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The Existence of Solutions for Second-Order Differential Inclusions under Almost Fisher-Type Multivalued F-Contractions in Metric Spaces

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Abstract. This paper presents novel fixed point (FP) theorems for a specific class of multivalued contractions, referred to as "Almost Fisher-type multivalued F-contractions" within complete metric spaces (MSs) endowed with a Γ-transitive binary relation $\Re$. These theorems establish the existence of FPs for such contractions and explore their intrinsic properties. Illustrative examples are provided to demonstrate the applicability and effectiveness of the proposed results. An application to second-order differential inclusions (SODIs) is given under these contractions.

2020 Mathematics Subject Classifications: 47H10, 47H09

Key Words and Phrases: Fixed point, metric space, almost Fisher-type contraction, binary relation, second-order differential inclusion

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1. Introduction and Preliminaries

In recent years, fixed point (FP) theory has undergone substantial development with various generalizations and refinements of the classical Banach contraction principle (BCP). In 1977, Jaggi [1] generalized the BCP in complete MSs with the condition: $\forall \varsigma_1, \varsigma_2 \in \Delta_{\Re}, \exists \lambda_1, \lambda_2 \in [0, \infty)$ with $\lambda_1 + \lambda_2 < 1$ such that $d(\Gamma\varsigma_1, \Gamma\varsigma_2) \leq \lambda_1 d(\varsigma_1, \varsigma_2) + \lambda_2 \frac{d(\varsigma_1, \Gamma\varsigma_1).d(\varsigma_2, \Gamma\varsigma_2)}{d(\varsigma_1, \varsigma_2)}$, where the map $\Gamma : \Delta_{\Re} \to \Delta_{\Re}$ has a unique fixed point (UFP) $\varsigma^* \in \Delta_{\Re}$. Karapinar [2] further enriched FP theory by introducing interpolative-type contractions, establishing connections with interpolation theory, as explored in related works, which was also discussed in [3, 4]. Karapinar and Fulga [5] introduced a hybrid contraction by combining Jaggi-type and interpolative-type contractions. In 2012, Wardowski [6] introduced the concept of F-contraction as an extension of the BCP. Subsequent works in [7, 8] applied

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these ideas to combine the results of Wardowski's cyclic contraction operators and admissible mappings of Geraghty F-contraction, producing new FP theorems. Ali et al. [9] discussed new FP results of dynamic process of integral Ciric-type F-contractions set-valued mappings in MSs. Nadler [10] earlier initiated the study of FPs for multivalued mappings. Notably, recent articles in the field of FP theory for multivalued mappings have been published, providing valuable assistance to researchers. Acar and Altun [11] extended multivalued F-contractions with δ-Distance and established FP results in complete MSs, (see [12, 13]). In 1980, Fisher [14] explored new results that generalized the BCP by employing a new rational inequality, $\forall\varsigma_1,\varsigma_2 \in \Delta_{\Re}, \exists\lambda_1,\lambda_2 \in [0,\infty)$ such that $d(\Gamma\varsigma_1,\Gamma\varsigma_2) \leq \lambda_1 d(\varsigma_1,\varsigma_2) + \lambda_2\frac{d(\varsigma_1,\Gamma\varsigma_1).d(\varsigma_2,\Gamma\varsigma_2)}{1+d(\varsigma_1,\varsigma_2)}$, broadening fixed point theory and inspiring further research on generalized contractive mappings. Consequently, Fisher-type F-contractions and Jaggi-type F-contractions are important generalizations of classical BCs within FP theory, particularly for multivalued mappings. By incorporating functional inequalities through auxiliary functions (F-functions), they provide greater flexibility in defining contraction conditions, making them applicable to a broader class of problems. These contractions have proven effective in establishing the existence of solutions to nonlinear integral equations (IEs), differential inclusions, and equilibrium problems. They play a vital role in optimization theory, control systems, fuzzy dynamics, game theory, and fractal-based image compression. Moreover, they are instrumental in best approximation problems in Banach spaces and in the study of non-expansive multivalued mappings in geodesic spaces. Extending FP theory to multivalued settings through Jaggi-type contractions equips researchers with powerful tools for analyzing complex systems beyond the scope of single-valued mappings. The development of new FP theorems with sophisticated contractive conditions on different spaces is essential for filling gaps in the existing literature. By introducing new FP theorems based on this contractive approach, our study intends to address the gaps in the literature and provide further insights into the theory of FPs for multivalued mappings. This research has the potential to enhance our understanding of the subject and pave the way for future developments in the field. This approach likely incorporates the almost Fisher-type multivalued F-contractions discussed earlier, within the framework of complete MSs endowed with a Γ-transitive binary relation $\Re$. Alam and Imdad [15] introduced the relation-theoretic contraction principle on MS endowed with an arbitrary binary relation. Recently, Tomar and Joshi [16] discussed the relation-theoretic contractions in F-MSs to demonstrate the existence of FP and solved a two-point boundary value problem arising in a hanging cable problem. Alam and Imdad [17, 18] generalized some metrical notions related to relation-theoretic setting and locally T-transitive binary relation with utilizing these notions to prove coincidence points and FP theorems for self-mappings on an MS. Authors in [19–21] extended the above results for set-valued mappings in MSs. Negi and Gairola [22] introduced the notion of generalized multivalued (ψ-$F_{\Re}$)-contraction in PMS endowed with an arbitrary binary relation and established a new FP theorem, see [23, 24]. In line with these advancements, we touch on some basics and concepts that are far famed in the literature:

Definition 1. *[6] Let $(\Delta_{\Re}, d)$ be a MS. A map $\Gamma : \Delta_{\Re} \to \Delta_{\Re}$ is said to be an F-contraction if there exists $\tau \in \mathbb{R}^+$ such that $\forall \varsigma_1, \varsigma_2 \in \Delta_{\Re}$,*

$$d(\Gamma\varsigma_1, \Gamma\varsigma_2) > 0 \Rightarrow \tau + F(\Gamma\varsigma_1, \Gamma\varsigma_2) \leq F(d(\varsigma_1, \varsigma_2)),$$

where Δ_w is the family of functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the assumptions below:

(F_1) *F is strictly increasing, i.e., $\forall \varsigma_1, \varsigma_2 \in (0, \infty)$, so that $\varsigma_1 < \varsigma_2$, then $F(\varsigma_1) < F(\varsigma_2)$;*

(F_2) *for $\{\varsigma_\jmath\}_{\jmath=1}^{\infty} \subseteq \mathbb{R}^+$, $\lim_{\jmath\to\infty} \varsigma_\jmath = 0 \Leftrightarrow \lim_{\jmath\to\infty} F(\varsigma_\jmath) = -\infty$;*

(F_3) *$\exists\ \ell \in (0, 1)$, so that $\lim_{\jmath\to\infty} \varsigma^{\ell} F(\varsigma_\jmath) = 0$.*

Theorem 1. *[6] Let $(\Delta_{\Re}, d)$ be a complete MS and $\Gamma : \Delta_{\Re} \to \Delta_{\Re}$ be an F-contraction map. If $\exists\ F \in \Delta_w$ and $\tau \in (0, \infty)$. Thus, Γ has a UFP.*

Remark 1. *[3] If F is right continuous and satisfies (F_2), then*

$$F(\inf A) = \inf F(A), \forall A \subset (0, \infty)\, with \inf(A) > 0.$$

Now, we introduce some primary debates and terminology about MS that are well known in the literature.

Definition 2. *[6] Let $(\Delta_{\Re}, d)$ be a MS. Then, we have*

(1) *A sequence $\{\varsigma_n\}$ converges to a point ς iff*

$$\lim_{n\to\infty} d(\varsigma_n, \varsigma) = 0.$$

(2) *A sequence $\{\varsigma_n\}$ in $\Delta_{\Re}$ is said to be a Cauchy sequence iff*

$$\lim_{n,m\to\infty} d(\varsigma_n, \varsigma_m) = 0.$$

(3) *An MS is said to be complete if every Cauchy sequence $\{\varsigma_n\}$ converges to a point ς such that*

$$\lim_{n\to\infty} d(\varsigma_n, \varsigma) = 0.$$

If $(\varsigma_1, \varsigma_2) \in \Re$, then it is said that ς_1 is related to ς_2. Here, we take $\Re$ as a binary relation on a nonempty subset $\Delta_{\Re}$ and $(\Delta_{\Re}, d)$ is a MS equipped with a binary relation $\Re$.

Definition 3. *A binary relation $\Re$ on $\Delta_{\Re} \neq 0$ is a subset of $\Delta_{\Re} \times \Delta_{\Re}$, we say that ς_1 is related to ς_2 (i.e.$\varsigma_1 \Re \varsigma_2$) if and only if $(\varsigma_1, \varsigma_2) \in \Re$.*

Definition 4. *[15] Let $\Gamma : \Delta_{\Re} \to \Delta_{\Re}$. Then, $\Re$ on $\Delta_{\Re}$ is designated as Γ-closed, if*

$$(\varsigma_1, \varsigma_2) \in \Re \Rightarrow (\Gamma\varsigma_1, \Gamma\varsigma_2) \in \Re, \varsigma_1, \varsigma_2 \in \Delta_{\Re}.$$

Definition 5. *[15] Let* $\Re$ *be a binary relation on* $\Delta_{\Re}$*. Then, a sequence* $\{\varsigma_n\} \subset \Delta_{\Re}$ *is designated as* $\Re$*-preserving if* $\forall n \in \mathbb{N}$*, then* $(\varsigma_n, \varsigma_{n+1}) \in \Re$.

Definition 6. *[18] Let* $\Gamma : \Delta_{\Re} \to \Delta_{\Re}$*. Then,* $\Re$ *on* $\Delta_{\Re}$ *is designated as* Γ*-transitive, if for any* $\varsigma_1, \varsigma_2, \varsigma_3 \in \Delta_{\Re}$,

$$(\Gamma\varsigma_1, \Gamma\varsigma_2), (\Gamma\varsigma_2, \Gamma\varsigma_3) \in \Re \Rightarrow (\Gamma\varsigma_1, \Gamma\varsigma_3) \in \Re.$$

Definition 7. *[25] An MS* $(\Delta_{\Re}, d)$ *is* $\Re$*-regular if for every* $\{\varsigma_n\}_{n\in\mathbb{N}} \subset \Delta_{\Re}$,

$$(\varsigma_n, \varsigma_{n+1}) \in \Re, \varsigma_n \to \varsigma \in \Delta_{\Re} \Rightarrow (\varsigma_n, \varsigma) \in \Re.$$

Definition 8. *[17] Let* $\Re$ *be a binary relation in an MS* $(\Delta_{\Re}, d)$ *and* $\varsigma^* \in \Delta_{\Re}$*. A map* $\Gamma : \Delta_{\Re} \to \Delta_{\Re}$ *is* $\Re$*-continuous at* ς^* *if for any* $\Re$*-preserving sequence* $\{\varsigma_n\} \to^d \varsigma^*$*, we have* $\Gamma\varsigma_n \to^d \Gamma\varsigma^*$. Γ *is* $\Re$*-continuous if it is* $\Re$*-continuous at each point of* $\Delta_{\Re}$.

Definition 9. *[22] Let* $(\Delta_{\Re}, d)$ *be a partial MS with a binary relation* $\Re$ *and* Γ *a multivalued-maps on* $\Delta_{\Re}$*. Then,*

(1) $\Re$ *is designated as* Γ*-closed, if for any* $\varsigma_1, \varsigma_2 \in \Delta_{\Re}$,

$$(\varsigma_1, \varsigma_2) \in \Re \Rightarrow (a, b) \in \Re$$

for some $a \in \Gamma\varsigma_1$ *and* $b \in \Gamma\varsigma_2$.

(2) Γ *is called* $\Re$*-continuous at* $\varsigma^* \in \Delta_{\Re}$*, if for any* $\Re$*-preserving sequence* $\{\varsigma_n\} \subset$ *with* $\{\varsigma_n\} \to^d \varsigma^*$*, we have* $\Gamma\varsigma_n \to^{H_d} \Gamma\varsigma^*$*, we say that* Γ *is* $\Re$*-continuous if it is* $\Re$*-continuous at each point of* $\Delta_{\Re}$.

(3) $\Re$ *is designated as* Γ*-transitive, if for any* $\varsigma_1, \varsigma_2, \varsigma_3 \in \Delta_{\Re}$*,* $a \in \Gamma\varsigma_1, b \in \Gamma\varsigma_2, c \in \Gamma\varsigma_3$*, we have*

$$(a, b) \in \Re, (b, c) \in \Re \Rightarrow (a, c) \in \Re.$$

Definition 10. *[19] Let* $\Delta_{\Re} \neq \emptyset$ *and* $\Gamma : \Delta_{\Re} \to CP(\Delta_{\Re})$*. A binary relation* $\Re$ *on an MS* $\Delta_{\Re}$ *is designated as* Γ*-transitive, if for any* $\varsigma_1, \varsigma_2, \varsigma_3 \in \Delta_{\Re}$*,* $a \in \Gamma\varsigma_1, b \in \Gamma\varsigma_2, c \in \Gamma\varsigma_3$*, we have*

$$(a, b) \in \Re, (b, c) \in \Re \Rightarrow (a, c) \in \Re.$$

Definition 11. *[26] Let* $(\Delta_{\Re}, d)$ *be a MS with a binary relation* $\Re$ *and* $\Gamma : \Delta_{\Re} \to CP(\Delta_{\Re})$*. Then,* $\Re$ *is called* Γ*-d-closed, if*

$$(\varsigma_1, \varsigma_2) \in \Re, r \in \Gamma\varsigma_1, s \in \Gamma\varsigma_2, d(r, s) \leq d(\varsigma_1, \varsigma_2) \Rightarrow (r, s) \in \Re.$$

Let $(\Delta_{\Re}, d)$ be a MS. we shall denote $CB(\Delta_{\Re})$ the family of all bounded and closed subsets of $\Delta_{\Re}$ and $K(\Delta_{\Re})$ the class of all compact subsets of $\Delta_{\Re}$.

Definition 12. *[10] Let $H : CB(\Delta_{\Re}) \times CB(\Delta_{\Re}) \to [0,\infty)$ be the Pompeiu-Hausdorff metric induced by d so that*

$$H(A,B) = \max\{\sup_{\varsigma_1 \in A} D(\varsigma_1, B), \sup_{\varsigma_2 \in B} D(A, \varsigma_2)\},$$

where $\forall \varsigma_1 \in \Delta_{\Re}$ and $A, B \in CB(\Delta_{\Re})$

$$D(\varsigma_1, B) = \inf_{\varsigma_2 \in B} d(\varsigma_1, \varsigma_2).$$

In addition, $CB(\Delta_{\Re}), H$ is known as a Pompeiu-Hausdorff MS. Here, we say that an element $a \in \Delta_{\Re}$ is an FP of a multivalued map $\Gamma : \Delta_{\Re} \to CB(\Delta_{\Re})$ if $a \in \Gamma a$.

Lemma 1. *[10] If $A, B \in CB(\Delta_{\Re})$, then $D(\varsigma, B) \leq H(A,B)$ for every $\varsigma \in A$.*

Lemma 2. *[10] If $A, B \in CB(\Delta_{\Re})$. For $\mu > 0, a \in A$ there is $e \in B$ such that $d(a,e) \leq H(A,B) + \mu$.*

Definition 13. *[1] Let $(\Delta_{\Re}, d)$ be an MS and $\Gamma : \Delta_{\Re} \to \Delta_{\Re}$ be a self-map, then Γ is called a Jaggi contraction if there are $\lambda_1, \lambda_2 \in [0,\infty)$ with $\lambda_1 + \lambda_2 < 1$ such that $\forall \varsigma_1, \varsigma_2 \in \Delta_{\Re}$*

$$d(\Gamma\varsigma_1, \Gamma\varsigma_2) \leq \lambda_1 d(\varsigma_1, \varsigma_2) + \lambda_2 \frac{d(\varsigma_1, \Gamma\varsigma_1).d(\varsigma_2, \Gamma\varsigma_2)}{d(\varsigma_1, \varsigma_2)}.$$

Theorem 2. *[1] Let $(\Delta_{\Re}, d)$ be a complete MS and $\Gamma : \Delta_{\Re} \to \Delta_{\Re}$ be a Jaggi contraction map. Then, Γ has a UFP in $\Delta_{\Re}$.*

Definition 14. *[2] Let $(\Delta_{\Re}, d)$ be an MS. We say that the self-mapping $\Gamma : \Delta_{\Re} \to \Delta_{\Re}$ is an interpolative Kannan-type contraction, if there exist $\lambda \in [0,\infty)$ and $\alpha \in (0,1)$, such that $\forall \varsigma_1, \varsigma_2 \in \Delta_{\Re}$ with $\varsigma_1 \neq \Gamma\varsigma_1$*

$$d(\Gamma\varsigma_1, \Gamma\varsigma_2) \leq \lambda [d(\varsigma_1, \Gamma\varsigma_1)]^{\alpha} . [d(\varsigma_2, \Gamma\varsigma_2)]^{1-\alpha}.$$

Theorem 3. *[2] Let $(\Delta_{\Re}, d)$ be a complete MS and $\Gamma : \Delta_{\Re} \to \Delta_{\Re}$ be an interpolative Kannan-type contraction map. Then, Γ has a UFP in $\Delta_{\Re}$.*

Definition 15. *[5] A self-mapping Γ on an MS $(\Delta_{\Re}, d)$ is called a Jaggi-type hybrid contraction if there is $\psi \in \Psi$ so that*

$$d(\Gamma\varsigma_1, \Gamma\varsigma_2) \leq \psi(M_J(\varsigma_1, \varsigma_2)),$$

where for $s \geq 0$, $\varsigma_1, \varsigma_2 \in \Delta_{\Re}$ and $\lambda_i \geq 0$, $i = 1, 2, ...$ with $\lambda_1 + \lambda_2 = 1$ and

$$M_J(\varsigma_1, \varsigma_2) = \begin{cases} \left[\lambda_1 \left(\frac{d(\varsigma_1, \Gamma\varsigma_1) d(\varsigma_2, \Gamma\varsigma_2)}{d(\varsigma_1, \varsigma_2)}\right)^s + \lambda_2 (d(\varsigma_1, \varsigma_2))^s\right]^{\frac{1}{s}}, \text{for } s > 0, & \varsigma_1 \neq \varsigma_2, \\ (d(\varsigma_1, \Gamma\varsigma_1))^{\lambda_1} (d(\varsigma_2, \Gamma\varsigma_2))^{\lambda_2} \text{ for } s = 0, & \varsigma_1, \varsigma_2 \in \Delta_{\Re} \backslash F_{\Gamma}(\Delta_{\Re}) \end{cases},$$

where $F_{\Gamma}(\Delta_{\Re}) = \{\varsigma \in \Delta_{\Re} : \Gamma\varsigma = \varsigma\}$.

Theorem 4. *[5] Let $(\Delta_{\Re}, d)$ be a complete MS and $\Gamma : \Delta_{\Re} \to \Delta_{\Re}$ be a continuous Jaggi-type hybrid contraction. Then, Γ has an FP in $\Delta_{\Re}$. Moreover, for any $\varsigma_0 \in \Delta_{\Re}$, the sequence $\{\Gamma\varsigma_0\}$ converges to ς.*

Definition 16. *[27] A self-mapping Γ on a graphical b-MS $(\Delta_{\Re}, d)$ with $s \geq 1$ is called a Fisher-type graph contraction for H_G on $(\Delta_{\Re}, d)$ if H_G is graph preserving and if there exist nonnegative constants λ_1, λ_2 with $\lambda_1 + \lambda_2 < \frac{1}{s}$ so that for every $\varsigma_1, \varsigma_2 \in \Delta_{\Re}$ with $(\varsigma_1, \varsigma_2) \in E(H_G)$, we have*

$$d(\Gamma\varsigma_1, \Gamma\varsigma_2) \leq \lambda_1 d(\varsigma_1, \varsigma_2) + \lambda_2 \frac{d(\varsigma_1, \Gamma\varsigma_1)\, d(\varsigma_2, \Gamma\varsigma_2)}{1 + d(\varsigma_1, \varsigma_2)}.$$

2. Main Result

In this part, we study the existence of FPs for an almost Fisher-type multivalued F-contraction mappings endowed with a Γ-transitive binary relation $\Re$ into a MS.

Definition 17. *Let $(\Delta_{\Re}, d)$ be a MS. A map $\Gamma : \Delta_{\Re} \to CB(\Delta_{\Re})$ is called an almost Fisher-type multivalued F-contraction endowed with a Γ-transitive binary relation $\Re$, If there exist $\tau \in \mathbb{R}^+$, $F \in \Delta_w$ and $\lambda_1, \lambda_2, L \geq 0$ with $\lambda_1 + \lambda_2 \leq 1$ such that for all $(\varsigma_1, \varsigma_2) \in \Re^* = \{(\varsigma_1, \varsigma_2) \in \Re : \varsigma_1, \varsigma_2 \in \Delta_{\Re} \backslash Fix(\Gamma)\}$, we have*

$$\tau + F(H(\Gamma\varsigma_1, \Gamma\varsigma_2)) \leq F(M_{\Re}(\varsigma_1, \varsigma_2)) + LN_{\Re}(\varsigma_1, \varsigma_2), \tag{1}$$

where

$$M_{\Re}(\varsigma_1, \varsigma_2) = \begin{cases} \left[\lambda_1 \left(\frac{D(\varsigma_1, \Gamma\varsigma_1) D(\varsigma_2, \Gamma\varsigma_2)}{1 + D(\varsigma_1, \varsigma_2)}\right)^{\beta} + \lambda_2 (d(\varsigma_1, \varsigma_2))^{\beta}\right]^{\frac{1}{\beta}} & \text{if } \beta > 0; \\ (D(\varsigma_1, \Gamma\varsigma_1))^{\lambda_1} (D(\varsigma_2, \Gamma\varsigma_2))^{\lambda_2} & \text{if } \beta = 0, \end{cases} \tag{2}$$

and

$$N_{\Re}(\varsigma_1, \varsigma_2) = \min\{D(\varsigma_1, \Gamma\varsigma_1), D(\varsigma_2, \Gamma\varsigma_2), D(\varsigma_1, \Gamma\varsigma_2), D(\varsigma_2, \Gamma\varsigma_1)\}. \tag{3}$$

Theorem 5. *Let $(\Delta_{\Re}, d)$ be an $\Re$-complete MS, and let $\Gamma : \Delta_{\Re} \to CB(\Delta_{\Re})$ be an almost Fisher-type multivalued F-contraction mapping endowed with a Γ-transitive binary relation $\Re$. Assume that*

$(T1)$ $\Delta_{\Re}(\Gamma, \Re) = \{\varsigma \in \Delta_{\Re} : (\varsigma, \Gamma\varsigma) \in \Re\} \neq \emptyset$;
$(T2)$ $\Re$ is Γ-closed;
$(T3)$ Γ is $\Re$-continuous or $(\Delta_{\Re}, d)$ is $\Re$-regular space.
Then, Γ has a FP $\varsigma^* \in \Delta_{\Re}$.

Proof. From $(T1)$ $\exists\, \varsigma_0 \in \Delta_{\Re}$ such that $(\varsigma_0, \Gamma\varsigma_0) \in \Re$ and from $(T2)$ we have $\left(\Gamma\varsigma_0, \Gamma^2\varsigma_0\right) \in \Re$. Since $\Gamma\varsigma_0 \neq \emptyset$ and closed $\exists\, \varsigma_1 \in \Delta_{\Re}$ such that $\varsigma_1 \in \Gamma\varsigma_0 \subset \Delta_{\Re}$ such that $(\varsigma_1, \Gamma\varsigma_1) \in \Re$ and from $(T2)$, we have $\left(\Gamma^2\varsigma_0, \Gamma^3\varsigma_0\right) \in \Re$. Continuing in this way, we construct a sequence $\{\varsigma_n\}$ by $\varsigma_n \in \Gamma\varsigma_{n-1} = \Gamma^n\varsigma_0$ $\forall n \in \mathbb{N}_0$. If $\varsigma_n \in \Gamma\varsigma_n$ for some $n \in \mathbb{N}_0$ then ς_n becomes a FP of Γ

and the proof is done. So, we assume that $\varsigma_n \notin \Gamma\varsigma_n \ \forall\ n \in \mathbb{N}_0$ then $D(\varsigma_n, \Gamma\varsigma_n) > 0$ and by Lemma 1.15, we have

$$0 < D(\varsigma_n, \Gamma\varsigma_n) \leq H(\Gamma\varsigma_{n-1}, \Gamma\varsigma_n) \ \ \forall\ n \in \mathbb{N}_0 \tag{4}$$

Since $(\varsigma_0, \Gamma\varsigma_0) \in \Re$ and from $(T2)$, we conclude that $\left(\Gamma^n\varsigma_0, \Gamma^{n+1}\varsigma_0\right) \in \Re\ \forall n \in \mathbb{N}_0$. That is

$$(\varsigma_n, \varsigma_{n+1}) \in \Re\ \forall n \in \mathbb{N}_0. \tag{5}$$

Now, from (4) and (5), we have $(\varsigma_{n-1}, \varsigma_n) \in \Re^*\ \forall n \in \mathbb{N}_0$.Utilizing (F_1) in (4) and applying Remark 1.3 together with (1), we get

$$\begin{aligned} F\left(d(\varsigma_n, \varsigma_{n+1})\right) &= F\left(D(\varsigma_n, \Gamma\varsigma_n)\right) \leq F\left(H(\Gamma\varsigma_{n-1}, \Gamma\varsigma_n)\right) \\ &\leq F\left(M_{\Re}(\varsigma_{n-1}, \varsigma_n)\right) + LN_{\Re}(\varsigma_{n-1}, \varsigma_n) - \tau, \end{aligned} \tag{6}$$

where if $\beta > 0$, then

$$\begin{aligned} M_{\Re}(\varsigma_{n-1}, \varsigma_n) &= \left[\lambda_1 \left(\frac{D(\varsigma_{n-1}, \Gamma\varsigma_{n-1})\, D(\varsigma_n, \Gamma\varsigma_n)}{1 + d(\varsigma_{n-1}, \varsigma_n)}\right)^{\beta} + \lambda_2 \left(d(\varsigma_{n-1}, \varsigma_n)\right)^{\beta}\right]^{\frac{1}{\beta}} \\ &= \left[\lambda_1 \left(\frac{d(\varsigma_{n-1}, \varsigma_n)\, d(\varsigma_n, \varsigma_{n+1})}{1 + d(\varsigma_{n-1}, \varsigma_n)}\right)^{\beta} + \lambda_2 \left(d(\varsigma_{n-1}, \varsigma_n)\right)^{\beta}\right]^{\frac{1}{\beta}} \\ &\leq \left[\lambda_1 \left(\frac{d(\varsigma_{n-1}, \varsigma_n)\, d(\varsigma_n, \varsigma_{n+1})}{d(\varsigma_{n-1}, \varsigma_n)}\right)^{\beta} + \lambda_2 \left(d(\varsigma_{n-1}, \varsigma_n)\right)^{\beta}\right]^{\frac{1}{\beta}} \\ &= \left[\lambda_1 \left(d(\varsigma_n, \varsigma_{n+1})\right)^{\beta} + \lambda_2 \left(d(\varsigma_{n-1}, \varsigma_n)\right)^{\beta}\right]^{\frac{1}{\beta}}. \end{aligned}$$

Assume that $d(\varsigma_{n-1}, \varsigma_n) \leq d(\varsigma_n, \varsigma_{n+1})$, then we get

$$\begin{aligned} M_{\Re}(\varsigma_{n-1}, \varsigma_n) &\leq \left[\lambda_1 \left(d(\varsigma_n, \varsigma_{n+1})\right)^{\beta} + \lambda_2 \left(d(\varsigma_n, \varsigma_{n+1})\right)^{\beta}\right]^{\frac{1}{\beta}} \\ &= \left[(\lambda_1 + \lambda_2) \left(d(\varsigma_n, \varsigma_{n+1})\right)^{\beta}\right]^{\frac{1}{\beta}} \\ &< d(\varsigma_n, \varsigma_{n+1}) \end{aligned} \tag{7}$$

and

$$N_{\Re}(\varsigma_{n-1}, \varsigma_n) = \min\left\{d(x_{n-1}, x_n), d(\varsigma_n, \varsigma_{n+1}), d(\varsigma_{n-1}, \varsigma_{n+1}), d(\varsigma_n, \varsigma_n)\right\} = 0.$$

Then from (6), we have

$$\begin{aligned} F\left(d(\varsigma_n, \varsigma_{n+1})\right) &\leq F\left(d(\varsigma_n, \varsigma_{n+1})\right) - \tau \\ &< F\left(d(\varsigma_n, \varsigma_{n+1})\right). \end{aligned}$$

A contradiction. Therefore, $d(\varsigma_{n-1},\varsigma_n) > d(x_n,x_{n+1})$, which implies that

$$\begin{aligned} M_{\Re}(\varsigma_{n-1},\varsigma_n) &\leq \left[\lambda_1\left(d\left(\varsigma_{n-1},\varsigma_n\right)\right)^{\beta}+\lambda_2\left(d\left(\varsigma_{n-1},\varsigma_n\right)\right)^{\beta}\right]^{\frac{1}{\beta}} \\ &= \left[\left(\lambda_1+\lambda_2\right)\left(d\left(\varsigma_{n-1},\varsigma_n\right)\right)^{\beta}\right]^{\frac{1}{\beta}} \\ &< d\left(\varsigma_{n-1},\varsigma_n\right). \end{aligned}$$

Then from (6), we have

$$\begin{aligned} F\left(d(\varsigma_n,\varsigma_{n+1})\right) &\leq F\left(d\left(\varsigma_{n-1},\varsigma_n\right)\right)-\tau \\ &< F\left(d\left(\varsigma_{n-1},\varsigma_n\right)\right). \end{aligned} \tag{8}$$

Thus, the sequence $\{d(\varsigma_n,\varsigma_{n+1})\}$ is decreasing and convergent. By induction on n, we obtain

$$F\left(d(\varsigma_n,\varsigma_{n+1})\right) \leq F\left(d\left(\varsigma_{n-1},\varsigma_n\right)\right)-\tau < ... < F\left(d\left(\varsigma_0,\varsigma_1\right)\right)-n\tau.$$

Taking limit as $n\to\infty$ above, we get

$$\lim_{n\to\infty} F\left(d(\varsigma_n,\varsigma_{n+1})\right) = -\infty.$$

By (F_2), we have

$$\lim_{n\to\infty} d(\varsigma_n,\varsigma_{n+1}) = 0. \tag{9}$$

We claim that $\{\varsigma_n\}$ is a Cauchy sequence, by supposing on the contrary that it is not. Then $\exists\ \varepsilon>0$ and subsequences $\{\varsigma_{l_n}\}$ and $\{\varsigma_{q_n}\}$ so that for $l_n > q_n > n$, we have

$$d(\varsigma_{l_n},\varsigma_{q_n}) \geq \varepsilon \text{ and } d(\varsigma_{l_n-1},\varsigma_{q_n}) < \varepsilon\ \forall\ n\in\mathbb{N}. \tag{10}$$

By triangle inequality, we have

$$\varepsilon \leq d(\varsigma_{l_n},\varsigma_{q_n}) \leq d(\varsigma_{l_n},\varsigma_{l_{n-1}}) + d(\varsigma_{l_{n-1}},\varsigma_{q_n}).$$

Taking limit as $n\to\infty$ and using (9) and (10), we get

$$\lim_{n\to\infty} d(\varsigma_{l_n},\varsigma_{q_n}) = \varepsilon. \tag{11}$$

Using triangle inequality again, we get

$$d(\varsigma_{l_m},\varsigma_{q_m+1}) \leq d(\varsigma_{l_m},\varsigma_{q_m}) + d(\varsigma_{q_m},\varsigma_{q_m+1}).$$

Taking limit as $m\to\infty$ above and using (9) and (11), we get

$$\lim_{m\to\infty} d(\varsigma_{l_m},\varsigma_{q_m+1}) \leq \varepsilon. \tag{12}$$

Similarly, we have

$$\varepsilon \leq d(\varsigma_{l_m},\varsigma_{q_m}) \leq d(\varsigma_{l_m},x_{q_{m+1}}) + d(\varsigma_{q_{m+1}},\varsigma_{q_m}).$$

Taking limit as $m \to \infty$ above and using (9) and (11), we get

$$\lim_{m\to\infty} d(\varsigma_{l_m}, \varsigma_{q_m+1}) \geq \varepsilon. \tag{13}$$

Therefore, from (12) and (13), we get

$$\lim_{m\to\infty} d(\varsigma_{l_m}, \varsigma_{q_m+1}) = \varepsilon.$$

By similar way, we conclude

$$\lim_{m\to\infty} d(\varsigma_{q_m}, \varsigma_{l_m+1}) = \varepsilon.$$

Next, we claim that

$$d(\varsigma_{l_n+1}, \varsigma_{q_n+1}) > 0 \ \forall \ n \in \mathbb{N}. \tag{14}$$

Arguing by contradiction, there exists $m \in \mathbb{N}$, such that

$$d(\varsigma_{l_m+1}, \varsigma_{q_m+1}) = 0. \tag{15}$$

Using triangle inequality, we have

$$\begin{aligned}\varepsilon \leq d(\varsigma_{l_m}, \varsigma_{q_m}) &\leq d(\varsigma_{l_m}, \varsigma_{l_m+1}) + d(\varsigma_{l_m+1}, \varsigma_{q_m}) \\ &\leq d(\varsigma_{l_m}, \varsigma_{l_m+1}) + d(\varsigma_{l_m+1}, \varsigma_{q_m+1}) + d(\varsigma_{q_m+1}, \varsigma_{q_m}).\end{aligned}$$

Taking limit as $m \to \infty$ above and from (9) and (15), we get a contradiction. Hence (14) hold true.

Now, we have

$$0 < d(\varsigma_{l_m+1}, \varsigma_{q_m+1}) = d(\varsigma_{l_m+1}, \Gamma\varsigma_{q_m}) \leq H(\Gamma\varsigma_{l_m}, \Gamma\varsigma_{q_m}). \tag{16}$$

Since $\{\varsigma_n\}$ is $\Re$-preserving sequence, then by Γ-transitivity of $\Re$ we have $(\varsigma_{l_n}, \varsigma_{q_n}) \in \Re$ and from (16), we have $(\varsigma_{l_n}, \varsigma_{q_n}) \in \Re^*$. Utilizing (F_1) in (16) and applying (1), we obtain

$$\begin{aligned}F\left(d(\varsigma_{l_m+1}, \varsigma_{q_m+1})\right) &\leq F\left(H(\Gamma\varsigma_{l_m}, \Gamma\varsigma_{q_m})\right) \\ &\leq F\left(M_{\Re}(\varsigma_{l_m}, \varsigma_{q_m})\right) + LN_{\Re}(\varsigma_{l_m}, \varsigma_{q_m}) - \tau,\end{aligned} \tag{17}$$

where

$$\begin{aligned}M_{\Re}(\varsigma_{l_m}, \varsigma_{q_m}) &= \left[\lambda_1 \left(\frac{D(\varsigma_{l_m}, \Gamma\varsigma_{l_m})D(\varsigma_{q_m}, \Gamma\varsigma_{q_m})}{1 + d(\varsigma_{l_m}, \varsigma_{q_m})}\right)^\beta + \lambda_2 \left(d(\varsigma_{l_m}, \varsigma_{q_m})\right)^\beta\right]^{\frac{1}{\beta}} \\ &= \left[\lambda_1 \left(\frac{d(\varsigma_{l_m}, \varsigma_{l_m+1})d(\varsigma_{q_m}, \varsigma_{q_m+1})}{1 + d(\varsigma_{l_m}, \varsigma_{q_m})}\right)^\beta + \lambda_2 \left(d(\varsigma_{l_m}, \varsigma_{q_m})\right)^\beta\right]^{\frac{1}{\beta}}\end{aligned}$$

and

$$N_{\Re}(\varsigma_{l_m}, \varsigma_{q_m}) = \min\left\{D(\varsigma_{l_m}, \Gamma\varsigma_{l_m}), D(\varsigma_{q_m}, \Gamma\varsigma_{q_m}), D(\varsigma_{l_m}, \Gamma\varsigma_{q_m}), D(\varsigma_{q_m}, \Gamma\varsigma_{l_m})\right\}$$

$$= \min\{d(\varsigma_{l_m}, \varsigma_{l_m+1}), d(\varsigma_{q_m}, \varsigma_{q_m+1}), d(\varsigma_{l_m}, \varsigma_{q_m+1}), d(\varsigma_{q_m}, \varsigma_{l_m+1})\}.$$

As F is continuous then taking limit as $m \to \infty$ in (17), we get

$$F(\varepsilon) \leq F\left(\sqrt[\beta]{\lambda_2}\varepsilon\right) + L(0) - \tau < F(\varepsilon),$$

which gives a contradiction. So, $\{\varsigma_n\}$ is $\Re$-Cauchy sequence in an $\Re$-complete MS such that $\exists\ \varsigma^* \in \Delta_\Re$ implies that $\lim_{n\to\infty} \varsigma_n = \varsigma^*$. To proof that $\varsigma^* \in \Gamma\varsigma^*$, we claim that $D(\varsigma^*, \Gamma\varsigma^*) > 0$ and by using (1), Lemma 1.15 and $(T3)$, we write

$$\begin{aligned} F(D(\varsigma^*, \Gamma\varsigma^*)) &\leq \lim_{n\to\infty} F(D(\varsigma_n, \Gamma\varsigma^*)) \\ &\leq \lim_{n\to\infty} F(H(\Gamma\varsigma_{n-1}, \Gamma\varsigma^*)) \\ &\leq \lim_{n\to\infty} F(M_\Re(\varsigma_{n-1}, \varsigma^*)) + \lim_{n\to\infty} LN_\Re(\varsigma_{n-1}, \varsigma^*) - \tau, \end{aligned} \tag{18}$$

where

$$\begin{aligned} M_\Re(\varsigma_{n-1}, \varsigma^*) &= \left[\lambda_1\left(\frac{D(\varsigma_{n-1}, \Gamma\varsigma^*)\, D(\varsigma^*, \Gamma\varsigma_{n-1})}{1 + D(\varsigma_{n-1}, \varsigma^*)}\right)^\beta + \lambda_2 d(\varsigma_{n-1}, \varsigma^*)^\beta\right]^{\frac{1}{\beta}} \\ &= \left[\lambda_1\left(\frac{D(\varsigma_{n-1}, \Gamma\varsigma^*)\, d(\varsigma^*, \varsigma_n)}{1 + d(\varsigma_{n-1}, \varsigma^*)}\right)^\beta + \lambda_2 D(\Gamma x_{n-2}, x^*)^\beta\right]^{\frac{1}{\beta}}, \end{aligned}$$

and

$$\begin{aligned} N_\Re(\varsigma_{n-1}, \varsigma^*) &= \min\{D(\varsigma_{n-1}, \Gamma\varsigma_{n-1}), D(\varsigma^*, \Gamma\varsigma^*), D(\varsigma_{n-1}, \Gamma\varsigma^*), D(\varsigma^*, \Gamma\varsigma_{n-1})\} \\ &= \min\{d(\varsigma_{n-1}, \varsigma_n), D(\varsigma^*, \Gamma\varsigma^*), D(\varsigma_{n-1}, \Gamma\varsigma^*), D(\varsigma^*, \varsigma_n)\}. \end{aligned}$$

Taking limit as $n \to \infty$ in (18) with continuity of F, we have

$$\begin{aligned} F(D(\varsigma^*, \Gamma\varsigma^*)) &\leq F\left(\sqrt[\beta]{\lambda_2} D(\varsigma^*, \Gamma\varsigma^*)\right) + L(0) - \tau \\ &< F(D(\varsigma^*, \Gamma\varsigma^*)). \end{aligned}$$

A contradiction, thence $D(\varsigma^*, \Gamma\varsigma^*) = 0$ and $\varsigma^* \in \Gamma\varsigma^*$ is a FP of Γ.

Now, if $\beta = 0$, then

$$\begin{aligned} M_\Re(\varsigma_{n-1}, \varsigma_n) &= D(\varsigma_{n-1}, \Gamma\varsigma_{n-1})^{\lambda_1} D(\varsigma_n, \Gamma\varsigma_n)^{\lambda_2} \\ &= d(\varsigma_{n-1}, \varsigma_n)^{\lambda_1} d(\varsigma_n, \varsigma_{n+1})^{\lambda_2}. \end{aligned} \tag{19}$$

Assume that $d(\varsigma_{n-1}, \varsigma_n) \leq d(\varsigma_n, \varsigma_{n+1})$, then (19) becomes

$$\begin{aligned} M_\Re(\varsigma_{n-1}, \varsigma_n) &\leq d(\varsigma_n, \varsigma_{n+1})^{\lambda_1} d(\varsigma_n, \varsigma_{n+1})^{\lambda_2} \\ &= d(\varsigma_n, \varsigma_{n+1})^{\lambda_1+\lambda_2} \end{aligned} \tag{20}$$

$$= d(\varsigma_n, \varsigma_{n+1}).$$

Thereafter, from (6) and (20), we get a contradiction. Thus, $d(\varsigma_{n-1}, \varsigma_n) > d(\varsigma_n, \varsigma_{n+1})$ and (19) becomes

$$M_{\Re}(\varsigma_{n-1}, \varsigma_n) < d(\varsigma_{n-1}, \varsigma_n) \tag{21}$$

Applying (21) in (6) and following the same steps from (8) till (17), we get

$$\begin{aligned} M_{\Re}(\varsigma_{l_m}, \varsigma_{q_m}) &= d(\varsigma_{l_m}, \Gamma\varsigma_{l_m})^{\lambda_1} d(\varsigma_{q_m}, \Gamma\varsigma_{q_m})^{\lambda_2} \\ &= d(\varsigma_{l_m}, \varsigma_{l_m+1})^{\lambda_1} d(\varsigma_{q_m}, \varsigma_{q_m+1})^{\lambda_2} \end{aligned}$$

and

$$\begin{aligned} N_{\Re}(\varsigma_{l_m}, \varsigma_{q_m}) &= \min\{d(\varsigma_{l_m}, \Gamma\varsigma_{l_m}), d(\varsigma_{q_m}, \Gamma\varsigma_{q_m}), d(\varsigma_{l_m}, \Gamma\varsigma_{q_m}), d(\varsigma_{q_m}, \Gamma\varsigma_{l_m})\} \\ &= \min\{d(\varsigma_{l_m}, \varsigma_{l_m+1}), d(\varsigma_{q_m}, \varsigma_{q_m+1}), d(\varsigma_{l_m}, \varsigma_{q_m+1}), d(\varsigma_{q_m}, \varsigma_{l_m+1})\}. \end{aligned}$$

As F is continuous then taking limit as $m \to \infty$ in (17), we get

$$F(\varepsilon) < F(0) + L(0) - \tau$$

which gives a contradiction again. So, $\{\varsigma_n\}$ is $\Re$-Cauchy sequence in an $\Re$-complete MS such that $\exists\ \varsigma^* \in \Delta_{\Re}$ implies that $\lim_{n\to\infty} \varsigma_n = \varsigma^*$. To proof that $\varsigma^* \in \Gamma\varsigma^*$, we claim that $D(\varsigma^*, \Gamma\varsigma^*) > 0$ and by using (1), Lemma 1.15 and $(T3)$, we write

$$\begin{aligned} F(D(\varsigma^*, \Gamma\varsigma^*)) &\leq \lim_{n\to\infty} F(D(\varsigma_n, \Gamma\varsigma^*)) \\ &\leq \lim_{n\to\infty} F(H(\Gamma\varsigma_{n-1}, \Gamma\varsigma^*)) \\ &\leq \lim_{n\to\infty} F(M_{\Re}(\varsigma_{n-1}, \varsigma^*)) + \lim_{n\to\infty} L N_{\Re}(\varsigma_{n-1}, \varsigma^*) - \tau, \end{aligned} \tag{22}$$

where

$$\begin{aligned} M_{\Re}(\varsigma_{n-1}, \varsigma^*) &= D(\varsigma_{n-1}, \Gamma\varsigma^*)^{\lambda_1} D(\varsigma^*, \Gamma\varsigma_{n-1})^{\lambda_2} \\ &= D(\varsigma_{n-1}, \Gamma\varsigma^*)^{\lambda_1} d(\varsigma^*, \varsigma_n)^{\lambda_2}, \end{aligned}$$

and

$$\begin{aligned} N_{\Re}(\varsigma_{n-1}, \varsigma^*) &= \min\{D(\varsigma_{n-1}, \Gamma\varsigma_{n-1}), D(\varsigma^*, \Gamma\varsigma^*), D(\varsigma_{n-1}, \Gamma\varsigma^*), D(\varsigma^*, \Gamma\varsigma_{n-1})\} \\ &= \min\{d(\varsigma_{n-1}, \varsigma_n), D(\varsigma^*, \Gamma\varsigma^*), D(\varsigma_{n-1}, \Gamma\varsigma^*), d(\varsigma^*, \varsigma_n)\}. \end{aligned}$$

Taking limit as $n \to \infty$ in (22) with continuity of F, we have

$$F(D(\varsigma^*, \Gamma\varsigma^*)) \leq F(0) + L(0) - \tau,$$

this implies that $D(\varsigma^*, \Gamma\varsigma^*) = 0$ and $\varsigma^* \in \Gamma\varsigma^*$ is a FP of Γ.

Example 1. *Let $\Delta_{\Re} = [0, \infty)$ equipped with a usual metric. Take a sequence $\{\varsigma_n\} \subset \Delta_{\Re}$ given by $\varsigma_n = \frac{n^2(n+1)^2}{4}$ $\forall$ $n \geq 1$. Set*

$$\Re = \{(\varsigma_n, \varsigma_n), (\varsigma_n, \varsigma_{n+1}), (\varsigma_n, \varsigma_{n+2}) : n = 1, 2, ...\},$$

and $\Gamma : \Delta_{\Re} \to CB(\Delta_{\Re})$ by

$$\Gamma x = \begin{cases} \{3x\} & \text{if } 0 \leq x \leq \varsigma_1 \\ \{0, \varsigma_1\} & \text{if } \varsigma_1 \leq x \leq \varsigma_2 \\ \left\{\varsigma_{n-1} + \left(\frac{\varsigma_n - \varsigma_{n-1}}{\varsigma_{n+1} - \varsigma_n}\right)(x - \varsigma_n)\right\} & \text{if } \varsigma_n \leq x \leq \varsigma_{n+1},\ n = 2, ... \end{cases}.$$

Then, $(\Delta_{\Re}, d)$ is a complete MS, $\Delta_{\Re}(\Gamma, \Re) \neq \emptyset$ as $\varsigma_1 = 1 \in \Delta_{\Re}$, $1 \in \Gamma\varsigma_1 = \Gamma 1 = \{0, 1\}$ and $(1, 1) \in \Re$. also it is easy to check that $\Re$ is Γ-transitive and Γ is $\Re$-continuous or $(\Delta_{\Re}, d)$ is $\Re$-regular space. Now, for $n = 2, ...$, we have

$$\begin{aligned} H(\Gamma\varsigma_n, \Gamma\varsigma_{n+1}) &= \max\left\{\sup_{r \in \Gamma\varsigma_n} D(r, \Gamma\varsigma_{n+1}), \sup_{s \in \Gamma\xi_{n+1}} D(\Gamma\varsigma_n, s)\right\} \\ &= \max\{d(\varsigma_{n-1}, \varsigma_n), d(\varsigma_{n-1}, \varsigma_n)\} \\ &= d(\varsigma_n, \varsigma_{n-1}). \end{aligned}$$

From (2) and (3) with $\beta \geq 0$, we get

$$M_{\Re}(\varsigma_n, \varsigma_{n+1}) = d(\varsigma_{n+1}, \varsigma_n) \text{ and } N_{\Re}(\varsigma_n, \varsigma_{n+1}) = 0.$$

Then, utilizing (1), we have

$$\begin{aligned} \tau + F(H(\Gamma\varsigma_n, \Gamma\varsigma_{n+1})) &= \tau + F(d(\varsigma_n, \varsigma_{n-1})) \\ &= \tau + \ln(d(\varsigma_n, \varsigma_{n-1})), \end{aligned} \tag{23}$$

and

$$\begin{aligned} &F(M_{\Re}(\varsigma_n, \varsigma_{n+1})) + LN_{\Re}(\varsigma_n, \varsigma_{n+1}) \\ &\leq F(d(\varsigma_{n+1}, \varsigma_n)) + L(0) \\ &= \ln(d(\varsigma_{n+1}, \varsigma_n)). \end{aligned} \tag{24}$$

From (23) and (24), we deduce that

$$\tau + \ln(d(\varsigma_n, \varsigma_{n-1})) \leq \ln(d(\varsigma_{n+1}, \varsigma_n)),$$

implies that

$$\tau \leq \ln\left(\frac{d(\varsigma_{n+1}, \varsigma_n)}{d(\varsigma_n, \varsigma_{n-1})}\right). \tag{25}$$

Let

$$f(n) = \ln\left(\frac{|\varsigma_{n+1} - \varsigma_n|}{|\varsigma_n - \varsigma_{n-1}|}\right). \tag{26}$$

Table 1: Iteration and f(n)

Iter	f(n)	Iter	f(n)
n=2	1.21	n=11	0.26
n=3	0.86	n=12	0.24
n=4	0.66	n=13	0.22
n=5	0.55	n=14	0.20
n=6	0.46	n=15	0.19
n=7	0.40	n=16	0.18
n=8	0.35	n=17	0.17
n=9	0.31	...	...
n=10	0.28	n=50	0.059

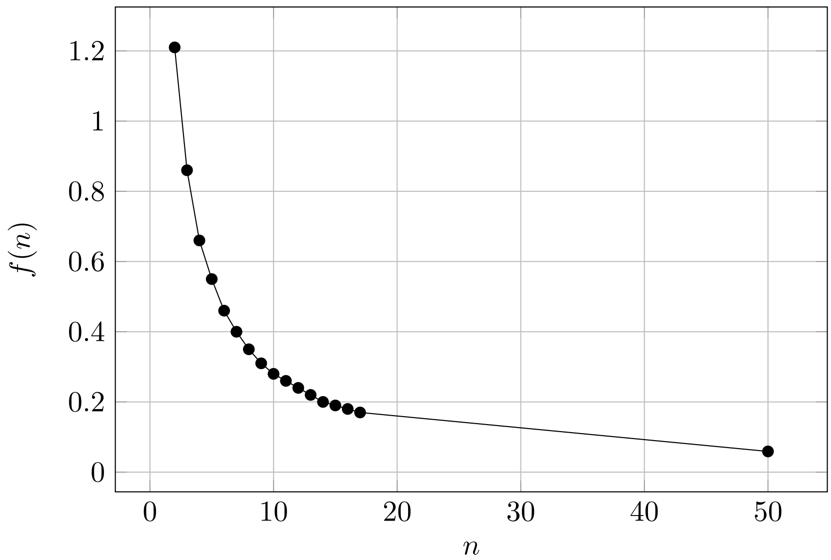

Figure 1: Behavior of $f(n)$ for $n \in [2, 50]$.

In view of $Table 1$ *and* $Figure 1$, *since the sequence* $\{f(n)\}_{n\geq 2}$ *is decreasing and discontinuous, the smallest value in (26) is* 0.059. *Therefore, the Eq (25) holds for* $0 < \varsigma < 0.059$. *So, the contraction (1) is satisfied for all* $\varsigma_1, \varsigma_2 \in \Delta_{\Re}$ *such that* $(\varsigma_1, \varsigma_2) \in \Re^*$. *Hence,* Γ *has infinite FPs.*

Example 2. *Let* $\Delta_{\Re} = \{1, 2, 3, 4\}$ *and* $d : \Delta_{\Re} \times \Delta_{\Re} \to [0, \infty)$ *be a metric on* $\Delta_{\Re}$ *defined as* $d(1,2) = 4,\ d(1,3) = 6,\ d(1,4) = 3,\ d(2,3) = 4,\ d(2,4) = 3,\ d(3,4) = 5,\ d(\varsigma_1, \varsigma_2) = d(\varsigma_2, \varsigma_1)$ *and* $d(\varsigma_1, \varsigma_1) = 0,\ \forall \varsigma_1, \varsigma_2 \in \Delta_{\Re}$. *We define a binary relation on* $\Delta_{\Re}$ *as* $\Re = \left\{ \begin{array}{c} (1,1), (1,2), (1,3), (1,4), (4,1), \\ (2,1), (2,2), (2,4), (3,1), (4,3) \end{array} \right\}$.

Consider a mapping $\Gamma : \Delta_{\Re} \to CB\left(\Delta_{\Re}\right)$ as

$$\Gamma\varsigma = \begin{cases} \{3,4\}, & \varsigma \in \{1,4\}; \\ \{3\}, & \varsigma = 2; \\ \{4\}, & \varsigma = 3. \end{cases}$$

Then, $\Delta_{\Re}(\Gamma, \Re) \neq \emptyset$ as $4 \in \Delta_{\Re}$, $4 \in \Gamma 3 = \{4\}$ and $(4,3) \in \Re$. also it is easy to check that $\Re$ is Γ-transitive and Γ is $\Re$-continuous or $(\Delta_{\Re}, d)$ is $\Re$-regular space. Now, for all $\lambda_1 = \frac{1}{5}$, $\lambda_2 = \frac{3}{5}$, $\beta = 2$, $L = \frac{1}{5}$ and $(\varsigma_1, \varsigma_2) \in \Re^* = \{(1,1),(1,2),(1,3),(3,1),(2,2),(2,1)\}$, we have for $(\varsigma_1, \varsigma_2) = (1,2)$

$$\begin{aligned} H(\Gamma 1, \Gamma 2) &= \max\left\{\sup_{a\in\Gamma 1} D(a, \Gamma 2), \sup_{b\in\Gamma 2} D(\Gamma 1, b)\right\} \\ &= \max\{d(4,3), d(3,3)\} \\ &= \max\{5, 0\} \\ &= 5, \end{aligned}$$

where

$$\begin{aligned} M_{\Re}(1,2) &= \left[\lambda_1\left(\frac{D(1,\Gamma 1)\, D(2,\Gamma 2)}{1 + d(1,2)}\right)^{\beta} + \lambda_2\left(d(1,2)\right)^{\beta}\right]^{\frac{1}{\beta}} \\ &= \left[\lambda_1\left(\frac{d(1,3)\, d(2,3)}{1 + d(1,2)}\right)^{\beta} + \lambda_2\left(d(1,2)\right)^{\beta}\right]^{\frac{1}{\beta}} \\ &= \left[\lambda_1\left(\frac{6\times 4}{1+4}\right)^{\beta} + \lambda_2 (4)^{\beta}\right]^{\frac{1}{\beta}} \\ &\leq \left[\frac{1}{3}\left(\frac{24}{5}\right)^{2} + \frac{2}{3}(4)^{2}\right]^{\frac{1}{2}} \\ &= \left(\frac{86}{75}\right)^{\frac{1}{2}} \times 4 = 4.283300908, \end{aligned}$$

and

$$\begin{aligned} N_{\Re}(1,2) &= \min\{D(1,\Gamma 1), D(2,\Gamma 2), D(1,\Gamma 2), D(2,\Gamma 1)\} \\ &= \min\{d(1,3), d(2,3), d(1,3), d(2,3)\} \\ &= \min\{6,4,6,4\} = 4. \end{aligned}$$

For $\beta = 0$, we get

$$\begin{aligned} M_{\Re}(1,2) &= D(1,\Gamma 1)^{\frac{1}{5}}\, D(2,\Gamma 2)^{\frac{3}{5}} \\ &= d(1,3)^{\frac{1}{5}}\, d(2,3)^{\frac{3}{5}} \end{aligned}$$

$$= 6^{\frac{1}{5}} \times 4^{\frac{3}{5}} \approx 3.3.$$

Therefore, $\tau + F(5) \leq F(4) + \frac{4}{5}$ and For $\beta = 0$, we get $\tau + F(5) \leq F(3.3) + \frac{4}{5}$.
For $(\varsigma_1, \varsigma_2) = (1, 3)$, we have $H(\Gamma 1, \Gamma 3) = 5$, $M_{\Re}(1, 3) = 5.488$, and $N_{\Re}(1, 3) = 0$. Therefore

$$\tau + F(5) \leq F(5.488) + 0$$

For $(\varsigma_1, \varsigma_2) = (1, 1)$, we have $H(\Gamma 1, \Gamma 1) = 5$, $M_{\Re}(1, 1) = 12\sqrt{3}$ and $N_{\Re}(1, 1) = 6$. Therefore

$$\tau + F(5) \leq F\left(12\sqrt{3}\right) + \frac{6}{5}.$$

For $(\varsigma_1, \varsigma_2) = (2, 2)$,we have $H(\Gamma 2, \Gamma 2) = 0$, where $M_{\Re}(2, 2) = \frac{16}{\sqrt{3}}$, and $N_{\Re}(2, 2) = 4$. Therefore

$$\tau + F(0) \leq F\left(\frac{16}{\sqrt{3}}\right) + \frac{4}{5}.$$

Hence, for all $(\varsigma_1, \varsigma_2) \in \Re^*$, we find that (1) is achieved and all the conditions of Theorem 5 are satisfied so that Γ has an FP $4 \in \Delta_{\Re}$.

3. Corollaries

Corollary 1. *Let $(\Delta_{\Re}, d)$ be a MS. A map $\Gamma : \Delta_{\Re} \to \Delta_{\Re}$ is called an almost Fisher-type F-contraction endowed with a Γ-transitive binary relation $\Re$, If there exist $\tau \in \mathbb{R}^+$, $F \in \Delta_w$ and $\lambda_1, \lambda_2, L \geq 0$ with $\lambda_1 + \lambda_2 \leq 1$ such that for all $(\varsigma_1, \varsigma_2) \in \Re^* = \{(\varsigma_1, \varsigma_2) \in \Re : \varsigma_1, \varsigma_2 \in \Delta_{\Re} \backslash Fix(\Gamma)\}$, we have*

$$\tau + F(d(\Gamma\varsigma_1, \Gamma\varsigma_2)) \leq F(M_{\Re}(\varsigma_1, \varsigma_2)) + LN_{\Re}(\varsigma_1, \varsigma_2), \tag{27}$$

where $M_{\Re}(\varsigma_1, \varsigma_2)$ and $N_{\Re}(\varsigma_1, \varsigma_2)$ are defined as in (2) and (3) respectively.

Hence, Γ has an FP $\varsigma^* \in \Delta_{\Re}$.

Corollary 2. *Let $(\Delta_{\Re}, d)$ be a MS. A map $\Gamma : \Delta_{\Re} \to CB(\Delta_{\Re})$ is called an almost Fisher-type multivalued F-contraction endowed with a Γ-transitive binary relation $\Re$, If there exist $\tau \in \mathbb{R}^+$, $F \in \Delta_w$ and $\lambda_1, \lambda_2, L \geq 0$ with $\lambda_1 + \lambda_2 \leq 1$ such that for all $(\varsigma_1, \varsigma_2) \in \Re^* = \{(\varsigma_1, \varsigma_2) \in \Re : \varsigma_1, \varsigma_2 \in \Delta_{\Re} \backslash Fix(\Gamma)\}$, we have*

$$\tau + F(H(\Gamma\varsigma_1, \Gamma\varsigma_2)) \leq F(M_{\Re}(\varsigma_1, \varsigma_2)) + Ld(\varsigma_2, \Gamma\varsigma_1), \tag{28}$$

Where $M_{\Re}(\varsigma_1, \varsigma_2)$ is defind as in (2).

Hence, Γ has an FP in $\Delta_{\Re}$.

Corollary 3. *Let $(\Delta_{\Re}, d)$ be a MS. A map $\Gamma : \Delta_{\Re} \to \Delta_{\Re}$ is called an almost Fisher-type F-contraction endowed with a Γ-transitive binary relation $\Re$, If there exist $\tau \in \mathbb{R}^+$, $F \in \Delta_w$ and $\lambda_1, \lambda_2, L \geq 0$ with $\lambda_1 + \lambda_2 \leq 1$ such that for all $(\varsigma_1, \varsigma_2) \in \Re^* = \{(\varsigma_1, \varsigma_2) \in \Re : \varsigma_1, \varsigma_2 \in \Delta_{\Re} \backslash Fix(\Gamma)\}$, we have*

$$\tau + F(d(\Gamma\varsigma_1, \Gamma\varsigma_2)) \leq F(M_{\Re}(\varsigma_1, \varsigma_2)) + Ld(\varsigma_2, \Gamma\varsigma_1), \tag{29}$$

Where $M_{\Re}(\varsigma_1, \varsigma_2)$ is defined as in (2).

Hence, Γ has an FP in $\Delta_{\Re}$.

Corollary 4. *Let $(\Delta_{\Re}, d)$ be a MS. A map $\Gamma : \Delta_{\Re} \to CB(\Delta_{\Re})$ is called an almost multivalued F-contraction endowed with a Γ-transitive binary relation $\Re$, If there exist $\tau \in \mathbb{R}^+$, $F \in \Delta_w$ and $\lambda, L \geq 0$ with $\lambda + L \leq 1$ such that for all $(\varsigma_1, \varsigma_2) \in \Re^* = \{(\varsigma_1, \varsigma_2) \in \Re : \varsigma_1, \varsigma_2 \in \Delta_{\Re} \backslash Fix(\Gamma)\}$, we have*

$$\tau + F\left(H(\Gamma\varsigma_1, \Gamma\varsigma_2)\right) \leq F\left(\lambda d(\varsigma_1, \varsigma_2)\right) + Ld\left(\varsigma_1, \varsigma_2\right). \tag{30}$$

Then, Γ has an FP in $\Delta_{\Re}$.

Corollary 5. *Let $(\Delta_{\Re}, d)$ be a MS. A map $\Gamma : \Delta_{\Re} \to \Delta_{\Re}$ is called Fisher-type F-contraction, if there exist $\tau \in \mathbb{R}^+$, $F \in \Delta_w$ and $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 \leq 1$ such that for all $\varsigma_1, \varsigma_2 \in \Delta_{\Re} \backslash Fix(\Gamma)$, we have*

$$\tau + F\left(d(\Gamma\varsigma_1, \Gamma\varsigma_2)\right) \leq F\left(M_{\Re}(\varsigma_1, \varsigma_2)\right),$$

where

$$M_{\Re}(\varsigma_1, \varsigma_2) = \begin{cases} \left[\lambda_1 \left(\frac{d(\varsigma_1, \Gamma\varsigma_1) d(\varsigma_2, \Gamma\varsigma_2)}{1 + d(\varsigma_1, \varsigma_2)}\right)^{\beta} + \lambda_2 \left(d(\varsigma_1, \varsigma_2)\right)^{\beta}\right]^{\frac{1}{\beta}} & \text{if } \beta > 0; \\ \left(d\left(\varsigma_1, \Gamma\varsigma_1\right)\right)^{\lambda_1} \left(d\left(\varsigma_2, \Gamma\varsigma_2\right)\right)^{\lambda_2} & \text{if } \beta = 0. \end{cases}$$

Hence, Γ has an FP in $\Delta_{\Re}$.

4. An application to second-order differential inclusions

In this section, we apply the previous theoretical results to study the existence of solutions for the following SODI. In line with [28–31], we consider the boundary value problem (BVP) on $[0, 1]$:

$$\begin{cases} \varsigma''(t) \in \Pi(t, \varsigma(t), \varsigma'(t)), & t \in [0, 1], \\ \varsigma(0) = 0, \quad \varsigma(1) = 0, \end{cases} \tag{31}$$

where $\Pi : [0, 1] \times \mathbb{R}^2 \to CB(\mathbb{R})$ has nonempty, closed and compact values and satisfies measurability hypotheses to ensure selections.

Let $\Delta_{\Re} = C^1([0, 1], \mathbb{R})$ with norm $\|\varsigma\| = \sup_t |\varsigma(t)| + \sup_t |\varsigma'(t)|$ and metric $d(\varsigma_1, \varsigma_2) = \|\varsigma_1 - \varsigma_2\|$. Denote by $G(t, s)$ the Green function for the homogeneous Dirichlet problem:

$$G(t, s) = \begin{cases} t(1 - s), & t \leq s, \\ s(1 - t), & t > s. \end{cases}$$

Define the multivalued operator $\Gamma : \Delta_{\Re} \to CB(\Delta_{\Re})$ by

$$\Gamma(\varsigma) = \left\{\Upsilon \in C^1([0, 1]) : \ \Upsilon(t) = \int_0^1 G(t, s) f(s)\, ds, \quad f(s) \in \Pi(s, \varsigma(s), \varsigma'(s))\right\}. \tag{32}$$

A fixed point $\varsigma^* \in \Gamma\varsigma^*$ is a solution of (31).
Compute

$$K = \sup_{t\in[0,1]} \int_0^1 G(t,s)\,ds = \frac{1}{8}, \qquad M = \sup_{t\in[0,1]} \int_0^1 \left|\partial_t G(t,s)\right| ds = \frac{1}{2},$$

so $K + M \leq 5/8$.

We assume Π satisfies the following pointwise control:

(P1) There exist measurable functions $a(s), b(s) \geq 0$ and constants $\lambda_1, \lambda_2 \geq 0$, $0 \leq \lambda_1 + \lambda_2 \leq 1$, and $\beta \in (0,1]$ such that $\forall\ s \in [0,1]$ and all $u_i \in \Pi(s, \varsigma_i(s), \varsigma_i'(s))$ (for $i = 1,2$), one has

$$|u_1 - u_2|^\beta \leq a(s)\left(|\varsigma_1(s) - \varsigma_2(s)|^\beta + |\varsigma_1'(s) - \varsigma_2'(s)|^\beta\right) + b(s)\,\mathcal{D}_s(\varsigma_1, \varsigma_2), \tag{33}$$

where $\mathcal{D}_s(\varsigma_1, \varsigma_2)$ stands for combinations of pointwise "distances-to-value-sets" (these will produce the $D(\cdot, \Gamma(\cdot))$-type terms appearing in (2)). This is an abstract but standard assumption, it generalizes the Hausdorff–Lipschitz condition and allows us to produce the more general $M_{\Re}$-term.

(P2) Fix $\varsigma_1, \varsigma_2 \in \Delta_{\Re}$. Let $f(\cdot)$ be a measurable selection from $\Pi(\cdot, \varsigma_1(\cdot), \varsigma_1'(\cdot))$ and $g(\cdot)$ from $\Pi(\cdot, \varsigma_2(\cdot), \varsigma_2'(\cdot))$. For $\eta = \int G(t,s)f(s) \in \Gamma(\varsigma_1)$ and $\zeta = \int G(t,s)g(s) \in \Gamma(\varsigma_2)$ we have, for any t,

$$|\eta(t) - \zeta(t)| \leq \int_0^1 G(t,s)\,|f(s) - g(s)|\,ds. \tag{34}$$

Theorem 6. *Under the two assumptions above, SODI (31) has at least one solution $\varsigma^* \in \Delta_{\Re}$ iff Γ has an FP.*

Proof. The set $\Delta_{\Re} = C^1([0,1], \mathbb{R})$ is a complete MS. Define the multivalued operator $\Gamma : \Delta_{\Re} \to CB(\Delta_{\Re})$ as in (32) and from (P2), we raise to the power $\beta \in (0,1]$ in (34) and use Jensen (or generalized Hölder) to obtain

$$|\eta(t) - \zeta(t)|^\beta \leq \int_0^1 G(t,s)^\beta\,|f(s) - g(s)|^\beta\,ds.$$

Taking supremum in t and using $C_1 = \sup_t \int_0^1 G(t,s)^\beta ds$, $C_2 = \sup_t \int_0^1 |\partial_t G(t,s)|^\beta ds$, we get

$$\|\eta - \zeta\|^\beta \leq (C_1 + C_2) \int_0^1 |f(s) - g(s)|^\beta\,ds.$$

From (P2) and applying the structural bound (33) to the integrand and integrate

$$\left(\int_0^1 |f - g|^\beta\right)^{1/\beta} \leq A\,d(\varsigma_1, \varsigma_2) + B\,M_{\Re}(\varsigma_1, \varsigma_2),$$

for appropriate constants $A, B \geq 0$ depending on $a(\cdot), b(\cdot), C_1, C_2$, where $M_{\Re}(\varsigma_1, \varsigma_2)$ denotes a combination of distances-to-value-sets of the type appearing in (2). Concretely one may choose

$$M_{\Re}(\varsigma_1, \varsigma_2) = \Big[\lambda_1 \Big(\frac{D(\varsigma_1, \Gamma\varsigma_1) D(\varsigma_2, \Gamma\varsigma_2)}{1 + D(\varsigma_1, \varsigma_2)}\Big)^{\beta} + \lambda_2 \big(d(\varsigma_1, \varsigma_2)\big)^{\beta}\Big]^{1/\beta},$$

Combining the previous two displayed bounds yields

$$H(\Gamma\varsigma_1, \Gamma\varsigma_2) \leq C^{1/\beta}\big(A\, d(\varsigma_1, \varsigma_2) + B\, M_{\Re}(\varsigma_1, \varsigma_2)\big),$$

where $C = C_1 + C_2$. Now choose

$$\tau = 0, \qquad F(t) = t \quad (t \geq 0), \qquad N_{\Re}(\varsigma_1, \varsigma_2) = d(\varsigma_1, \varsigma_2),$$

and define the constant $L = C^{1/\beta} A$ and note that $F(M_{\Re}(\varsigma_1, \varsigma_2))$ can absorb the other piece $C^{1/\beta} B\, M_{\Re}$. With these choice the bound becomes exactly of the form

$$\tau + F\big(H(\Gamma\varsigma_1, \Gamma\varsigma_2)\big) \leq F\big(M_{\Re}(\varsigma_1, \varsigma_2)\big) + L\, N_{\Re}(\varsigma_1, \varsigma_2),$$

which is equation (1).

We now show that the conditions (T1)–(T3) in Theorem 5 can be satisfied by natural choices.

- Choose the binary relation $\Re = \Delta_{\Re} \times \Delta_{\Re}$ (the universal relation). It is clearly transitive and closed.
- (T1) The set $\Delta_{\Re}(\Gamma, \Re) = \{\varsigma \in \Delta_{\Re} : (\varsigma, \Gamma\varsigma) \in \Re\}$ is equal to $\Delta_{\Re}$ (hence nonempty) because $\Re$ is universal.
- (T2) $\Re$ is Γ-closed trivially.
- (T3) is satisfied because under the compactness of values and the structural bounds we can show upper semicontinuity of Γ in the Hausdorff metric; hence Γ is $\Re$-continuous or the space $(\Delta_{\Re}, d)$ is $\Re$-regular.

Hence, with the above matching of parameters and the structural pointwise control (33), all hypotheses of Theorem 2 are satisfied and yields existence of a fixed point $\varsigma^* \in \Delta_{\Re}$ of Γ, which is a solution of (31).

Example 3. *Take*

$$\Pi(t, u, v) = \big[\tfrac{1}{4}(u + v),\ \tfrac{1}{2}(u + v)\big], \qquad t \in [0, 1].$$

Pointwise one has for any $u_i \in \Pi(t, u_i^*, v_i^*)$

$$|u_1 - u_2| \leq \tfrac{1}{2}(|u_1^* - u_2^*| + |v_1^* - v_2^*|),$$

so setting $\beta = 1$ *and integrating the previous estimates yields the linear contraction estimate*

$$H(\Gamma\varsigma_1, \Gamma\varsigma_2) \leq (K + M)\tfrac{1}{2}\, d(\varsigma_1, \varsigma_2),$$

with $(K + M)\frac{1}{2} = \frac{5}{16} < 1$. *In the view of Definition 1 this corresponds to taking* $M_{\Re}$ *negligible (or explicitly zero) and* $L = (K + M) \cdot \frac{1}{2}$*; the inequality (1) is thus satisfied and Theorem 2 gives existence of the BVP's solution.*

5. Conclusions

In this article, we have introduced and studied a new class of contractions, namely almost Fisher-type multivalued F-contractions equipped with a Γ-transitive binary relation. Within this framework, we proved several fixed-point results which extend and unify a number of existing theorems in the literature on multivalued contractions. The obtained results highlight the flexibility of the proposed approach, as they recover well-known outcomes as special cases and, at the same time, yield genuinely new contributions. To illustrate the applicability of our theory, we provided explicit examples and developed an application to second-order differential inclusions. This application shows how the abstract fixed point results can be effectively employed to guarantee the existence of solutions to nonlinear problems. In particular, it demonstrates that the almost Fisher-type multivalued F-contraction framework can serve as a powerful tool in the analysis of integral and differential inclusions.

Data Availability Statement

All data generated or analyzed during this study are included in this manuscript.

Conflict of Interests

The authors declare that they have no competing interests.

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Authors' contributions

MM was responsible for conceptualization, methodology, software, original draft preparation and editing. MA was responsible for supervision, investigation, and resources. AH was responsible for validation, formal analysis, data curation, and review. HA was responsible for project administration, resources, and funding acquisition. All authors have read and agreed to be responsible for the content and conclusions of the article.

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