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A New Approach to Vague Soft Rough Topological Spaces

Raed Hatamleh[1], Haitham Qawaqneh[2], Nasir Odat[1], Abdallah Al-Husban[3], Arif Mehmood[4,*], Alaa M. Abd El-latif[5], Walid Abdelfattah[5], M. I. Elashiry[5], Abdelhalim Hasnaoui[5]

[1] *Department of Mathematics, Faculty of Science, Jadara University, P.O. Box 733, Irbid 21110, Jordan*
[2] *Al-Zaytoonah University of Jordan, Amman 11733, Jordan.*
[3] *Department of Mathematics, Faculty of Science and Technology, Irbid National University, P.O. Box: 2600 Irbid, Jordan*
[4] *Department of Mathematics, Institute of Numerical Sciences, Gomal University, Dera Ismail Khan 29050, KPK, Pakistan*
[5] *Department of Mathematics, College of Science, Northern Border University, Arar 91431, Saudi Arabia*

Abstract. In this particular piece of work, the new hybrid concept of vague soft rough set theory is introduced. It is a combination of rough set theory, soft set theory, and vague set theory. Based on this new concept, some definitions and operations are introduced. Furthermore, lower and upper vague soft approximations, vague soft rough positive, vague soft negative, and vague soft boundaries are discussed with examples. In addition, several theorems are presented in terms of lower and upper vague soft approximations, supported by examples for better understanding. Finally, an entirely new mathematical structure known as the vague soft rough topological structure is introduced. The related definitions of open sets, closed sets, closure, and interior, as well as their relationships in vague soft rough topological spaces, are addressed. To enhance understanding of this study, numerous examples are provided.

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Key Words and Phrases: Rough sets, vague soft rough sets, vague soft rough topology. vague soft rough open sets, vague soft rough close sets

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*Corresponding author.
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Email addresses: raed@jadara.edu.jo (R. Hatamleh), h.alqawaqneh@zuj.edu.jo (H. Qawaqneh), nodat@jadara.edu.jo (N. Odat), dralhosban@inu.edu.jo (A. Al-Husban), mehdaniyal@gmail.com (A. Mehmood),
alaa.ali@nbu.edu.sa (A. M. Abd El-latif), walid.abdelfattah@nbu.edu.sa (W. Abdelfattah), mustafa.elashiry@nbu.edu.sa (M. I. Elashiry), abdllhalim.hasanawa@nbu.edu.sa (A. Hasnaoui)

1. Introduction

Zadeh [1] introduced the idea of fuzzy set theory (FST) . Pawlak [2] originated rough set theory (RST). Molodtsov [3] established soft set theory (SST). Maji et al. [4] strengthened SST more influential by showing its applications in practical problems. Maji et al. [5] bridged the holes that exist in [2]. Work on the journey continued because the idea of SST was still in its infancy. Pei and Miao [6] and Chen [7] polished the work of Maji et al. The (IVNSR), which is an advancement over relations such as soft, fuzzy soft, intuitionistic fuzzy soft set theory (IFST), etc., was developed by Broumi et al. [8]. The most valuable structure in mathematics, known as soft topology (ST), was introduced by Cagman et al. [9]. The identical structure was also worked on by Shabir and Naz [10]. Developed ST with additional points known as soft points by Bayramov and Gunduz [11]. The idea of IFST was developed by Atanassov [12]. Untouched findings remained in some cases. Fundamental concepts were touched upon and the concept of intuitionistic fuzzy topology (IFT) was introduced by Bayramov and Gunduz [13]. Type-2 soft sets are a complex structure that Hayat et al. [14] explored. Traditional relationship between a vertex and its neighbors was established by Hayat et al. [15]. The concept of soft set was introduced by Hayat et al. [16] along with TOPSIS and the Shannon entropy. The notion of bipolar soft sets and its foundations were developed by Shabir and Naz [17].

A new access to the bipolar soft set was established by Karaaslan and Karatas [18]. Bipolar soft topology was organized by Ozturk [19]. Numerous soft semi-compact spaces of the journal type were discussed by Al-Shami et al. [20]. Soft pre-open set was used by Al-Shami and EL-Shafei [21] to handle soft compact and Lindelof spaces. Feng etal. [22] made a marriage of rough sets with soft sets. Soft set theory is used, for the first time, to generalize Pawak's rough set model. The authors presented the fundamental properties of SRA and SRS. New types of soft sets such as full soft sets, the concept of soft rough equal relations is presented and concerning properties are examined in between the lines. Li et al,[23] supposed a new kind of SS. Based on them, the authors, proposed soft approximations and explored their characteristics. Soft rough sets are defined and their topological structures are obtained. At the end, the relationship between SRS and topologies has been investigated carefully. Liu et al. [24] proposed a new hybrid models that combined together fuzzy set, soft sets and rough sets. The beauty of these models lie in the fact that it reduces the uncertainties. These models are verified by providing excellent examples. Liu et al. [25] suggested extremely a new hybrid model called N-soft rough sets, which is attachment of rough sets with N-soft sets. On top of that, approximation operators and some useful properties relative to N- soft rough approximation space are exhibited. Alkhazaleh and Marei [26] pointed out some flaws in [22] and showed that these models do work with application of real-life problems. These models were modified and the verification was addressed with real-life problems. Safty et al. [27] presented a new technique for creating a soft approximation as a modification and generalization of Zhaowen et al. approach. Comparisons were pictured out between our approach and previous study. Besides, an application on corona virus has been presented. On the top of that, construction of algorithm and proposed model was made and its application to decision-making problem were pictured out. Zhang et al. [28] installed the concept of IFS and IFSRS. Demirtas et al. [29] presented the notion of inverse soft rough sets by applying the concept of inverse soft sets and soft rough sets. Besides, different approaches were used to discuss the relationships between them. On the top of that development of algorithm was also sorted out to apply it to decision making problems. F. Afzal et al. [30] characterized of bipolar vague soft s-open Sets. Alkhazaleh and Salleh [33] introduced the concept of FSES and in continuation, launched the basic operations related to this new theory. This reference [33] became source of motivation for this study. Abd El-latif [34] introduced a generalized fuzzy soft rough model. The fuzzy soft IP-upper and fuzzy soft IP-lower approximations are two novel fuzzy soft rough approximations that are offered along with their associated qualities. Furthermore, in comparison to what is currently known in the literature, the author was able to lower the fuzzy soft border region. The structure of fuzzy soft ideal rough topologies caused by fuzzy soft sets and fuzzy ideals was finally determined by the author. Atef et al. [35] introduced the concept of complementary soft neighborhoods and presents three types of covering soft rough set (CSR) models. Soft computing techniques introduced in modelling complex data structures have developed tremendously in the past few years. To illustrate this point, we could take several examples of using deep learning in medical image analysis [36], the investigation of IoT-based learning techniques [37], or the development of fractional-order PID controllers to manage the robotic systems [38]. Furthermore, conformable fractional Pareto distribution has found usage in modelling the uncertain systems [**?**]. This effort is fueled by such developments, and our contribution suggests an alternative, a new direction on vague, soft, rough topological spaces that seeks to represent better the imprecise and vague information that is of dynamical space of decision-making. It looks at these models' fundamental characteristics and how they relate to one another. The article also presents the $\triangle$-topological spaces ($\triangle - TS$) approach to CSR, which examines topological features and their interactions, including $\triangle$-open sets, $\triangle$-closed sets, $\triangle$-interior, $\triangle$-closure, $\triangle$-boundary, $\triangle$-neighborhoods, and $\triangle$-limit points. Lastly, utilizing the constructed topologies, a method is suggested to handle uncertainty and resolve Multi-Group Decision Making (MGDM) problems. Ali. Et al. [40] developed new kinds of soft rough sets models by using the concept of near open sets, where the accuracy of approximations is enhanced significantly This article introduced the concept of near soft rough approximations, called "JSR-approximations," for each $J \in \{P, S, \gamma, \alpha, \beta\}$, generalizing several previously introduced concepts. It discusses the properties and relationships of these approximations and compares them with earlier methods. An algorithm is provided for decision-making problems, and its performance is tested on hypothetical data to compare it with existing methods.

The layout of this paper is systematized as follows, Section 2, implies some basic definitions including soft set, rough set, vague soft sets and some operations on vague soft sets which are necessary for the up-coming section. In section 3, lower and UVSA, vague soft rough positive, vague soft negative and vague soft boundary zone are addressed with examples. In addition to this, few theorems are presented in terms of lower and upper vague soft approximations sense and these theorems are supported with examples. In section 4, we defined and studied, vague soft rough topology. Few results are studied on the basis of interior and closure and the linked between interior and closures is also exhibited in few theorems. For better understanding these results are supported with examples. On the top of that, future work and the conclusion of this research work is summarized in section 5.

2. Preliminaries

In this section, we present some basic definitions.

Definition 1. *[21]. Let X be an initial universe, E be a set parameter and $P(X)$ denotes the power set of X. A pair (F, E) is named as soft set , with mapping given by $F : E \longrightarrow P(X)$.*

Definition 2. *[23]. Let X be initial universe, E be a parameter's set. Lower, upper and boundary approximation of E are define as*

$$\check{R}_{(E)} = \bigcup_{x \in X} (\check{R}(x) : \check{R}(x) \subseteq E)$$

$$\check{R}^{-}(E) = \bigcup_{x \in X} (\check{R}(x) : \check{R}(x) \cap E \neq \phi)$$

and

$$B\check{R}(E) = \check{R}^{-}(E) \backslash \check{R}_{-}(E)$$

with $\check{R} \subseteq X \times X$ which indicates over information about element of X.

Definition 3. *[32]. A vague set F on universe of discourse X is defined as,*

$$F = \{(x, \mu_F(x), w_F(x)) : x \in X\},$$

Where $\mu, w : X \longrightarrow]-0, +1[$ and $\leq \mu_F(x), w_F(x) \leq 2^{+}$.

Definition 4. *[31]. Let X be initial universe set and E be a set of parameters. Let $FS(X)$ denotes set of all vague set then, a vague soft set $(\tilde{F}, E)$ over X is a set defined by a set valued function F representing a mapping $\tilde{F} : E \longrightarrow FS(X)$ where $\tilde{F}$ is called approximate function of vague soft set $(\tilde{F}, E)$. That is*

$$(\tilde{F}, E) = \{(e, (x, \mu_{\tilde{F}(e)}(x), w_{\tilde{F}(e)}(x)) : x \in X), e \in E\}$$

With $\mu_{\tilde{F}(e)}(x), w_{\tilde{F}(e)}(x) \in [0, 1]$, respectively called truth membership and falsity-membership function of$\tilde{F}(e)$. Since supremum of each $\mu, w is 1$ so inequality $0 \leq \mu_{\tilde{F}(e)}(x) + w_{\tilde{F}(e)}(x)) \leq 2$ We denoted set of all vague soft sets of X by $FSS(X, E)$.

Definition 5. *[31] Let $(\tilde{F}, E)$ be a vague soft set over universe set X. The complement of $(\tilde{F}, E)$ is denoted by $(\tilde{F}, E)^c$ and is defined by:*

$$(\tilde{F}, E)^c = \{(e, (x, w_{\tilde{F}(e)}(x), \mu_{\tilde{F}(e)}(x)) : x \in X), e \in E\}$$

Obvious that, $((\tilde{F}, E)^c)^c = (\tilde{F}, E)$

Definition 6. *[31] Let$(\tilde{F}_1, E)$ and $(\tilde{F}_2, E)$ be two vague soft sets. $(\tilde{F}_1, E)$ is said to be a vague soft subset of $(\tilde{F}_2, E)$ if*

$$\mu_{\tilde{F}_1(e)}(x) \leq \mu_{\tilde{F}_2(e)}(x), \quad w_{\tilde{F}_1(e)}(x) \geq w_{\tilde{F}_2(e)}(x) \quad \forall e \in E, \forall x \in X$$

It is denoted by, $(\tilde{F}_1, E) \subseteq (\tilde{F}_2, E)$. $(\tilde{F}_1, E)$ is said to be vague soft equal to $(\tilde{F}_2, E)$ if $(\tilde{F}_1, E)$ is a vague soft subset of $(\tilde{F}_2, E)$ and $(\tilde{F}_2, E)$ is a vague soft subset of $(\tilde{F}_1, E)$. It is denoted by, $(\tilde{F}_1, E) = (\tilde{F}_2, E)$.

Definition 7. *[32] Let$(\tilde{F}_1, E)$ and $(\tilde{F}_2, E)$ be two vague soft sets. Then their union is denoted by $(\tilde{F}_1, E) \cup (\tilde{F}_2, E) = (\tilde{F}_3, E)$ and is defined by:*

$$(\tilde{F}_3, E) = \{(e, (x, \mu_{\tilde{F}(e)}(x), w_{\tilde{F}(e)}(x)) : x \in X), e \in E\}$$

Where.

$$\mu_{\tilde{F}(e)}(x) = \max\{\mu_{\tilde{F}_1(e)}(x), \mu_{\tilde{F}_2(e)}(x)\}$$
$$w_{\tilde{F}(e)}(x) = \min\{w_{\tilde{F}_1(e)}(x), w_{\tilde{F}_2(e)}(x)\}$$

Definition 8. *[32] Let$(\tilde{F}_1, E)$ and $(\tilde{F}_2, E)$ be two vague soft sets. Then their intersection is denoted by $(\tilde{F}_1, E) \cap (\tilde{F}_2, E) = (\tilde{F}_3, E)$ and is defined by:*

$$(\tilde{F}_3, E) = \{(e, (x, \mu_{\tilde{F}(e)}(x), w_{\tilde{F}(e)}(x)) : x \in X), e \in E\}$$

Where.

$$\mu_{\tilde{F}(e)}(x) = \min\{\mu_{\tilde{F}_1(e)}(x), \mu_{\tilde{F}_2(e)}(x)\}$$
$$w_{\tilde{F}(e)}(x) = \max\{w_{\tilde{F}_1(e)}(x), w_{\tilde{F}_2(e)}(x)\}$$

Definition 9. *[32] Let$(\tilde{F}_1, E)$ and $(\tilde{F}_2, E)$ be two vague soft sets over the universe set X. Then, $(\tilde{F}_1, E) \setminus (\tilde{F}_2, E) = (\tilde{F}_3, E)$ and is introduced as $(\tilde{F}_3, E) = (\tilde{F}_1, E) \cap (\tilde{F}_2, E)^c$:*

$$(\tilde{F}, E) = \{(e, \langle x, \mu_{\tilde{F}(e)}(x), w_{\tilde{F}(e)}(x) \rangle : x \in X), e \in E\}$$

Where.

$$\mu_{\tilde{F}(e)}(x) = \min\{\mu_{\tilde{F}_1(e)}(x), \mu_{\tilde{F}_2(e)}(x)\}$$
$$w_{\tilde{F}(e)}(x) = \max\{w_{\tilde{F}_1(e)}(x), w_{\tilde{F}_2(e)}(x)\}$$

Definition 10. *[32] 1. A vague soft set $(\tilde{F}, E)$ over the universe set X is said to be a null vague soft set if:*

$$\mu_{\tilde{F}_3(e)}(x) = 0, \quad w_{\tilde{F}_3(e)}(x) = 1, \quad \forall e \in E, \forall x \in X.$$

It is denoted by $0_{(X,E)}$.

2. A vague soft set $(\tilde{F}, E)$ is said to be an absolute vague soft set if:

$$\mu_{\tilde{F}_3(e)}(x) = 1, \quad w_{\tilde{F}_3(e)}(x) = 0,, \quad \forall e \in E, \forall x \in X.$$

It is symbolized as $1_(X, E)$.

3. Characterization of Few Structures in Terms of Vague Soft Rough Sets

In this section, lower and upper vague soft approximations, vague soft rough positive, vague soft negative and vague soft boundary zone are addressed with examples. In addition to this, few theorems are presented in terms of lower and upper vague soft approximations sense and these theorems are supported with examples.

Definition 11. *Let X be a non-empty universal set and E be a set of parameters. . Let $FS(X)$ be the set of all vague set over X. Then $(\tilde{F}, E)$ is a vague soft set over the universe set X with $\tilde{F}$ is a mapping given by $\tilde{F} : E \longrightarrow FS(X)$. Then $(X, \tilde{F}, E)$ is vague soft approximation (VSA) space. The lower and upper vague soft approximation of $\acute{\eta} \subseteq FSS(X, E)$ concerning $(X, \tilde{F}, E)$ are denoted by $\underline{apr}_{FS}(\acute{\eta})$ and $\overline{apr}_{FS}(\acute{\eta})$ respectively, define by;*

$$\underline{apr}_{FS}(\acute{\eta}) = \left\{ \left(e, \left(\frac{x}{\underline{\mu}_{e_i}(x), \underline{w}_{e_i}(x)} \right) \right), \forall e_i \in E, \forall x \in X \right\}$$

$$\overline{apr}_{FS}(\acute{\eta}) = \left\{ \left(e, \left(\frac{x}{\overline{\mu}_{e_i}(x), \overline{w}_{e_i}(x)} \right) \right), \forall e_i \in E, \forall x \in X \right\}$$

$$\underline{\mu}_{e_i}(x) = \{\wedge \mu_{e_i}(x) : \mu_{e_i}(x) \in \acute{\eta} \cap (\tilde{F}_i, E); \forall (\tilde{F}_i, E) \subseteq \acute{\eta}, \forall e_i \in E, \forall x \in X\}$$
$$\underline{w}_{e_i}(x) = \{\vee w_{e_i}(x) : w_{e_i}(x) \in \acute{\eta} \cap (\tilde{F}_i, E); \forall (\tilde{F}_i, E) \subseteq \acute{\eta}, \forall e_i \in E, \forall x \in X\}$$
$$\overline{\mu}_{e_i}(x) = \{\vee \mu_{e_i}(x) : \mu_{e_i}(x) \in \acute{\eta} \cup (\tilde{F}_i, E); \forall (\tilde{F}_i, E) \subseteq \acute{\eta}, \forall e_i \in E, \forall x \in X\}$$
$$\overline{w}_{e_i}(x) = \{\wedge w_{e_i}(x) : w_{e_i}(x) \in \acute{\eta} \cup (\tilde{F}_i, E); \forall (\tilde{F}_i, E) \subseteq \acute{\eta}, \forall e_i \in E, \forall x \in X\}$$

Where $\wedge$ and $\vee$ mean min *and* max *operators, respectively. Since $\underline{apr}_{FS}(\acute{\eta})$ and $\overline{apr}_{FS}(\acute{\eta})$ are two VSS of $FSS(X, E)$. If $\underline{apr}_{FS}(\acute{\eta}) = \overline{apr}_{FS}(\acute{\eta})$ then $\acute{\eta}$ is said to be vague soft definable set; otherwise it is $VSRS$.*

Example 1. *Let's assume $X = \{x_1, x_2, x_3, x_4\}$ represents the students in a classroom or program. Let $E = \{e_1, e_2, e_3\}$ represent education-related attributes or criteria. We consider following vague soft sets;*

$$(\tilde{F}_1, E) = \begin{bmatrix} \left((e_1, (x_1, 3\times10^{-1}, 7\times10^{-1})), (x_3, 6\times10^{-1}, 4\times10^{-1}), (x_4, 7\times10^{-1}, 2\times10^{-1})\right), \\ \left((e_2, (x_2, 6\times10^{-1}, 5\times10^{-1})), (x_3, 4\times10^{-1}, 7\times10^{-1}), (x_4, 2\times10^{-1}, 2\times10^{-1})\right) \end{bmatrix}$$

$$(\tilde{F}_2, E) = \begin{bmatrix} \left((e_1, (x_1, 8\times10^{-1}, 5\times10^{-1})), (x_3, 3\times10^{-1}, 3\times10^{-1})\right), \\ \left((e_2, (x_1, 8\times10^{-1}, 2\times10^{-1})), (x_3, 7\times10^{-1}, 2\times10^{-1})\right), \\ \left((e_3, (x_2, 2\times10^{-1}, 5\times10^{-1})), (x_4, 7\times10^{-1}, 2\times10^{-1})\right) \end{bmatrix}$$

$$(\tilde{F}_3, E) = \begin{bmatrix} \left((e_1, (x_3, 8\times10^{-1}, 3\times10^{-1})), (x_4, 6\times10^{-1}, 3\times10^{-1})\right), \\ \left((e_2, (x_2, 7\times10^{-1}, 3\times10^{-1}), (x_3, 8\times10^{-1}, 1\times10^{-1}), (x_4, 7\times10^{-1}, 1\times10^{-1}))\right), \\ \left((e_3, (x_2, 4\times10^{-1}, 2\times10^{-1}), (x_3, 8\times10^{-1}, 3\times10^{-1}), (x_4, 6\times10^{-1}, 2\times10^{-1}))\right) \end{bmatrix}$$

$$\acute{\eta} = \begin{bmatrix} ((e_1, (x_1, 7\times10^{-1}, 4\times10^{-1}), (x_2, 8\times10^{-1}, 2\times10^{-1}), (x_3, 8\times10^{-1}, 2\times10^{-1}), \\ (x_4, 9\times10^{-1}, 1\times10^{-1}))), \\ \left((e_2, (x_2, 7\times10^{-1}, 3\times10^{-1}), (x_3, 8\times10^{-1}, 1\times10^{-1}), (x_4, 7\times10^{-1}, 1\times10^{-1}))\right), \\ \left((e_3, (x_2, 4\times10^{-1}, 2\times10^{-1}), (x_3, 8\times10^{-1}, 3\times10^{-1}), (x_4, 6\times10^{-1}, 2\times10^{-1}))\right) \end{bmatrix}$$

The lower and upper vague soft approximation of $\acute{\eta}$ *are calculated as;*

$$\underline{apr}_{FS}(\acute{\eta}) = \begin{bmatrix} \left((e_1, (x_1, 3\times10^{-1}, 7\times10^{-1}))\right), \\ \left((e_2, (x_2, 4\times10^{-1}, 7\times10^{-1})), (x_3, 4\times10^{-1}, 7\times10^{-1})\right) \end{bmatrix}$$

$$\overline{apr}_{FS}(\acute{\eta}) = \begin{bmatrix} ((e_1, (x_1, 8\times10^{-1}, 3\times10^{-1}), (x_2, 8\times10^{-1}, 2\times10^{-1}), (x_3, 8\times10^{-1}, 2\times10^{-1}), \\ (x_4, 9\times10^{-1}, 1\times10^{-1}))), \\ \left((e_2, (x_2, 7\times10^{-1}, 3\times10^{-1}), (x_3, 8\times10^{-1}, 1\times10^{-1}), (x_4, 6\times10^{-1}, 1\times10^{-1}))\right), \\ \left((e_3, (x_2, 4\times10^{-1}, 2\times10^{-1}), (x_3, 8\times10^{-1}, 3\times10^{-1}), (x_4, 6\times10^{-1}, 1\times10^{-1}))\right) \end{bmatrix}$$

Remark 1. *For any* $\acute{\eta} \subseteq \tilde{F}SS(X,E)$*, the sets of* $POS_{FS}(\acute{\eta}) = \underline{apr}_{FS}(\acute{\eta})$*,* $FEG_{FS}(\acute{\eta}) = \overline{apr}_{FS}(\acute{\eta})$ *ve* $Bnd_{FS}(\acute{\eta}) = \underline{apr}_{FS}(\acute{\eta}) \backslash \overline{apr}_{FS}(\acute{\eta})$ *are called vague soft rough positive, vague soft rough boundary regions of considered set* $(\acute{\eta})$*, respectively.*

Example 2. *We consider Example 1. Then we can write vague soft rough positive, negative and boundary region as follow;*

$$POS_{FS}(\acute{\eta}) = \underline{apr}_{FS}(\acute{\eta}) = \begin{bmatrix} \left((e_1, (x_1, 5\times10^{-1}, 5\times10^{-1}))\right), \\ \left((e_2, (x_3, 4\times10^{-1}, 7\times10^{-1})), (e_1, (x_4, 2\times10^{-1}, 2\times10^{-1}))\right) \end{bmatrix}$$

$$FEG_{FS}(\acute{\eta}) = \overline{apr}_{FS}(\acute{\eta}) = \begin{bmatrix} ((e_1, (x_1, 8\times10^{-1}, 3\times10^{-1}), (x_2, 8\times10^{-1}, 2\times10^{-1}), \\ (x_3, 8\times10^{-1}, 2\times10^{-1}), (x_4, 9\times10^{-1}, 1\times10^{-1}))), \\ ((e_2, (x_2, 7\times10^{-1}, 3\times10^{-1}), (x_3, 8\times10^{-1}, 1\times10^{-1}), \\ (x_4, 6\times10^{-1}, 1\times10^{-1}))), \\ ((e_3, (x_2, 4\times10^{-1}, 2\times10^{-1}), (x_3, 8\times10^{-1}, 1\times10^{-1}), \\ (x_4, 6\times10^{-1}, 1\times10^{-1}))) \end{bmatrix}$$

$$Bnd_{FS}(\acute{\eta}) = \overline{apr}_{FS}(\acute{\eta}) \backslash \underline{apr}_{FS}(\acute{\eta}) = \begin{bmatrix} ((e_1, (x_1, 8\times10^{-1}, 3\times10^{-1}), (x_2, 8\times10^{-1}, 2\times10^{-1}), \\ (x_3, 8\times10^{-1}, 2\times10^{-1}), (x_4, 9\times10^{-1}, 1\times10^{-1}))), \\ ((e_2, (x_2, 7\times10^{-1}, 3\times10^{-1}), (x_3, 7\times10^{-1}, 4\times10^{-1}), \\ (x_4, 2\times10^{-1}, 2\times10^{-1}))), \\ ((e_3, (x_2, 4\times10^{-1}, 2\times10^{-1}), (x_3, 8\times10^{-1}, 3\times10^{-1}), \\ (x_4, 6\times10^{-1}, 1\times10^{-1}))) \end{bmatrix}$$

Obviously, since it is clear $\overline{apr}_{FS}(\acute{\eta}) \neq \underline{apr}_{FS}(\acute{\eta})$*. So* $\acute{\eta}$ *is vague set in approximation space* $(X, \tilde{F}, E)$*.*

Theorem 1. *Let $(\tilde{F}, E)$ be a vague soft set over X. $(X, \tilde{F}, E)$ be a VSRA space and $\acute{\eta}, \kappa \in FSS(X, E)$, then we have*

(i) $\underline{apr}_{FS}(\acute{\eta}) \subseteq \acute{\eta} \subseteq \overline{apr}_{FS}(\acute{\eta})$.

(ii) $\underline{apr}_{FS}(0_{(X,E)}) = 0_{(X,E)}, \quad \overline{apr}_{FS}(1_{(X,E)}) = 1_{(X,E)}$.

(iii) $If\ \acute{\eta} \subseteq \kappa,\ then\ \underline{apr}_{FS}(\acute{\eta}) \subseteq \underline{apr}_{FS}(\kappa)$.

(iv) $If\ \acute{\eta} \subseteq \kappa,\ then\ \overline{apr}_{FS}(\acute{\eta}) \subseteq \overline{apr}_{FS}(\kappa)$.

(v) $If\ \acute{\eta} \cap \kappa,\ then\ \underline{apr}_{FS}(\acute{\eta}) \cap \underline{apr}_{FS}(\kappa)$.

(vi) $If\ \acute{\eta} \cap \kappa,\ then\ \overline{apr}_{FS}(\acute{\eta}) \cap \overline{apr}_{FS}(\kappa)$.

(vii) $If\ \acute{\eta} \cap \kappa,\ then\ \underline{apr}_{FS}(\acute{\eta}) \cap \underline{apr}_{FS}(\kappa)$.

(viii) $If\ \acute{\eta} \cup \kappa,\ then\ \overline{apr}_{FS}(\acute{\eta}) \cup \overline{apr}_{FS}(\kappa)$.

Proof. (i). From $definition$ 11, it is seen $\underline{apr}_{FS}(\acute{\eta}) \subseteq \acute{\eta}$. Also from definition of vague soft upper approximation,

$$\forall(\tilde{F}_i, E) \cap \acute{\eta} \neq (0_{(X,E)}), \quad \overline{\mu}, \overline{v}, \overline{w} \in \acute{\eta} \cup (\tilde{F}_i, E)$$

Hence $\acute{\eta} \subseteq \overline{apr}_{FS}(\acute{\eta})$. Thus $\underline{apr}_{FS}(\acute{\eta}) \subseteq \acute{\eta} \subseteq \overline{apr}_{FS}(\acute{\eta})$.
(ii). From $definition$11, proof of (ii) is straightforward.
(iii). Let $\acute{\eta} \subseteq \kappa$ and $(\tilde{F}_i, E) \subseteq (\acute{\eta}), i = (1, 2)$. Then $\underline{apr}_{FS}(\acute{\eta}) = \acute{\eta} \cap (\cap_{i=1}^{2})(\tilde{F}_i, E)$. In addition $(\tilde{F}_i, E) \subseteq (\acute{\eta})$ then $(\tilde{F}_i, E) \subseteq (\kappa)$. Hence,

$$(\kappa) = \kappa \cap (\cap_{i=1}^{2})(\tilde{F}_i, E) \ \Rightarrow \underline{apr}_{FS}(\acute{\eta}) \subseteq \underline{apr}_{FS}(\kappa)$$

(iv). Let $\acute{\eta} \subseteq \kappa$ and $(\tilde{F}_i, E) \subseteq (\acute{\eta}) \neq 0, i = (1, 2)$. Then $\overline{apr}_{FS}(\acute{\eta}) = \acute{\eta} \cup (\cup_{i=1}^{2})(\tilde{F}_i, E)$. In addition $(\tilde{F}_i, E) \subseteq (\acute{\eta})$ then $(\tilde{F}_i, E) \subseteq (\kappa)$. Hence,

$$(\kappa) = \kappa \cup (\cup_{i=1}^{2})(\tilde{F}_i, E) \ \Rightarrow \overline{apr}_{FS}(\acute{\eta}) \subseteq \overline{apr}_{FS}(\kappa)$$

(v). Let $x^{e_i}_{(\alpha,\gamma)} \in \underline{apr}_{FS}(\acute{\eta} \cap \kappa)$, there exist $(\tilde{F}, E)$, such that $x^{e_i}_{(\alpha,\gamma)} \in (\tilde{F}, E) \subseteq \underline{apr}_{FS}(\acute{\eta} \cap \kappa)$, $x^{e_i}_{(\alpha,\gamma)} \in (\tilde{F}, E) \subseteq \acute{\eta}$ and $x^{e_i}_{(\alpha,\gamma)} \in (\tilde{F}, E) \subseteq \kappa$, so $x^{e_i}_{(\alpha,\gamma)} \in \underline{apr}_{FS}(\acute{\eta})$ and $(\tilde{F}, E) \subseteq \underline{apr}_{FS}(\kappa)$,

$$\Rightarrow x^{e_i}_{(\alpha,\gamma)} \in \underline{apr}_{FS}(\acute{\eta}) \cap \underline{apr}_{FS}(\kappa)$$

Thus, $\underline{apr}_{FS}(\acute{\eta} \cap \kappa) \subseteq \underline{apr}_{FS}(\acute{\eta}) \cap \underline{apr}_{FS}(\kappa)$.
(vi). Let $x^{e_i}_{(\alpha,\gamma)} \in \overline{apr}_{FS}(\acute{\eta} \cap \kappa)$, $\forall\ (\tilde{F}, E)$, such that, $x^{e_i}_{(\alpha,\gamma)} \in (\tilde{F}, E) \cap (\acute{\eta} \cap \kappa) \neq 0_{(X,E)}, (\tilde{F}, E) \cap (\acute{\eta}) \neq 0_{(X,E)}\ and\ (\tilde{F}, E) \cap (\kappa) \neq 0_{(X,E)}$. So, $x^{e_i}_{(\alpha,\gamma)} \in \overline{apr}_{FS}(\acute{\eta})$ and $x^{e_i}_{(\alpha,\gamma)} \in \overline{apr}_{FS}(\kappa)$

$$\Rightarrow \overline{apr}_{FS}(\acute{\eta}) \cap \overline{apr}_{FS}(\kappa)$$

(vii). Let $x^{e_i}_{(\alpha,\gamma)} \notin \underline{apr}_{FS}(\acute{\eta} \cap \kappa)$, there exist $(\tilde{F}, E)$, such that $x^{e_i}_{(\alpha,\gamma)} \in (\tilde{F}, E) \subsetneq \underline{apr}_{FS}(\acute{\eta} \cap \kappa)$, $\Rightarrow (\tilde{F}, E) \subsetneq \acute{\eta}$ and $x^{e_i}_{(\alpha,\gamma)} \in (\tilde{F}, E) \subsetneq \kappa$. Therefore, $x^{e_i}_{(\alpha,\gamma)} \notin \underline{apr}_{FS}(\acute{\eta})$ and $x^{e_i}_{(\alpha,\gamma)} \notin \underline{apr}_{FS}(\kappa)$,

$$\Rightarrow x^{e_i}_{(\alpha,\gamma)} \notin \underline{apr}_{FS}(\acute{\eta}) \cap \underline{apr}_{FS}(\kappa)$$

Thus, $\underline{apr}_{FS}(\acute{\eta} \cap \kappa) \subseteq \underline{apr}_{FS}(\acute{\eta}) \cup \underline{apr}_{FS}(\kappa)$.
(viii). Let $x^{e_i}_{(\alpha,\gamma)} \in \overline{apr}_{FS}(\acute{\eta} \cap \kappa)$, $\forall$ $(\tilde{F}, E)$, such that, $x^{e_i}_{(\alpha,\gamma)} \in (\tilde{F}, E) \cup (\acute{\eta} \cap \kappa) \neq 0_{(X,E)}$, it folllows that $(\tilde{F}, E) \cap (\acute{\eta}) \neq 0_{(X,E)} and (\tilde{F}, E) \cap (\kappa) \neq 0_{(X,E)}$. So, $x^{e_i}_{(\alpha,\gamma)} \in \overline{apr}_{FS}(\acute{\eta})$ or $x^{e_i}_{(\alpha,\gamma)} \in \overline{apr}_{FS}(\kappa)$. Hence,

$$\Rightarrow x^{e_i}_{(\alpha,\gamma)} \in \overline{apr}_{FS}(\acute{\eta}) \cap \overline{apr}_{FS}(\kappa)$$

. Thus, $\overline{apr}_{FS}(\acute{\eta} \cap \kappa) \subseteq \overline{apr}_{FS}(\acute{\eta}) \cup \overline{apr}_{FS}(\kappa)$.

Theorem 2. *Let $(\tilde{F}, E)$ be a vague soft set over X. $(X, \tilde{F}, E)$ and $\acute{\eta}, \kappa \subseteq FSS(X, E)$. Then following characteristics followed,*

(i) $\underline{apr}_{FS}[\underline{apr}_{FS}(\acute{\eta})] = \underline{apr}_{FS}(\acute{\eta})$.

(ii) $\overline{apr}_{FS}[\overline{apr}_{FS}(\acute{\eta})] \supseteq \overline{apr}_{FS}(\acute{\eta})$.

(iii) $\overline{apr}_{FS}(\acute{\eta}) \supseteq \overline{apr}_{FS}[\overline{apr}_{FS}(\acute{\eta})]$.

(iv) $\overline{apr}_{FS}[\overline{apr}_{FS}(\acute{\eta})] \supseteq \underline{apr}_{FS}(\acute{\eta})$.

(v) $\underline{apr}_{FS}(\acute{\eta}^c) \supseteq [\overline{apr}_{FS}(\acute{\eta})]^c$.

Proof. (i). Let $x^{e_i}_{(\alpha,\gamma)} \in \underline{apr}_{FS}(\acute{\eta})$. Then we have, $x^{e_i}_{(\alpha,\gamma)} \in (\tilde{F}, E) \subseteq \underline{apr}_{FS}(\acute{\eta})$. So, $x^{e_i}_{(\alpha,\gamma)} \in \underline{apr}_{FS}[\underline{apr}_{FS}(\acute{\eta})]$. So, $\underline{apr}_{FS}[\underline{apr}_{FS}(\acute{\eta})] = \underline{apr}_{FS}(\acute{\eta})$. From *theorem*1 $\underline{apr}_{FS}(\acute{\eta}) \subseteq \acute{\eta}$. Using (iii) of *theorem*3.5, we obtain $\underline{apr}_{FS}[\underline{apr}_{FS}(\acute{\eta})] \subseteq \underline{apr}_{FS}(\acute{\eta})$. Hence, $\underline{apr}_{FS}[\underline{apr}_{FS}(\acute{\eta})] = \underline{apr}_{FS}(\acute{\eta})$.
(ii). Let $H = \overline{apr}_{FS}(\acute{\eta})$. Using (i) of *theorem* 1, we got $H \subseteq \overline{apr}_{FS}(\acute{\eta})$. Hence, $\overline{apr}_{FS}[\overline{apr}_{FS}(\acute{\eta})] \supseteq \overline{apr}_{FS}(\acute{\eta})$.
(iii). Let $H = \overline{apr}_{FS}(\acute{\eta})$. Using (i) of *theorem*1, we got $\underline{apr}_{FS}(\acute{\eta}) \subseteq H$. Hence, $\underline{apr}_{FS}[\overline{apr}_{FS}(\acute{\eta})] \supseteq \overline{apr}_{FS}(\acute{\eta})$.
(iv). Let $H = \underline{apr}_{FS}(\acute{\eta})$. Using (i) of *theorem*1, we got $\overline{apr}_{FS}(\acute{\eta}) \supseteq H$. Hence, $\overline{apr}_{FS}(\acute{\eta}) \subseteq \underline{apr}_{FS}[\underline{apr}_{FS}(\acute{\eta})]\overline{apr}_{FS}(\acute{\eta})$.
(v). Let $x^{e_i}_{(\alpha,\gamma)} \notin \underline{apr}_{FS}(\acute{\eta}^c)$ and $x^{e_i}_{(\alpha,\gamma)} \notin (\tilde{F}, E)$. We have that, $(\tilde{F}, E) \subseteq \acute{\eta}^c$ and $(\tilde{F}, E) \cap \acute{\eta}^c = 0_{(X,E)}$. Thus $(\tilde{F}, E) \cap \acute{\eta}^c \neq 0_{(X,E)}$, where $x^{e_i}_{(\alpha,\gamma)} \in \overline{apr}_{FS}(\acute{\eta}^c)$ but $x^{e_i}_{(\alpha,\gamma)} \notin [\underline{apr}_{FS}(\acute{\eta})]^c$. Therefore $\underline{apr}_{FS}(\acute{\eta}^c) \supseteq [\overline{apr}_{FS}(\acute{\eta})]^c$.

4. Vague Soft Rough Topological Spaces (VSRTS) in Term of Open Sets

In this section, VSRTS with some important properties is studied. Few results are studied on the basis of interior and closure and the linked between interior and closures is also exhibited in few theorems. For better understanding these results are supported with examples.

Definition 12. *Let X be a universe and $(X, \tilde{F}, E)$ be a VSAS. Then, vague soft rough topology is defined as;*

$$\tilde{\tau}_{FR}(\acute{\eta}) = \Big(0_{(X,E)}, 1_{(X,E)}, \underline{apr}_{FS}(\acute{\eta}), \overline{apr}_{FS}(\acute{\eta}), Bnd_{FS}(\acute{\eta})\Big)$$

Where $\acute{\eta} \in FSS(X, E)$ and $tilde\tau_{FR}(\acute{\eta})$ satisfies the following conditions

(i) $0_{(X,E)}$ and $1_{(X,E)}$ belongs to $tilde\tau_{FR}(\acute{\eta})$.

(ii) Union of members of any number of $tilde\tau_{FR}(\acute{\eta})$ belongs to $tilde\tau_{FR}(\acute{\eta})$.

(iii) Intersection of members of finite number of $tilde\tau_{FR}(\acute{\eta})$ belongs to $tilde\tau_{FR}(\acute{\eta})$.

The topology defined by $tilde\tau_{FR}(\acute{\eta})$ on X is called vague soft rough topology on X and $[X, \tilde{\tau}_{FR}(\acute{\eta}), E]$ is said to be vague soft rough topological space.

Example 3. *Let $X = \{x_1, x_2, x_3, x_4\}$: representing students and $E = \{e_1, e_2, e_3\}$: representing educational attributes. Then we can construct the following vague soft sets as;*

$$(\tilde{F}_1, E) = \begin{bmatrix} \big((e_1, (x_1, 4\times10^{-1}, 8\times10^{-1}), (x_3, 6\times10^{-1}, 5\times10^{-1}))\big), \\ \big((e_2, (x_1, 5\times10^{-1}, 4\times10^{-1}), (x_3, 3\times10^{-1}, 5\times10^{-1}), (x_4, 3\times10^{-1}, 5\times10^{-1}))\big) \end{bmatrix}$$

$$(\tilde{F}_2, E) = \begin{bmatrix} \big((e_1, (x_1, 7\times10^{-1}, 3\times10^{-1}), (x_2, 7\times10^{-1}, 4\times10^{-1}), (x_4, 4\times10^{-1}, 8\times10^{-1}))\big), \\ \big((e_2, (x_1, 7\times10^{-1}, 1\times10^{-1}), (x_3, 4\times10^{-1}, 1\times10^{-1}))\big) \end{bmatrix}$$

$$(\tilde{F}_3, E) = \begin{bmatrix} \big((e_1, (x_1, 7\times10^{-1}, 1\times10^{-1}), (x_3, 4\times10^{-1}, 1\times10^{-1}))\big), \\ \big((e_2, (x_1, 7\times10^{-1}, 5\times10^{-1}), (x_3, 4\times10^{-1}, 3\times10^{-1}))\big), \\ \big((e_3, (x_1, 5\times10^{-1}, 4\times10^{-1}), (x_3, 4\times10^{-1}, 1\times10^{-1}), (x_4, 3\times10^{-1}, 3\times10^{-1}))\big) \end{bmatrix}$$

For another vague soft set $\acute{\eta} \subset FSS(X, E)$ with respect to $(X, \tilde{F}, E)$;

$$\acute{\eta} = \begin{bmatrix} ((e_1, (x_1, 7\times10^{-1}, 2\times10^{-1}), (x_2, 9\times10^{-1}, 1\times10^{-1}), (x_3, 6\times10^{-1}, 3\times10^{-1}), \\ (x_4, 5\times10^{-1}, 2\times10^{-1}))), \\ ((e_2, (x_2, 6\times10^{-1}, 1\times10^{-1}), (x_2, 8\times10^{-1}, 3\times10^{-1}), (x_3, 5\times10^{-1}, 1\times10^{-1}), \\ (x_4, 4\times10^{-1}, 1\times10^{-1}))), \\ \big((e_3, (x_1, 7\times10^{-1}, 1\times10^{-1}), (x_2, 5\times10^{-1}, 1\times10^{-1}), (x_4, 3\times10^{-1}, 2\times10^{-1}))\big) \end{bmatrix}$$

The lower and upper vague soft approximation of $\acute{\eta}$ are calculated as

$$\underline{apr}_{FS}(\acute{\eta}) = \big[\big((e_1, (x_3, 5 \times 10^{-1}, 5 \times 10^{-1})), (e_1(x_2, 3 \times 10^{-1}, 5 \times 10^{-1}))\big)\big]$$

$$\overline{apr}_{FS}(\acute{\eta}) = \begin{bmatrix} ((e_1, (x_1, 7 \times 10^{-1}, 2 \times 10^{-1}), (x_2, 9 \times 10^{-1}, 1 \times 10^{-1}), (x_3, 6 \times 10^{-1}, 3 \times 10^{-1}), \\ (x_4, 5 \times 10^{-1}, 2 \times 10^{-1}))), \\ ((e_2, (x_2, 6 \times 10^{-1}, 1 \times 10^{-1}), (x_2, 8 \times 10^{-1}, 3 \times 10^{-1}), (x_3, 5 \times 10^{-1}, 1 \times 10^{-1}), \\ (x_4, 4 \times 10^{-1}, 1 \times 10^{-1}))), \\ \big((e_3, (x_1, 7 \times 10^{-1}, 1 \times 10^{-1}), (x_2, 5 \times 10^{-1}, 1 \times 10^{-1}), (x_4, 3 \times 10^{-1}, 2 \times 10^{-1}))\big) \end{bmatrix}$$

$$Bnd_{FS}(\acute{\eta}) = \begin{bmatrix} ((e_1, (x_1, 7 \times 10^{-1}, 2 \times 10^{-1}), (x_2, 9 \times 10^{-1}, 1 \times 10^{-1}), (x_3, 5 \times 10^{-1}, 5 \times 10^{-1}), \\ (x_4, 5 \times 10^{-1}, 2 \times 10^{-1}))), \\ ((e_2, (x_2, 6 \times 10^{-1}, 1 \times 10^{-1}), (x_2, 8 \times 10^{-1}, 3 \times 10^{-1}), (x_3, 5 \times 10^{-1}, 1 \times 10^{-1}), \\ (x_4, 4 \times 10^{-1}, 3 \times 10^{-1}))), \\ \big((e_3, (x_1, 7 \times 10^{-1}, 1 \times 10^{-1}), (x_2, 5 \times 10^{-1}, 1 \times 10^{-1}), (x_4, 3 \times 10^{-1}, 2 \times 10^{-1}))\big) \end{bmatrix}$$

Then

$$\tilde{\tau}_{FR}(\acute{\eta}) = \Big(0_{(X,E)}, 1_{(X,E)}, \underline{apr}_{FS}(\acute{\eta}), \overline{apr}_{FS}(\acute{\eta}), Bnd_{FS}(\acute{\eta})\Big)$$

is VSRT on X.

Definition 13. *Let $(X, \tilde{\tau}_{FR}(\acute{\eta}), E)$ be a VSRTS. Then, each number of $\tilde{\tau}_{FR}(\acute{\eta})$ are vague soft rough open sets (VSROSs). A vague soft rough set is said to be a vague soft closed set if its complement belong to X, $\tilde{\tau}_{FR}(\acute{\eta})$.*

Theorem 3. *Consider $(X, \tilde{\tau}_{FR}(\acute{\eta}), E)$ as VSRTS. Then,*

(i) $0_{(X,E)}$ *and* $1_{(X,E)}$ *are vague soft rough closed sets (VSROSs).*

(ii) The intersection of any number of vague soft rough closed sets(VSRCSs) is a vague soft rough closed set over X.

Finite union of VSRCS is VSRCS over X.

Proof. Trivial.

Definition 14. *Let $(X, \tilde{\tau}_{FR}(\acute{\eta}), E)$ be VSRTS and $\omega \in FSS(X, E)$. Then*

$$\tilde{\tau}_{FR}(\acute{\eta}) = \Big(\omega \cap (\tilde{F}_i, E) : (\tilde{F}_i, E) \in \tilde{\tau}_{FR}(\acute{\eta}) fori \in I\Big)$$

is vague soft rough subspace topology on ω and $(X, \tilde{\tau}_{FR}(\acute{\eta}), E)$ is vague soft rough subspace of $[X, \tilde{\tau}_{FR}(\acute{\eta}), E]$.

Definition 15. *Let $(X, \tilde{\tau}_{FR}(\acute{\eta}), E)$ be a vague soft rough topological space over X and $\omega \in FSS(X, E)$. Then the vague soft rough interior of ω is vague soft union of all vague soft open subsets of ω and we denoted it as $Int_{FSR}(\omega)$.Clearly, $Int_{FSR}(\omega)$ is the largest vague soft rough open set contained by (ω).*

Example 4. *Let consider example* 4.2. *Suppose that any* $\omega \in FSS(X,E)$ *be defined as follow;*

$$\omega = \begin{bmatrix} ((e_1,(x_1,8\times10^{-1},1\times10^{-1}),(x_2,9\times10^{-1},1\times10^{-1}),(x_3,6\times10^{-1},4\times10^{-1}),\\ (x_4,6\times10^{-1},1\times10^{-1}))),\\ ((e_2,(x_2,6\times10^{-1},1\times10^{-1}),(x_2,9\times10^{-1},2\times10^{-1}),(x_3,6\times10^{-1},1\times10^{-1}),\\ (x_4,5\times10^{-1},2\times10^{-1}))),\\ ((e_3,(x_1,8\times10^{-1},1\times10^{-1}),(x_2,5\times10^{-1},3\times10^{-1}),(x_4,6\times10^{-1},1\times10^{-1}),\\ (x_4,4\times10^{-1},2\times10^{-1}))) \end{bmatrix}$$

Then

$$0_{(X,E)}, 1_{(X,E)}, \underline{apr}_{FS}(\acute{\eta}), \overline{apr}_{FS}(\acute{\eta}), Bnd_{FS}(\acute{\eta} \subseteq \omega$$

Therefore,

$$Int_{FSR}(\omega) = 0_{(X,E)} \cup \underline{apr}_{FS}(\acute{\eta}) \cup Bnd_{FS}(\acute{\eta} = Bnd_{FS}(\acute{\eta}$$

Theorem 4. *Let* $(X, \tilde{\tau}_{FR}(\acute{\eta}), E)$ *be a vague soft rough topological space over* X *and* $\omega, \pounds \in FSS(X,E)$. *Then*

(i) $Int_{FSR}[0_{(X,E)}] = 0_{(X,E)}$ *and* $Int_{FSR}[1_{(X,E)}] = 1_{(X,E)}$,

(ii) $Int_{FSR}(\omega) \subseteq \omega$,

(iii) ω *is a vague soft rough open set* $\Leftrightarrow Int_{FSR}(\omega) = \omega$,

(iv) $\omega \subseteq \pounds \Rightarrow Int_{FSR}(\omega) \subseteq Int_{FSR}(\pounds)$,

(v) $Int_{FSR}(\omega) \cap Int_{FSR}(\pounds) = Int_{FSR}(\omega \cap \pounds)$,

(vi) $Int_{FSR}(\omega) \cup Int_{FSR}(\pounds) = Int_{FSR}(\omega \cup \pounds)$.

Proof. (i) and (ii) are obvious.
(iii). Suppose that $Int_{FSR}(\omega) = \omega$. Since $Int_{FSR}(\omega)$ is a vague soft rough open set ω is a vague soft rough open set. Conversely, if ω is a vague soft rough open set, then the largest vague set rough open set contained in ω is ω itself. Thus $Int_{FSR}(\omega) = \omega$.
(iv). Suppose that $\omega \subseteq \pounds$. By the condition (ii), $Int_{FSR}(\omega) \subseteq \omega \subseteq \pounds$. Since $Int_{FSR}(\omega)$ is the largest vague a in ω and so $Int_{FSR}(\omega) \subseteq Int_{FSR}(\pounds)$.
(v). Since $\omega \cap \pounds \subseteq \omega$ and $\omega \cap \pounds \subseteq \pounds$ then $Int_{FSR}(\omega \cap \pounds) \subseteq Int_{FSR}(\omega)$ and $Int_{FSR}(\omega \cap \pounds) \subseteq Int_{FSR}(\pounds)$. Hence $Int_{FSR}(\omega \cap \pounds) \subseteq Int_{FSR}(\omega) \cap Int_{FSR}(\pounds)$. On the other hand, since $Int_{FSR}(\omega) \subseteq Int_{FSR}(\omega)$ and $Int_{FSR}(\pounds) \subseteq Int_{FSR}(\pounds)$ then $Int_{FSR}(\omega) \cap Int_{FSR}(\pounds) \subseteq \omega \cap \pounds$. Besides $Int_{FSR}(\omega \cap \pounds) \subseteq \omega \cap \pounds$ and it is the largest vague soft rough open set. Therefore, $Int_{FSR}(\omega) \cap Int_{FSR}(\pounds) \subseteq Int_{FSR}(\omega \cap \pounds)$. Thus $Int_{FSR}(\omega) \cap Int_{FSR}(\pounds) = Int_{FSR}(\omega \cap \pounds)$.
(vi). Since $\omega \subseteq \omega \cup \pounds$ and $\pounds \subseteq \omega \cup \pounds$ then $Int_{FSR}(\omega) \subseteq Int_{FSR}(\omega \cup \pounds)$ and $Int_{FSR}(\pounds) \subseteq Int_{FSR}(\omega \cup \pounds)$. Therefore, $Int_{FSR}(\omega) \cup Int_{FSR}(\pounds) = Int_{FSR}(\omega \cup \pounds)$.

Definition 16. *Let* $(X, \tilde{\tau}_{FR}(\acute{\eta}), E)$ *be a vague soft rough topological space over* X *and* $\omega \in FSS(X,E)$. *Then the vague soft rough closure of* ω *is vague soft intersection of all vague soft closed subsets of* ω *and we denoted it as* $cl_{FSR}(\omega)$.

Example 5. *Let consider the vague soft rough topology $\tilde{\tau}_{FR}(\acute{\eta})$. Suppose that any $\omega \in FSS(X, E)$ be defined as follow;*

$$\omega = \begin{bmatrix} ((e_1, (x_1, 1\times 10^{-1}, 8\times 10^{-1},), (x_2, 1\times 10^{-1}, 9\times 10^{-1}), (x_3, 2\times 10^{-1}, 7\times 10^{-1}), \\ (x_4, 2\times 10^{-1}, 6\times 10^{-1}))), \\ ((e_2, (x_2, 1\times 10^{-1}, 7\times 10^{-1}), (x_2, 2\times 10^{-1}, 9\times 10^{-1}), (x_3, 1\times 10^{-1}, 7\times 10^{-1}), \\ (x_4, 1\times 10^{-1}, 7\times 10^{-1}))), \\ ((e_3, (x_1, 1\times 10^{-1}, 8\times 10^{-1}), (x_2, 3\times 10^{-1}, 6\times 10^{-1}), (x_4, 1\times 10^{-1}, 6\times 10^{-1}), \\ (x_4, 2\times 10^{-1}, 4\times 10^{-1}))) \end{bmatrix}$$

Obviously $(0_{(X,E)})^c, (1_{(X,E)})^c, (\underline{apr}_{FS}(\acute{\eta}))^c, (\overline{apr}_{FS}(\acute{\eta}))^c, (Bnd_{FS}(\acute{\eta})^c$ are all vague soft rough closed sets over $(X, \tilde{\tau}_{FR}(\acute{\eta}), E)$. We can calculate these sets as follows,

$$(0_{(X,E)})^c = 0_{(X,E)}, (1_{(X,E)})^c = 1_{(X,E)}$$

$$(\underline{apr}_{FS}(\acute{\eta}))^c = \begin{bmatrix} ((e_1, (x_1, 1\times 10^{-1}, 0\times 10^{-1},), (x_2, 1\times 10^{-1}, 1\times 10^{-1}), (x_3, 5\times 10^{-1}, 5\times 10^{-1}), \\ (x_4, 1\times 10^{-1}, 0\times 10^{-1}))), \\ ((e_2, (x_2, 1\times 10^{-1}, 0\times 10^{-1}), (x_2, 1\times 10^{-1}, 0\times 10^{-1}), (x_3, 1\times 10^{-1}, 0\times 10^{-1}), \\ (x_4, 5\times 10^{-1}, 3\times 10^{-1}))), \\ ((e_3, (x_1, 1\times 10^{-1}, 0\times 10^{-1}), (x_2, 1\times 10^{-1}, 0\times 10^{-1}), (x_4, 1\times 10^{-1}, 0\times 10^{-1}), \\ (x_4, 1\times 10^{-1}, 0\times 10^{-1}))) \end{bmatrix}$$

$$(\overline{apr}_{FS}(\acute{\eta}))^c = \begin{bmatrix} ((e_1, (x_1, 7\times 10^{-1}, 2\times 10^{-1},), (x_2, 1\times 10^{-1}, 9\times 10^{-1}), (x_3, 3\times 10^{-1}, 6\times 10^{-1}), \\ (x_4, 2\times 10^{-1}, 5\times 10^{-1}))), \\ ((e_2, (x_2, 1\times 10^{-1}, 6\times 10^{-1}), (x_2, 3\times 10^{-1}, 8\times 10^{-1}), (x_3, 1\times 10^{-1}, 5\times 10^{-1}), \\ (x_4, 1\times 10^{-1}, 6\times 10^{-1}))), \\ ((e_3, (x_1, 1\times 10^{-1}, 7\times 10^{-1}), (x_2, 1\times 10^{-1}, 0\times 10^{-1}), (x_4, 1\times 10^{-1}, 8\times 10^{-1}), \\ (x_4, 2\times 10^{-1}, 3\times 10^{-1}))) \end{bmatrix}$$

$$(Bnd_{FS}(\acute{\eta}))^c = \begin{bmatrix} ((e_1, (x_1, 7\times 10^{-1}, 2\times 10^{-1},), (x_2, 1\times 10^{-1}, 9\times 10^{-1}), (x_3, 5\times 10^{-1}, 5\times 10^{-1}), \\ (x_4, 2\times 10^{-1}, 5\times 10^{-1}))), \\ ((e_2, (x_2, 1\times 10^{-1}, 6\times 10^{-1}), (x_2, 3\times 10^{-1}, 8\times 10^{-1}), (x_3, 1\times 10^{-1}, 5\times 10^{-1}), \\ (x_4, 3\times 10^{-1}, 4\times 10^{-1}))), \\ ((e_3, (x_1, 1\times 10^{-1}, 7\times 10^{-1}), (x_2, 1\times 10^{-1}, 0\times 10^{-1}), (x_4, 1\times 10^{-1}, 8\times 10^{-1}), \\ (x_4, 2\times 10^{-1}, 3\times 10^{-1}))) \end{bmatrix}$$

Then $(\omega) \subseteq (0_{(X,E)})^c, (1_{(X,E)})^c, (\underline{apr}_{FS}(\acute{\eta}))^c, (\overline{apr}_{FS}(\acute{\eta}))^c, (Bnd_{FS}(\acute{\eta})^c$. Therefore,

$$cl_{FSR}(\omega) \subseteq (0_{(X,E)})^c \cap (\underline{apr}_{FS}(\acute{\eta}))^c \cap (\overline{apr}_{FS}(\acute{\eta}))^c \cap (Bnd_{FS}(\acute{\eta})^c = (\overline{apr}_{FS}(\acute{\eta})^c$$

Theorem 5. *Let $(X, \tilde{\tau}_{FR}(\acute{\eta}), E)$ be a vague soft rough topological space over X and $\omega, \pounds \in FSS(X, E)$. Then* *(i)* $cl_{FSR}[0_{(X,E)}] = 0_{(X,E)}$ *and* $cl_{FSR}[1_{(X,E)}] = 1_{(X,E)}$,

(ii) $cl_{FSR}(\omega) \subseteq \omega$,

(iii) ω *is a vague soft rough open set* $\Leftrightarrow cl_{FSR}(\omega) = \omega$,

(iv) $cl_{FSR}[cl_{FSR}(\omega)] = cl_{FSR}(\omega)$

(v) $\omega \subseteq \pounds \Rightarrow cl_{FSR}(\omega) \subseteq cl_{FSR}(\pounds)$,

(vi) $cl_{FSR}(\omega) \cup Int_{FSR}(\pounds) = cl_{FSR}(\omega \cup \pounds)$,

(vii) $cl_{FSR}(\omega) \cap cl_{FSR}(\pounds) = cl_{FSR}(\omega \cap \pounds)$.

Proof. (i) and (ii) are obvious.
(iii). Consider $cl_{FSR}(\omega) = \omega$. Since $cl_{FSR}(\omega)$ is a VSRCS so, ω is a VSRCS. Conversely suppose that, if ω be a VSCS over ω. Then ω is VSRC superset ω. So that, $\omega = cl_{FSR}(\omega)$.
(iv). Let $cl_{FSR}(\omega) = \pounds$. Then ω is VSRCS. Hence, by the condition (iii), $cl_{FSR}(\pounds) = \pounds$. Therefore, $cl_{FSR}[cl_{FSR}(\omega)] = cl_{FSR}(\omega)$.
(v). We know that $\omega \subseteq cl_{FSR}(\omega)$ and $\pounds \subseteq cl_{FSR}(\pounds)$, and so $\omega \subseteq \pounds \subseteq cl_{FSR}(\omega)$. Since $cl_{FSR}(\omega)$ is the smallest vague containing ω. So $cl_{FSR}(\omega) \subseteq cl_{FSR}(\pounds)$.
(vi). Since $\omega \subseteq \omega \cup \pounds$ and $\pounds \subseteq \omega \cup \pounds$ then $cl_{FSR}(\omega) \subseteq cl_{FSR}(\omega \cup \pounds)$ and $cl_{FSR}(\pounds) \subseteq cl_{FSR}(\omega \cup \pounds)$. Hence $cl_{FSR}(\omega) \cup Int_{FSR}(\pounds) \subseteq cl_{FSR}(\omega \cup \pounds)$. On the other hand, since $\omega \subseteq cl_{FSR}(\omega)$ and $\pounds \subseteq cl_{FSR}(\pounds)$ then $\omega \cup \pounds \subseteq cl_{FSR}(\omega) \cup cl_{FSR}(\pounds)$. Besides $\omega \cup \pounds \subseteq cl_{FSR}(\omega \cup \pounds)$ and it is the smallest vague soft rough closed set that containing $\omega \cup \pounds$. Therefore, $cl_{FSR}(\omega \cup \pounds) \subseteq cl_{FSR}(\omega) \cup cl_{FSR}(\pounds)$. Hence $cl_{FSR}(\omega \cup \pounds) = cl_{FSR}(\omega) \cup cl_{FSR}(\pounds)$.
(vii). Since $\omega \cap \pounds \subseteq cl_{FSR}(\omega) \cap cl_{FSR}(\pounds)$ and $\omega \cap \pounds \subseteq cl_{FSR}(\pounds)$ and $cl_{FSR}(\omega \cap \pounds)$ is the smallest vague soft rough closed set that containing $\omega \cap \pounds$. Therefore, $cl_{FSR}(\omega \cap \pounds) = cl_{FSR}(\omega) \cap cl_{FSR}(\pounds)$.

5. Conclusion

In both pure and applied mathematics, rough set theory has many intriguing applications. Our approach involves the creation of a novel theory known as vague soft rough set theory. The idea of lower and upper vague soft approximations, vague soft rough positive, vague soft negative, and vague soft border is required by this theory. All of these advancements are made and are positively handled with pertinent, thorough, and many examples. The definitions of open sets, closed sets, closure, and interior in vague soft rough topological spaces are also addressed. Finally, a completely new mathematical structure called as vague soft rough topological structure is demonstrated. Additionally, some results are discussed, and examples are provided to help the reader understand this study. We will attempt to illustrate vague soft rough topological spaces in relation to soft points of the spaces in our upcoming work. In vague soft rough topological spaces, we shall define new open sets known as pre-open sets, semi-open sets, alpha open sets, and beta open sets.

Building upon the foundations of vague soft rough topological spaces, future research will explore the generalization of open sets—including pre-open, semi-open, -open, and -open sets—by drawing inspiration from recent advancements in hybrid soft computing and neutrosophic topologies. Specifically, the framework of quadri-partitioned neutrosophic soft topological spaces [41] and bipolar vague soft expert sets [40] can be adapted to define these open sets, leveraging their robust capabilities for handling uncertainty and multi-attribute decision-making. Furthermore, fixed-point theorems for 3-self mappings [42] could provide powerful tools to analyze the stability and convergence of sequences within these generalized topological spaces. Additionally, weighted aggregation operators developed for interval-valued Pythagorean neutrosophic sets [43] and trigonometric neutrosophic sets [44] may refine the membership and non-membership boundaries of these open sets, thereby enhancing their practical applicability in domains such as medical diagnosis [40]. Finally, AI-assisted fuzzy soft relations [39] offer a promising pathway to model dynamic interactions among these sets, particularly in emerging applications like wearable health technologies.

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References

[1] L. A. Zadeh, *Fuzzy Sets*, Information and Control, vol. 8, pp. 338–353, 1965.

[2] Z. Pawlak, *Rough Sets*, International Journal of Computer Science, vol. 11, pp. 341–356, 1982.

[3] D. A. Molodtsov, *Soft Set Theory-First Results*, Computers and Mathematics with Applications, vol. 37, pp. 19–31, 1999.

[4] P. K. Maji, R. Biswas, and A. R. Roy, *An Application of Soft Sets in a Decision Making Problem*, Computers and Mathematics with Applications, vol. 44, pp. 1077–1083, 2002.

[5] P. K. Maji and A. R. Roy, *Soft Set Theory*, Computers and Mathematics with Applications, vol. 45, pp. 555–562, 2003.

[6] D. Pie and D. Miao, *From Soft Sets to Information Systems*, Granular Computing IEEE International Conference, vol. 2, pp. 617–621, 2005.

[7] D. Chen, *The Parametrization Reduction of Soft Sets and Its Applications*, Computers and Mathematics with Applications, pp. 757–763, 2005.

[8] S. Broumi, I. Deli, and F. Smarandache, *Relations on Interval Valued Neutrosophic Soft Sets*, Journal of New Theory, vol. 5, pp. 1–20, 2014.

[9] N. Cagman, S. Enginoglu, and F. Citak, *Fuzzy Soft Set Theory and Its Application*, Iranian Journal of Fuzzy Systems, vol. 8, pp. 137–147, 2011.

[10] M. Shabir and M. Naz, *On Soft Topological Spaces*, Computers and Mathematics with Applications, vol. 61, pp. 1786–1799, 2011.

[11] S. Bayramov and C. Gunduz, *A New Approach to Separability and Compactness in Soft Topological Spaces*, TWMS Journal of Pure and Applied Mathematics, vol. 9, pp. 82–93, 2018.

[12] K. T. Atanassov, *Intuitionistic Fuzzy Sets*, Fuzzy Sets and Systems, vol. 20, pp. 87–96, 1986.

[13] S. Bayramov and C. Gunduz, *On Intuitionistic Fuzzy Soft Topological Spaces*, TWMS Journal of Pure and Applied Mathematics, vol. 5, pp. 66–79, 2014.

[14] K. Hayat, M. I. Ali, B. Y. Cao, and F. Karaaslan, *New Results on Type-2 Soft Sets*, Hacettepe Journal of Mathematics, vol. 47, pp. 855–876, 2018.

[15] K. Hayat, M. I. Ali, B. Y. Cao, F. Karaaslan, and X. P. Yang, *A New Type-2 Soft Set: Type-2 Soft Graphs and Their Applications*, Advances in Fuzzy Systems, vol. 2017, Article ID 6162753, 2017.

[16] K. Hayat, M. I. Ali, B. Y. Cao, F. Karaaslan, X. P. Yang, and M. H. Shah, *Design Concept Evaluation Using Soft Sets Based on Acceptable and Satisfactory Levels: An Integrated TOPSIS and Shannon Entropy*, Soft Computing, vol. 24, pp. 2229–2263, 2020.

[17] M. Shabir and M. Naz, *On Bipolar Soft Sets*, arXiv, Retrieved from `https://arxiv.org/abs/1303.1344`, 2013.

[18] F. Karaaslan and S. Karatas, *A New Approach to Bipolar Soft Sets and Its Applications*, Discrete Mathematics, Algorithms and Applications, vol. 7, 1550054, 2015.

[19] T. Y. Ozturk, C. G. Aras, and S. Bayramov, *A New Approach to Operations on Neutrosophic Soft Sets and to Neutrosophic Soft Topological Spaces*, Communications in Mathematics and Applications, vol. 10, pp. 481–493, 2019.

[20] T. M. Al-shami, M. E. El-Shafei, and M. Abo-Elhamayel, *Seven Generalized Types of Soft Semi-Compact Spaces*, The Korean Journal of Mathematics, vol. 27, no. 3, pp. 661–690, 2019.

[21] T. M. Al-shami and M. E. El-Shafei, *On Soft Compact and Soft Lindelöf Spaces via Soft Pre-Open Sets*, Annals of Fuzzy Mathematics and Informatics, vol. 17, no. 1, pp. 79–100, 2019.

[22] F. Feng, X. Liu, V. L. Fotea, and Y. B. Jun, *Soft Sets and Soft Rough Sets*, Information Sciences, vol. 181, pp. 1125–1137, 2011, `https://doi.org/10.1016/j.ins.2010.11.004`.

[23] Z. Li and N. Gao, *Soft Rough Sets and Topologies*, Intelligent Computing Theories and Applications, vol. 7390, pp. 103–110, 2012, `http://doi.org/10.1007/978-3-642-31576-3_14`.

[24] Y. Liu, L. Martinez, and K. Qin, *A Comparative Study of Some Soft Rough Sets*, Symmetry, vol. 9, 252, 2017, `https://doi.org/10.3390/sym9110252`.

[25] Z. Di, L. Yu, and A. Shuang, *N-Soft Rough Sets and Its Applications*, Journal of Intelligent and Fuzzy Systems, vol. 40, pp. 565–573, 2021.

[26] S. Alkhazaleh and E. A. Marei, *New Soft Rough Set Approximations*, International Journal of Fuzzy Logic and Intelligent Systems, vol. 21, pp. 123–134, 2021.

[27] M. A. El Safty, S. Al Zahrani, M. K. El-Bably, and M. El Sayed, *Soft ζ-Rough Set and Its Applications in Decision Making of Coronavirus*, Computer, Material and Continua Tech Science Press, vol. 70, pp. 267–285, 2022, DOI:10.32604/cmc.2022.019345.

[28] H. Zhang, L. Shu, and S. Liao, *Intuitionistic Fuzzy Soft Rough Set and Its Application in Decision Making*, Abstract and Applied Analysis, Article ID 287314, 13 pages, 2014, http://dx.doi.org/10.1155/2014/287314.

[29] N. Demirtas, S. Hussain, and O. Dalkilic, *New Approach of Inverse Soft Rough Sets and Their Applications in a Decision Making*, Journal of Applied Mathematics and Informatics, vol. 38, pp. 335–349, 2020, https://doi.org/10.14317/jami.2020.335.

[30] F. Afzal, A. Mehmood, S. Al Ghour, M. Zafar, H. Sakidin, and S. Gul, *Characterization of Bipolar Vague Soft S-Open Sets*, Journal of Function Spaces, vol. 2022, Article ID 5964872, 13 pages, 2022, https://doi.org/10.1155/2022/5964872.

[31] A. Mehmood, S. Abdullah, and C. Park, *A New Approach to Vague Soft Bi-Topological Spaces*, CMES, vol. 131, pp. 412–427, 2022.

[32] A. Mehmood, F. Afzal, S. Abdullah, M. I. Khan, and S. Gul, *An Absolutely New Attempt to Vague Soft Bi-Topological Spaces Basis at Newly Defined Operations Regarding Vague Soft Points*, Mathematical Problems in Engineering, vol. 2021, Article ID 4408754, 11 pages, 2021.

[33] S. Alkhazaleh and A. R. Salleh, *Fuzzy Soft Expert Set and Its Application*, Applied Mathematics, vol. 5, pp. 1349–1368, 2014.

[34] A. M. Abd El-latif, *New Generalized Fuzzy Soft Rough Approximations Applied to Fuzzy Topological Spaces*, Journal of Intelligent & Fuzzy Systems, vol. 35, pp. 2123–2136, 2018, DOI:10.3233/JIFS-172076.

[35] M. Atef, S. Nada, and A. Nawar, *Covering Soft Rough Sets and Its Topological Properties with Application*, Soft Computing, vol. 27, pp. 4451–4461, 2023, https://doi.org/10.1007/s00500-023-07812-x.

[36] M. Elbes, T. Kanan, M. Alia, and M. Ziad, *COVID-19 Detection Platform from X-Ray Images Using Deep Learning*, International Journal of Advances in Soft Computing and Its Applications, vol. 14, no. 1, pp. 197–211, 2022.

[37] T. Kanan, M. Elbes, K. Abu Maria, and M. Alia, *Exploring the Potential of IoT-Based Learning Environments in Education*, International Journal of Advances in Soft Computing and Its Applications, vol. 15, no. 2, 2023.

[38] I. M. Batiha, S. A. Njadat, R. M. Batyha, A. Zraiqat, A. Dababneh, and Sh. Momani, *Design Fractional-Order PID Controllers for Single-Joint Robot Arm Model*, International Journal of Advances in Soft Computing and Its Applications, vol. 14, no. 2, pp. 96–114, 2022.

[39] M. I. Ali, M. K. El-Bably, and El-S. A. Abo-Tabl, *Topological Approach to Gen-*

eralized Soft Rough Sets via Near Concepts, Soft Computing, vol. 26, pp. 499–509, 2022.

[40] R. Hatamleh, A. Al-Husban, S. A. M. Zubair, M. Elamin, M. M. Saeed, E. Abdolmaleki, and A. M. Khattak, "AI-Assisted Wearable Devices for Promoting Human Health and Strength Using Complex Interval-Valued Picture Fuzzy Soft Relations," *European Journal of Pure and Applied Mathematics*, vol. 18, no. 1, pp. 5523–5523, 2025.

[41] M. M. Saeed, R. Hatamleh, A. A. Abubaker, A. Al-Husban, J. J. Hamja, G. Nordo, C. L. Armada, T. Fujita, and A. Mehmood, "A Comprehensive Study of Bipolar Vague Soft Expert P-Open Sets in Bipolar Vague Soft Expert Topological Spaces with Applications to Cancer Diagnosis," *European Journal of Pure and Applied Mathematics*, vol. 18, no. 2, pp. 5900–5900, 2025.

[42] M. M. Saeed, R. Hatamleh, A. M. AbdEl-latif, A. A. H. Al-Husban, H. M. Attaalfadeel, T. Fujita, and A. M. Khattak, "A Breakthrough Approach to Quadri-Partitioned Neutrosophic Soft Topological Spaces," *European Journal of Pure and Applied Mathematics*, vol. 18, no. 2, pp. 5845–5845, 2025.

[43] M. Noorwali, R. Hatamleh, A. A. Abubaker, A. Al-Husban, J. J. Hamja, G. Nordo, and A. M. Khattak, "Characterization and Application of Fixed-Point Theorems for 3-Self Mappings in Generalized Metric Spaces," *European Journal of Pure and Applied Mathematics*, vol. 18, no. 2, pp. 5838–5838, 2025.

[44] R. Hatamleh, A. S. Heilat, M. Palanikumar, and A. Al-Husban, "Characterization of Interaction Aggregating Operators Setting Interval-Valued Pythagorean Neutrosophic Set," *Neutrosophic Sets and Systems*, vol. 81, pp. 285–305, 2025.

[45] R. Hatamleh, A. S. Heilat, M. Palanikumar, and A. Al-Husban, "Different Operators via Weighted Averaging and Geometric Approach Using Trigonometric Neutrosophic Interval-Valued Set and Its Extension," *Neutrosophic Sets and Systems*, vol. 80, pp. 194–213, 2025.