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Gelfand-Tsetlin modules for Lie algebras of rank 2

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Abstract. We explicitly construct families of simple modules for all simple Lie algebras of rank 2 on which a certain commutative subalgebra acts diagonally with a simple spectrum. In type A, these modules are the well-known generic Gelfand-Tsetlin modules.

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1. Introduction

Let $\mathfrak{g}$ be a simple finite-dimensional simple Lie algebra over the complex numbers and let $\mathfrak{h}$ be a fixed Cartan subalgebra of $\mathfrak{g}$. A $\mathfrak{g}$-module M is *weight* (with respect to $\mathfrak{h}$) if $\mathfrak{h}$ is diagonalizable on M, that is

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda \,,$$

where $hv = \lambda(h)v$, for any $v \in M_\lambda$ and $h \in \mathfrak{h}$. The subspace M_λ is called a weight subspace of weight λ, if $M_\lambda \neq 0$.

Simple weight modules were studied extensively in the last 50 years. Classical results of Fernando [1] and Mathieu [2] provided a complete classification of simple weight modules with finite-dimensional weight subspaces. On the other hand, the classification of simple weight modules with infinite-dimensional weight subspaces is still an open problem. The most progress has been achieved in the case of Lie algebras of type A, where simple Gelfand-Tsetlin modules were classified (see [3–6] and references therein). These are weight modules with diagonalizable action of a certain commutative subalgebra of the universal enveloping algebra $U(\mathfrak{g})$, called the Gelfand-Tsetlin subalgebra. Generically, such simple Gelfand-Tsetlin modules have infinite-dimensional weight subspaces. In particular, in the case of $\mathfrak{sl}(2)$ we obtain in this way all simple weight modules. They depend on two parameters and have

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1-dimensional weight subspaces. In the case of $\mathfrak{g} = \mathfrak{sl}(3)$ a complete description of simple Gelfand-Tsetlin modules was given in [7].

The original approach to the study of weight modules was based on the reduction to the study of simple modules over the centralizer $U_0(\mathfrak{g})$ of the Cartan subalgebra $\mathfrak{h}$ in the universal enveloping algebra $U(\mathfrak{g})$: if M is a simple weight $\mathfrak{g}$-module, then M_λ is a simple $U_0(\mathfrak{g})$-module, for any weight λ of M. Hence, every simple weight $\mathfrak{g}$-module corresponds to a simple (not unique) $U_0(\mathfrak{g})$-module and, in its turn, any simple $U_0(\mathfrak{g})$-module corresponds to a unique simple weight $\mathfrak{g}$-module. This approach was successfully used in the case of $\mathfrak{g} = \mathfrak{sl}(3)$ [8–15], etc.

As the structure of $U_0(\mathfrak{g})$ is rather complicated (there are two commuting generators for $\mathfrak{sl}(2)$, and there are six generators for $\mathfrak{sl}(3)$ with three polynomial relations and five commuting generators among them), there were essentially no attempts beyond the $\mathfrak{sl}(3)$ case.

The goal of the paper is to revise the centralizer approach and to construct new simple weight modules with infinite-dimensional weight subspaces for all simple Lie algebras of rank 2. We explicitly construct simple generic modules in the category of Γ-pointed modules for a commutative subalgebra Γ of the centralizer $U_0(\mathfrak{g})$. In type A, Γ-pointed modules are the celebrated Gelfand-Tsetlin modules, and similar constructions can be viewed analogously in other types.

The structure of the paper is the following. In Section 2 we discuss the structure of the centralizer $U_0(\mathfrak{g})$ of the Cartan subalgebra $\mathfrak{h}$ in the universal enveloping algebra $U(\mathfrak{g})$, prove that $U_0(\mathfrak{g})$ is finitely generated and finitely presented. We give a generating set of elements and describe an algorithm for computing all relations between them. In Section 4 we consider the Lie algebra of type A_2. Our approach is a suitable modification of [9] and [14], and we recover a construction of generic torsion free A_2-modules with infinite-dimensional weight spaces obtained in [14] and [7]. They are *tame* Gelfand-Tsetlin modules with diagonalizable action of the Gelfand-Tsetlin subalgebra. In Section 5 we consider the Lie algebra $\mathfrak{g}$ of type C_2 and give the generators and the defining relations of the centralizer $U_0(\mathfrak{g})$. We construct two 4-parameter families of simple torsion free C_2-modules with infinite-dimensional weight spaces. These modules are Γ-pointed, where Γ is the 4-generated Gelfand-Tsetlin subalgebra of $U_0(\mathfrak{g})$ which has a simple spectrum on such representations. Finally, in Section 6 we construct a 3-parameter family of simple torsion free G_2-modules with infinite-dimensional weight spaces. These modules are Γ-pointed with respect to a 4-generated Gelfand-Tsetlin subalgebra Γ of $U_0(\mathfrak{g})$ which has a simple spectrum on such representations.

We hope to use the defined representations to construct new simple modules for all simple finite-dimensional and Affine Lie algebras via the parabolic induction.

2. Cartan centralizers

Let $\Delta = \{\alpha_1, \ldots, \alpha_{k_1}\}$ be the root system of $(\mathfrak{g}, \mathfrak{h})$, and let $\pi = \{\beta_1, \ldots, \beta_{k_0}\}$ be a basis of Δ. With respect to the basis π, we have the decomposition of Δ into positive and negative roots: $\Delta = \Delta^+ \cup \Delta^-$. Let W be the Weyl group of the root system Δ. Choose a basis $G = G_0 \cup G_1$ of the Lie algebra $\mathfrak{g}$, where $G_0 = \{h_\beta \in \mathfrak{h} \,|\, \beta \in \pi\}$ and $G_1 = \{e_\alpha \in \mathfrak{g}_\alpha \setminus \{0\} \,|\, \alpha \in \Delta\}$. Set $f_\alpha = e_{-\alpha}$.

Denote by $U_0(\mathfrak{g})$ the centralizer of the Cartan subalgebra $\mathfrak{h}$ in the universal enveloping algebra $U(g)$. For each $i \in \mathbb{N}$ denote by $U^{(i)}(\mathfrak{g})$ the vector subspace of $U(\mathfrak{g})$ spanned by the monomials $x_1 x_2 \cdots x_j$, where $x_1, \ldots, x_j \in G$ and $j \leq i$. Then we get an increasing sequence of subspaces

$$U^{(1)}(\mathfrak{g}) \subset U^{(2)}(\mathfrak{g}) \subset \cdots \subset U^{(i)}(\mathfrak{g}) \subset \cdots,$$

which defines a canonical filtration of $U(\mathfrak{g})$. The canonical filtration of $U_0(\mathfrak{g})$ is the sequence of subspaces

$$U_0^{(1)}(\mathfrak{g}) \subset U_0^{(2)}(\mathfrak{g}) \subset \cdots \subset U_0^{(i)}(\mathfrak{g}) \subset \cdots,$$

where $U_0^{(i)}(\mathfrak{g}) = U^{(i)}(\mathfrak{g}) \cap U_0(\mathfrak{g})$.

For any monomial $X = x_1 x_2 \cdots x_j$ denote by $T_0(X)$ the set of monomials obtained from X by permuting the variables x_i. We will treat each monomial as an element of the algebra $U(\mathfrak{g})$, considering it as a monomial with coefficient 1. Define the degree function $\deg(y)$ as follows: $\deg(y) = i$ if $y \in U^{(i)}(\mathfrak{g})$, but $y \notin U^{(i-1)}(\mathfrak{g})$.

Now, fix some order on the set G:

$$x_1 \leq x_2 \leq \cdots \leq x_k, \tag{2.1}$$

where k is the dimension of $\mathfrak{g}$. Define standard monomials of $U(\mathfrak{g})$ with respect to this order as follows:

$$X_S = x_1^{s_1} x_2^{s_2} \cdots x_k^{s_k}.$$

where $S = (s_1, \ldots, s_k)$ is a k-tuple of nonnegative integers, with at least one $s_i \neq 0$.

For every monomial X define a lexicographical order on the set $T_0(X)$ with respect to the order (2.1): if $X_1 = x_{i_1} x_{i_2} \cdots x_{i_n}$ and $X_2 = x_{j_1} x_{j_2} \cdots x_{j_n}$, with $X_1, X_2 \in T_0(X)$, then $X_1 < X_2$ if either $x_{i_1} < x_{j_1}$ or there exists an index $1 < s \leq n$ such that $x_{i_s} < x_{j_s}$ and $x_{i_t} = x_{j_t}$, for $1 \leq t < s$.

Denote by P the set of all standard monomials and by $P^{(i)}$ the set of all standard monomials of degree i. Set $P_0 = P \cap U_0(\mathfrak{g})$ and $P_0^{(i)} = P^{(i)} \cap U_0(\mathfrak{g})$. In particular, $T_0(P) = \bigcup_{X \in P} T_0(X)$ will denote the set of all monomials in the algebra $U(\mathfrak{g})$.

The following lemma is an immediate consequence of the PBW theorem.

Lemma 2.1.

1. The set of the following standard monomials $\bar{P}^{(i)} = P^{(1)} \cup P^{(2)} \cup \cdots \cup P^{(i)}$ forms a basis for the vector space $U^{(i)}(\mathfrak{g})$.

2. The set of the standard monomials $\bar{P}_0^{(i)} = P_0^{(1)} \cup P_0^{(2)} \cup \cdots \cup P_0^{(i)}$ forms a basis for the vector space $U_0^{(i)}(\mathfrak{g})$.

3. If $a \in U^{(i)}(\mathfrak{g})$ and $b \in U^{(j)}(\mathfrak{g})$, then $ab \in U^{(i+j)}(\mathfrak{g})$ and $ab - ba \in U^{(i+j-1)}(\mathfrak{g})$.

4. If $X \in P_0^{(i)}$ and $X_1, X_2 \in T_0(X)$, then $X_1 - X_2 \in U_0^{(i-1)}(\mathfrak{g})$.

5. For every monomial $X \in T_0(P)$, the minimal element of the set $T_0(X)$ with respect to the above lexicographical order is a standard monomial.

6. Let P' be a set of monomials of degree $\leq i$, such that for every monomial $X \in \bar{P}^{(i)}$, there exist a unique $X' \in P'$ satisfying $T_0(X) = T_0(X')$. Then P' is a basis of $U^{(i)}(\mathfrak{g})$ and $P'_0 = P' \cap U_0^{(i)}(\mathfrak{g})$ is a basis of $U_0^{(i)}(\mathfrak{g})$.

Denote by $\hat{P}_0^{(i)} \subset P_0^{(i)}$ the subset of monomials of degree i generated by G_1, and set $\hat{P}_0 = \bigcup_i \hat{P}_0^{(i)}$.

Let $X \in T_0(P)$. Then $X = h_{\beta_1} \cdots h_{\beta_m} e_{\alpha_1} \cdots e_{\alpha_n}$, for some $\alpha_1, \dots, \alpha_n \in \Delta$, $\beta_1, \dots, \beta_m \in \pi$. Define two lists of roots associated with X as follows: $L_0(X) = (\beta_1, \dots, \beta_m)$ and $L_1(X) = (\alpha_1, \dots, \alpha_m)$. Extend this definition to the set of all monomials as follows: if $X' \in T_0(X)$, then $L_0(X') = L_0(X)$, $L_1(X') = L_1(X)$.

For any monomial $X \in T_0(P)$, the sum of all roots in the list $L_1(X)$ is called the weight of the list $L_1(X)$. Clearly, the weight of the list $L_1(X)$ is zero if and only if $X \in T_0(P_0)$. Denote by $B(\Delta)$ the set of all zero-weight lists of roots. Furthermore, if $L_0(X) = L_0(Y)$ and $L_1(X) = L_1(Y)$, for some monomials X, Y, then $T_0(X) = T_0(Y)$.

A list $L_1(X)$ for $X \in \bar{P}_0$ is called decomposable if there exist $X_1, X_2 \in \bar{P}_0$ such that $L_1(X) = L_1(X_1) \sqcup L_1(X_2)$ (disjoint union of two lists). In this case, it follows that $T_0(X) = T_0(X_1 X_2)$. Conversely, if no such decomposition exists, the list is called indecompossable. Note that decompositions are not unique for certain monomials. For example, in the algebra A_2 (see Section 4.1), $(\alpha_1, \dots, \alpha_6) = (\alpha_1, \alpha_6) \sqcup (\alpha_2, \alpha_5) \sqcup (\alpha_3, \alpha_4) = (\alpha_1, \alpha_2, \alpha_4) \sqcup (\alpha_3, \alpha_5, \alpha_6)$.

Denote by $B_1(\Delta) \subset B(\Delta)$ the set of all zero-weight indecomposable lists. Define the action of the Weyl group W on the list of roots $a = (\alpha_1, \alpha_2, \dots, \alpha_n)$ as follows:

$$w(a) = (w\alpha_1, w\alpha_2, \dots, w\alpha_n), \quad \text{with} \in W.$$

Clearly, if a is indecomposable, then $w(a)$ is also indecomposable. Moreover, if a has a zero weight, then $w(a)$ also has a zero weight. The simplest examples of indecomposable lists are those containing only one positive or only one negative root.

Define the set of *primitive* lists $B_2(\Delta) \subseteq B_1(\Delta)$ as follows: $r \in B_2(\Delta)$ if there exists $w \in W$ such that the list $w(r)$ contains only one negative or only one positive root.

Let $M_k = \{(n_1, n_2, \dots, n_k) \,|\, n_i \in \mathbb{N}\}$ be the set of vectors with nonnegative integer coordinates. Define a partial order on the set M_k as follows:

$$(n_1, n_2, \dots, n_k) \le (m_1, m_2, \dots, m_k) \quad \text{if } n_i \le m_i\,, \quad \text{for all } i = 1, \dots, k.$$

We will be using the following result [16, Lemma 2.6.2].

Lemma 2.2. If $S_k \subset M_k$ is any infinite subset, then there exist two elements $r_1, r_2 \in S_k$ such that $r_1 \le r_2$.

We have the following properties of indecomposable lists of roots.

Lemma 2.3. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra with root system Δ. Then

1. The set of indecomposable lists $B_1(\Delta)$ is finite.
2. If $\mathfrak{g} \in \{A_2, C_2, G_2\}$, then all indecomposable lists are primitive and hence $B_1(\Delta) = B_2(\Delta)$.

Proof. Define the function σ on the set $B(\Delta)$ as follows: $\sigma(\alpha_1, \alpha_2, \dots, \alpha_m) = (n_1, n_2, \dots, n_{|\Delta|})$, where n_i is the number of occurrences of the i-th root of Δ (with respect to the ordering in (2.1) in the list $(\alpha_1, \alpha_2, \dots, \alpha_m)$. It is clear that a list r_1 is a sublist of a list r_2 if and only if $\sigma(r_1) \le \sigma(r_2)$. Now, the proof of the first statement follows from the previous lemma.

The second statement for the case A_2 is obvious. We will now prove the second statement for the case C_2. Let $\pi = \{\beta_1, \beta_2\}$ be a basis of the root system. For convenience, we will represent the roots of Δ as vectors in this basis: $(i,j) = i\beta_1 + j\beta_2$. Then $\Delta = \{(1,0),(0,1),(1,1),(2,1),(-1,0),(0,-1),(-1,-1),(-2,-1)\}$. Let $v = (i,j) \in \Delta$ be any root. The action of the Weyl group is given by $w_1(v) = (2j-i,j)$ and $w_2(v) = (i,i-j)$, where w_1 and w_2 are simple reflections.

We will prove that non-primitive indecomposable lists do not exist for C_2. By the definition of a primitive list, any list consisting of two or three roots is primitive. Suppose there exists a non-primitive indecomposable list r with more than three roots. First, observe that if $(2,1),(-2,-1) \notin r$, then r cannot contain more than three roots.

Consider the case $(2,1) \in r$. (The case $(-2,-1) \in r$ is miror of first case.) Then necessarily $(-2,-1) \notin r$. Consider the list $w_1(r)$. Then $w_1(2,1) = (0,1) \in w_1(r)$. First, suppose $(2,1),(-2,-1) \notin w_1(r)$. In this case, $w_1(r)$ is primitive, and consequently, so is r. Second, if $(2,1) \in w_1(r)$, then both $(2,1)$ and $(0,1)$ belong to $w_1(r)$. Finally, if $(-2,-1) \in w_1(r)$, then we obtain $(2,1),(0,1) \in w_1w_2(r)$.

Now, suppose there exists $w \in W$ such that $(2,1),(0,1) \in r' = w(r)$. The possible negative roots in the list r' are $(-1,-1)$ and $(-1,0)$. Since the second coordinate of the sum of all positive roots is at least two, r' must contain at least two occurrences of $(-1,-1)$. Thus we have $r' = \{(2,1),(0,1),(-1,-1),(-1,-1)\}$. Now, applying w_2, we obtain $w_2(r') =$
$= \{(2,1),(0,-1),(-1,0),(-1,0)\}$, which is primitive. Consequently r is also primitive.

The proof of the second statement for G_2 is similar to the case of C_2. For the sets of all primitive lists in all three cases, see Sections 4-6. $\square$

Define the following order on the set of roots Δ, derived from (2.1). For $\alpha_1, \alpha_2 \in \Delta$

$$\alpha_1 \leq \alpha_2 \text{ if and only if } e_{\alpha_1} \leq e_{\alpha_2}. \tag{2.2}$$

A monomial X is called *perfect* if either $X = h_\beta \in G_0$, or $X = e_{\alpha_1} \cdot e_{\alpha_2} \cdot \dots \cdot e_{\alpha_n}$ and the associated list $L_1(X) = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is both indecomposable and ordered according to (2.2), i.e., $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n$.

Since the set of all indecomposable lists is finite, the set of all perfect monomials is also finite. Let $I(\mathfrak{g}) = \{p_1, \dots, p_q\}$ denote the set of all perfect monomials with degree greater than one. Then $I(\mathfrak{g}) \cup G_0$ is a generating set of $U_0(\mathfrak{g})$ consisting of all perfect monomials.

Denote by d_0 the maximal degree of perfect monomials in $I(\mathfrak{g})$.

Let $J(\mathfrak{g}) = \{c_1, \dots, c_q\}$ be the set of new indeterminates, $r_i = c_i - p_i$, $i = 1, \dots, q$ and $D = \langle r_1, \dots, r_q \rangle$ the 2-sided ideal of $U_0(\mathfrak{g})[c_1, \dots, c_q]$ generated by these elements. Define

$$\hat{U}_0(\mathfrak{g}) = U_0(\mathfrak{g})[c_1, \dots, c_q]/D.$$

Clearly, $\hat{U}_0(\mathfrak{g}) \cong U_0(\mathfrak{g})$ and $\hat{U}_0(\mathfrak{g})$ is generated by $J(\mathfrak{g}) \cup G_0$ with the ideal of relations K inherited from $U_0(\mathfrak{g})$. Define the degrees of new indeterminates c_i by: $\deg(c_i) = \deg(p_i)$.

Denote by $Q_0(\mathfrak{g})$ the set of all monomials generated by $J(\mathfrak{g}) \cup G_0$ and set $Q_0^{(i)}(\mathfrak{g}) = Q_0(\mathfrak{g}) \cap U_0^{(i)}(\mathfrak{g})$. Define the function from $Q_0(\mathfrak{g})$ to $I(\mathfrak{g}) \cup G_0$ as follows:

$$\omega : Q_0(\mathfrak{g}) \to I(\mathfrak{g}) \cup G_0, \ \omega(c_i) = p_i, \ \omega(h) = h, \ h \in G_0.$$

Let us fix some order on the set $J(\mathfrak{g}) \cup G_0$:

$$x_1 \leq x_2 \leq \dots \leq x_{q+k_0}, \ x_i \in J(\mathfrak{g}) \cup G_0, \quad i = 1, \dots, q + k_0\,. \tag{2.3}$$

For any monomial $Y \in Q_0(\mathfrak{g})$, denote by $T_1(Y)$ the set of all monomials $X \in Q_0(\mathfrak{g})$ such that $T_0(\omega(Y)) = T_0(\omega(X))$. Note that all monomials in the sets $T_0(\omega(Y))$ and $T_1(Y)$ have the same degree. Extend function L_1 on the set $J(\mathfrak{g})$ as follows: $L_1(x) = L_1(\omega(x))$, $x \in Q_0(\mathfrak{g})$.

Define a lexicographical order on the set $T_1(Y)$ with respect to the order given in (2.3) as follows: if $Y_1 = a_1a_2\cdots a_n$, $Y_2 = b_1b_2\cdots b_m$, for $a_i, b_j \in J_0(\mathfrak{g}) \cup G_0$ and $Y_1, Y_2 \in T_1(Y)$, then $Y_1 < Y_2$ if there exists an index $1 \leq s \leq \min(n, m)$, such that $a_s < b_s$ and $a_t = b_t$, for all $0 < t < s$. Note that neither Y_1 nor Y_2 can be a prefix of the other, which ensure that our definition is well-defined and that any two distinct elements are comparable.

A monomial $Y \in Q_0(\mathfrak{g})$ is called *semi-perfect* if it is the minimal element of the set $T_1(Y)$ under this lexicographical order. Denote by $S(\mathfrak{g})$ the set of all semi-perfect monomials, and set $S^{(i)}(\mathfrak{g}) = S(\mathfrak{g}) \cap \hat{U}_0^{(i)}(\mathfrak{g})$.

Lemma 2.4.

1. For any $i \geq 1$ and $X \in P_0^{(i)}$ there exists a unique semi-perfect monomial $Y \in S^{(i)}(\mathfrak{g})$ such that $T_0(X) = T_0(\omega(Y))$.

2. Let $X = a_1a_2\cdots a_m$ with $a_i \in J_0(\mathfrak{g}) \cup G_0$, be a semi-perfect monomial. Then, for any $s \in \{1, \ldots, m\}$, the monomial $X' = a_1\cdots a_{s-1}a_{s+1}\cdots a_m$ is also semi-perfect.

3. The set of semi-perfect monomials $S^{(i)}(\mathfrak{g})$ is a basis of the vector space $\hat{U}_0^{(i)}(\mathfrak{g})$.

Proof. The one-to-one correspondence between standard monomials in $U_0^{(i)}(\mathfrak{g})$ and semi-perfect monomials in $\hat{U}_0^{(i)}(\mathfrak{g})$ follows from Lemma 2.1 and the definition of semi-perfect monomials. Since $\hat{U}_0^{(i)}(\mathfrak{g})$ is a finite-dimensional vector space and the set of semi-perfect monomials $S^{(i)}(\mathfrak{g})$ satisfy the condition for the set P' in the part 6 of the Lemma 2.1, the lemma follows. □

Since the set of semi-perfect monomials $S^{(i)}(\mathfrak{g})$ forms a basis of the algebra $U_0^{(i)}(\mathfrak{g})$, any element $x \in \hat{U}_0(\mathfrak{g})$ can be expressed as a linear combination of semi-perfect elements. We need to determine how to multiply any two semi-perfect monomials and express their product as a linear combination of elements of $S(\mathfrak{g})$.

If an element $X \in \hat{U}_0(\mathfrak{g})$ is expressed as a linear combination of semi-perfect monomials, we say that X is in a *normal form*, denoted by $\mathcal{N}(X)$. The process of converting a given element into its normal form is called *normalization*.

Define the set of relations as follows:

$$K' = \{c_{i_1}c_{i_2}\cdots c_{i_m} - \mathcal{N}(c_{i_1}c_{i_2}\cdots c_{i_m}) \,|\, m \leq d_0, c_i \in J(\mathfrak{g})\}. \tag{2.4}$$

Additionally, define the length function on $K' \subseteq K$ as: $\mathrm{Len}(c_{i_1}\cdots c_{i_m} - \mathcal{N}(c_{i_1}\cdots c_{i_m})) = m$.

Thus, K' is the set of all relations with length less or equal to d_0. We are now ready to state the main result regarding the Cartan centralizers of the universal enveloping algebras.

Theorem 2.5. The ideal of relations K is generated by the set K'.

Proof. We prove that the product of semi-perfect elements can be expressed as a linear combination of semi-perfect elements using only the relations K'. This follows from the reduction algorithm described below.

We will show that any monomial, written as product of perfect monomials in arbitrary order, can be transformed into a linear combination of semi-perfect monomial.

Let $X = a_1 a_2 \cdots a_m \in Q_0^{(i)}(\mathfrak{g})$ be a monomial of degree i that is a product of perfect elements. By Lemmas 2.1 and 2.4, there exists a semi-perfect element $Y = b_1 b_2 \cdots b_n \in Q_0^{(i)}(\mathfrak{g})$, such that $T_1(X) = T_1(Y)$ and $X - Y \in \hat{U}_0^{(i-1)}(\mathfrak{g})$.

The proof proceeds by induction on the degree of the monomials. The statement is obvious for the case $i = 1$. Assume that the statement holds for all elements in $\hat{U}_0^{(i-1)}(\mathfrak{g})$, that is, any such element can be transformed into the normal form.

We will show that, using only relations K', X can be transformed into a sum $X = Y + Y_1$, where Y is above semi-perfect monomial and $Y_1 \in \hat{U}_0^{(i-1)}(\mathfrak{g})$. Denote $Y' = b_2 \cdots b_n$. There are two cases to consider.

1. Let $b_1 \in G_0$. Since $L_0(X) = L_0(Y)$, by Lemma 2.4, it follows that $b_1 = a_s$ for some $1 \le s \le m$. As b_1 belongs to the center of $\hat{U}_0(\mathfrak{g})$, we can rewrite X as $X = b_1 a_1 a_2 \cdots a_{s-1} a_{s+1} \cdots a_m = b_1 X'$. Thus, we obtain

$$X = Y + b_1(X' - Y')\,.$$

   It is easy to see that $T_1(X') = T_1(Y')$, and Y' is semi-perfect. By the induction hypothesis, we can apply our algorithm to the pair X', Y'. This gives $(X' - Y') = X'' \in \hat{U}_0^{(i-2)}$, implying that $b_1 X'' \in \hat{U}_0^{(i-1)}$. Therefore, the statement follows by the induction hypothesis.

2. Let $b_1 \in J_0(\mathfrak{g})$. Since $L_1(X) = L_1(Y)$, it follows that $L_1(b_1)$ is a sublist of $L_1(X)$. Let $\{z_1, z_2, \ldots, z_t\} \subset \{a_1, a_2, \ldots, a_m\}$ be a minimal set of perfect monomials such that $L_1(b_1) \sqsubset L_1(z_1 z_2 \cdots z_t)$. Set $Z = z_1 z_2 \cdots z_t$. Since the cardinality of any indecomposable list is bounded by d_0, we have $t \le d_0$.

   Using the commutation relations for two perfect elements we can transform X into the following form: $X = Z X_1 + X_2$, where $T_1(X) = T_1(Z X_1)$ and $X_2 \in \hat{U}_0^{(i-1)}(\mathfrak{g})$.

   Denote $n = \deg(Z)$, then $\deg(X_1) = i - n$. There exists a semi-perfect monomial Z_1 such that $T_1(Z) = T_1(Z_1)$. Since $t \le d_0$, we can apply the normalization process to the monomial Z. We obtain $Z = Z_1 + Z_2$, where $Z_2 \in \hat{U}_0^{(n-1)}(\mathfrak{g})$.

   Since Z_1 is a semi-perfect monomial, the first perfect element z_1' in its decomposition $Z_1 = z_1' z_2' \cdots z_s' = z_1' Z_1'$ is the minimal perfect element with respect to the order (2.3). This means that for every perfect monomial x such that $L_1(x) \sqsubseteq L_1(Z_1)$, we have $z_1' \le x$. Since $L_1(b_1), L_1(z_1) \sqsubseteq L_1(Z) \sqsubseteq L_1(Y)$, it follows that $z_1' = b_1$.

   Now we have

$$\begin{aligned} X &= Z X_1 + X_2 = (Z_1 + Z_2) X_1 + X_2 = (b_1 Z_1' + Z_2) X_1 + X_2 \\ &= Y - b_1 Y' + (b_1 Z_1' + Z_2) X_1 + X_2 = Y + b_1(Z_1' X_1 - Y') + Z_2 X_1 + X_2\,. \end{aligned}$$

   It is easy to see that the pair of monomials $X_3 = Z_1' X_1$ and Y' satisfies the initial conditions of our lemma, with degree of the monomials less than i.

   Since $\deg(Y') = i - k,\;\; T_1(X_3) = T_1(Y')$ and Y' is semi-perfect, we can apply, by the induction hypothesis, our algorithm to the pair X_3, Y'

   This gives $X_3 = Y' + X_4$, where $X_4 \in \hat{U}_0^{(i-k-1)}(\mathfrak{g})$. Finally, we have $X = Y + Y_1$, where $Y_1 = b_1 X_4 + Z_2 X_1 + X_2 \in \hat{U}_0^{(i-1)}(\mathfrak{g})$.

The statement follows by the induction hypothesis. Note that we used relations K' twice: for commuting perfect elements and for normalizing Z.

$\square$

Let A be an associative algebra defined by a set of generators S and a set of relation R (as polynomials in the generators of S). The set of generators S can be partitioned into three subsets, $S = S_1 \cup S_2 \cup S_3$, where S_1 generates the center of the algebra A and $S_1 \cup S_2$ generates a commutative subalgebra of A.

The width of a monomial X, denoted by $\mathrm{width}(X)$, is defined as the number of occurrences of variables from the set S_3 in the monomial. The width of a polynomial P is then defined as the maximal width among all monomials that make up the polynomial.

We can decompose the set of relation R into a union of subsets as follows: $R = R_0 \cup R_1 \cup \cdots$, where $R_i = \{Y \in R \,|\, \mathrm{width}(Y) = i\}$, for $i = 0, 1, 2, \ldots$.

Note that this decomposition is not unique and may vary depending on the choice of generators S and the subset S_2. Our goal will be to identify the "best" set of generators S and "best" decomposition, such that the cardinality of the set S_3 is minimal.

3. Category of Γ-pointed modules

Let Γ be a commutative subalgebra of $U_0(\mathfrak{g})$ such that $\mathfrak{h} \subseteq \Gamma \subset U_0(\mathfrak{g})$. By $\mathrm{Hom}(\Gamma, \mathbb{C})$ we denote the set of all characters of Γ, that is the set of all $\mathbb{C}$-algebra homomorphisms from Γ to $\mathbb{C}$.

Let M be a Γ-module. For each $\chi \in \mathrm{Hom}(\Gamma, \mathbb{C})$ we set

$$M_\chi = \{v \in M;\, av = \chi(a)v\,,\ \forall a \in \Gamma\},$$

and call it the Γ-weight space of M with weight χ. When $M_\chi \neq \{0\}$, we say that χ is a Γ-weight of M and the elements of M_χ are called Γ-weight vectors of weight χ. If a Γ-module M satisfies

$$M = \bigoplus_{\chi \in \mathrm{Hom}(\Gamma, \mathbb{C})} M_\chi\,,$$

then we call M a Γ-weight module. The dimension of the vector space $M_\chi \neq 0$ will be called the Γ-*multiplicity* of χ in M. Module is called Γ-*pointed* if Γ-multiplicity of any character χ equals 1, that is Γ separates the basis elements of M. In particular, M is a *tame* module with diagonalizable action of Γ. A weight module M is *torsion free* provided all root vectors of $\mathfrak{g}$ act injectively on M.

In particular, if $\Gamma = U(\mathfrak{h})$, then Γ-weight module is a classical weight module.

Suppose $\mathfrak{g}$ is of type A and Γ is a Gelfand-Tsetlin subalgebra of $\mathfrak{g}$ [17]. Then $\Gamma \subset U_0(\mathfrak{g})$ and every generic Gelfand-Tsetlin $\mathfrak{g}$-module is Γ-pointed. We refer to [17] for details. Clearly, every finite-dimensional $\mathfrak{g}$-module is also Γ-pointed. A family of simple Γ-pointed modules in type A was studied in [18]. On the other hand, there exist Gelfand-Tsetlin modules which are not pointed.

4. Construction of simple weight A_2-modules

In this section we consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}(3)$. Even though this case is well understood we give some details to illustrate our approach.

4.1. Centralizer of the Cartan subalgebra of A_2

Let $\Delta = \{\alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2, \alpha_4 = -\alpha_1 - \alpha_2, \alpha_5 = -\alpha_2, \alpha_6 = -\alpha_1\}$ be the root system of $\mathfrak{g}$. Fix a Chevalley basis G: $e_{10} = E_{12}, f_{10} = E_{21}$, $e_{01} = E_{23}$, $f_{01} = E_{32}$, $e_{11} = E_{13}$, $f_{11} = E_{31}$, $h_{10} = E_{11} - E_{22}$, $h_{01} = E_{22} - E_{33}$.

Let us define the following order on the elements of G:

$$h_{01} < h_{10} < f_{01} < f_{10} < f_{11} < e_{11} < e_{10} < e_{01}.$$

We have the following set of indecomposable lists of roots: $\{\alpha_1, \alpha_6\}$, $\{\alpha_2, \alpha_5\}$, $\{\alpha_3, \alpha_4\}$, $\{\alpha_1, \alpha_2, \alpha_4\}$, $\{\alpha_3, \alpha_5, \alpha_6\}$, and the following set of perfect monomials:

$$h_1 = h_{01}, h_2 = h_{10}, c_1 = f_{01}e_{01}, c_2 = f_{10}e_{10}, c_3 = f_{11}e_{11}, c_4 = f_{11}e_{10}e_{01}, c_5 = f_{01}f_{10}e_{11}.$$

Define the order on the set of perfect monomials: $h_1 < h_2 < c_1 < c_2 < c_3 < c_4 < c_5$.

Note that any monomial containing both variables c_4 and c_5 is not semi-perfect since $m' = c_1c_2c_3$ has the same associated list of roots as $m = c_4c_5$, but $c_1 < c_4$. Applying the relations between the generators we easily obtain the following statement.

Proposition 4.1.

- The following set of monomials is a basis of $\hat{U}_0(\mathfrak{g})$:

$$P = \{h_1^{s_1} h_2^{s_2} c_1^{s_3} c_2^{s_4} c_3^{s_5} c_4^{s_6} \mid s_1, \ldots, s_6 \in \mathbb{N}\} \cup \{h_1^{s_1} h_2^{s_2} c_1^{s_3} c_2^{s_4} c_3^{s_5} c_5^{s_6} \mid s_1, \ldots, s_6 \in \mathbb{N}\}.$$

- The set $\tilde{K} = \{X \mid X \in K', \mathrm{Len}(X) = 2\}$ is a generating set of the ideal of relations K.

The following $\tilde{K}$ is the list of all relations of length two:

$$c_jh_i = h_ic_j, i = 1, 2, j = 1, \ldots, 5 \tag{4.1}$$
$$c_2c_1 = -c_5 + c_4 + c_1c_2, \tag{4.2}$$
$$c_3c_1 = c_5 - c_4 + c_1c_3, \tag{4.3}$$
$$c_4c_1 = -2c_5 + (2 + h_1)c_4 + c_1c_4 - c_1c_3 + c_1c_2, \tag{4.4}$$
$$c_5c_1 = -h_1c_5 + c_1c_5 + c_1c_3 - c_1c_2, \tag{4.5}$$
$$c_3c_2 = -c_5 + c_4 + c_2c_3, \tag{4.6}$$
$$c_4c_2 = h_2c_3 + h_2c_4 + c_2c_4 + c_2c_3 - c_1c_2, \tag{4.7}$$
$$c_5c_2 = -h_2c_3 - h_2c_5 + c_2c_5 - c_2c_3 + c_1c_2, \tag{4.8}$$
$$c_4c_3 = 2c_5 - (h_2 + h_1 + 2)c_4 + c_3c_4 - h_2c_3 - c_2c_3 + c_1c_3, \tag{4.9}$$
$$c_5c_3 = (h_2 + h_1)c_5 + c_3c_5 + h_2c_3 + c_2c_3 - c_1c_3, \tag{4.10}$$
$$c_5c_4 = -(2h_2 + h_1)c_5 - c_3c_5 - 2h_2c_3 + c_2c_5 - 2c_2c_3 - c_1c_5 + c_1c_2c_3 + (h_2 + h_1 + 2)c_1c_2, \tag{4.11}$$
$$c_4c_5 = -h_1c_5 - c_3c_5 + h_1h_2c_3 + c_2c_5 + h_1c_2c_3 - c_1c_5 + h_2c_1c_3 + c_1c_2c_3. \tag{4.12}$$

The following Casimir elements generate the center of the universal enveloping algebra [14]:

$$z_1 = c_3 + c_2 + c_1 + \frac{1}{3}(h_2^2 + 3h_2 + h_1^2 + 3h_1 + h_2h_1) \tag{4.13}$$

$$z_2 = c_5 + c_4 + \frac{1}{3}(h_1 - h_2)c_3 - \frac{1}{3}(6 + 2h_1 + h_2)c_2 + \frac{1}{3}(h_1 + 2h_2)c_1 \\ + \frac{1}{27}(-h_2 - 3 + h_1)(6 + 2h_1 + h_2)(h_1 + 2h_2) \tag{4.14}$$

Using the relations (4.1)-(4.14), we can reduce the number of generators of $U_0(\mathfrak{g})$: applying (4.2), (4.13), and (4.14) we can exclude c_3, c_4, c_5 from all other relations. As a result we will get a generating set $S = \{h_1, h_2, z_1, z_2, c_1, c_2\}$ of $U_0(\mathfrak{g})$ with the following decomposition: $S_1 = \{h_1, h_2, z_1, z_2\}$, $S_2 = \{c_1\}$, $S_3 = \{c_2\}$. For the set of relations we have $R = R_1 \cup R_2$, where R_1 consist of the relations obtained from (4.1)-(4.5), and R_2 consists of the relations obtained from (4.6)-(4.12).

4.2. Generic torsion free A_2-modules

Let Γ be the commutative subalgebra of $U_0(\mathfrak{g})$ generated by the elements h_1, h_2, z_1, z_2, c_1. This is a Gelfand-Tsetlin subalgebra of $\mathfrak{sl}(3)$. We give a construction of a family of Γ-pointed modules $V(a_1, a_2, a_3, \xi, \mu)$ which depend on five complex parameters with the restriction $a_3 \notin \mathbb{Z}$. These are essentially the universal generic Gelfand-Tsetlin modules initially constructed in [17] and [7] using a Gelfand-Tsetlin basis [19].

Fix arbitrary $a_1, a_2, a_3, \xi, \mu \in \mathbb{C}$ such that $a_3 \notin \mathbb{Z}$. To simplify formulas we define the following set of indexed variables:

$$\begin{aligned}
&h_{ij}^{(1)} = a_1 + 2i - j,\ h_{ij}^{(2)} = a_2 - i + 2j,\ s_{j,k} = a_3 - j + 2k - 1,\\
&S_{ijk}^{+} = \frac{1}{2}(s_{j,k} + h_{i,j}^{(1)}) = \frac{1}{2}(a_1 + a_3 - 1) + i - j + k,\\
&S_{ik}^{-} = \frac{1}{2}(s_{0,k} - h_{i,0}^{(1)}) = \frac{1}{2}(-a_1 + a_3 - 1) - i + k,\\
&T_k^{+} = \frac{1}{2}(s_{0,k} + \frac{1}{3}(h_{0,0}^{(1)} + 2h_{0,0}^{(2)})) = \frac{1}{6}(a_1 + 2a_2 + 3a_3) + k - \frac{1}{2},\\
&T_{jk}^{-} = \frac{1}{2}(s_{j,k} - \frac{1}{3}(h_{0,j}^{(1)} + 2h_{0,j}^{(2)})) = \frac{1}{6}(-a_1 - 2a_2 + 3a_3) - j + k - \frac{1}{2},\\
&Q_{jk}^{-} = -\mu + T_{j,k-2}^{-}\xi - T_{j,k}^{-}T_{j,k-1}^{-}T_{j,k-2}^{-} = -(T_{j,k-1}^{-} - t_1)(T_{j,k-1}^{-} - t_2)(T_{j,k-1}^{-} - t_3),\\
&Q_k^{+} = \mu + T_k^{+}\xi - T_k^{+}T_{k-1}^{+}T_{k-2}^{+} = (T_{k-1}^{+} + t_1)(T_{k-1}^{+} + t_2)(T_{k-1}^{+} + t_3).\\
&\quad \text{where } t_1, t_2, t_3 \text{ are three roots of the equation } t^3 - t(\xi + 1) + \mu + \xi = 0
\end{aligned} \tag{4.15}$$

Define an action of the Lie algebra $\mathfrak{g}$ on the vector space $V(a_1, a_2, a_3, \xi, \mu) = \mathrm{span}_{\mathbb{C}}\{\mathbf{v}_{ijk} \mid i, j, k \in \mathbb{Z}\}$ as follows:

$$\begin{aligned}
&z_1(\mathbf{v}_{ijk}) = \xi\mathbf{v}_{ijk},\ z_2(\mathbf{v}_{ijk}) = \mu\mathbf{v}_{ijk},\ h_{01}(\mathbf{v}_{ijk}) = h_{ij}^{(1)}\mathbf{v}_{ijk},\ h_{10}(\mathbf{v}_{ijk}) = h_{ij}^{(2)}\mathbf{v}_{ijk},\\
&e_{01}(\mathbf{v}_{ijk}) = S_{ijk}^{+}\mathbf{v}_{i+1,j,k},\ f_{01}(\mathbf{v}_{ijk}) = S_{ik}^{-}\mathbf{v}_{i-1,j,k},\\
&e_{10}(\mathbf{v}_{ijk}) = Q_{jk}^{-}\mathbf{v}_{i,j+1,k} + \frac{S_{i,k}^{-}}{s_{j+1,k}s_{jk}}\mathbf{v}_{i,j+1,k+1},\\
&f_{10}(\mathbf{v}_{ijk}) = Q_k^{+}\mathbf{v}_{i,j-1,k-1} + \frac{S_{i,j,k}^{+}}{s_{j+1,k}s_{jk}}\mathbf{v}_{i,j-1,k}.
\end{aligned} \tag{4.16}$$

We have the following statement (cf. [7, 14]).

Theorem 4.2. Let $a_1, a_2, a_3, \xi, \mu \in \mathbb{C}$ and $a_3 \notin \mathbb{Z}$. Then

1. The space $V(a_1, a_2, a_3, \xi, \mu)$ is a torsion free Γ-pointed $\mathfrak{g}$-module if and only if S^+_{ijk} and S^-_{ik} are non-zero, for all $i, j, k \in \mathbb{Z}$.
2. The module $V(a_1, a_2, a_3, \xi, \mu)$ is simple if and only if S^+_{ijk}, S^-_{ik}, Q^-_{jk}, Q^+_k are non-zero, for all $i, j, k \in \mathbb{Z}$.

Proof. The fact that $V(a_1, a_2, a_3, \xi, \mu)$ is a $\mathfrak{g}$-module was shown in [14] and [7]. It is Γ-pointed as Γ separates the basis elements. It follows from formulas above that the module $V(a_1, a_2, a_3, \xi, \mu)$ is torsion free if and only if S^+_{ijk} and S^-_{ik} are different from zero for all $i, j, k \in \mathbb{Z}$. Note that for every $\mathfrak{h}$-weight λ of $V(a_1, a_2, a_3, \xi, \mu)$, the λ-weight subspace is infinite-dimensional with basis $\mathbf{v}_{ijk}$, where i, j are determined by λ and k runs through $\mathbb{Z}$. In this basis the operator c_1 is presented by an infinite diagonal matrix and the operator c_2 is presented by a 3-diagonal matrix (b_{st}), with $b_{st} = 0$, for all s, t such that $|s - t| > 1$. This is a consequence of relations above. The simplicity of $V(a_1, a_2, a_3, \xi, \mu)$ results in the simplicity of each λ-weight subspace as $U_0(\mathfrak{g})$-module. This implies the required conditions in item 2. $\square$

Simple subquotients of the module $V(a_1, a_2, a_3, \xi, \mu)$ give all simple generic Gelfand-Tsetlin $\mathfrak{g}$-modules with finite or infinite weight multiplicities [20]. Moreover, since any simple subquotient V' of $V(a_1, a_2, a_3, \xi, \mu)$ is torsion free and we can choose any weight from the weight lattice as part of the parameter set, we have

Corollary 4.3. If V' is a simple generic Gelfand-Tsetlin $\mathfrak{g}$-module, then it is isomorphic to a subquotient of $V(a_1, a_2, a_3, \xi, \mu)$ for some suitable parameters such that $0 \leq Re\, a_1 < 1$, $0 \leq Re\, a_2 < 3$, $0 < Re\, a_3 < 2$, $a_3 \neq 1$, where $Re\, a$ stands for the real part of a.

4.3. Subquotients of $V(a_1, a_2, a_3, \xi, \mu)$

As we saw in the previous section, the Γ-pointed module $V(a_1, a_2, a_3, \xi, \mu)$ is simple when S^+_{ijk}, S^-_{ik}, Q^-_{jk}, Q^+_k are non-zero, for all $i, j, k \in \mathbb{Z}$. If one or more these coefficients are equal to zero, then a system of sub modules arises. Below, we present two examples of such sub modules. Let $I = \mathbb{Z}^3 \subset \mathbb{R}^3$ be the weight lattice (the lattice of indices) of the module $V(a_1, a_2, a_3, \xi, \mu)$.

Case 1. Assume that $S^+_{i_0,j_0,k_0} = 0$ for some $i_0, j_0, k_0 \in \mathbb{Z}$. Then, it follows that $a_1 + a_3 \in 2\mathbb{Z} + 1$. Consider the function $F(i, j, k) = S^+_{ijk}$ defined on the set I. The subset of zero points $P(F) = \{(i, j, k) \,|\, F(i, j, k) = 0\}$ is called the splitting hyperplane of the function F. The splitting hyperplane $F(i, j, k)$ divides the set I into two subsets, called the positive and the negative components, given by

$$I^+_F = \{(i, j, k) \,|\, F(i, j, k) > 0\} \quad \text{and} \quad I^-_F = \{(i, j, k) \,|\, F(i, j, k) \leq 0\}.$$

Using (4.16), we observe that, for any $(i, j, k) \in I^-_F$, the action of the elements $e_{01}, f_{01}, e_{10}, f_{10}$ on $\mathbf{v}_{ijk}$ results in elements that remain in the subspace $V' = \mathrm{span}_{\mathbb{C}}\{\mathbf{v}_{ijk} \,|\, (i, j, k) \in I^-_F\} \subset V$. Thus V' forms a submodule of V.

It is simple if none of S^-_{ik}, Q^+_{jk} or Q^-_{jk} vanish for any choice of indices.

Case 2. Assume that $Q^+_k = 0$ for some $k \in \mathbb{Z}$. As we can see, Q^+_k is the product of three first degree polynomials $F_m(i, j, k) = (T^+_k + t_m)$, with $m = 1, 2, 3$, in the variable k. Let us consider the case when $Q_k^+ = 0$ has three distinct roots. We need to find possible values of the parameters $a_1, a_2, a_3, t_1, t_2, t_3$ such that the system of three equations: $F_m(i, j, k_m) = 0$, $m = 1, 2, 3$ has integer solutions for the variables k_m. We find that $Q_k^+ = 0$, for $k = k_1, k_2, k_3 \in \mathbb{Z}$ if the following conditions hold: $t_m = \frac{1}{3}(k_1 + k_2 + k_3) - k_m$, $m = 1, 2, 3$ and $a_3 = 3 - \frac{1}{3}(2k_1 + 2k_2 + 2k_3 + a_1 + 2a_2)$. Suppose that $k_1 > k_2 > k_3$. Then we have three splitting hyperplanes $P_m = \{(i, j, k) \,|\, F_m(i, j, k) = 0\}$ and $I = I_{F_m}^+ \cup I_{F_m}^-$, where

$$I_{F_m}^+ = \{(i, j, k) \,|\, F_m(i, j, k) \geq 0\}\,, I_{F_m}^- = \{(i, j, k) \,|\, F_m(i, j, k) < 0\}\,, \text{for } m = 1, 2, 3\,.$$

Again, using (4.16) we immediately obtain

Proposition 4.4. The subspaces $V'_m = \operatorname{span}_{\mathbb{C}}\{\mathbf{v}_{ijk} \,|\, (i, j, k) \in I_{F_m}^+\} \subset V(a_1, a_2, a_3, \xi, \mu)$ are submodules of $V(a_1, a_2, a_3, \xi, \mu)$ satisfying the inclusion $V'_1 \subset V'_2 \subset V'_3$.

If none of S_{ijk}^+ or S_{ik}^- or Q_{jk}^- are zero for all choices of indices, then V'_1 and V'_2/V'_1 and V'_3/V'_2 and $V(a_1, a_2, a_3, \xi, \mu)/V'_3$ are simple, torsion-free modules. All non-zero weight spaces of these modules have dimensions respectively: ∞ for V'_1 and $V(a_1, a_2, a_3, \xi, \mu)/V'_3$, $k_1 - k_2$ for V'_2/V'_1, and $k_2 - k_3$ for V'_3/V'_2.

A similar construction of a family of torsion-free submodules can be obtained in the case when $Q_{j,k}^- = 0$. Combining the conditions under which one or more coefficients are equal to zero we can obtain various distinct modules. The full classification of $\mathfrak{sl}(3)$ modules was obtained in [7] using the Gelfand-Tsetlin tableaux technique.

5. Construction of simple weight C_2-modules

In this section we use the technique developed in the previous section to construct simple Gelfand-Tsetlin modules for the Lie algebra of type C_2.

5.1. The centralizer for C_2

Consider the root system of C_2:

$$\Delta = \{\alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2, \alpha_4 = 2\alpha_1 + \alpha_2, \alpha_5 = -\alpha_4, \alpha_6 = -\alpha_3, \alpha_7 = -\alpha_2, \alpha_8 = -\alpha_1\}$$

and the Chevalley basis: $e_{10} = E_{12} - E_{43}$, $f_{10} = E_{21} - E_{34}$, $e_{01} = E_{31}$, $f_{01} = E_{13}$, $h_{10} = (E_{11} - E_{22} - E_{33} + E_{44})$, $h_{01} = -E_{11} + E_{33}$, $e_{11} = -E_{32} - E_{41}$, $e_{21} = 2E_{42}$, $f_{11} = -E_{14} - E_{23}$, $f_{21} = 2E_{24}$ with the order $h_{10} < h_{01} < f_{10} < f_{01} < f_{11} < f_{21} < e_{21} < e_{11} < e_{01} < e_{10}$.

The following is a complete set of indecomposable lists of roots: $\{\alpha_1, \alpha_8\}$, $\{\alpha_2, \alpha_7\}$, $\{\alpha_3, \alpha_6\}$, $\{\alpha_4, \alpha_5\}$, $\{\alpha_1, \alpha_2, \alpha_6\}$, $\{\alpha_3, \alpha_7, \alpha_8\}$, $\{\alpha_1, \alpha_3, \alpha_5\}$, $\{\alpha_4, \alpha_6, \alpha_8\}$, $\{\alpha_1, \alpha_1, \alpha_2, \alpha_5\}$, $\{\alpha_4, \alpha_7, \alpha_8, \alpha_8\}$, $\{\alpha_3, \alpha_3, \alpha_5, \alpha_7\}$, $\{\alpha_2, \alpha_4, \alpha_6, \alpha_6\}$.

Let us choose the following generators of $U_0 = U_0(C_2)$:

$$\begin{aligned}
&h_1 = h_{01},\ h_2 = h_{10},\ c_1 = f_{01}e_{01},\ c_2 = f_{21}e_{21},\ c_3 = f_{10}e_{10},\ c_4 = f_{11}e_{11},\\
&c_5 = f_{11}e_{01}e_{10},\ c_6 = f_{10}f_{01}e_{11},\ c_7 = f_{21}e_{11}e_{10},\ c_8 = f_{10}f_{11}e_{21},\\
&c_9 = f_{21}e_{01}e_{10}e_{10},\ c_{10} = f_{10}f_{10}f_{01}e_{21},\ c_{11} = f_{01}f_{21}e_{11}e_{11},\ c_{12} = f_{11}f_{11}e_{21}e_{01}.
\end{aligned}$$

Order the set of perfect monomials as follows:

$$h_1 < h_2 < c_1 < c_2 < c_3 < c_4 < c_5 < c_6 < c_7 < c_8 < c_9 < c_{10} < c_{11} < c_{12}. \tag{5.1}$$

As in type A_2 we have

Proposition 5.1.

1) The following set of monomials is a basis of U_0:

$$
\begin{aligned}
P = & \{h_1^{s_1} h_2^{s_2} c_1^{s_3} c_2^{s_4} c_3^{s_5} c_4^{s_6} c_m^{s_m} c_n^{s_n} \mid s_1, s_2, \dots, s_6, s_m, s_n \in \mathbb{N} \text{ and} \\
& (m,n) \in \{(5,9),(5,12),(6,10),(6,11),(7,9),(7,11),(8,10),(8,12)\}\}.
\end{aligned} \tag{5.2}
$$

2) The set $\tilde{K} = \{X | X \in K', \mathrm{Len}(X) = 2\}$ is a generating set of the ideal of relations K.

Proof. Let $X = c_m c_n c_k \in Q_0(C_2)$, where $4 < m < n < k \leq 12$ be a monomial. Manual calculation shows that for any triple m, n, k satisfying above condition there is $1 \leq p \leq 4$ such that $L_1(c_p)$ is sublist of $L_1(X)$. For example, if $(m, n, k) = (5, 6, 7)$, then $L_1(c_1) \sqsubset L_1(X)$. This means that no semi-perfect element can contain above three perfect monomials.

Now, let $X = c_m c_n \in Q_0(C_2)$, where $4 < m < n \leq 12$ be a monomial. One can easily see that for any pair m, n which is not in the (5.2), there exists $1 \leq p \leq 4$ such that $L_1(c_p)$ is a sublist of $L_1(X)$. For example, if $(m, n) = (9, 12)$, then $L_1(c_2) \sqsubset L_1(X)$. It follows that the only possible perfect monomials in semi-perfect monomials are the ones given above, proving 1. In particular this means that we can take $d_0 = 2$ in the definition of K' (cf. (2.4)) and in Theorem 2.5 with sort order defined in (5.1).

□

Denote $h_3 = h_1 + h_2$ and let $U_0' = U_0[h_1^{-1}, h_3^{-1}]$. It will be convenient for us to work with U_0'.

The list of all relations is rather big, so we will give only those of them that are used in our calculations. The following list of relations is used to express $c_{12}, \dots, c_5$ via c_4, c_3, c_2, c_1, h_1, h_3, z_1 in U_0':

$$c_{12} = -c_{10} - c_2 - 2c_8 + [c_1, c_8], \tag{5.3}$$

$$c_{11} = -c_9 - 2c_7 + c_2 - [c_1, c_7], \tag{5.4}$$

$$c_{10} = c_9 - c_8 + c_7 + \frac{1}{2}[c_5, c_2] + [c_1, c_8] \tag{5.5}$$

$$c_9 = \frac{1}{2h_1}([c_1, [c_1, c_7]] - 2c_1c_2 - 2c_7c_1 - 2c_1c_7) + \frac{1}{2}[c_7, c_1] - 2c_7 - c_2, \tag{5.6}$$

$$c_8 = c_7 + \frac{1}{2}[c_2, c_3] \tag{5.7}$$

$$c_7 = \frac{1}{16h_3}(-[c_2, [c_2, c_3]] + 8c_3c_2 - 8c_2c_4 - 8h_1c_2) + \frac{1}{4}[c_3, c_2] - \frac{1}{2}c_2, \tag{5.8}$$

$$c_6 = c_5 + [c_3, c_1] \tag{5.9}$$

$$c_5 = -c_4 + c_1c_3 + \frac{1}{2h_1}(-[c_1, c_1, c_3] - c_3c_1(h_1 - 2) - 2c_1c_4). \tag{5.10}$$

The following relations are used to find relations between the elements c_4, c_3, c_2, c_1, h_1, h_3, z_1.

$$c_5c_1 = -c_6 + (h_1 + 1)c_5 + h_1c_4 + c_1c_5 + c_1c_4 - c_1c_3 \tag{5.11}$$

$$c_7c_2 = -4h_3c_7 + c_2c_7 - 2c_2c_4 + 2c_2c_3 + (-2h_3 - 2h_1)c_2 \tag{5.12}$$

$$c_5c_3 = c_9 + c_8 - 2c_6 + (h_3 - h_1)c_5 + c_3c_5 - c_3c_4 + 2c_1c_3 \tag{5.13}$$

$$c_7c_3 = -2c_9 - 2c_8 + (h_3 - h_1)c_7 + 4c_6 + c_3c_7 + 2c_3c_4 - c_2c_3 \tag{5.14}$$

$$\begin{aligned} c_6c_5 = {} & c_{10} + 2c_8 - c_7 + (h_3 - h_1 - 2)c_6 - c_4c_5 \\ & - c_3c_6 - 2c_3c_4 - c_1c_8 + 2c_1c_6 + c_1c_3c_4 + (h_3 + h_1 + 2)c_1c_3. \end{aligned} \tag{5.15}$$

The center of the universal enveloping algebra of $\mathfrak{g}$ is generated by the following Casimir elements:

$$z_1 = 4c_1 + c_2 + 2c_3 + 2c_4 + 2h_1^2 + 2h_3^2 + 2h_1 + 4h_3, \tag{5.16}$$

$$\begin{aligned} z_2 = {} & 2c_{12} + 2c_{11} - 2c_{10} - 2c_9 + (2h_1 + 1)c_8 + (2h_1 - 1)c_7 \\ & + (4h_3 + 6)c_6 + (4h_3 + 10)c_5 - c_4^2 + (-2h_1h_3 - 4h_1 + 2h_3 + 6)c_4 - 2c_3c_4 \\ & - c_3^2 + (2h_1h_3 + 4h_1 + 2h_3 + 6)c_3 - (h_1 - 1)(h_1 + 1)c_2 - 4c_1c_2 \\ & - 4(h_2 + 3)(h_2 + 1)c_1 - h_1(h_3 + 3)(h_3 + 1)(h_1 + 2). \end{aligned}$$

As in type A, our goal is to find the "best" choice of a generating set S of U_0, such that the cardinality of the set S_3 is minimal.

Lemma 5.2.

1. The set $S = \{h_1, h_2, z_1, z_2, c_1, c_2, c_3\}$ is a generating set of the centralizer U_0' with the following decomposition: $S_1 = \{h_1, h_2, z_1, z_2\}$, $S_2 = \{c_1, c_2\}$, $S_3 = \{c_3\}$.
2. The decomposition of the set of relations is the following: $R = R_1 \cup R_2$, where R_1 consists of two relations obtained from (5.11) and (5.12), while R_2 consists of three relations obtained from (5.13), (5.14), and (5.15).

Proof. Using the (5.3)-(5.10) and (5.16) for the Casimir element z_1 we can exclude the generators $c_{12}, \dots, c_4$ from all other relations. As the result, we will get the generating set $S = \{h_1, h_2, z_1, c_1, c_2, c_3\}$. The rest can be verified by direct computations. □

Remark. We can not claim that S is a generating set of U_0. Nevertheless, any U_0-module M with a non-zero scalar action of h_1 is a U_0'-module.

5.2. Construction of torsion free C_2-modules

Let Γ be the commutative subalgebra of $U_0(C_2)$ generated by the elements h_1, h_2, z_1, z_2, c_1. We construct two families of Γ-pointed modules, each depending on four complex parameters.

Fix arbitrary complex number a_1, a_2, a_3, a_4, η, and define the following set of indexed variables:

$$\begin{aligned} & h_{ij}^{(1)} = a_1 + 2i - j,\ h_{ij}^{(2)} = a_2 - 2i + 2j,\ s_{jk} = a_3 - j + 2k - 1,\ Q_{jk}^{\pm} = \frac{\eta}{s_{jk}} \pm 1, \\ & S_{ijk}^{+} = \frac{1}{2}(a_1 + a_3 + 2i - 2j + 2k - 1),\ S_{ik}^{-} = \frac{1}{2}(-a_1 + a_3 - 2i + 2k - 1), \\ & T_k^{+} = \frac{1}{2}(a_1 + a_2 + a_4 + 2k - 1),\ T_{jk}^{-} = \frac{1}{2}(-a_1 - a_2 + a_4 - 2j + 2k - 1), \\ & \quad \text{where}\ i, j, k \in \mathbb{Z}. \end{aligned} \tag{5.17}$$

Consider the vector space $V(a_1,a_2,a_3,a_4,\eta) = \mathrm{span}_{\mathbb{C}}\{\mathbf{v}_{ijk} \,|\, i,j,k \in \mathbb{Z}\}$ and define the following operators on $V(a_1,a_2,a_3,a_4,\eta)$ (using the same letters as for the generators of $\mathfrak{g}$):

$$\begin{aligned}
h_1(\mathbf{v}_{ijk}) &= h^{(1)}_{ij}\mathbf{v}_{ijk},\ h_2(\mathbf{v}_{ijk}) = h^{(2)}_{ij}\mathbf{v}_{ijk},\\
e_{01}(\mathbf{v}_{ijk}) &= S^+_{ijk}\mathbf{v}_{i+1,j,k},\ f_{01}(\mathbf{v}_{ijk}) = S^-_{ik}\mathbf{v}_{i-1,j,k},\\
e_{10}(\mathbf{v}_{ijk}) &= S^-_{ik}Q^+_{j+1,k}\mathbf{v}_{i,j+1,k+1} + T^-_{j+1,k}Q^-_{j+1,k}\mathbf{v}_{i,j+1,k},\\
f_{10}(\mathbf{v}_{ijk}) &= S^+_{ijk}Q^+_{j+1,k}\mathbf{v}_{i,j-1,k} + T^+_{k-1}Q^-_{j+1,k}\mathbf{v}_{i,j-1,k-1},\\
e_{11}(\mathbf{v}_{ijk}) &= -S^+_{ijk}Q^+_{j+1,k}\mathbf{v}_{i+1,j+1,k+1} + T^-_{j+1,k}Q^-_{j+1,k}\mathbf{v}_{i+1,j+1,k},\\
f_{11}(\mathbf{v}_{ijk}) &= S^-_{i,k}Q^+_{j+1,k}\mathbf{v}_{i-1,j-1,k} - T^+_{k-1}Q^-_{j+1,k}\mathbf{v}_{i-1,j-1,k-1},\\
e_{21}(\mathbf{v}_{ijk}) &= 2T^-_{j+1,k}\mathbf{v}_{i+1,j+2,k+1},\ f_{21}(\mathbf{v}_{ijk}) = 2T^+_{k-1}\mathbf{v}_{i-1,j-2,k-1}.
\end{aligned} \tag{5.18}$$

These formulas define a Γ-module structure on $V(a_1,a_2,a_3,a_4,\eta)$, but at this point we can not claim a $\mathfrak{g}$-module structure.

Lemma 5.3. The subalgebra Γ has a simple spectrum on $V(a_1,a_2,a_3,a_4,\eta)$, and hence separates the basis elements $\mathbf{v}_{ijk}$, if and only if $a_3 \notin \mathbb{Z}$.

Proof. We need to show that Γ acts with different characters on basis elements $\mathbf{v}_{i,j,k}$. It is sufficient to consider vectors from the same weight space. Suppose

$$h_1(\mathbf{v}_{ijk}) = (a_1+2i-j)\mathbf{v}_{ijk} = \lambda\mathbf{v}_{ijk},\ h_2(\mathbf{v}_{ijk}) = (a_2-2i+2j)\mathbf{v}_{ijk} = \mu\mathbf{v}_{ijk},$$

for some fixed λ and μ. Then i and j are uniquely determined: $j = \lambda+\mu-a_1-a_2$ and $i = \frac{1}{2}(\lambda+j-a_1)$. Hence, basis elements of this weight subspace differ by the third index. Consider $\mathbf{v}_{ijk_1}$ and $\mathbf{v}_{ijk_2}$ for arbitrary integer i, j, k_1, k_2. We have

$$c_1(\mathbf{v}_{ijk}) = f_{01}e_{01}(\mathbf{v}_{ijk}) = f_{01}(S^+_{i,j,k}\mathbf{v}_{i+1jk}) = S^-_{i+1,k}S^+_{i,j,k}(\mathbf{v}_{ijk}) \tag{5.19}$$

Suppose that c_1 has the same value on $\mathbf{v}_{ijk_1}$ and $\mathbf{v}_{ijk_2}$. Then we have the equation $0 = S^-_{i+1,k_1}S^+_{i,j,k_1} - S^-_{i+1,k_2}S^+_{i,j,k_2}$ which simplifies to $0 = \frac{1}{4}(k_1-k_2)(a_3-j+k_1+k_2-2)$. Since $k_1 \neq k_2$, it follows that $a_3 = j-k_1-k_2+2$, which must be an integer. □

Denote $V_1(a_1,a_2,a_3,\eta) = V(a_1,a_2,a_3,a_3,\eta)$ and $V_2(a_1,a_2,a_3,a_4) = V(a_1,a_2,a_3,a_4,0)$ and $\xi = 2(\eta+1)(\eta-2)$.

Theorem 5.4. Let $V = V(a_1,a_2,a_3,a_4,\eta)$. Then

1. V is a $\mathfrak{g}$-module if and only if $V = V_1(a_1,a_2,a_3,\eta)$ with $a_3 \notin \mathbb{Z}$, or $V = V_2(a_1,a_2,a_3,a_4)$.
2. The $\mathfrak{g}$-module V is a torsion free Γ-pointed $\mathfrak{g}$-module if and only if $S^+_{ijk}S^-_{ik}T^-_{jk}T^+_k \neq 0$ for all $i,j,k \in \mathbb{Z}$.
3. The torsion free Γ-pointed $\mathfrak{g}$-module V is simple if and only if $Q^+_{jk}Q^-_{jk} \neq 0$ for all $j,k \in \mathbb{Z}$.
4. If V' is a simple torsion free Γ-pointed $\mathfrak{g}$-module with a basis parametrized by the lattice $\mathbb{Z}^3$ and with separating action of Γ on basis elements, then $V' \simeq V(a'_1,a'_2,a'_3,a'_4,\eta')$ for some suitable choice of parameters such that $0 \leq Re\,a'_1 < 1$, $0 \leq Re\,a'_2 < 2$, $0 < Re\,a'_3 < 2$, where $Re\,a$ denotes the real part of a.

5. The action of the Casimir elements in the case $a_3 = a_4$:

$$z_1(\mathbf{v}_{ijk}) = \xi \mathbf{v}_{ijk},\ z_2(\mathbf{v}_{ijk}) = -\frac{1}{4}\xi(\xi+4)\mathbf{v}_{ijk}$$

   and in the case $a_3 \neq a_4$: $z_1(\mathbf{v}_{ijk}) = ((a_3 - a_4)^2 - 4)\mathbf{v}_{ijk}$, $z_2(\mathbf{v}_{ijk}) = 0$.

Proof. The formulas (5.17) and (5.18) are well defined if and only if $V = V_1(a_1, a_2, a_3, \eta)$ with $a_3 \notin \mathbb{Z}$, or $V = V_2(a_1, a_2, a_3, a_4)$. The statement that V is a $\mathfrak{g}$-module follows by checking all defining relations of the Lie algebra. The relations 5.11-5.15 give only two alternatives $V_1(a_1, a_2, a_3, \eta)$ and $V_2(a_1, a_2, a_3, a_4)$. It follows immediately from the formulas of the action of $\mathfrak{g}$ that $V(a_1, a_2, a_3, a_4, \eta)$ is a torsion free module if and only if $S^+_{i,j,k}$, $S^-_{i,k}$, T^-_{jk} and T^+_{k-1} are different from zero, for all $i, j, k \in \mathbb{Z}$. Note that Γ has a simple spectrum on $V(a_1, a_2, a_3, a_4, \eta)$ and hence, the action of Γ separates the basis elements by Lemma 5.3. In particular, $V(a_1, a_2, a_3, a_4, \eta)$ is Γ-pointed. This implies the second statement.

Suppose the module $V(a_1, a_2, a_3, a_4, \eta)$ is torsion free. Similarly to case of A_2, using the relations in $U_0(\mathfrak{g})$ we get that c_1, c_2 are presented by infinite diagonal matrices, while the element c_3 is presented by a 3-diagonal matrix on every weight subspace of $V(a_1, a_2, a_3, a_4, \eta)$. Using this fact and the separating action of Γ on basis elements, it is easy to see that conditions $Q^+_{jk} Q^-_{jk} \neq 0$, for all $i, j, k \in \mathbb{Z}$ are necessary and sufficient to guarantee that any element of $V(a_1, a_2, a_3, a_4, \eta)$ generates the whole module, which is equivalent to the simplicity of the module. This implies the third statement. Let V' be a simple torsion free Γ-pointed C_2-module with a basis $\{\mathbf{v}'_{ijk},\ i, j, k \in \mathbb{Z}\}$ such that Γ acts by different characters on the basis elements $\mathbf{v}'_{ijk}$. Fix one basis element $\mathbf{v}'_{ijk}$ and apply the generators of the centralizer $U_0(\mathfrak{g})$. One can see directly from the action that $U_0(\mathfrak{g})\mathbf{v}'_{ijk}$ will be equal to the whole weight space of V', which $\mathbf{v}'_{ijk}$ belongs to. Moreover, the action of the generators of Γ will determine uniquely the corresponding parameters a_1, a_2, a_3, a_4, η. Hence, we get a non-zero homomorphism θ of $U_0(\mathfrak{g})$-modules $U_0(\mathfrak{g})\mathbf{v}_{ijk}$ and $U_0(\mathfrak{g})\mathbf{v}'_{ijk}$: $\theta(\mathbf{v}_{ijk}) = \mathbf{v}'_{ijk}$. Moreover, $U_0(\mathfrak{g})\mathbf{v}'_{ijk}$ is a simple $U_0(\mathfrak{g})$-module as V' is simple. Hence, θ is surjective. It extends to a surjective homomorphism from $V(a_1, a_2, a_3, a_4, \eta)$ to V'. Comparing the bases of both modules we conclude the isomorphism $V' \simeq V(a_1, a_2, a_3, a_4, \eta)$. Since V' is torsion free we can choose any weight of the weight lattice as part of the parameter set. This proves the fourth statement.

The last statement follows by direct computation. □

Hence, Theorem 5.4 provides two 4-parameter families of simple torsion free Γ-pointed $\mathfrak{g}$-modules. If $V(a_1, a_2, a_3, a_4, \eta)$ is a torsion free Γ-pointed $\mathfrak{g}$-module which is not simple, then all its simple subquotients are torsion free Γ-pointed modules. They exhaust all *generic* simple torsion free Γ-pointed $\mathfrak{g}$-modules, which are analogs of generic simple Gelfand-Tsetlin modules in type A.

6. Gelfand-Tsetlin modules for G_2

In this section we extend the results of previous sections to the of the Lie algebra of type G_2.

6.1. Construction of Centralizer of G_2

Define the root system for G_2 (for convenience will use notation $\beta_{i,j}$ for $\alpha_{-i,-j}$):

$$\Delta = \{\alpha_{01}, \alpha_{10}, \alpha_{11}, \alpha_{21}, \alpha_{31}, \alpha_{32}, \beta_{32}, \beta_{31}, \beta_{21}, \beta_{11}, \beta_{10}, \beta_{01}\},$$

where $\alpha_{ij} = i\alpha_{10} + j\alpha_{01}$. Fix a Chevalley basis: $e_{01} = E_{31} + E_{64}$, $f_{01} = E_{13} + E_{46}$, $e_{10} = 2E_{17} + E_{23} - E_{45} - E_{76}$, $f_{10} = E_{32} - E_{54} - 2E_{67} + E_{71}$, $e_{11} = -E_{21} + 2E_{37} - E_{65} + E_{74}$, $f_{11} = -E_{12} + 2E_{47} - E_{56} + E_{73}$, $e_{21} = E_{14} + 2E_{27} + E_{36} + E_{75}$, $f_{21} = E_{41} + 2E_{57} + E_{63} + E_{72}$, $e_{31} = E_{15} + E_{26}$, $f_{31} = E_{51} + E_{62}$, $e_{32} = -E_{24} + E_{35}$, $f_{32} = -E_{42} + E_{53}$, $h_{01} = -E_{11} + E_{33} - E_{44} + E_{66}$, $h_{10} = 2E_{11} + E_{22} - E_{33} + E_{44} - E_{55} - 2E_{66}$, $h_{31} = E_{11} + E_{22} - E_{55} - E_{66}$, $h_{21} = E_{11} + 2E_{22} + E_{33} - E_{44} - 2E_{55} - E_{66}$.

Here we are using a standard realization of G_2 with matrix units E_{ij}. All indecomposable lists of roots of G_2 can be obtained from primitive lists of roots by Lemma 2.3.2.

Then we obtain the following description of perfect monomials (we omit the details).

Lemma 6.1. The following is the set of all perfect monomials:

$$\begin{gathered}
h_1 = h_{01},\ h_2 = h_{21},\ c_1 = f_{01}e_{01},\ c_2 = f_{10}e_{10},\ c_3 = f_{11}e_{11},\ c_4 = f_{11}e_{10}e_{01},\ c_5 = f_{01}f_{10}e_{11},\\
c_6 = f_{21}e_{21},\ c_7 = f_{21}e_{11}e_{10},\ c_8 = f_{10}f_{11}e_{21},\ c_9 = f_{21}e_{10}^2e_{01},\ c_{10} = f_{01}f_{10}^2e_{21},\ c_{11} = f_{11}^2e_{21}e_{01},\\
c_{12} = f_{01}f_{21}e_{11}^2,\ c_{13} = f_{31}e_{31},\ c_{14} = f_{31}e_{21}e_{10},\ c_{15} = f_{10}f_{21}e_{31},\ c_{16} = f_{31}e_{11}e_{10}^2,\ c_{17} = f_{10}^2f_{11}e_{31},\\
c_{18} = f_{31}e_{10}^3e_{01},\ c_{19} = f_{01}f_{10}^3e_{31},\ c_{20} = f_{32}e_{32},\ c_{21} = f_{32}e_{31}e_{01},\ c_{22} = f_{32}e_{21}e_{11},\ c_{23} = f_{01}f_{31}e_{32},\\
c_{24} = f_{11}f_{21}e_{32},\ c_{25} = f_{32}e_{21}e_{10}e_{01},\ c_{26} = f_{11}f_{21}e_{31}e_{01},\ c_{27} = f_{32}e_{11}^2e_{10},\ c_{28} = f_{01}f_{31}e_{21}e_{11},\\
c_{29} = f_{01}f_{10}f_{21}e_{32},\ c_{30} = f_{10}f_{11}^2e_{32},\ c_{31} = f_{32}e_{11}e_{10}^2e_{01},\ c_{32} = f_{10}f_{11}^2e_{31}e_{01},\ c_{33} = f_{01}f_{31}e_{11}^2e_{10},\\
c_{34} = f_{01}f_{10}^2f_{11}e_{32},\ c_{35} = f_{32}e_{10}^3e_{01}^2,\ c_{36} = f_{01}^2f_{10}^3e_{32},\ c_{37} = f_{11}^3e_{32}e_{01},\ c_{38} = f_{01}f_{32}e_{11}^3,\\
c_{39} = f_{11}^3e_{31}e_{01}^2,\ c_{40} = f_{01}^2f_{31}e_{11}^3,\ c_{41} = f_{11}f_{31}e_{32}e_{10},\ c_{42} = f_{21}^2e_{32}e_{10},\ c_{43} = f_{10}f_{32}e_{31}e_{11},\\
c_{44} = f_{21}^2e_{31}e_{11},\ c_{45} = f_{10}f_{32}e_{21}^2,\ c_{46} = f_{11}f_{31}e_{21}^2,\ c_{47} = f_{21}^2e_{31}e_{10}e_{01},\ c_{48} = f_{01}f_{10}f_{31}e_{21}^2,\\
c_{49} = f_{11}f_{32}e_{21}^2e_{01},\ c_{50} = f_{01}f_{21}^2e_{32}e_{11},\ c_{51} = f_{21}f_{31}e_{32}e_{10}^2,\ c_{52} = f_{10}^2f_{32}e_{31}e_{21},\ c_{53} = f_{21}f_{32}e_{31}e_{11}^2,\\
c_{54} = f_{11}^2f_{31}e_{32}e_{21},\ c_{55} = f_{31}^2e_{32}e_{10}^3,\ c_{56} = f_{10}^3f_{32}e_{31}^2,\ c_{57} = f_{31}f_{32}e_{21}^3,\ c_{58} = f_{2,1}^3e_{32}e_{31},\\
c_{59} = f_{21}^3e_{31}^2e_{01},\ c_{60} = f_{01}f_{31}^2e_{21}^3,\ c_{61} = f_{32}^2e_{21}^3e_{01},\ c_{62} = f_{32}^2e_{31}e_{11}^3,\ c_{63} = f_{11}^3f_{31}e_{32}^2,\ c_{64} = f_{01}f_{21}^3e_{32}^2.
\end{gathered}$$

Define the order on the set of perfect monomials as follows: $h_1 < h_2 < c_1 < c_2 < \cdots < c_{63} < c_{64}$. The following proposition can be easily checked.

Proposition 6.2. The centralizer subalgebra $U_0(G_2)$ is generated by a finite set of monomials $\{h_1, h_2, c_1, \ldots, c_{64}\}$ with a finite number of relations $\{r_1, \ldots, r_m\}$, where r_i is a polynomial in $h_1, h_2, c_1, \ldots, c_{64}$ of length ≤ 4.

We will use the following quadratic Casimir element of $U_0(\mathfrak{g})$:

$$z_1 = 3c_1 + c_2 + c_3 + c_6 + 3c_{13} + 3c_{20} + h_{01}^2 + h_{01}h_{10} + h_{10}^2 + 4h_{01} + 5h_{10}.$$

Tedious computations show that using the relations, the elements $c_{64}, \ldots, c_5$ can be written in terms of c_4, c_3, c_2, c_1, h_1, h_2, z_1.

The Lie algebra G_2 contains the subalgebra $\hat{\mathfrak{g}}$ of type A_2 generated by the following elements h_{01}, h_{31}, e_{01}, f_{01}, e_{31}, f_{31}, e_{32}, f_{32}. We will use the results from the previous sections just adding "hat" to all generators and to all formulas. Consider a natural embedding $\varphi : \hat{\mathfrak{g}} \to \mathfrak{g}$:

$$\varphi(\hat{h}_{01}) = h_{01},\ \varphi(\hat{h}_{10}) = h_{31},\ \varphi(\hat{e}_{01}) = e_{01},\ \varphi(\hat{f}_{01}) = f_{01},\ \varphi(\hat{e}_{10}) = e_{31},$$
$$\varphi(\hat{f}_{10}) = f_{31},\ \varphi(\hat{e}_{11}) = e_{32},\ \varphi(\hat{f}_{11}) = f_{32}$$

and extend it to the embedding of the universal enveloping algebras.

The images of the Casimir elements $\hat{z}_1$, $\hat{z}_2$ of $U(\hat{\mathfrak{g}})$ in $U(\mathfrak{g})$ are:

$$
\begin{aligned}
Z_1 &= \varphi(\hat{z}_1) = c_{20} + c_{13} + c_1 + \frac{1}{3}(h_{31}^2 + 3h_{31} + h_{01}^2 + 3h_{01} + h_{31}h_{01})\\
Z_2 &= \varphi(\hat{z}_2) = c_{23} + c_{21} + \frac{1}{3}(h_{01} - h_{31})c_{20} - \frac{1}{3}(6 + 2h_{01} + h_{31})c_{13} + \frac{1}{3}(h_{01} + 2h_{31})c_1\\
&\quad + \frac{1}{27}(-h_{31} - 3 + h_{01})(6 + 2h_{01} + h_{31})(h_{01} + 2h_{31}).
\end{aligned}
$$

6.2. Torsion free G_2-modules

Let Γ be commutative subalgebra of $U_0(G_2)$ generated by elements h_1, h_2, z_1, c_1. This our Gelfand-Tsetlin subalgebra for G_2. We will give a construction of a 3-parameter family of Γ-pointed modules with separating action of Γ on basis elements.

Fix $a_1, a_2, a_3 \in \mathbb{C}$ such that $a_3 \notin \mathbb{Z}$. Define the following set of indexed variables:

$$
\begin{aligned}
&h_{01}(i,j) = a_1 + 2i - j,\;\; h_{10}(i,j) = \frac{1}{2}(h_{21}(i,j) - 3h_{01}(i,j)) = \frac{1}{2}(a_2 - 3a_1) - 3i + 2j,\\
&h_{21}(i,j) = a_2 + j,\;\; h_{11}(i,j) = \frac{1}{2}(h_{21}(i,j) + 3h_{01}(i,j)) = \frac{1}{2}(a_2 + 3a_1) + 3i - j,\\
&h_{31}(i,j) = \frac{1}{2}(h_{21}(i,j) - h_{01}(i,j)) = \frac{1}{2}(a_2 - a_1) - i + j,\\
&h_{32}(i,j) = \frac{1}{2}(h_{21}(i,j) + h_{01}(i,j)) = \frac{1}{2}(a_2 + a_1) + i,\;\; s_{jk} = a_3 - j + 2k - 1,\\
&\quad S^+_{ijk} = \frac{1}{2}(s_{jk} + h_{01}(i,j)) = \frac{1}{2}(a_1 + a_3 + 2i - 2j + 2k - 1),\\
&\quad S^-_{ik} = \frac{1}{2}(s_{0k} - h_{01}(i,0)) = \frac{1}{2}(-a_1 + a_3 - 2i + 2k - 1),\\
&\quad T^+_{jk} = \frac{1}{2}(s_{jk} + \frac{1}{3}h_{2,1}(0,j)) = \frac{1}{6}(a_1 + 2a_2 + 3a_3 - 2j + 6k - 3),\\
&\quad T^-_{jk} = \frac{1}{2}(s_{jk} - \frac{1}{3}h_{2,1}(0,j)) = \frac{1}{6}(-a_1 - 2a_2 + 3a_3 - 4j + 6k - 3),\\
&\quad A^+_{jk} = \frac{T^-_{j-1,k-1}T^+_{jk}T^+_{j+1,k}}{9s_{jk}s_{j+1,k}},\;\; A^-_{jk} = \frac{T^-_{j-1,k-1}T^-_{jk}T^+_{j+1,k}}{9s_{jk}s_{j+1,k}},\\
&\quad B^+_{jk} = \frac{T^+_{j-1,k}T^+_{jk}T^+_{j+1,k}}{27s_{jk}s_{j+1,k}},\;\; B^-_{jk} = \frac{T^-_{j-1,k-1}T^-_{jk}T^-_{j+1,k+1}}{27s_{jk}s_{j+1,k}},\;\; i,j,k \in \mathbb{Z}.
\end{aligned}
\tag{6.1}
$$

Define the action of the Lie algebra $\mathfrak{g}$ on $V(a_1, a_2, a_3) = \mathrm{span}_{\mathbb{C}}\{\mathbf{v}_{ijk} \,|\, i,j,k \in \mathbb{Z}\}$ as follows:

$$
\begin{aligned}
&h_{01}(\mathbf{v}_{ijk}) = h_{01}(i,j)\mathbf{v}_{ijk},\;\; h_{10}(\mathbf{v}_{ijk}) = h_{10}(i,j)\mathbf{v}_{ijk},\;\; h_{11}(\mathbf{v}_{ijk}) = h_{11}(i,j)\mathbf{v}_{ijk},\;\; h_{21}(\mathbf{v}_{ijk}) = h_{21}(i,j)\mathbf{v}_{ijk},\\
&h_{31}(\mathbf{v}_{ijk}) = h_{31}(i,j)\mathbf{v}_{ijk},\;\; h_{32}(\mathbf{v}_{ijk}) = h_{32}(i,j)\mathbf{v}_{ijk},\;\; e_{01}(\mathbf{v}_{ijk}) = S^+_{ijk}\mathbf{v}_{i+1,j,k},\;\; f_{01}(\mathbf{v}_{ijk}) = S^-_{ik}\mathbf{v}_{i-1,j,k},\\
&e_{21}(\mathbf{v}_{ijk}) = T^+_{j+1,k}\mathbf{v}_{i+1,j+2,k+1},\;\; f_{21}(\mathbf{v}_{ijk}) = T^-_{j-1,k-1}\mathbf{v}_{i-1,j-2,k-1},\;\; e_{10}(\mathbf{v}_{ijk}) = 3\mathbf{v}_{i,j+1,k} + A^+_{jk}S^-_{ik}\mathbf{v}_{i,j+1,k+1},\\
&f_{10}(\mathbf{v}_{ijk}) = -3\mathbf{v}_{i,j-1,k-1} - A^-_{jk}S^+_{ijk}\mathbf{v}_{i,j-1,k},\;\; e_{11}(\mathbf{v}_{ijk}) = -3\mathbf{v}_{i+1,j+1,k} + A^+_{jk}S^+_{i+1,j+1,k}\mathbf{v}_{i+1,j+1,k+1},\\
&f_{11}(\mathbf{v}_{ijk}) = -3\mathbf{v}_{i-1,j-1,k-1} + A^-_{jk}S^-_{i+1,k+1}\mathbf{v}_{i-1,j-1,k},\;\; e_{31}(\mathbf{v}_{ijk}) = \mathbf{v}_{i+1,j+3,k+1} - B^+_{jk}S^-_{i+1,k+1}\mathbf{v}_{i+1,j+3,k+2},\\
&f_{31}(\mathbf{v}_{,jk}) = \mathbf{v}_{i-1,j-3,k-2} - B^-_{jk}S^+_{i+1,j+1,k}\mathbf{v}_{i-1,j-3,k-1},\;\; e_{32}(\mathbf{v}_{,jk}) = -\mathbf{v}_{i+2,j+3,k+1} - B^+_{jk}S^+_{i+1,j+1,k}\mathbf{v}_{i+2,j+3,k+2},\\
&f_{32}(\mathbf{v}_{ijk}) = \mathbf{v}_{i-2,j-3,k-2} + B^-_{jk}S^-_{i+1,k+1}\mathbf{v}_{i-2,j-3,k-1},\;\; z_1(\mathbf{v}_{ijk}) = \frac{14}{3}\mathbf{v}_{ijk}.
\end{aligned}
$$

Lemma 6.3. Subalgebra Γ has a simple spectrum on $V(a_1, a_2, a_3)$, and hence separates the basis elements $\mathbf{v}_{ijk}$, if and only if $a_3 \notin \mathbb{Z}$.

Proof. It is sufficient to consider vectors from the same weight space. Suppose

$$h_1(\mathbf{v}_{ijk}) = (a_1 + 2i - j)\mathbf{v}_{ijk} = \lambda \mathbf{v}_{ijk}, \ h_2(\mathbf{v}_{ijk}) = (a_2 + j)\mathbf{v}_{ijk} = \mu \mathbf{v}_{ijk},$$

for some λ and μ. Then $j = \mu - a_2$ and $i = \frac{1}{2}(\lambda + \mu - a_1 - a_2)$. Hence, basis elements of this weight subspace differ by the third index.

Consider $\mathbf{v}_{ijk_1}$ and $\mathbf{v}_{ijk_2}$ for arbitrary integers i, j, k_1, k_2. We have

$$c_1(\mathbf{v}_{ijk}) = f_{01}e_{01}(\mathbf{v}_{ijk}) = f_{01}(S^+_{i,j,k}\mathbf{v}_{i+1jk}) = S^-_{i+1,k}S^+_{i,j,k}(\mathbf{v}_{ijk}).$$

Since the formulas for $S^-_{i,k}$ and $S^+_{i,j,k}$ are the same for C_2 (5.17) and G_2 (6.1), we obtain similar results as in (5.19) and conclude that c_1 separates the basis elements $\mathbf{v}_{ijk_1}$ and $\mathbf{v}_{ijk_2}$. This implies the statement. □

Theorem 6.4. For any complex a_1, a_2, a_3 such that $a_3 \notin \mathbb{Z}$, the above formulas define the G_2-module structure on the space $V(a_1, a_2, a_3)$.

Proof. Follows by checking that the defining relations of the Lie algebra G_2 are satisfied on $V(a_1, a_2, a_3)$. We omit the details. □

Theorem 6.5. Let $a_1, a_2, a_3 \in \mathbb{C}$ and $a_3 \notin \mathbb{Z}$. Then

1. $V(a_1, a_2, a_3)$ is a torsion free simple Γ-pointed G_2-module if and only if $S^+_{ijk}S^-_{ik}T^+_{jk}T^-_{jk} \neq 0$, for all $i, j, k \in \mathbb{Z}$.
2. If V' is a simple torsion free Γ-pointed G_2-module with a basis parametrized by the lattice $\mathbb{Z}^3$ and with separating action of Γ on basis elements, then it is isomorphic to $V(a_1, a_2, a_3)$ for suitable parameters a_1, a_2, a_3 such that $0 \leq Re\, a_1 < 1$, $0 \leq Re\, a_2 < 2$, $0 < Re\, a_3 < 2$ and $a_3 \neq 1$.

Proof. It follows immediately from the formulas of the action of G_2 that $V(a_1, a_2, a_3)$ is a torsion free module if and only if $S^+_{ijk} \neq 0$, $S^-_{ik} \neq 0$, $T^+_{jk} \neq 0$ and $T^-_{jk} \neq 0$, for all $i, j, k \in \mathbb{Z}$. Note that Γ has a simple spectrum on $V(a_1, a_2, a_3)$ and hence, the action of Γ separates the basis elements by Lemma 6.3. In particular, $V(a_1, a_2, a_3)$ is Γ-pointed. Using this fact, it is easy to see that conditions $S^+_{ijk}S^-_{ik}T^+_{jk}T^-_{jk} \neq 0$, for all $i, j, k \in \mathbb{Z}$ are necessary and sufficient to guarantee that any element of $V(a_1, a_2, a_3)$ generates the whole module, and hence the simplicity of the module.

Suppose that V' is a simple torsion free Γ-pointed G_2-module with a basis $\{v'_{ijk}, \ i, j, k \in \mathbb{Z}\}$ such that Γ acts by different characters on the basis elements v'_{ijk}. Then the same argument as in the proof of Theorem 5.4 shows that $V' \simeq V(a_1, a_2, a_3)$ for some choice of complex parameters a_1, a_2, a_3 with $a_3 \notin \mathbb{Z}$. Since module V' is torsion free, we can choose any weight of the weight lattice and hence the parameters can be chosen in such a way that their real parts satisfy the inequalities $0 \leq Re\, a_1 < 1$, $0 \leq Re\, a_2 < 3$, $0 < Re\, a_3 < 2$. This completes the proof.

□

Consider the following standard embedding $\theta : A_2 \to G_2$: $\theta(\hat{e}_{01}) = e_{01}$, $\theta(\hat{f}_{01}) = f_{01}$, $\theta(\hat{e}_{10}) = e_{31}$, $\theta(\hat{f}_{10}) = f_{31}$, $\theta(\hat{e}_{11}) = e_{32}$, $\theta(\hat{f}_{11}) = f_{32}$, $\theta(\hat{h}_{01}) = h_{01}$, $\theta(\hat{h}_{10}) = h_{31}$, and denote by $\hat{\mathfrak{g}}$ its image. Consider the restriction $\hat{V}(a_1, a_2, a_3)$ of $V(a_1, a_2, a_3)$ on $\hat{\mathfrak{g}}$. The images Z_1 (respectively Z_2) of the Casimir elements of A_2 act on $\hat{V}(a_1, a_2, a_3)$ by $\frac{-8}{9}$ (respectively $\frac{8}{9}$). We have the following decomposition of $\hat{V}(a_1, a_2, a_3)$.

Theorem 6.6. Suppose that $V(a_1, a_2, a_3)$ is torsion free. The A_2-module $\hat{V}(a_1, a_2, a_3)$ decomposes into a direct sum of three torsion free submodules $\hat{V}(a_1, a_2, a_3) = \hat{V}(1) \oplus \hat{V}(2) \oplus \hat{V}(3)$, where

$$\hat{V}(m) = \mathrm{span}_{\mathbb{C}}\{\mathbf{v}_{i,3j+m,k} \,|\, i, j, k \in \mathbb{Z}\} \simeq W = V(a_1^{(m)}, a_2^{(m)}, a_3, \frac{-8}{9}, \frac{8}{9}),$$

$a_1^{(m)} = a_1 - m$, $a_2^{(m)} = \frac{1}{2}(a_2 - a_1) + m$, for $m = 0, 1, 2$.

Proof. Consider the subspace $\hat{V}(m) = \mathrm{span}_{\mathbb{C}}\{\mathbf{v}_{i,3j+m,k} \,|\, i, j, k \in \mathbb{Z}\}$ of $\hat{V}(a_1, a_2, a_3)$, for $m = 0, 1, 2$. Clearly, $\hat{V}(m)$ is a $\hat{\mathfrak{g}}$-submodule of $\hat{V}(a_1, a_2, a_3)$. The $\hat{\mathfrak{g}}$-module $W = \mathrm{span}_{\mathbb{C}}\{w_{ijk} \,|\, i, j, k \in \mathbb{Z}\}$ is defined by the formulas (4.15), (4.16), and we have a homomorphism of $\hat{\mathfrak{g}}$-modules $\psi_m : W \to \hat{V}(m)$, such that $\psi_m : w_{ijk} \mapsto v_{i+j,3j+m,k+j}$, and for any $x \in \hat{\mathfrak{g}}$ holds $\psi_m(x(w_{ijk})) = \theta(x)(\psi_m(w_{ijk})) = \theta(x)(v_{i+j,3j+m,k+j})$, for $i, j, k \in \mathbb{Z}$ and $m = 0, 1, 2$. Since ψ_m is a linear isomorphism, the statement follows. □

7. Conclusion

For all simple Lie algebras of rank 2 we constructed families of simple modules with infinite-dimensional weight spaces. These modules admit a diagonalizable action of a certain commutative subalgebra with a simple spectrum. In type A_2, this commutative subalgebra is a famous Gelfand-Tsetlin subalgebra and the corresponding modules are generic Gelfand-Tsetlin modules. In type C_2 we construct two 4-parameter families of simple modules which are analogs of generic Gelfand-Tsetlin modules, while in type G_2 we construct a 3-parameter family of such simple modules.

Detailed computations are available at
https://icm.sustech.edu.cn/en/people/faculty/vyacheslav-futorny-en .

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