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Integral Representations of Generalizations of Pell Numbers and Their Companion Numbers

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Abstract. This paper discusses a one-parameter generalization of Pell numbers that preserves the recurrence relation with arbitrary initial conditions. We introduce generalized Pell-Lucas-like numbers, which are simple associations of generalized Pell numbers. Consequently, we give some new and well-known identities. Furthermore, we propose integral representations of these numbers associated with generalized Pell and Pell-Lucas-like numbers. Our results not only generalize the integral representations of the Pell and Pell-Lucas numbers but also apply to all the companion numbers of generalized Pell numbers.

2020 Mathematics Subject Classifications: 11B39, 11B37

Key Words and Phrases: Generalized Pell number, generalized Pell-Lucas-like number, integral representation

1. Introduction

Recall that Pell numbers P_n are defined by the recurrence relations

$$P_0 = 0, P_1 = 1, \quad \text{and} \quad P_n = 2P_{n-1} + P_{n-2}, \quad n \geq 2,$$

and its associated numbers or Pell-Lucas numbers Q_n are defined by the recurrence relations

$$Q_0 = 2, Q_1 = 1, \quad \text{and} \quad Q_n = 2Q_{n-1} + Q_{n-2}, \quad n \geq 2.$$

The Binet's formulas for the Pell and Pell-Lucas numbers are related to the silver ratio $\varphi = 1 + \sqrt{2}$. There are some generalizations of Pell and Pell-Lucas numbers defined in different ways.

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In 2007, Falcón and Plaza [1] introduced the first kind of one-parameter generalization of Fibonacci and Pell numbers as follows: The *k-Fibonacci numbers* $F_{k,n}$ are defined by the recurrence relations

$$F_{k,0} = 0, F_{k,1} = 1, \quad \text{and} \quad F_{k,n} = kF_{k,n-1} + F_{k,n-2}, \quad n \geq 2,$$

where k and n are non-negative integers with $k \neq 0$. The associated numbers of k-Fibonacci numbers introduced in 2011 by Falcón [2] as follows: The *k-Lucas numbers* $L_{k,n}$ are defined by the recurrence relations

$$L_{k,0} = 2, L_{k,1} = k, \quad \text{and} \quad L_{k,n} = kL_{k,n-1} + L_{k,n-2}, \quad n \geq 2.$$

In 2013, Catarino [3] introduced the second kind of one-parameter generalization of Pell numbers as follows: The *k-Pell numbers* $P_{k,n}$ are defined by the recurrence relations

$$P_{k,0} = 0, P_{k,1} = 1, \quad \text{and} \quad P_{k,n} = 2P_{k,n-1} + kP_{k,n-2}, \quad n \geq 2,$$

Subsequently, Catarino and Vasco [4] introduced the association of Pell numbers as follows: *The k-Pell-Lucas numbers* $Q_{k,n}$ are defined by the recurrence relations

$$Q_{k,0} = 2, Q_{k,1} = 2, \quad \text{and} \quad Q_{k,n} = 2Q_{k,n-1} + kQ_{k,n-2}, \quad n \geq 2,$$

In 2019, Trojnar-Spenlina and Włoch [5] introduced the third kind of one-parameter generalization of Pell numbers as follows: The *generalized Pell numbers* $\mathcal{P}_{k,n}$ are defined by the recurrence relations

$$\mathcal{P}_{k,0} = 0, \mathcal{P}_{k,1} = 1, \quad \text{and} \quad \mathcal{P}_{k,n} = k\mathcal{P}_{k,n-1} + (k-1)\mathcal{P}_{k,n-2}, \quad n \geq 2, \tag{1}$$

Subsequently, the association of generalized Pell numbers $\mathcal{Q}_{k,n}$, so-called *generalized Pell-Lucas numbers*, are defined by the recurrence relations

$$\mathcal{Q}_{k,0} = 2, \mathcal{Q}_{k,1} = 2, \quad \text{and} \quad \mathcal{Q}_{k,n} = k\mathcal{Q}_{k,n-1} + (k-1)\mathcal{Q}_{k,n-2}, \quad n \geq 2,$$

In the paper [6], the other associated numbers of generalized Pell numbers, which are the so-called *generalized modified Pell numbers* $q_{k,n}$, are defined by the recurrence relations

$$q_{k,0} = 1, q_{k,1} = 1, \quad \text{and} \quad q_{k,n} = kq_{k,n-1} + (k-1)q_{k,n-2}, \quad n \geq 2,$$

It is known that $\mathcal{Q}_{k,n} = 2q_{k,n}$.

Recall that a pair of Lucas sequences $(\{U_n\}, \{V_n\})$ [7] is defined by the formulas

$$U_0 = 0, U_1 = 1, \quad \text{and} \quad U_n = \alpha U_{n-1} - \beta U_{n-2}, \quad n \geq 2,$$

$$V_0 = 2, V_1 = \alpha, \quad \text{and} \quad V_n = \alpha V_{n-1} - \beta V_{n-2}, \quad n \geq 2.$$

where α and β are integers such that the discriminant $\Delta = \alpha^2 + 4\beta \neq 0$. In this case, $\{U_n\}$ and $\{V_n\}$ are called the Lucas sequences of the first and second kinds, respectively. Note that $(\{F_{k,n}\}, \{L_{k,n}\})$ and $(\{P_{k,n}\}, \{Q_{k,n}\})$ are included in the more general definition by assuming $(\alpha, \beta) = (k, -1)$ and $(\alpha, \beta) = (2, -k)$, respectively. However, the third kind of one-parameter generalization of Pell numbers $(\{\mathcal{P}_{k,n}\}, \{\mathcal{Q}_{k,n}\})$ and $(\{\mathcal{P}_{k,n}\}, \{q_{k,n}\})$ are not a pair of Lucas sequences. If we define $\mathcal{L}_{k,n}$ by

$$\mathcal{L}_{k,0} = 2, \mathcal{L}_{k,1} = k, \quad \text{and} \quad \mathcal{L}_{k,n} = k\mathcal{L}_{k,n-1} + (k-1)\mathcal{L}_{k,n-2}, \quad n \geq 2,$$

then it is a convenient Lucas sequence of the second kind such that $(\{\mathcal{P}_{k,n}\}, \{\mathcal{L}_{k,n}\})$ is a pair of Lucas sequences with $(\alpha, \beta) = (k, 1-k)$ to consider it to be an associated number of $\mathcal{P}_{k,n}$. Here $\mathcal{L}_{k,n}$ is called *generalized Pell-Lucas-like*. The tables presented below contain initial terms of the sequences $\{\mathcal{L}_{k,n}\}$ for selected values k (Table 1).

Table 1: Initial terms of the generalized Pell-Lucas-like numbers $\{\mathcal{L}_{k,n}\}$.

n	0	1	2	3	4	5	6	7	8	9
$\mathcal{L}_{2,n}$	2	2	6	14	34	82	198	478	1154	2786
$\mathcal{L}_{3,n}$	2	3	13	45	161	573	2041	7269	25889	92205
$\mathcal{L}_{4,n}$	2	4	22	100	466	2164	10054	46708	216994	1008100
$\mathcal{L}_{5,n}$	2	5	33	185	1057	6025	34353	195865	1116737	6367145
$\mathcal{L}_{6,n}$	2	6	46	306	2066	13926	93886	632946	4267106	28767366

For $k = 2$, we can see that the classical Pell–Lucas numbers are obtained. Moreover, sequences $\{\mathcal{L}_{2,n}\}$, $\{\mathcal{L}_{3,n}\}$, and $\{\mathcal{L}_{4,n}\}$ are listed in The Online Encyclopaedia of Integer Sequences [8] under the symbols A002203, A206776, and A080042, respectively.

In this paper, we study all the third kind of one-parameter generalization of Pell numbers that preserve the recurrence relation (1) with arbitrary initial conditions. We see that the generalized Pell-Lucas-like is a simple association of generalized Pell numbers. Consequently, we give some new and well-known identities. Furthermore, we propose integral representations of these numbers associated with generalized Pell and Pell-Lucas-like numbers.

2. The companion numbers of generalized Pell numbers

In this section, we point out the third kind of generalized Pell numbers to study a generalization of the Pell numbers with one parameter positive integer $k \geq 2$ which is called *the companion generalized Pell number*, denoted by $\mathcal{GP}_{k,n} = \mathcal{GP}_{k,n}(a, b)$, defined by a recurrence relation

$$\mathcal{G}_{k,0} = a, \mathcal{GP}_{k,1} = b, \quad \text{and} \quad \mathcal{GP}_{k,n} = k\mathcal{GP}_{k,n-1} + (k-1)\mathcal{GP}_{k,n-2}, \quad n \geq 2, \qquad (2)$$

where a and b are arbitrary non-negative integers such that $a + b \neq 0$. Note that $\mathcal{GP}_{k,n}$ correspond to special cases of Horadam numbers [9]. The first terms $\mathcal{GP}_{k,n}$ are:

$$\mathcal{GP}_{k,0} = a$$
$$\mathcal{GP}_{k,1} = b$$

$$
\begin{aligned}
\mathcal{GP}_{k,2} &= (a+b)k - a \\
\mathcal{GP}_{k,3} &= (a+b)k^2 - (a-b)k - b \\
\mathcal{GP}_{k,4} &= (a+b)k^3 + 2bk^2 - 2(a+b)k + a \\
\mathcal{GP}_{k,5} &= (a+b)k^4 + (a+3b)k^3 - 2(2a+b)k^2 + (a-2b)k + b \\
\mathcal{GP}_{k,6} &= (a+b)k^5 + 2(a+2b)k^4 - (5a+b)k^3 - (a+6b)k^2 + 3(a+b)k - a.
\end{aligned}
$$

Some particular cases of the previous definition are

(i) $\mathcal{P}_{k,n} = \mathcal{GP}_{k,n}(0,1)$,

(ii) $\mathcal{L}_{k,n} = \mathcal{GP}_{k,n}(2,k)$,

(iii) $\mathcal{Q}_{k,n} = \mathcal{GP}_{k,n}(2,2)$,

(iv) $q_{k,n} = \mathcal{GP}_{k,n}(1,1)$,

(v) generalized Pell numbers introduced in [10], $H_n^{a,b} = \mathcal{GP}_{2,n}(a,b)$,

(vi) $P_n = \mathcal{P}_{2,n} = \mathcal{GP}_{2,n}(0,1)$,

(vii) $Q_n = \mathcal{L}_{2,n} = \mathcal{Q}_{2,n} = \mathcal{GP}_{2,n}(2,2)$, and

(viii) $q_n = \mathcal{GP}_{2,n}(1,1)$.

The Binet's formula for $\mathcal{GP}_{k,n}$ is given in the following theorem.

Theorem 1 (Binet's formulas)**.** *Let k and n be non-negative integers with $k \geq 2$ and $\Delta_k = \sqrt{k^2+4k-4}$. Then*

$$\mathcal{GP}_{k,n} = \frac{2b - ak + a\Delta_k}{2\Delta_k}\sigma_k^n + \frac{ak - 2b + a\Delta_k}{2\Delta_k}\frac{(1-k)^n}{\sigma_k^n}, \tag{3}$$

where $\sigma_k = \frac{k+\Delta_k}{2}$.

Proof. The recurrence relation (2) generates a characteristic equation of the form

$$r^2 - kr + 1 - k = 0. \tag{4}$$

Since $\Delta_k^2 = k^2 + 4k - 4 = k^2 + 4(k-1) > 0$ for $k \geq 2$, this equation has two roots,

$$r_1 = \frac{k+\Delta_k}{2} = \sigma_k$$

and

$$r_2 = \frac{k-\Delta_k}{2} = \frac{(k-\Delta_k)(k+\Delta_k)}{2(k+\Delta_k)} = \frac{2(1-k)}{(k+\Delta_k)} = \frac{1-k}{\sigma_k}.$$

Note that $r_1 + r_2 = k$, $r_1 - r_2 = \Delta_k$, and $r_1 r_2 = 1 - k$. Therefore, the general term $\mathcal{GP}_{k,n}$ can be expressed as

$$\mathcal{GP}_{k,n} = \alpha r_1^n + \beta r_2^n = \alpha \sigma_k^n + \beta \frac{(1-k)^n}{\sigma_k^n}$$

for some coefficients α and β. Since $\mathcal{GP}_{k,0} = a$ and $\mathcal{GP}_{k,1} = b$, we get

$$\alpha + \beta = a \quad \text{and} \quad \alpha r_1 + \beta r_2 = b.$$

It can be shown that

$$\alpha = \frac{2b - ak + a\Delta_k}{2\Delta_k} \quad \text{and} \quad \beta = \frac{ak - 2b + a\Delta_k}{2\Delta_k}.$$

Therefore, (3) has been proved.

If $(a, b) \in \{(0,1), (2,k), (2,2), (1,1)\}$, then we have the following:

Corollary 1. *Let k and n be non-negative integers with $k \geq 2$ and $\Delta_k = \sqrt{k^2 + 4k - 4}$. Then*

(i) $\mathcal{P}_{k,n} = \frac{1}{\Delta_k}\left(\sigma_k^n - \frac{(1-k)^n}{\sigma_k^n}\right)$,

(ii) $\mathcal{L}_{k,n} = \sigma_k^n + \frac{(1-k)^n}{\sigma_k^n}$,

(iii) $\mathcal{Q}_{k,n} = \left(\frac{2-k+\Delta_k}{\Delta_k}\right)\sigma_k^n + \left(\frac{k-2\Delta_k}{\Delta_k}\right)\frac{(1-k)^n}{\sigma_k^n}$, *and*

(iv) $q_{k,n} = \left(\frac{2-k+\Delta_k}{2\Delta_k}\right)\sigma_k^n + \left(\frac{k-2\Delta_k}{2\Delta_k}\right)\frac{(1-k)^n}{\sigma_k^n}$.

Remark 1. *As in Corollary 1, we get the following:*

- *(i) and (iii) are presented in [5, Corollary 2.3];*
- *(iv) is presented in [6, Theorem 2.3];*
- *the Binet's formula of $\mathcal{L}_{k,n}$ is simpler than that of $\mathcal{Q}_{k,n}$ and $q_{k,n}$.*

If $k = 2$, then we have the following:

Corollary 2. *Let n be a non-negative integer. Then*

$$H_n^{a,b} = \frac{b - a + \sqrt{2}a}{2\sqrt{2}}(1+\sqrt{2})^n + \frac{a - b - \sqrt{2}a}{2\sqrt{2}}(1-\sqrt{2})^n. \tag{5}$$

Proof. Notice that $\mathcal{GP}_{2,n} = H_n^{a,b}$, $\Delta_2 = 2\sqrt{2}$ and $\sigma_2 = 1 + \sqrt{2}$. Setting $k = 2$ in (3), we get (5) which completes the proof.

Theorem 2. *Let k and n be non-negative integers with $k \geq 2$ and $\Delta_k = \sqrt{k^2 + 4k - 4}$. Then the following hold:*

(i) $\mathcal{L}_{k,n} + \Delta_k \mathcal{P}_{k,n} = 2\sigma_k^n$;

(ii) $\mathcal{L}_{k,n} - \Delta_k \mathcal{P}_{k,n} = 2\frac{(1-k)^n}{\sigma_k^n}$;

(iii) $\mathcal{L}_{k,n}^2 - \Delta_k^2 \mathcal{P}_{k,n}^2 = 4(1-k)^n$.

Proof. The conclusions follow from (i) and (ii) of Corollary 1.

Next, we present that the companion generalized Pell numbers are associated with the generalized Pell and generalized Pell-Lucas-like numbers in the following results.

Theorem 3. *Let k and n be non-negative integers with $k \geq 2$. Then*

$$\mathcal{GP}_{k,n} = \frac{a}{2}\mathcal{L}_{k,n} + \frac{2b-ak}{2}\mathcal{P}_{k,n}.$$

Proof. It follows from (3), (i) and (ii) of Theorem 2 that

$$\begin{aligned}\mathcal{GP}_{k,n} &= \left(\frac{2b-ak+a\Delta_k}{2\Delta_k}\right)\sigma_k^n + \left(\frac{ak-2b+a\Delta_k}{2\Delta_k}\right)\frac{(1-k)^n}{\sigma_k^n} \\ &= \left(\frac{2b-ak+a\Delta_k}{2\Delta_k}\right)\left(\frac{\mathcal{L}_{k,n}+\Delta_k \mathcal{P}_{k,n}}{2}\right) + \left(\frac{ak-2b+a\Delta_k}{2\Delta_k}\right)\left(\frac{\mathcal{L}_{k,n}-\Delta_k \mathcal{P}_{k,n}}{2}\right) \\ &= \frac{a}{2}\mathcal{L}_{k,n} + \frac{2b-ak}{2}\mathcal{P}_{k,n}.\end{aligned}$$

This completes the proof.

If $(a,b) \in \{(2,2),(1,1)\}$, then we have the following:

Corollary 3. *Let k and n be non-negative integers with $k \geq 2$. Then*

(i) $\mathcal{Q}_{k,n} = \mathcal{L}_{k,n} - (k-2)\mathcal{P}_{k,n}$;

(ii) $q_{k,n} = \frac{1}{2}\mathcal{L}_{k,n} - \frac{(k-2)}{2}\mathcal{P}_{k,n}$.

If $k = 2$, then we have the following:

Corollary 4. *Let n be a non-negative integer. Then* $H_n^{a,b} = \frac{a}{2}Q_n + (b-a)P_n$.

From Theorem 2 and Corollary 3, we have the following:

Corollary 5 ([11, Lemma 2.1])**.** *Let k and n be non-negative integers with $k \geq 2$ and $\Delta_k = \sqrt{k^2+4k-4}$. Then the following hold:*

(i) $\mathcal{Q}_{k,n} + (k-2+\Delta_k)\,\mathcal{P}_{k,n} = 2\sigma_k^n$;

(ii) $\mathcal{Q}_{k,n} + (k-2-\Delta_k)\,\mathcal{P}_{k,n} = 2\frac{(1-k)^n}{\sigma_k^n}$;

(iii) $(\mathcal{Q}_{k,n} + (k-2)\mathcal{P}_{k,n})^2 - \Delta_k \mathcal{P}_{k,n}^2 = 4(1-k)^n$.

Theorem 4. *Let k, m and n be non-negative integers with $k \geq 2$ and $\Delta_k = \sqrt{k^2+4k-4}$. Then the following hold:*

(i) $2\mathcal{P}_{k,m+n} = \mathcal{P}_{k,m}\mathcal{L}_{k,n} + \mathcal{P}_{k,n}\mathcal{L}_{k,m}$;

(ii) $2\mathcal{L}_{k,m+n} = \mathcal{L}_{k,m}\mathcal{L}_{k,n} + \Delta_k^2\mathcal{P}_{k,m}\mathcal{P}_{k,n}$.

Proof.

(i) Using (i) and (ii) of Corollary 1, we have

$$\begin{aligned}\mathcal{P}_{k,m}\mathcal{L}_{k,n} &= \left[\frac{1}{\Delta_k}\left(\sigma_k^m - \frac{(1-k)^m}{\sigma_k^m}\right)\right]\left(\sigma_k^n + \frac{(1-k)^n}{\sigma_k^n}\right)\\ &= \frac{1}{\Delta_k}\left(\sigma_k^{m+n} + \frac{(1-k)^n\sigma_k^m}{\sigma_k^n} - \frac{(1-k)^m\sigma_k^n}{\sigma_k^m} - \frac{(1-k)^{m+n}}{\sigma_k^{m+n}}\right)\end{aligned}$$

and

$$\begin{aligned}\mathcal{P}_{k,n}\mathcal{L}_{k,m} &= \left[\frac{1}{\Delta_k}\left(\sigma_k^n - \frac{(1-k)^n}{\sigma_k^n}\right)\right]\left(\sigma_k^m + \frac{(1-k)^m}{\sigma_k^m}\right)\\ &= \frac{1}{\Delta_k}\left(\sigma_k^{m+n} + \frac{(1-k)^m\sigma_k^n}{\sigma_k^n} - \frac{(1-k)^n\sigma_k^m}{\sigma_k^n} - \frac{(1-k)^{m+n}}{\sigma_k^{m+n}}\right).\end{aligned}$$

So, we get

$$\begin{aligned}\mathcal{P}_{k,m}\mathcal{L}_{k,n} + \mathcal{P}_{k,n}\mathcal{L}_{k,m} &= \frac{2}{\Delta_k}\left(\sigma_k^{m+n} - \frac{(1-k)^{m+n}}{\sigma_k^{m+n}}\right)\\ &= 2\mathcal{P}_{k,m+n}.\end{aligned}$$

(ii) Using (i) of Corollary 1, we have

$$\begin{aligned}\mathcal{P}_{k,m}\mathcal{P}_{k,n} &= \left[\frac{1}{\Delta_k}\left(\sigma_k^m - \frac{(1-k)^m}{\sigma_k^m}\right)\right]\left[\frac{1}{\Delta_k}\left(\sigma_k^n - \frac{(1-k)^n}{\sigma_k^n}\right)\right]\\ &= \frac{1}{\Delta_k^2}\left(\sigma_k^{m+n} - \frac{(1-k)^n\sigma_k^m}{\sigma_k^n} - \frac{(1-k)^m\sigma_k^n}{\sigma_k^m} + \frac{(1-k)^{m+n}}{\sigma_k^{m+n}}\right).\end{aligned}$$

By using (ii) of Corollary 1, we have

$$\begin{aligned}\mathcal{L}_{k,m}\mathcal{L}_{k,n} &= \left(\sigma_k^m + \frac{(1-k)^m}{\sigma_k^m}\right)\left(\sigma_k^n + \frac{(1-k)^n}{\sigma_k^n}\right)\\ &= \sigma_k^{m+n} + \frac{(1-k)^n\sigma_k^m}{\sigma_k^n} + \frac{(1-k)^m\sigma_k^n}{\sigma_k^m} + \frac{(1-k)^{m+n}}{\sigma_k^{m+n}}.\end{aligned}$$

This implies that

$$\begin{aligned}\mathcal{L}_{k,m}\mathcal{L}_{k,n} + (k^2+4k-4)\mathcal{P}_{k,m}\mathcal{P}_{k,n} &= 2\left(\sigma_k^{m+n} + \frac{(1-k)^{m+n}}{\sigma_k^{m+n}}\right)\\ &= 2\mathcal{L}_{k,m+n}.\end{aligned}$$

Hence, (i) and (ii) complete the proof.

Corollary 6 ([11, Lemma 2.2]). *Let k, m and n be non-negative integers with $k \geq 2$ and $\Delta_k = \sqrt{k^2+4k-4}$. Then the following hold:*

(i) $2\mathcal{P}_{k,m+n} = \mathcal{P}_{k,m}\mathcal{Q}_{k,n} + \mathcal{P}_{k,n}\mathcal{Q}_{k,m} + 2(k-2)\mathcal{P}_{k,m}\mathcal{P}_{k,n}$;

(ii) $2\mathcal{Q}_{k,m+n} = \mathcal{Q}_{k,m}\mathcal{Q}_{k,n} + 8(k-1)\mathcal{P}_{k,m}\mathcal{P}_{k,n}$.

Proof. It follows from (i) of Theorem 4 and (i) of Corollary 3 that

$$\begin{aligned}
2\mathcal{P}_{k,m+n} &= \mathcal{P}_{k,m}\mathcal{L}_{k,n} + \mathcal{P}_{k,n}\mathcal{L}_{k,m} \\
&= \mathcal{P}_{k,m}\left(\mathcal{Q}_{k,n} + (k-2)\mathcal{P}_{k,n}\right) + \mathcal{P}_{k,n}\left(\mathcal{Q}\mathcal{L}_{k,m} + (k-2)\mathcal{P}_{k,m}\right) \\
&= \mathcal{P}_{k,m}\mathcal{Q}_{k,n} + \mathcal{P}_{k,n}\mathcal{Q}_{k,m} + 2(k-2)\mathcal{P}_{k,m}\mathcal{P}_{k,n}.
\end{aligned}$$

Since

$$\begin{aligned}
\mathcal{L}_{k,m}\mathcal{L}_{k,n} &= \left(\mathcal{Q}_{k,m} + (k-2)\mathcal{P}_{k,m}\right)\left(\mathcal{Q}_{k,n} + (k-2)\mathcal{P}_{k,n}\right) \\
&= \mathcal{Q}_{k,m}\mathcal{Q}_{k,n} + (k-2)\mathcal{P}_{k,m}\mathcal{Q}_{k,n} + (k-2)\mathcal{P}_{k,n}\mathcal{Q}_{k,m} + (k-2)^2\mathcal{P}_{k,m}\mathcal{P}_{k,n} \\
&= \mathcal{Q}_{k,m}\mathcal{Q}_{k,n} + (k-2)\left[\mathcal{P}_{k,m}\mathcal{Q}_{k,n} + \mathcal{P}_{k,n}\mathcal{Q}_{k,m} + 2(k-2)\mathcal{P}_{k,m}\mathcal{P}_{k,n}\right] \\
&\quad - (k-2)^2\mathcal{P}_{k,m}\mathcal{P}_{k,n} \\
&= \mathcal{Q}_{k,m}\mathcal{Q}_{k,n} + 2(k-2)\mathcal{P}_{k,m+n} - (k-2)^2\mathcal{P}_{k,m}\mathcal{P}_{k,n},
\end{aligned}$$

we get

$$\begin{aligned}
2\mathcal{Q}_{k,m+n} &= 2\mathcal{L}_{k,m+n} - 2(k-2)\mathcal{P}_{k,m+n} \\
&= \mathcal{L}_{k,m}\mathcal{L}_{k,n} + \left(k^2+4k-4\right)\mathcal{P}_{k,m}\mathcal{P}_{k,n} - 2(k-2)\mathcal{P}_{k,m+n} \\
&= \mathcal{Q}_{k,m}\mathcal{Q}_{k,n} - (k-2)^2\mathcal{P}_{k,m}\mathcal{P}_{k,n} + (k^2+4k-4)\mathcal{P}_{k,m}\mathcal{P}_{k,n} \\
&= \mathcal{Q}_{k,m}\mathcal{Q}_{k,n} + 8(k-1)\mathcal{P}_{k,m}\mathcal{P}_{k,n}.
\end{aligned}$$

Hence, (i) and (ii) complete the proof.

Theorem 5 (Asymptotic behavior). *Let k be a positive integer with $k \geq 2$. Then*

$$\lim_{n\to\infty} \frac{\mathcal{GP}_{k,n+1}}{\mathcal{GP}_{k,n}} = \sigma_k.$$

Proof. By using (3), we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\mathcal{GP}_{k,n+1}}{\mathcal{GP}_{k,n}} &= \lim_{n\to\infty} \frac{\left(\frac{2b-ak+a\Delta_k}{2\Delta_k}\right)\sigma_k^{n+1} + \left(\frac{ak-2b+a\Delta_k}{2\Delta_k}\right)\frac{(1-k)^{n+1}}{\sigma_k^{n+1}}}{\left(\frac{2b-ak+a\Delta_k}{2\Delta_k}\right)\sigma_k^{n} + \left(\frac{ak-2b+a\Delta_k}{2\Delta_k}\right)\frac{(1-k)^{n}}{\sigma_k^{n}}} \\
&= \lim_{n\to\infty} \frac{\left(\frac{2b-ak+a\Delta_k}{2\Delta_k}\right)\sigma_k + \left(\frac{ak-2b+a\Delta_k}{2\Delta_k}\right)\frac{(1-k)^{n}}{\sigma_k^{2n}}\cdot\frac{(1-k)}{\sigma_k}}{\left(\frac{2b-ak+a\Delta_k}{2\Delta_k}\right) + \left(\frac{ak-2b+a\Delta_k}{2\Delta_k}\right)\frac{(1-k)^{n}}{\sigma_k^{2n}}}. \qquad (6)
\end{aligned}$$

Since σ_k is the root of (4), we have $\sigma_k^2 = k\sigma_k + (k-1) > k-1$ and so $\left|\frac{1-k}{\sigma_k^2}\right| < 1$. Then

$$\lim_{n\to\infty} \frac{(1-k)^n}{\sigma_k^{2n}} = \lim_{n\to\infty} \left(\frac{1-k}{\sigma_k^2}\right)^n = 0.$$

This together with (6) gives

$$\lim_{n\to\infty} \frac{\mathcal{GP}_{k,n+1}}{\mathcal{GP}_{k,n}} = \sigma_k.$$

This completes the proof.

If $(a,b) \in \{(2,k),(0,1),(2,2),(1,1)\}$, then we have the following:

Corollary 7 ([5, Lemma 2.4])**.** *Let k be a positive integer with $k \geq 2$. Then*

$$\lim_{n\to\infty} \frac{\mathcal{L}_{k,n+1}}{\mathcal{L}_{k,n}} = \lim_{n\to\infty} \frac{\mathcal{P}_{k,n+1}}{\mathcal{P}_{k,n}} = \lim_{n\to\infty} \frac{\mathcal{Q}_{k,n+1}}{\mathcal{Q}_{k,n}} = \lim_{n\to\infty} \frac{q_{k,n+1}}{q_{k,n}} = \sigma_k.$$

If $k = 2$, then we have the following:

Corollary 8. $\lim_{n\to\infty} \frac{H_{n+1}^{a,b}}{H_n^{a,b}} = 1 + \sqrt{2}$.

Theorem 6 (Catalan's identities)**.** *Let k, n, and r be non-negative integers with $k \geq 2$, $\Delta_k = \sqrt{k^2+4k-4}$, and $n \geq r$. Then*

$$\mathcal{GP}_{k,n-r}\mathcal{GP}_{k,n+r} - \mathcal{GP}_{k,n}^2 = \frac{1}{4}(2b - ak + a\Delta_k)(ak - 2b + a\Delta_k)(1-k)^{n-r}\mathcal{P}_{k,r}^2.$$

Proof. Let $\alpha = \frac{2b-ak+a\Delta_k}{2\Delta_k}$ and $\beta = \frac{ak-2b+a\Delta_k}{2\Delta_k}$. By using (3), we have

$$\begin{aligned}
\mathcal{GP}_{k,n-r}\mathcal{GP}_{k,n+r} &= \left(\alpha\sigma_k^{n-r} + \beta\frac{(1-k)^{n-r}}{\sigma_k^{n-r}}\right)\left(\alpha\sigma_k^{n+r} + \beta\frac{(1-k)^{n+r}}{\sigma_k^{n+r}}\right) \\
&= \alpha^2\sigma_k^{2n} + \beta^2\frac{(1-k)^{2n}}{\sigma_k^{2n}} + \alpha\beta(1-k)^{n-r}\left(\sigma_k^{2r} + \frac{(1-k)^{2r}}{\sigma_k^{2r}}\right)
\end{aligned}$$

and

$$\mathcal{GP}_{k,n}^2 = \left(\alpha\sigma_k^n + \beta\frac{(1-k)^n}{\sigma_k^n}\right)^2 = \alpha^2\sigma_k^{2n} + \beta^2\frac{(1-k)^{2n}}{\sigma_k^{2n}} + \alpha\beta(1-k)^{n-r}\left(2(1-k)^r\right).$$

Then

$$\begin{aligned}
&\mathcal{GP}_{k,n-r}\mathcal{GP}_{k,n+r} - \mathcal{GP}_{k,n}^2 \\
&= \alpha\beta(1-k)^{n-r}\left(\sigma_k^{2r} + \frac{(1-k)^{2r}}{\sigma_k^{2r}} - 2(1-k)^r\right) \\
&= \left(\frac{2b-ak+a\Delta_k}{2\Delta_k}\right)\left(\frac{ak-2b+a\Delta_k}{2\Delta_k}\right)(1-k)^{n-r}\left(\sigma_k^r - \frac{(1-k)^r}{\sigma_k^r}\right)^2
\end{aligned}$$

$$= \frac{1}{4}(2b - ak + a\Delta_k)(ak - 2b + a\Delta_k)\,(1-k)^{n-r}\left[\frac{1}{\Delta_k}\left(\sigma_k^r + \frac{(1-k)^r}{\sigma_k^r}\right)\right]^2$$
$$= \frac{1}{4}(2b - ak + a\Delta_k)(ak - 2b + a\Delta_k)(1-k)^{n-r}\mathcal{P}_{k,r}^2.$$

This completes the proof.

If $(a, b) \in \{(2, k), (0, 1), (2, 2), (1, 1)\}$, then we have the following:

Corollary 9 ([5, Theorem 2.6])**.** *Let k, n, and r be non-negative integers with $k \geq 2$, $\Delta_k = \sqrt{k^2 + 4k - 4}$, and $n \geq r$. Then*

(i) $\mathcal{L}_{k,n-r}\mathcal{L}_{k,n+r} - \mathcal{L}_{k,n}^2 = \Delta_k^2(1-k)^{n-r}\mathcal{P}_{k,r}^2$;

(ii) $\mathcal{P}_{k,n-r}\mathcal{P}_{k,n+r} - \mathcal{P}_{k,n}^2 = -(1-k)^{n-r}\mathcal{P}_{k,r}^2$;

(iii) $\mathcal{Q}_{k,n-r}\mathcal{Q}_{k,n+r} - \mathcal{Q}_{k,n}^2 = -8(1-k)^{n-r+1}\mathcal{P}_{k,r}^2$;

(iv) $q_{k,n-r}q_{k,n+r} - q_{k,n}^2 = -2(1-k)^{n-r+1}\mathcal{P}_{k,r}^2$.

If $k = 2$, then we have the following:

Corollary 10. *Let n and r be non-negative integers with $n \geq r$. Then*

$$H_{n-r}^{a,b}H_{n+r}^{a,b} - (H_n^{a,b})^2 = (b - a + \sqrt{2}a)(a - b - \sqrt{2}a)(-1)^{n-r}P_r^2.$$

Note that $r = 1$ in Theorem 6, the Catalan's identities give Cassini's identities as follows:

Theorem 7 (Cassini's identities)**.** *Let k, n, and r be non-negative integers with $k \geq 2$, $\Delta_k = \sqrt{k^2 + 4k - 4}$, and $n \geq r$. Then*

$$\mathcal{GP}_{k,n-1}\mathcal{GP}_{k,n+1} - \mathcal{GP}_{k,n}^2 = \frac{1}{4}(2b - ak + a\Delta_k)(ak - 2b + a\Delta_k)(1-k)^{n-1}.$$

If $(a, b) \in \{(2, k), (0, 1), (2, 2), (1, 1)\}$, then we have the following:

Corollary 11. *Let k, n, and r be non-negative integers with $k \geq 2$, $\Delta_k = \sqrt{k^2 + 4k - 4}$, and $n \geq r$. Then*

(i) $\mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1} - \mathcal{L}_{k,n}^2 = \Delta_k^2(1-k)^{n-1}$;

(ii) $\mathcal{P}_{k,n-1}\mathcal{P}_{k,n+1} - \mathcal{P}_{k,n}^2 = -(1-k)^{n-1}$;

(iii) $\mathcal{Q}_{k,n-1}\mathcal{Q}_{k,n+1} - \mathcal{Q}_{k,n}^2 = -8(1-k)^n$;

(iv) $q_{k,n-1}q_{k,n+1} - q_{k,n}^2 = -2(1-k)^n$.

If $k = 2$, then we have the following:

Corollary 12. *Let n be a non-negative integer. Then*

$$H_{n-1}^{a,b}H_{n+1}^{a,b} - (H_n^{a,b})^2 = (b - a + \sqrt{2}a)(a - b - \sqrt{2}a)(-1)^{n-1}.$$

3. The integral representation of the generalized Pell numbers

There are several ways to represent the special numbers. One of them is the integral representation; see, for example, [11–22]. The integral representations for Pell and Pell-Lucas numbers are studied by the second author in [17] as follows:

$$P_{\ell n} = \frac{nP_\ell}{2^n} \int_{-1}^{1} (Q_\ell + 2\sqrt{2}P_\ell x)^{n-1} dx$$

and

$$Q_{\ell n} = \frac{1}{2^n} \int_{-1}^{1} (Q_\ell + 2\sqrt{2}(n+1)P_\ell x)(Q_\ell + 2\sqrt{2}P_\ell x)^{n-1} dx,$$

where ℓ and n are non-negative integers. Subsequently, the authors [11, 18, 19] give new integral representations for the general of Pell and Pell-Lucas numbers such as the k-Fibonacci, k-Lucas, k-Pell, k-Pell-Lucas, and the k-Pell-Lucas-like numbers.

In this section, we obtain new integral representations for the companion generalized Pell numbers. We start with the following theorem for the generalized Pell number $\mathcal{P}_{k,\ell n}$ by employing other known relations between the two numbers $\mathcal{P}_{k,\ell}$ and $\mathcal{L}_{k,\ell}$.

Theorem 8. *Let k, ℓ and n be non-negative integers with $k \geq 2$ and $\Delta_k = \sqrt{k^2 + 4k - 4}$. Then*

$$\mathcal{P}_{k,\ell n} = \frac{n\mathcal{P}_{k,\ell}}{2^n \Delta_k} \int_{-\Delta_k}^{\Delta_k} \left(\mathcal{L}_{k,\ell} + \mathcal{P}_{k,\ell}\, x\right)^{n-1} dx. \tag{7}$$

Proof. For $n = 0$ or $\ell = 0$, we have done. Let us assume that $\ell, n > 0$. Using integration by substitution, we get

$$\begin{aligned}\int_{-\Delta_k}^{\Delta_k} (\mathcal{L}_{k,\ell} + \mathcal{P}_{k,\ell} x)^{n-1} dx &= \frac{1}{n\,\mathcal{P}_{k,\ell}} \Big[(\mathcal{L}_{k,\ell} + \mathcal{P}_{k,\ell} x)^n \Big]_{-\Delta_k}^{\Delta_k} \\ &= \frac{1}{n\mathcal{P}_{k,\ell}} \left[(\mathcal{L}_{k,\ell} + \Delta_k \mathcal{P}_{k,\ell})^n - (\mathcal{L}_{k,\ell} - \Delta_k \mathcal{P}_{k,\ell})^n\right].\end{aligned}$$

It follows from (i) and (ii) of Theorem 2 with n replaced with ℓ that

$$\begin{aligned}\int_{-\Delta_k}^{\Delta_k} (\mathcal{L}_{k,\ell} + \mathcal{P}_{k,\ell} x)^{n-1} dx &= \frac{1}{n\mathcal{P}_{k,\ell}} \left[\left(2\sigma_k^\ell\right)^n - \left(2\frac{(1-k)^\ell}{\sigma_k^\ell}\right)^n \right] \\ &= \frac{2^n \Delta_k}{n\mathcal{P}_{k,\ell}} \left[\frac{1}{\Delta_k} \left(\sigma_k^{\ell n} - \frac{(1-k)^{\ell n}}{\sigma_k^{\ell n}} \right) \right].\end{aligned}$$

By using (i) of Corollary 1 with replace n by ℓn, we have

$$\int_{-\Delta_k}^{\Delta_k} (\mathcal{L}_{k,\ell} + \mathcal{P}_{k,\ell} x)^{n-1} dx = \frac{2^n \Delta_k\, \mathcal{P}_{k,\ell n}}{n\mathcal{P}_{k,\ell}}.$$

Then (7) which completes the proof.

Remark 2. *As in Theorem 8, equation* (7) *is equivalent to*

$$\mathcal{P}_{k,\ell n} = \frac{n\mathcal{P}_{k,\ell}}{2^n} \int_{-1}^{1} (\mathcal{L}_{k,\ell} + \Delta_k \mathcal{P}_{k,\ell}\, t)^{n-1}\, dt.$$

In fact, substituting $t = \frac{x}{\Delta_k}$ *produces* $dx = \Delta_k dt$ *and the integration limits are changed to* -1 *and* 1*, respectively.*

The generalized Pell number $P_{k,\ell n}$ by employing the two numbers $\mathcal{P}_{k,\ell}$ and $\mathcal{Q}_{k,\ell}$ is presented as follows:

Corollary 13 ([11, Theorem 2.3])**.** *Let* k*,* ℓ *and* n *be non-negative integers with* $k \geq 2$ *and* $\Delta_k = \sqrt{k^2 + 4k - 4}$*. Then*

$$\mathcal{P}_{k,\ell n} = \frac{n\mathcal{P}_{k,\ell}}{2^n \Delta_k} \int_{-\Delta_k}^{\Delta_k} (\mathcal{Q}_{k,\ell} + (k - 2 + x)\, \mathcal{P}_{k,\ell})^{n-1}\, dx. \tag{8}$$

Proof. From (i) of Corollary 3, we have $\mathcal{L}_{k,\ell} = \mathcal{Q}_{k,\ell} + (k-2)\,\mathcal{P}_{k,\ell}$. This together with (7) that (8) holds.

The integral representations of the generalized Pell number for even and odd orders are shown as follows:

Theorem 9. *Let* k *and* n *be non-negative integers with* $k \geq 2$ *and* $\Delta_k = \sqrt{k^2 + 4k - 4}$*.*

(i) The generalized Pell number $\mathcal{P}_{k,2n}$ *can be represented by*

$$\mathcal{P}_{k,2n} = \frac{nk}{2^n \Delta_k} \int_{-\Delta_k}^{\Delta_k} \left(k^2 + 2k - 2 + kx\right)^{n-1} dx. \tag{9}$$

(ii) The generalized Pell number $\mathcal{P}_{k,2n+1}$ *can be represented by*

$$\mathcal{P}_{k,2n+1} = \frac{1}{2^{n+1}\Delta_k} \int_{-\Delta_k}^{\Delta_k} \big(2k - 2 + (n+1)(k^2 + kx)\big) \left(k^2 + 2k - 2 + k\,x\right)^{n-1} dx.$$

Proof.

(i) Notice that $\mathcal{P}_{k,2} = k$ and $\mathcal{L}_{k,2} = k^2 + 2k - 2$. Setting $\ell = 2$ in (7), we have

$$\mathcal{P}_{k,2n} = \frac{n\mathcal{P}_{k,2}}{2^n \Delta_k} \int_{-\Delta_k}^{\Delta_k} (\mathcal{L}_{k,2} + \mathcal{P}_{k,2}x)^{n-1}\, dx = \frac{nk}{2^n \Delta_k} \int_{-\Delta_k}^{\Delta_k} \left(k^2 + 2k - 2 + kx\right)^{n-1} dx.$$

(ii) Re-indexing n by $n+1$ in (9), we get

$$\mathcal{P}_{k,2n+2} = \frac{(n+1)k}{2^{n+1}\Delta_k} \int_{-\Delta_k}^{\Delta_k} \left(k^2 + 2k - 2 + kx\right)^{n} dx. \tag{10}$$

Using $\mathcal{P}_{k,2n+2} = k\mathcal{P}_{k,2n+1} + (k-1)\mathcal{P}_{k,2n}$ with (9) and (10), we obtain

$$\begin{aligned}\mathcal{P}_{k,2n+1} &= \frac{1}{k}\mathcal{P}_{k,2n+2} - \frac{k-1}{k}\mathcal{P}_{k,2n} \\ &= \frac{(n+1)}{2^{n+1}\Delta_k}\int_{-\Delta_k}^{\Delta_k} \left(k^2+2k-2+kx\right)^n dx - \frac{n(k-1)}{2^n\Delta_k}\int_{-\Delta_k}^{\Delta_k} \left(k^2+2k-2+kx\right)^{n-1} dx \\ &= \frac{1}{2^{n+1}\Delta_k}\int_{-\Delta_k}^{\Delta_k} \left(2k-2+(n+1)(k^2+kx)\right)\left(k^2+2k-2+k\,x\right)^{n-1} dx.\end{aligned}$$

This completes the proof.

Setting $k = 2$ in Theorem 8 and Remark 2, we have the following corollary.

Corollary 14 ([17, Theorem 3.1]). *Let ℓ and n be non-negative integers. Then*

$$P_{\ell n} = \frac{nP_\ell}{2^{n+1}\sqrt{2}}\int_{-2\sqrt{2}}^{2\sqrt{2}} (Q_\ell + P_\ell\, x)^{n-1} dx = \frac{nP_\ell}{2^n}\int_{-1}^{1} (Q_\ell + 2\sqrt{2}\, P_\ell\, x)^{n-1} dx.$$

Next, we provide integral representations for the generalized Pell-Lucas-like number $\mathcal{L}_{k,\ell n}$ based on the two numbers $\mathcal{P}_{k,\ell}$ and $\mathcal{L}_{k,\ell}$.

Theorem 10. *Let k, ℓ and n be non-negative integers with $k \geq 2$ and $\Delta_k = \sqrt{k^2+4k-4}$. Then*

$$\mathcal{L}_{k,\ell n} = \frac{1}{2^n\Delta_k}\int_{-\Delta_k}^{\Delta_k} \left(\mathcal{L}_{k,\ell} + (n+1)\mathcal{P}_{k,\ell}x\right)\left(\mathcal{L}_{k,\ell} + \mathcal{P}_{k,\ell}x\right)^{n-1} dx. \tag{11}$$

Proof. For $n = 0$ or $\ell = 0$, it is easy to see that (11) holds. We assume now that $\ell, n > 0$. We will solve (11) using integration by parts. Let u and v be such that

$$u(x) = \mathcal{L}_{k,\ell} + (n+1)\mathcal{P}_{k,\ell}x \quad \text{and} \quad dv = \left(\mathcal{L}_{k,\ell} + \mathcal{P}_{k,\ell}x\right)^{n-1} dx.$$

Then

$$\begin{aligned}I &= \frac{1}{2^n\Delta_k}\int_{-\Delta_k}^{\Delta_k} \left(\mathcal{L}_{k,\ell} + (n+1)\mathcal{P}_{k,\ell}x\right)\left(\mathcal{L}_{k,\ell} + \mathcal{P}_{k,\ell}x\right)^{n-1} dx \\ &= \frac{1}{n2^n\Delta_k\,\mathcal{P}_{k,\ell}}\Big[\left(\mathcal{L}_{k,\ell} + (n+1)\mathcal{P}_{k,\ell}x\right)\left(\mathcal{L}_{k,\ell} + \mathcal{P}_{k,\ell}x\right)^n\Big]_{-\Delta_k}^{\Delta_k} \\ &\quad - \frac{(n+1)}{n2^n\Delta_k}\int_{-\Delta_k}^{\Delta_k} \left(\mathcal{L}_{k,\ell} + \mathcal{P}_{k,\ell}x\right)^n dx. \end{aligned} \tag{12}$$

Replacing n by $n+1$ in (7) becomes

$$\mathcal{P}_{k,\ell n+\ell} = \frac{(n+1)\mathcal{P}_{k,\ell}}{2^{n+1}\Delta_k}\int_{-\Delta_k}^{\Delta_k} \left(\mathcal{L}_{k,\ell} + \mathcal{P}_{k,\ell}x\right)^n dx$$

and so

$$\frac{2\mathcal{P}_{k,\ell n+\ell}}{n\mathcal{P}_{k,\ell}} = \frac{(n+1)}{n2^n\Delta_k}\int_{-\Delta_k}^{\Delta_k} \left(\mathcal{L}_{k,\ell} + \mathcal{P}_{k,\ell}x\right)^n dx.$$

This together with (12) gives

$$\begin{aligned} I &= \frac{1}{n2^n\Delta_k\,\mathcal{P}_{k,\ell}}\Big[\,(\mathcal{L}_{k,\ell}+(n+1)\Delta_k\mathcal{P}_{k,\ell})\,(\mathcal{L}_{k,\ell}+\Delta_k\mathcal{P}_{k,\ell})^n\,\Big] \\ &\quad -\frac{1}{n2^n\Delta_k\,\mathcal{P}_{k,\ell}}\Big[\,(\mathcal{L}_{k,\ell}-(n+1)\Delta_k\mathcal{P}_{k,\ell})\,(\mathcal{L}_{k,\ell}-\Delta_k\mathcal{P}_{k,\ell})^n\,\Big]-\frac{2\mathcal{P}_{k,\ell n+\ell}}{n\mathcal{P}_{k,\ell}}. \end{aligned} \tag{13}$$

Applying (i) and (ii) of Theorem 2 to (13) gives

$$\begin{aligned} I &= \frac{1}{n2^n\Delta_k\,\mathcal{P}_{k,\ell}}\Big[2^n\sigma_k^{\ell n}\,(\mathcal{L}_{k,\ell}+(n+1)\Delta_k\mathcal{P}_{k,\ell})\,\Big] \\ &\quad -\frac{1}{n2^n\Delta_k\,\mathcal{P}_{k,\ell}}\Big[2^n\frac{(1-k)^{\ell n}}{\sigma_k^{\ell n}}\,(\mathcal{L}_{k,\ell}-(n+1)\Delta_k\mathcal{P}_{k,\ell})\,\Big]-\frac{2\mathcal{P}_{k,\ell n+\ell}}{n\mathcal{P}_{k,\ell}} \\ &= \frac{1}{n\mathcal{P}_{k,\ell}}\left[\frac{1}{\Delta_k}\left(\sigma_k^{\ell n}-\frac{(1-k)^{\ell n}}{\sigma_k^{\ell n}}\right)\mathcal{L}_{k,\ell}+\left(\sigma_k^{\ell n}+\frac{(1-k)^{\ell n}}{\sigma_k^{\ell n}}\right)(n+1)\,\mathcal{P}_{k,\ell}-2\mathcal{P}_{k,\ell n+\ell}\right]. \end{aligned}$$

Using (i) and (ii) of Corollary 1, and (i) of Theorem 4, it follows that

$$\begin{aligned} I &= \frac{1}{n\mathcal{P}_{k,\ell}}\left[\mathcal{P}_{k,\ell n}\mathcal{L}_{k,\ell}+(n+1)\,\mathcal{P}_{k,\ell}\mathcal{L}_{k,\ell n}-2\mathcal{P}_{k,\ell n+\ell}\right] \\ &= \frac{1}{n\mathcal{P}_{k,\ell}}\left[\mathcal{P}_{k,\ell n}\mathcal{L}_{k,\ell}+\mathcal{P}_{k,\ell}\mathcal{L}_{k,\ell n}-2\mathcal{P}_{k,\ell n+\ell}\right]+\mathcal{L}_{k,\ell n} \\ &= \mathcal{L}_{k,\ell n}. \end{aligned}$$

This completes the proof.

Now, new integral representations for the companion generalized Pell numbers associated with the generalized Pell and generalized Pell-Lucas-like numbers are presented as follows:

Theorem 11. *Let k, ℓ and n be non-negative integers with $k \geq 2$ and $\Delta_k = \sqrt{k^2+4k-4}$. The companion generalized Pell numbers $\mathcal{GP}_{k,\ell n}$ are represented by*

$$\mathcal{GP}_{k,\ell n} = \frac{1}{2^{n+1}\Delta_k}\int_{-\Delta_k}^{\Delta_k}(a\mathcal{L}_{k,\ell}+(2b-ak)n\mathcal{P}_{k,\ell}+a(n+1)\mathcal{P}_{k,\ell}x)\,(\mathcal{L}_{k,\ell}+\mathcal{P}_{k,\ell}x)^{n-1}\,dx.$$

Proof. From Theorem 3, we obtain

$$\mathcal{GP}_{k,\ell n} = \frac{a}{2}\mathcal{L}_{k,\ell n}+\frac{2b-ak}{2}\,\mathcal{P}_{k,\ell n}. \tag{14}$$

Applying the integral representations of $\mathcal{P}_{k,\ell n}$ and $\mathcal{L}_{k,\ell n}$ from Theorems 8 and 10 to (14), this completes the proof.

Remark 3. *As in Theorems 3 and 11, we have the following results.*

(i) If $a = 0$, then $\mathcal{G}\mathcal{P}_{k,n} = bP_{k,n}$ and

$$\mathcal{G}\mathcal{P}_{k,\ell n} = \frac{bn\mathcal{P}_{k,\ell}}{2^n}\int_{-\Delta_k}^{\Delta_k}(\mathcal{L}_{k,\ell} + \mathcal{P}_{k,\ell}x)^{n-1}dx.$$

(ii) If $ak = 2b$, then $\mathcal{G}\mathcal{P}_{k,n} = \frac{a}{2}\mathcal{L}_{k,n}$ and

$$\mathcal{G}\mathcal{P}_{k,\ell n} = \frac{a}{2^{n+1}}\int_{-\Delta_k}^{\Delta_k}(\mathcal{L}_{k,\ell} + (n+1)\mathcal{P}_{k,\ell}x)(\mathcal{L}_{k,\ell} + (k+1)P_{k,\ell}x)^{n-1}dx.$$

Setting $(a, b) = (2, k)$ in Theorem 11 and using (i) of Corollary 3, we have the following corollary.

Corollary 15 ([11, Theorem 2.6])**.** *Let k, ℓ and n be non-negative integers with $k \geq 2$ and $\Delta_k = \sqrt{k^2 + 4k - 4}$. Then*

$$\mathcal{Q}_{k,\ell n} = \frac{1}{2^n\Delta_k}\int_{-\Delta_k}^{\Delta_k}(\mathcal{Q}_{k,\ell} + (k-2+x-n(k-2x))\mathcal{P}_{k,\ell})(\mathcal{Q}_{k,\ell} + (k-2+x)\mathcal{P}_{k,\ell})^{n-1}dx.$$

Setting $k = 2$ in Theorem 11, we have the following corollary.

Corollary 16. *Let ℓ and n be non-negative integers. Then*

$$H_{\ell n}^{a,b} = \frac{1}{2^{n+2}\sqrt{2}}\int_{-2\sqrt{2}}^{2\sqrt{2}}(aQ_\ell + 2(b-a)nP_\ell + a(n+1)P_\ell x)\,(Q_\ell + P_\ell x)^{n-1}\,dx.$$

Setting $k = 2$ in Theorem 10 or $(a, b) = (2, 2)$ in Corollary 16, we have the following corollary.

Corollary 17 ([17, Theorem 3.4])**.** *Let ℓ and n be non-negative integers. Then*

$$Q_{\ell n} = \frac{1}{2^{n+1}\sqrt{2}}\int_{-2\sqrt{2}}^{2\sqrt{2}}(\mathcal{Q}_\ell + (n+1)\mathcal{P}_\ell x)\,(\mathcal{Q}_\ell + \mathcal{P}_\ell x)^{n-1}\,dx.$$

Finally, both $\mathcal{P}_{k,\ell n}$ and $\mathcal{L}_{k,\ell n}$ are then used to establish integral representations for $\mathcal{P}_{k,\ell n+r}$ and $\mathcal{L}_{k,\ell n+r}$ as the following theorems.

Theorem 12. *Let k, ℓ, n and r be non-negative integers with $k \geq 2$ and $\Delta_k = \sqrt{k^2 + 4k - 4}$. Then*

$$\mathcal{P}_{k,\ell n+r} = \frac{1}{2^{n+1}\Delta_k}\int_{-\Delta_k}^{\Delta_k}(n\mathcal{P}_{k,\ell}\mathcal{L}_{k,r} + \mathcal{P}_{k,r}\mathcal{L}_{k,\ell} + (n+1)\mathcal{P}_{k,\ell}\mathcal{P}_{k,r}x)\,(\mathcal{L}_{k,\ell} + \mathcal{P}_{k,\ell}x)^{n-1}dx.$$

Proof. Using (i) of Theorem 4 with m and n replaced by ℓn and r respectively, we get

$$\mathcal{P}_{k,\ell n+r} = \frac{1}{2}\mathcal{P}_{k,\ell n}\mathcal{L}_{k,r} + \frac{1}{2}\mathcal{P}_{k,r}\mathcal{L}_{k,\ell n}.$$

Applying the integral representations of $\mathcal{P}_{k,\ell n}$ and $\mathcal{L}_{k,\ell n}$ from Theorems 8 and 10, we obtain

$$\begin{aligned}
&\mathcal{P}_{k,\ell n+r}\\
&= \frac{1}{2}\left(\frac{n\mathcal{P}_{k,\ell}}{2^n\Delta_k}\int_{-\Delta_k}^{\Delta_k}(\mathcal{L}_{k,\ell}+\mathcal{P}_{k,\ell}x)^{n-1}dx\right)\mathcal{L}_{k,r}\\
&\quad+\frac{1}{2}\mathcal{P}_{k,r}\left(\frac{1}{2^n\Delta_k}\int_{-\Delta_k}^{\Delta_k}(\mathcal{L}_{k,\ell}+(n+1)\mathcal{P}_{k,\ell}x)(\mathcal{L}_{k,\ell}+\mathcal{P}_{k,\ell}x)^{n-1}dx\right)\\
&= \frac{1}{2^{n+1}\Delta_k}\int_{-\Delta_k}^{\Delta_k}(n\mathcal{P}_{k,\ell}\mathcal{L}_{k,r}+\mathcal{P}_{k,r}\mathcal{L}_{k,\ell}+(n+1)\mathcal{P}_{k,\ell}\mathcal{P}_{k,r}x)\,(\mathcal{L}_{k,\ell}+\mathcal{P}_{k,\ell}x)^{n-1}dx.
\end{aligned}$$

This completes the proof.

Setting $k=2$ in Theorem 12, we have the following corollary.

Corollary 18 ([17], Theorem 3.5)**.** *Let ℓ, n and r be non-negative integers. Then*

$$P_{\ell n+r} = \frac{1}{2^{n+2}\sqrt{2}}\int_{-2\sqrt{2}}^{2\sqrt{2}}(nP_\ell Q_r + P_r Q_\ell + (n+1)\,P_\ell P_r x)\,(Q_\ell + P_\ell x)^{n-1}dx.$$

Theorem 13. *Let k, ℓ, n and r be non-negative integers with $k\geq 2$ and $\Delta_k=\sqrt{k^2+4k-4}$. Then*

$$\mathcal{L}_{k,\ell n+r} = \frac{1}{2^{n+1}\Delta_k}\int_{-\Delta_k}^{\Delta_k}\left(n\Delta_k^2\mathcal{P}_{k,\ell}\mathcal{P}_{k,r}+\mathcal{L}_{k,\ell}\mathcal{L}_{k,r}+(n+1)\mathcal{P}_{k,\ell}\mathcal{L}_{k,r}x\right)(\mathcal{L}_{k,\ell}+\mathcal{P}_{k,\ell}x)^{n-1}dx.$$

Proof. Using (ii) of Theorem 4 with m and n replaced by ℓn and r respectively, we get

$$\mathcal{L}_{k,\ell n+r} = \frac{1}{2}\mathcal{L}_{k,\ell n}\mathcal{L}_{k,r}+\frac{\Delta_k^2}{2}\mathcal{P}_{k,\ell n}\mathcal{P}_{k,r}.$$

This together with Theorems 8 and 10 gives that the proof is finish.

Setting $k=2$ in Theorem 13, we have the following corollary.

Corollary 19 ([17], Theorem 3.6)**.** *Let ℓ, n and r be non-negative integers. Then*

$$Q_{\ell n+r} = \frac{1}{2^{n+2}\sqrt{2}}\int_{-2\sqrt{2}}^{2\sqrt{2}}(8nP_\ell P_r + Q_\ell Q_r + (n+1)\,P_\ell Q_r x)\,(Q_\ell + P_\ell x)^{n-1}dx.$$

Remark 4. *The integral representations for the companion generalized Pell numbers $\mathcal{GP}_{k,\ell n+r}$ are established by applying Theorems 3, 12 and 13.*

4. Conclusions

This paper presents a comprehensive study on one-parameter generalizations of Pell numbers and their associated sequences, introducing generalized Pell-Lucas-like numbers and their integral representations. The paper further extends known identities, derives Binet-type formulas, and proposes several new integral formulations that encompass and generalize classical results. Our results not only generalize the integral representations of the Pell and Pell-Lucas numbers but also apply to all the companion numbers of generalized Pell numbers.

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