2,3,3,4,4,\ldots,n,n\} & ,i=2,j=2 \\ \{2,1,2,3\ldots,n-1,3,3,j-2,j-2,3,3,4,4,\ldots,n,n\} & ,i=2,j=3 \\ \vdots \\ \vdots \\ \vdots \\ \{n,n-1,\ldots,1,n+1,n+1,n,n,n-1,n-1,\ldots,j-1,j\} & ,i=n\ ,j=1 \\ \{n,n-1,\ldots,1,n+1,n+1,n,n,n-1,n-1,\ldots,j-1,j-2\} & ,i=n\ ,j=2 \\ \{n,n-1,\ldots,1,n+1,n+1,n,n,n-1,n-1,\ldots,j-2,j-2\} & ,i=n\ ,j=3 \end{cases}$$

Connectivity: The subgraph induced by R includes the cycle C_n and two vertices from each triangle connected to v_i. Since each u_{i1}, u_{i2} is adjacent to v_i, the induced subgraph is connected.

Minimality: Suppose a smaller connected resolving set R' exists with $|R'| < 3n$. Then at least one v_i, u_{i1}, or u_{i2} is missing. Omitting any of these may cause failure to resolve the triangle or disconnect the induced subgraph. Hence, R is minimal.

Therefore, $\text{cdim}(C_n \odot C_3) = 3n$.

In the following theorems, we study and investigate the metric dimension of $P_n \bigodot P_3$, $C_n \bigodot P_3$, $P_n \bigodot C_3$, $C_n \bigodot C_3$ and $Ki_{m,n}$.

Theorem 5: Let G be $P_n \bigodot P_3$ graphs, then $M.D\left(P_n \bigodot P_3\right) = n$.

Proof. Label the vertices of P_n as $v_1, v_2, \ldots, v_n$. For each v_i, attach a copy of P_3 with vertices u_{i1}, u_{i2}, u_{i3}, where each u_{ij} is adjacent to v_i, as show in Figure 9.

Let $R = \{u_{i1} \mid 1 \leq i \leq n\}$. We claim that R is a resolving set.

Each v_i is uniquely identified by its distance to u_{i1} (which is 1), and to all other u_{j1} (which are at least 2). Similarly, within each P_3 copy, the vertices u_{i2} and u_{i3} are distinguishable by their distances to u_{i1} (which are 2), while u_{i1} is in R.

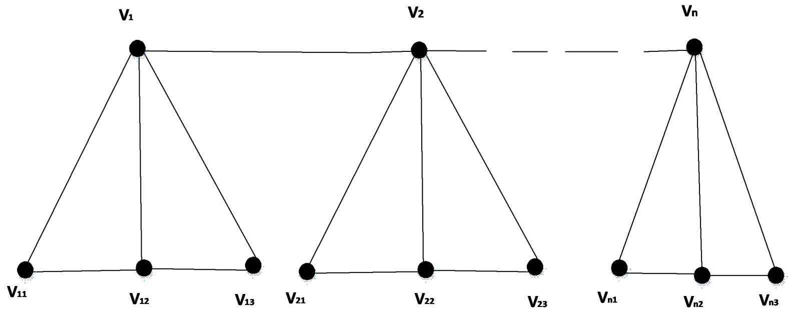

Figure 9. $P_n \bigodot P_3$ graph.

$$\boldsymbol{w}\left(\boldsymbol{v_i}, \boldsymbol{R}\right) = \begin{cases} \{1, 2, 3, \ldots, n\} & , i = 1 \\ \{2, 1, 2, \ldots, n-1\} & , i = 2 \\ \{3, 2, 1, 2, \ldots, n-2\} & , i = 3 \\ \vdots \\ \vdots \\ \vdots \\ \{n-2, \ldots, 2, 1, 2, 3\} & , i = n-2 \\ \{n-1, \ldots, 2, 1, 2, 3\} & , i = n-1 \\ \{n, \ldots, 3, 2, 1\} & . i = n \end{cases}$$

$$\boldsymbol{w}\left(\boldsymbol{u_{ij}}, \boldsymbol{R}\right) = \begin{cases} \{j-1, 3, 4, \ldots, n, n+1\} & , i = 1 \\ \{3, j-1, 3, 4, \ldots, n-1, n\} & , i = 2 \\ \{4, 3, j-1, 3, 4, \ldots, n-2, n-1\} & , i = 3 \\ \vdots \\ \vdots \\ \vdots \\ \{n-1, n-2, 4, 3, j-1, 3, 4\} & , i = n-2 \\ \{n, n-1, \ldots, 4, 3, j-1, 3\} & , i = n-1 \\ \{n+1, n, \ldots, 4, 3, j-1\} & . i = n \end{cases}$$

Thus, all vertices are uniquely identified by their distance vectors to R, and $|R| = n$.

To show minimality, note that if any u_{i1} is removed from R, then the vertices in the i-th P_3 copy are no longer distinguishable. Hence, $\dim(G) = n$.

Theorem 6: Let G be $C_n \bigodot P_3$ graph, then $M.D\left(C_n \bigodot P_3\right) = n,\ n \geq 3$.

Proof. Label the cycle C_n as $v_1, v_2, \ldots, v_n$ in cyclic order. For each v_i, attach a copy of P_3 with vertices u_{i1}, u_{i2}, u_{i3}, each adjacent to v_i, as show in Figure 10.

Let $R = \{u_{i1} \mid 1 \leq i \leq n\}$. We claim that R is a resolving set.

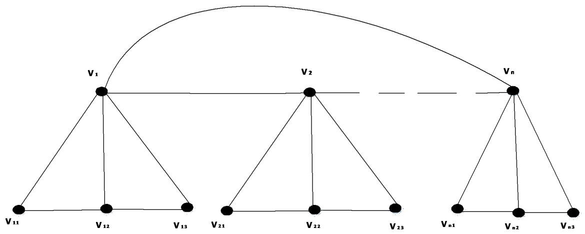

Figure 10. The $C_n \bigodot P_3$ graph.

$$\boldsymbol{w}\left(\boldsymbol{v_i},\boldsymbol{R}\right)=\begin{cases}\{1,2,\ldots,2\} & ,i=1\\ \{2,1,2,\ldots,2\} & ,i=2\\ \{3,2,1,\ldots,3\} & ,i=3\\ \vdots \\ \vdots \\ \vdots \\ \{2,3,n-2,\ldots,2,1\} & .i=n\end{cases}$$

$$\boldsymbol{w}\left(\boldsymbol{u_{ij}},\boldsymbol{R}\right)=\begin{cases}\{j-1,3,4,\ldots,3\} & ,i=1\\ \{3,j-1,3,\ldots,n-2,n-1,4\} & ,i=2\\ \{4,3,j-1,3,4,\ldots,n-2,n-1\} & ,i=3\\ \vdots \\ \vdots \\ \vdots \\ \{3,4,\ldots,4,3,j-1\} & .i=n\end{cases}$$

Each v_i is uniquely identified by its distance to u_{i1} (which is 1), and to other u_{j1} (which vary due to the cycle structure). Each u_{ij} in the i-th P_3 copy is distinguishable by its distance to u_{i1}.

If any u_{i1} is removed, then the corresponding P_3 copy cannot be resolved. Hence, R is minimal and $\dim(G) = n$.

Theorem 7: Let G be $P_n \bigodot C_3$ graph, then $M.D\left(P_n \bigodot C_3\right) = 2n,\ n \geq 1$.

Proof. Label the path P_n as $v_1, v_2, \ldots, v_n$. For each v_i, attach a copy of C_3 with vertices u_{i1}, u_{i2}, u_{i3} forming a triangle, and connect each to v_i, as show in Figure 11.

Let $R = \{u_{i1}, u_{i2} \mid 1 \leq i \leq n\}$. We claim that R is a resolving set.

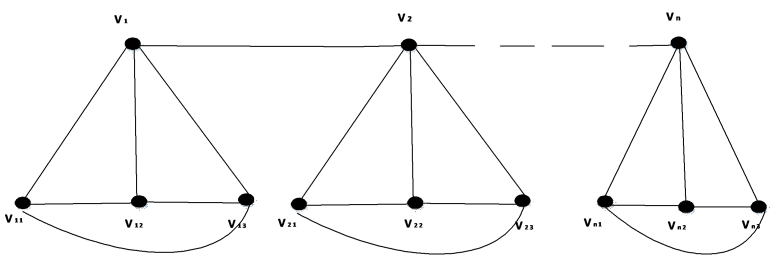

Figure 11. The $P_n \bigodot C_3$ graph.

$$\boldsymbol{w}\left(\boldsymbol{v_i},\boldsymbol{R}\right)=\begin{cases}\{1,1,2,2,3,3,\ldots,n,n\} & ,i=1\\ \{2,2,1,1,2,2\ldots,n-1,n-1\} & ,i=2\\ \{3,3,2,2,1,1,\ldots,n-2,n-2\} & ,i=3\\ \vdots\\ \vdots\\ \vdots\\ \{n,n.n-1,n-1,\ldots,1,1\} & .i=n\end{cases}$$

$$\boldsymbol{w}\left(\boldsymbol{u_{ij}},\boldsymbol{R}\right)=\begin{cases}\{j-1,j,3,3,4,4,\ldots,n,n,n+1,n+1\} & ,i=1,j=1\\ \{j-1,j-2,3,3,4,4,\ldots,n,n,n+1,n+1\} & ,i=1,j=2\\ \{j-2,j-2,3,3,4,4,\ldots,n,n,n+1,n+1\} & ,i=1,j=3\\ \{3,3,j-1,j,3,3,4,4,\ldots,n,n\} & ,i=2,j=1\\ \{3,3,j-1,j-2,3,3,4,4,\ldots,n,n\} & ,i=2,j=2\\ \{3,3,j-2,j-2,3,3,4,4,\ldots,n,n\} & ,i=2,j=3\\ \vdots\\ \vdots\\ \vdots\\ \{n+1,n+1,n,n,n-1,n-1,j-1,j\} & ,i=n\ ,j=1\\ \{n+1,n+1,n,n,n-1,n-1,j-1,j-2\} & ,i=n\ ,j=2\\ \{n+1,n+1,n,n,n-1,n-1,j-2,j-2\} & ,i=n\ ,j=3\end{cases}$$

Each v_i is uniquely identified by its distances to u_{i1} and u_{i2} (both 1), and to other u_{jk} (which are at least 2). Each u_{i3} is distinguishable from u_{i1} and u_{i2} by its distances to them (1 or 2).

Removing any u_{i1} or u_{i2} would make the i-th C_3 copy unresolved. Hence, R is minimal and $\dim(G)=2n$.

Theorem 8: Let G be $C_n\bigodot C_3$ graph, then $M.D\left(C_n\bigodot C_3\right)=2n$, $n\geq 3$.

Proof. Label the cycle C_n as $v_1,v_2,\ldots,v_n$. For each v_i, attach a copy of C_3 with vertices u_{i1},u_{i2},u_{i3} forming a triangle, and connect each to v_i, as show in Figure 12.

Let $R=\{u_{i1},u_{i2}\mid 1\leq i\leq n\}$. We claim that R is a resolving set.

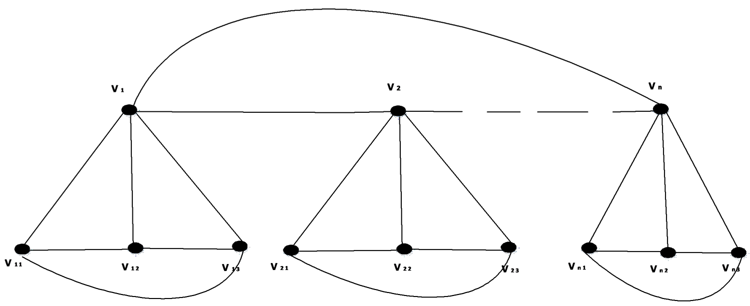

Figure 12. The $C_n\bigodot C_3$ graph.

$$\boldsymbol{w}\left(\boldsymbol{v_i},\boldsymbol{R}\right)=\begin{cases}\{1,1,2,2,3,3,\ldots,n,n\} & ,i=1\\ \{2,2,1,1,2,2\ldots,n-1,n-1\} & ,i=2\\ \{3,3,2,2,1,1,\ldots,n-2,n-2\} & ,i=3\\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \{n,n,n-1,n-1,\ldots,1,1\} & .i=n\end{cases}$$

$$\boldsymbol{w}\left(\boldsymbol{u_{ij}},\boldsymbol{R}\right)=\begin{cases}\{0,1,\ldots,n,n,n+1,n+1\} & ,i=1,j=1\\ \{1,0,\ldots,n,n,n+1,n+1\} & ,i=1,j=2\\ \{1,1,\ldots,n,n,n+1,n+1\} & ,i=1,j=3\\ \{3,3,j-1,j,3,3,4,4,\ldots,n,n\} & ,i=2,j=1\\ \{3,3,j-1,j-2,3,3,4,4,\ldots,n,n\} & ,i=2,j=2\\ \{3,3,j-2,j-2,3,3,4,4,\ldots,n,n\} & ,i=2,j=3\\ \vdots \\ \vdots \\ \vdots \\ \{n+1,n+1,n,n,n-1,n-1,\ldots,j-1,j\} & ,i=n\ ,j=1\\ \{n+1,n+1,n,n,n-1,n-1,\ldots,j-1,j-2\} & ,i=n\ ,j=2\\ \{n+1,n+1,n,n,n-1,n-1,\ldots,j-2,j-2\} & ,i=n\ ,j=3\end{cases}$$

Each v_i is uniquely identified by its distances to u_{i1} and u_{i2} (both 1), and to other u_{jk} (which vary due to the cycle). Each u_{i3} is distinguishable from u_{i1} and u_{i2} by its distances to them.

Removing any u_{i1} or u_{i2} would make the i-th C_3 copy unresolved. Hence, R is minimal and $\dim(G) = 2n$.

Theorem 9: Let G be $Ki_{m,n}$ graph, then $M.D.\left(Ki_{m,n}\right) = 2$, if m is odd.

Proof. We labelling $Ki_{m,n}$, as show in Figure 13. It is clear that the number of vertices is $n+m$. Let $\boldsymbol{R}=\{\boldsymbol{v_1},\boldsymbol{v_{n+2}}\}$ be any resolving set of $Ki_{m,n}$.

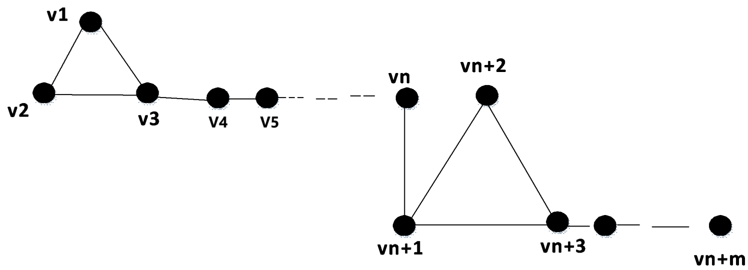

Figure 13. The $Ki_{m,n}$ graph.

Being
For ($i = 1; i \leq 2; i = i + 1$) **do**

$$r(v_i, R) = \{i - 1, n\}$$

End
For ($i = 3; i \leq n + m; i = i + 1$) **do**

$$r(v_i, R) = \{i - 2, |n + 2 - i|\}$$

End

We aim to show that the metric dimension of G is 2. That is, there exists a resolving set $R \subseteq V(G)$ with $|R| = 2$ such that for every pair of distinct vertices $x, y \in V(G)$, the distance vectors $r(x|R)$ and $r(y|R)$ are distinct.

Let us choose $R = \{u_1, v_1\}$, where $u_1 \in U$ and $v_1 \in V$. We will show that R resolves all vertices in G.

Case 1: $x, y \in U$, $x \neq y$.

Then $d(x, u_1) = 0$ if $x = u_1$, and 2 otherwise (since x and u_1 are in the same partite set and not adjacent, but both are adjacent to all vertices in V). Also, $d(x, v_1) = 1$ for all $x \in U$ since each x is adjacent to v_1. Thus, the distance vectors differ at the first coordinate if $x = u_1$, and otherwise they differ due to symmetry and the odd cardinality of U.

Case 2: $x, y \in V$, $x \neq y$.

Similarly, $d(x, v_1) = 0$ if $x = v_1$, and 2 otherwise. Also, $d(x, u_1) = 1$ for all $x \in V$. Hence, the distance vectors differ at the second coordinate if $x = v_1$, and otherwise they are distinguishable.

Case 3: $x \in U$, $y \in V$.

Then $d(x, u_1) \in \{0, 2\}$ and $d(y, u_1) = 1$, so the first coordinates differ. Hence, $r(x|R) \neq r(y|R)$.

Therefore, $R = \{u_1, v_1\}$ is a resolving set, and $\dim(G) \leq 2$.

To prove minimality, suppose there exists a resolving set R' with $|R'| = 1$. Then all vertices in one partite set are equidistant from the single vertex in R', and hence cannot be distinguished. Therefore, no single vertex can resolve G, and $\dim(G) \geq 2$.

Thus, $\dim(G) = 2$.

3. Algorithm for Connected Metric Dimension

In this section, we propose an algorithm that determines a connected Metric Dimension for an arbitrary graph G. this algorithm provides a basic framework for computing the connected metric dimension. For more complex graphs, heuristic or metanephritic approaches like Genetic Algorithms or Binary Equilibrium Optimization Algorithm can be used to find optimal or near-optimal solutions efficiently.

Algorithm: Connected Metric Dimension

Input: A connected graph $G = (V, E)$
Output: A connected resolving set (S) with the smallest possible size
Initialization: Let (S) be an empty set. Let (U) be the set of all vertices in (V). Step 1: **Select Initial Vertex**: Choose an arbitrary vertex $(v \in V)$ and add it to (S). Remove (v) from (U). Step 2: **Iterative Selection**: **While** (U) is not empty: **For** each vertex $(u \in U)$: **Calculate** the distance from u to all vertices in S. Step 3: **Select** the vertex $(u \in U)$ that maximizes the number of vertices in U that can be uniquely identified by their distances to the vertices in (S). **Add** (u) to (S). **Remove** (u) from (U). Step 4: **Ensure Connectivity**: **Check if** the sub graph induced by (S) is connected. **If not**, add the necessary vertices from $(V \setminus S)$ to (S) to make it connected. Step 5: **Optimization**: Attempt to remove any redundant vertices from (S) while maintaining its properties as a connected resolving set. Step 6: **Output**: **Return** the set (S).

Example: Path with 4 Vertices and One Triangle

Let G be a path P_4 with vertices $v_1 - v_2 - v_3 - v_4$ and a triangle u_1, u_2, u_3 attached to v_2.

- Step 1: Choose v_2 as initial vertex $\Rightarrow S = \{v_2\}$
- Step 2: Add v_1 (distinguishes v_1 and v_3) $\Rightarrow S = \{v_2, v_1\}$
- Step 3: Add u_1 (distinguishes triangle vertices) $\Rightarrow S = \{v_2, v_1, u_1\}$
- Step 4: Ensure connectivity: v_2 connects v_1 and u_1 $\Rightarrow$ connected
- Step 5: Check redundancy: all vertices resolved, S is minimal

Correctness

The algorithm ensures that:

- Each vertex is uniquely identified by its distance vector to S
- The induced subgraph $G[S]$ is connected

Completeness

The algorithm terminates when all vertices are resolved and S is connected. It guarantees a valid connected resolving set.

Time Complexity

Let $n = |V|$, $m = |E|$:

- Distance computation: $O(n^2)$ using BFS for each vertex
- Selection loop: $O(n^2)$ comparisons
- Connectivity check: $O(n + m)$
- Redundancy check: $O(n^2)$

Total Time Complexity: $O(n^2)$ for sparse graphs

4. CONCLUSION

We introduced metric and connected metric dimension of $P_n \bigodot C_3$, $P_n \bigodot P_3, C_n \bigodot C_3$and $C_n \bigodot P_3$.

Graph	Metric Dimension (dim)	Connected Metric Dimension (cdim)
$P_n \odot P_3$	n	$2n$
$P_n \odot C_3$	$2n$	$3n$
$C_n \odot P_3$	n	$2n$
$C_n \odot C_3$	$2n$	$3n$
$Ki_{m,n}$ (m odd)	2	–

Table 1: Summary of Metric and Connected Metric Dimensions for Various Graphs

In the future, we will apply all of this proofs for another graphs. The third contribution is that proposing a new approximate algorithm which finds a minimum connected metric dimension

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Data Availability Statement

All data generated or analyzed during this study are included in this published article.

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