EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6459
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Algebraic Aspects of Bipolar Fuzzy Soft Boolean Rings

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Abstract. This work introduces the concept of bipolar fuzzy soft Boolean rings (BFSBRs), an algebraic structure that unifies soft sets (SSs) and bipolar fuzzy sets (BFSs) within Boolean ring (BR) systems. It develops essential definitions, operations, and properties for modeling dual-sided uncertainty in decision-making contexts. Beyond its theoretical contributions, the framework is designed to support early-stage mathematical research by science classroom students. It encourages deeper engagement with abstract mathematical thinking and fosters collaboration between students, teachers, and research mentors, thus promoting a more inclusive and research-oriented learning environment.

2020 Mathematics Subject Classifications: 16Y30, 08A72, 06D72

Key Words and Phrases: Boolean ring, fuzzy set, bipolar fuzzy set, fuzzy soft set, soft set, bipolar fuzzy soft Boolean ring, bipolar fuzzy soft ideal

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6459

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1. Introduction

Zadeh [1] proposed the theory of a fuzzy set (FS) of a set. This concept has now been generalized in various ways. Lee [2] was the first to suggest the idea of BVFSs. Fuzzy sets with membership degrees ranging from $[0, 1]$ to $[-1, 1]$ are known as BVFSs. If the membership degree is zero, the elements have no relationship to the associated attribute. Elements with a membership degree between 0 and 1 partially meet the requirement. If the membership degree is $[-1, 0]$, then elements partially satisfy the implicit counter attribute.

Molodtsov [3] was the first to present the idea of SS theory as a new mathematical tool. Some scholars then looked at the algebraic aspects of this idea. First, Aktas and Cagman [4] employed SS to define the concept of soft groups and identify their fundamental characteristics. Soft rings were defined and their basic concepts were introduced by Acar et al. [5]. Çelik et al. [6] looked at some new characteristics of soft rings and defined some new binary relations on SSs.

Bipolar fuzzy sets were first proposed by Zhang [7] in 1994 as a generalization of fuzzy sets [1]. Naz and Shabir [8] developed the notions of fuzzy bipolar soft sets and bipolar fuzzy soft sets (BFSSs). They outlined the unique intersection and union of the two concepts and demonstrated their equivalency. Bipolar fuzzy soft Lie subalgebras were first proposed by Akram [9], who also investigated some of their characteristics. The ideas of an SS and a BF-set were blended by Abdullah et al. [10]. In addition, they presented the concept of a BFSS and outlined some of its key characteristics. BFSSs and their unique union and intersection were also studied by Aslam et al. [11]. The notion of BFS K-algebras was presented by Akram et al. [12]. Akram et al. [13] introduced the concept of bipolar fuzzy soft Γ-semigroups, combining BFSs and SSs within Γ-semigroup structures. Their work laid foundational properties and extended algebraic models for handling dual-sided uncertainty in structured environments. Yang and Li [14] introduced bipolar-value fuzzy soft sets as a hybrid model combining the strengths of bipolar fuzzy logic and soft set theory.

Some authors have examined the algebraic properties of fuzzy soft sets (FSSs). First, FSSs were defined, and some findings were established by Maji et al. [15]. Fuzzy soft groups were determined by Liu et al. [16]. Fuzzy soft Boolean rings were represented by Rao et al. [17]. Rao et al. [18] explored the structure of soft Boolean near-rings, extending classical near-ring theory through the lens of SS theory. Aygünoğlu and Aygün [19] introduced the concept of fuzzy soft groups. Feng et al. [20] introduced the concept of soft semirings. İnan and Öztürk [21] introduced the concepts of fuzzy soft rings and $(\in, \in \vee q)$-fuzzy soft subrings. Jun [22] developed the theory of soft BCK/BCI-algebras. Kazancı et al. [23] proposed the concept of soft BCH-algebras. Rao et al. [24] introduced the concept of soft intersection Boolean near-rings, highlighting their structural properties and practical relevance. Rao et al. [25] proposed the $(\in, \in \vee q_k)$-fuzzy soft Boolean near-rings as an extension of fuzzy soft algebraic structures. Rao et al. [26] developed fuzzy soft Boolean near-rings and their idealistic variants, enriching the algebraic foundation for soft computing by formalizing new structures and operations under fuzzy logic. Rao et al. [27] presented intuitionistic fuzzy soft Boolean rings, combining intuitionistic fuzzy logic with soft ring structures to better handle dual uncertainty. Rao et al. [28] introduced $(\in, \in \vee q_k)$-intuitionistic fuzzy soft Boolean near-rings, enhancing soft algebraic frameworks by integrating intuitionistic fuzzy logic with generalized membership concepts.

In this work, we introduce the notions of BFSBRs and BFSIs within the framework of Boolean rings. We investigate their fundamental algebraic properties, aiming to provide a theoretical contribution to soft algebraic structures and an accessible platform for students engaging in early-stage mathematical research.

2. Preliminaries

This section outlines the fundamental concepts necessary for developing BFSBRs. We briefly review BFSs, SSs, and their integration into BFSSs, along with basic operations. These preliminaries provide the algebraic foundation for the constructions and results that follow.

Definition 1. *[7] In the universe G, a bipolar fuzzy set (BFS) U is an entity with the form $U = \{(\rho, J_U^+(\rho), J_U^-(\rho)) : \rho \in G\}$. In this case, $J_U^+(\rho)$ denotes how satisfied an element ρ is with the property that corresponds to a BFS $U = \{(\rho, J_U^+(\rho), J_U^-(\rho)) : \rho \in G\}$, and $J_U^-(\rho)$ denotes how satisfied ρ is with an implicit counter property of $U = \{(\rho, J_U^+(\rho), J_U^-(\rho)) : \rho \in G\}$. The BFS $U = \{(\rho, J_U^+(\rho), J_U^-(\rho)) : \rho \in G\}$ has been denoted by the symbol $U = (J_U^+, J_U^-)$ for simplicity's sake.*

Definition 2. *[7] For two BFSs $U = (J_U^+, J_U^-)$ and $V = (J_V^+, J_V^-)$ in the universe G,*
(i) $U \subseteq V$ indicates that $J_U^+(\rho) \geq J_V^+(\rho)$ and $J_U^-(\rho) \leq J_V^-(\rho)$ for all $\rho \in G$
(ii) $U \cup V = \{(\rho, \max\{J_U^+(\rho), J_V^+(\rho)\}, \min\{J_U^-(\rho), J_V^-(\rho)\}) : \rho \in G\} = (J_U^+(\rho) \cup J_V^+(\rho), J_U^-(\rho) \cap J_V^-(\rho))$
(iii) $U \cap V = \{(\rho, \min\{J_U^+(\rho), J_V^+(\rho)\}, \max\{J_U^-(\rho), J_V^-(\rho)\}) : \rho \in G\} = (J_U^+(\rho) \cap J_V^+(\rho), J_U^-(\rho) \cup J_V^-(\rho))$.

Remark 1. *[7] For the above definitions, we can verify that*
(i) $J_{U \cup V}^+(\rho) = J_U^+(\rho) \cup J_V^+(\rho)$
(ii) $J_{U \cup V}^-(\rho) = J_U^-(\rho) \cap J_V^-(\rho)$
(iii) $J_{U \cap V}^+(\rho) = J_U^+(\rho) \cap J_V^+(\rho)$
(iv) $J_{U \cap V}^-(\rho) = J_U^-(\rho) \cup J_V^-(\rho)$.

Definition 3. *[3] Let E be the collection of parameters and G be the initial universe. The pair (J, U) is called a soft set (SS) over G if U is a non-empty subset of E. The mapping $J : U \to P(G)$ is used to define J.*

Definition 4. *[15] Let E be the collection of parameters and G be the initial universe set. The pair (J, U) is referred to as a fuzzy soft set (FSS) over G if U is a non-empty subset of E and $P(FS(G))$ is the collection of all FSs of G. J is a mapping that is given by the expression $J : U \to P(FS(G))$.*

Definition 5. *[14] Let E be the collection of parameters and G be the initial universe set. Let $BF(G)$ be the set of all BFSs of G and $U \subseteq E$. When J is provided by mapping $J : U \to BF(G)$, then a pair (J, U) is referred to as a bipolar fuzzy soft set (BFSS) over G. It is defined as*

$$(J, U) = \{(\rho, J_u^+(\rho), J_u^-(\rho)) : \rho \in G \quad \& \quad u \in U\}.$$

For any $u \in U$, $J(u) = \{(\rho, J_u^+(\rho), J_u^-(\rho)) : \rho \in G\} = \{J_U^+(\rho), J_U^-(\rho)\}$.

Definition 6. *[14] The complement of a BFSS (J, U) is symbolized by $(J, U)^c$ and it is described as $(J, U)^c = \{(\rho, 1 - J_U^+(\rho), -1 - J_U^-(\rho)) : \rho \in G\}$.*

Definition 7. *[14] For any two BFSSs (J, U) and (K, V) over G, then we say that (J, U) is a BFS subset of (K, V) if $U \subseteq V$ and $J(u) \subseteq K(u)$, $\forall u \in U$. This is written as $(J, U) \subseteq (K, V)$.*

Definition 8. *[8, 14] Let (J, U) and (K, V) be two BFSSs over G. Then (J, U) and (K, V), indicated by $(J, U) \wedge (K, V)$ is known as $(J, U) \wedge (K, V) = (L, W)$ where $W = U \times V$ and $L(\rho, \tau) = J(\rho) \cap K(\tau)$, $\forall (\rho, \tau) \in W = U \times V$.*

Definition 9. *[8, 14] Let (J, U) and (K, V) be two BFSSs over the universe G. Then (J, U) OR (K, V), indicated by $(J, U) \vee (K, V)$ is known as $(J, U) \vee (K, V) = (L, W)$ where $W = U \times V$ and $L(\rho, \tau) = J(\rho) \cup K(\tau)$, $\forall (\rho, \tau) \in W = U \times V$.*

Definition 10. *[8, 14] Let (J, U) and (K, V) be two BFSSs over the universe G. Then their extended union is a BFSS over G indicated by $(J, U) \cup_e (K, V)$ and is known as $(J, U) \cup_e (K, V) = (L, W)$, where $W = U \cup V$ and $L : W \to BF(G)$ is provided as,*

$$L(\omega) = \begin{cases} J(\omega) & \text{if } \omega \in U - V \\ K(\omega) & \text{if } \omega \in V - U \\ \max\{J(\omega), K(\omega)\} & \text{if } \omega \in U \cap V \end{cases} \quad \text{for all } \omega \in W.$$

Definition 11. *[8, 14] Let (J, U) and (K, V) be two BFSSs over the universe G. Then their extended intersection is a BFSS over G indicated by $(J, U) \cap_e (K, V)$ and is known as $(J, U) \cap_e (K, V) = (L, W)$, where $W = U \cup V$ and $L : W \to BF(G)$ is provided as,*

$$L(\omega) = \begin{cases} J(\omega) & \text{if } \omega \in U - V \\ K(\omega) & \text{if } \omega \in V - U \\ \min\{J(\omega), K(\omega)\} & \text{if } \omega \in U \cap V \end{cases} \quad \text{for all } \omega \in W.$$

Definition 12. *[14] Let (J, U) and (K, V) be two BFSSs over the universe G such that $U \cap V \neq \emptyset$. The restricted union of (J, U) and (K, V) is described to be a BFSS (L, W) over G, where $W = U \cap V$ and for $L(\omega) = J(\omega) \cup K(\omega)$, $\forall\ \omega \in W$. This is indicated by $(L, W) = (J, U) \cup_R (K, V)$.*

Definition 13. *[14] Let (J, U) and (K, V) be two BFSSs over the universe G such that $U \cap V \neq \emptyset$. The restricted intersection of (J, U) and (K, V) is described to be a BFSS (L, W) over G, where $W = U \cap V$ and for $L(\omega) = J(\omega) \cup K(\omega)$, $\forall\ \omega \in W$. This is indicated by $(L, W) = (J, U) \cap_R (K, V)$.*

3. Bipolar Fuzzy Soft Boolean Rings

In this section, we introduce the concept of BFSBRs as a natural extension of SS and BFS theories within BR structures. We define the core properties of BFSBRs and demonstrate how they preserve algebraic behavior under fundamental operations. In this section, $\Re$ will indicate a BR, and its role within the framework will be made explicit. This approach enables the modeling of structured uncertainty in positive and negative forms, laying the groundwork for further theoretical development.

Definition 14. *A BFSS (J,U) over $\Re$ is called a bipolar fuzzy soft Boolean ring (BFSBR) over $\Re$ if*

(i) $J^+(\kappa+\xi) \geq A\{J^+(\kappa), J^+(\xi)\}$ *(A indicates* $\min$*)*
(ii) $J^-(\kappa+\xi) \leq B\{J^-(\kappa), J^-(\xi)\}$ *(B indicates* $\max$*)*
(iii) $J^+(\kappa\xi) \geq A\{J^+(\kappa), J^+(\xi)\}$
(iv) $J^-(\kappa\xi) \leq B\{J^-(\kappa), J^-(\xi)\}$ *for all* $\kappa, \xi \in \Re$.

Example 1. *The binary operations $+$ and $*$ can be applied to the non-empty set $\Re = \{0, \kappa, \xi, \tau\}$ in the following ways: Let $U = \{e_1, e_2, e_3\}$ be the parameters collection. Then,*

$+$	0	κ	ξ	τ
0	0	κ	ξ	τ
κ	κ	0	τ	ξ
ξ	ξ	τ	0	κ
τ	τ	ξ	κ	0

$*$	0	κ	ξ	τ
0	0	0	0	0
κ	0	κ	0	κ
ξ	0	0	ξ	ξ
τ	0	κ	ξ	τ

specify a BFSS (J,U) on a BR $\Re$. Thus,

$$\begin{aligned}
J(e_1) &= \{(0,0.7,-0.2),(\kappa,0.5,-0.3),(\xi,0.1,-0.4),(\tau,0.1,-0.4)\}\\
J(e_2) &= \{(0,0.3,-0.3),(\kappa,0.3,-0.3),(\xi,0.7,-0.4),(\tau,0.9,-0.4)\}\\
J(e_3) &= \{(0,0.9,-0.3),(\kappa,0.7,-0.3),(\xi,0.4,-0.4),(\tau,0.4,-0.4)\}.
\end{aligned}$$

Consequently, the BFSS (J,U) is clearly a BFSBR.

Theorem 1. *If (J,U) and (K,V) are two BFSBRs over $\Re$, then, $(J,U) \wedge (K,V)$ is also a BFSBR over $\Re$.*

Proof. Let (J,U) and (K,V) be two BFSBRs over $\Re$. Then as defined $(J,U) \wedge (K,V)$, where $W = U \times V$ and $L(\rho,\tau) = J(\rho) \cap K(\tau)$, $\forall(\rho,\tau) \in W = U \times V$, as (J,U) and (K,V) are BFSBRs over $\Re$. Thus, for $\kappa, \xi \in \Re$,

$$\begin{aligned}
L^+_{(\rho,\tau)}(\kappa+\xi) &= A\{J^+_\rho(\kappa+\xi), K^+_\tau(\kappa+\xi)\}\\
&\geq A\{A\{J^+_\rho(\kappa), J^+_\rho(\xi)\}, A\{K^+_\tau(\kappa), K^+_\tau(\xi)\}\}\\
&= A\{A\{J^+_u(\kappa), K^+_\tau(\kappa)\}, A\{J^+_\rho(\xi), K^+_\tau(\xi)\}\}\\
&= A\{L^+_{(\rho,\tau)}(\kappa), L^+_{(\rho,\tau)}(\xi)\},
\end{aligned}$$

$$\begin{array}{rcl}
L^{-}_{(\rho,\tau)}(\kappa+\xi) & = & B\{J^{-}_{\rho}(\kappa+\xi), K^{-}_{\tau}(\kappa+\xi)\} \\
& \leq & B\{B\{J^{-}_{\rho}(\kappa), J^{-}_{\rho}(\xi)\}, B\{K^{-}_{\tau}(\kappa), K^{-}_{\tau}(\xi)\}\} \\
& = & B\{B\{J^{-}_{\rho}(\kappa), K^{-}_{\tau}(\kappa)\}, B\{J^{-}_{\rho}(\xi), K^{-}_{\tau}(\xi)\}\} \\
& = & B\{L^{-}_{(\rho,\tau)}(\kappa), L^{-}_{(\rho,\tau)}(\xi)\}, \\
L^{+}_{(\rho,\tau)}(\kappa\xi) & = & A\{J^{+}_{\rho}(\kappa\xi), K^{+}_{\tau}(\kappa\xi)\} \\
& \geq & A\{A\{J^{+}_{\rho}(\kappa), J^{+}_{\rho}(\xi)\}, A\{K^{+}_{\tau}(\kappa), K^{+}_{\tau}(\xi)\}\} \\
& = & A\{A\{J^{+}_{u}(\kappa), K^{+}_{\tau}(\kappa)\}, A\{J^{+}_{\rho}(\xi), K^{+}_{\tau}(\xi)\}\} \\
& = & A\{L^{+}_{(\rho,\tau)}(\kappa), L^{+}_{(\rho,\tau)}(\xi)\}, \\
L^{-}_{(\rho,\tau)}(\kappa\xi) & = & B\{J^{-}_{\rho}(\kappa\xi), K^{-}_{\tau}(\kappa\xi)\} \\
& \leq & B\{B\{J^{-}_{\rho}(\kappa), J^{-}_{\rho}(\xi)\}, B\{K^{-}_{\tau}(\kappa), K^{-}_{\tau}(\xi)\}\} \\
& = & B\{B\{J^{-}_{\rho}(\kappa), K^{-}_{\tau}(\kappa)\}, B\{J^{-}_{\rho}(\xi), K^{-}_{\tau}(\xi)\}\} \\
& = & B\{L^{-}_{(\rho,\tau)}(\kappa), L^{-}_{(\rho,\tau)}(\xi)\}.
\end{array}$$

Therefore, $(L,W) = (J,U) \wedge (K,V)$ is a BFSBR $\Re$.

Theorem 2. *If (J,U) and (K,V) are two BFSBRs over $\Re$, then $(J,U) \vee (K,V)$ is also a BFSBR over $\Re$.*

Proof. Let (J,U) and (K,V) be two BFSBRs over $\Re$. Then as defined $(J,U) \vee (K,V) = (L,W)$, where $W = U \times V$ and $L(\rho,\tau) = J(\rho) \cup K(\tau)$, $\forall(\rho,\tau) \in W = U \times V$, as (J,U) and (K,V) are BFSBR over $\Re$. Thus, for $\kappa, \xi \in \Re$,

$$\begin{array}{rcl}
L^{+}_{(\rho,\tau)}(\kappa+\xi) & = & B\{J^{+}_{\rho}(\kappa+\xi), K^{+}_{\tau}(\kappa+\xi)\} \\
& \geq & B\{A\{J^{+}_{\rho}(\kappa), J^{+}_{\rho}(\xi)\}, A\{K^{+}_{\tau}(\kappa), K^{+}_{\tau}(\xi)\}\} \\
& \geq & A\{B\{J^{+}_{\rho}(\kappa), K^{+}_{\tau}(\kappa)\}, B\{J^{+}_{\rho}(\xi), K^{+}_{\tau}(\xi)\}\} \\
& = & A\{L^{+}_{(\rho,\tau)}(\kappa), L^{+}_{(\rho,\tau)}(\xi)\}, \\
L^{-}_{(\rho,\tau)}(\kappa+\xi) & = & A\{J^{-}_{\rho}(\kappa+\xi), K^{-}_{\tau}(\kappa+\xi)\} \\
& \leq & A\{B\{J^{-}_{\rho}(\kappa), J^{-}_{\rho}(\xi)\}, B\{K^{-}_{\tau}(\kappa), K^{-}_{\tau}(\xi)\}\} \\
& \leq & B\{A\{J^{-}_{\rho}(\kappa), K^{-}_{\tau}(\kappa)\}, A\{J^{-}_{\rho}(\xi), K^{-}_{\tau}(\xi)\}\} \\
& = & B\{L^{-}_{(\rho,\tau)}(\kappa), L^{-}_{(\rho,\tau)}(\xi)\}, \\
L^{+}_{(\rho,\tau)}(\kappa\xi) & = & B\{J^{+}_{\rho}(\kappa\xi), K^{+}_{\tau}(\kappa\xi)\} \\
& \geq & B\{A\{J^{+}_{\rho}(\kappa), J^{+}_{\rho}(\xi)\}, A\{K^{+}_{\tau}(\kappa), K^{+}_{\tau}(\xi)\}\} \\
& \geq & A\{B\{J^{+}_{\rho}(\kappa), K^{+}_{\tau}(\kappa)\}, B\{J^{+}_{\rho}(\xi), K^{+}_{\tau}(\xi)\}\} \\
& = & A\{L^{+}_{(\rho,\tau)}(\kappa), L^{+}_{(\rho,\tau)}(\xi)\}, \\
L^{-}_{(\rho,\tau)}(\kappa\xi) & = & A\{J^{-}_{\rho}(\kappa\xi), K^{-}_{\tau}(\kappa\xi)\}
\end{array}$$

$$
\begin{aligned}
&\leq A\{B\{J_{\rho}^{-}(\kappa), J_{\rho}^{-}(\xi)\}, B\{K_{\tau}^{-}(\kappa), K_{\tau}^{-}(\xi)\}\} \\
&\leq B\{A\{J_{\rho}^{-}(\kappa), K_{\tau}^{-}(\kappa)\}, A\{J_{\rho}^{-}(\xi), K_{\tau}^{-}(\xi)\}\} \\
&= B\{L_{(\rho,\tau)}^{-}(\kappa), L_{(\rho,\tau)}^{-}(\xi)\}.
\end{aligned}
$$

Therefore, $(L, W) = (J, U) \vee (K, V)$ is a BFSBR over $\Re$.

Example 2. *Consider a decision-making scenario where two decision states,* 0 *and* 1*, represent binary investment options (e.g.,* $0 =$ *reject,* $1 =$ *approve). Let* $\Re = \{0, 1\}$ *be the BR with XOR and AND as addition and multiplication, respectively. Define two BFSBRs* $(J, \{p_1\})$ *and* $(K, \{p_2\})$ *over the parameter set* $\{p_1, p_2\}$*, where* p_1 *represents risk tolerance and* p_2 *expected impact. Assume the bipolar fuzzy soft membership values are:*

$$J_{p_1}^{+}(0) = 0.9, \quad J_{p_1}^{+}(1) = 0.9, \quad J_{p_1}^{-}(0) = -0.1, \quad J_{p_1}^{-}(1) = -0.1,$$

$$K_{p_2}^{+}(0) = 0.8, \quad K_{p_2}^{+}(1) = 0.8, \quad K_{p_2}^{-}(0) = -0.3, \quad K_{p_2}^{-}(1) = -0.3.$$

Using the join operation $(L, \{(p_1, p_2)\}) = (J, \{p_1\}) \vee (K, \{p_2\})$*, we obtain*

$$L_{(p_1,p_2)}^{+}(0) = \max\{0.9, 0.8\} = 0.9, \quad L_{(p_1,p_2)}^{-}(0) = \min\{-0.1, -0.3\} = -0.3,$$

$$L_{(p_1,p_2)}^{+}(1) = \max\{0.9, 0.8\} = 0.9, \quad L_{(p_1,p_2)}^{-}(1) = \min\{-0.1, -0.3\} = -0.3.$$

These values satisfy the axioms of a BFSBR over $\Re$*, and thus, under this analysis, both decision states provide high positive support with equal and acceptable negative uncertainty, highlighting the model's capability in supporting symmetric decision scenarios.*

Theorem 3. *If* (J, U) *and* (K, V) *are two BFSBRs over* $\Re$*, then* $(J, U) \cap (K, V)$ *is also a BFSBR over* $\Re$*.*

Proof. Let (J, U) and (K, V) be two BFSBRs over $\Re$. Then $(J, U) \cap (K, V) = (L, W)$, where $W = U \cap V$ and $L(\omega) = J(\omega) \cap K(\omega)$, $\forall\ \omega \in W$. Now,

$$
\begin{aligned}
L_{\omega}^{+}(\kappa + \xi) &= A\{J_{\omega}^{+}(\kappa + \xi), K_{\omega}^{+}(\kappa + \xi)\} \\
&\geq A\{A\{J_{\omega}^{+}(\kappa), J_{\omega}^{+}(\xi)\}, A\{K_{\omega}^{+}(\kappa), K_{\omega}^{+}(\xi)\}\} \\
&= A\{A\{J_{\omega}^{+}(\kappa), K_{\omega}^{+}(\kappa)\}, A\{J_{\omega}^{+}(\xi), K_{\omega}^{+}(\xi)\} \\
&= A\{L_{\omega}^{+}(\kappa), L_{\omega}^{+}(\xi)\}, \\
L_{\omega}^{-}(\kappa + \xi) &= B\{J_{\omega}^{-}(\kappa + \xi), K_{\omega}^{-}(\kappa + \xi)\} \\
&\leq B\{B\{J_{\omega}^{-}(\kappa), J_{\omega}^{-}(\xi)\}, B\{K_{\omega}^{-}(\kappa), eK_{\omega}^{-}(\xi)\}\} \\
&= B\{B\{J_{\omega}^{-}(\kappa), K_{\omega}^{-}(\kappa)\}, B\{J_{\omega}^{-}(\xi), K_{\omega}^{-}(\xi)\}\} \\
&= B\{L_{\omega}^{-}(\kappa), L_{\omega}^{-}(\xi)\}, \\
L_{\omega}^{+}(\kappa\xi) &= A\{J_{\omega}^{+}(\kappa\xi), K_{\omega}^{+}(\kappa\xi)\} \\
&\geq A\{A\{J_{\omega}^{+}(\kappa), J_{\omega}^{+}(\xi)\}, A\{K_{\omega}^{+}(\kappa), K_{\omega}^{+}(\xi)\}\} \\
&= A\{A\{J_{\omega}^{+}(\kappa), K_{\omega}^{+}(\kappa)\}, A\{J_{\omega}^{+}(\xi), K_{\omega}^{+}(\xi)\}
\end{aligned}
$$

$$\begin{array}{rcl} & = & A\{L^+_\omega(\kappa), L^+_\omega(\xi)\}, \\ L^-_\omega(\kappa\xi) & = & B\{J^-_\omega(\kappa\xi), K^-_\omega(\kappa\xi)\} \\ & \leq & B\{B\{J^-_\omega(\kappa), J^-_\omega(\xi)\}, B\{K^-_\omega(\kappa), eK^-_\omega(\xi)\}\} \\ & = & B\{B\{J^-_\omega(\kappa), K^-_\omega(\kappa)\}, B\{J^-_\omega(\xi), K^-_\omega(\xi)\}\} \\ & = & B\{L^-_\omega(\kappa), L^-_\omega(\xi)\}. \end{array}$$

Therefore, $(L, W) = (J, U) \cap (K, V)$ is a BFSBR over $\Re$.

Theorem 4. *If (J, U) and (K, V) are two BFSBRs over $\Re$, then $(J, U) \cup (K, V)$ is also a BFSBR over $\Re$.*

Proof. Let (J, U) and (K, V) be two BFSBRs over $\Re$. Then $(J, U) \cup (K, V) = (L, W)$, where $W = U \cup V$ and

$$L(\omega) = \begin{cases} J(\omega) & \text{if } \omega \in U - V \\ K(\omega) & \text{if } \omega \in V - U \\ J(\omega) \vee K(\omega) & \text{if } \omega \in U \cap V \end{cases} \quad \text{for all } \omega \in W.$$

Then we have the following cases:
Case 1: If $\omega \in U - V$, then

$$\begin{array}{rcl} L^+_\omega(\kappa + \xi) & = & J^+_\omega(\kappa + \xi) \\ & \geq & A\{J^+_\omega(\kappa), J^+_\omega(\xi)\} \\ & = & A\{L^+_\omega(\kappa), L^+_\omega(\xi)\}, \\ L^-_\omega(\kappa + \xi) & = & J^-_\omega(\kappa + \xi) \\ & \leq & B\{J^-_\omega(\kappa), J^-_\omega(\xi)\} \\ & = & B\{L^-_\omega(\kappa), L^-_\omega(\xi)\}, \\ L^+_\omega(\kappa\xi) & = & J^+_\omega(\kappa\xi) \\ & \geq & A\{J^+_\omega(\kappa), J^+_\omega(\xi)\} \\ & = & A\{L^+_\omega(\kappa), L^+_\omega(\xi)\}, \\ L^-_\omega(\kappa\xi) & = & J^-_\omega(\kappa\xi) \\ & \leq & B\{J^-_\omega(\kappa), J^-_\omega(\xi)\} \\ & = & B\{L^-_\omega(\kappa), L^-_\omega(\xi)\}. \end{array}$$

Case 2: If $\omega \in V - U$, then

$$\begin{array}{rcl} L^+_\omega(\kappa + \xi) & = & K^+_\omega(\kappa + \xi) \\ & \geq & A\{K^+_\omega(\kappa), K^+_\omega(\xi)\} \\ & = & A\{L^+_\omega(\kappa), L^+_\omega(\xi)\}, \\ L^-_\omega(\kappa + \xi) & = & K^-_\omega(\kappa + \xi) \end{array}$$

$$\begin{aligned}
&\leq B\{K_\omega^-(\kappa), K_\omega^-(\xi)\}\\
&= B\{L_\omega^-(\kappa), L_\omega^-(\xi)\},\\
L_\omega^+(\kappa\xi) &= K_\omega^+(\kappa\xi)\\
&\geq A\{K_\omega^+(\kappa), K_\omega^+(\xi)\}\\
&= A\{L_\omega^+(\kappa), L_\omega^+(\xi)\},\\
L_\omega^-(\kappa\xi) &= K_\omega^-(\kappa\xi)\\
&\leq B\{K_\omega^-(\kappa), K_\omega^-(\xi)\}\\
&= B\{L_\omega^-(\kappa), L_\omega^-(\xi)\}.
\end{aligned}$$

Case 3: If $\omega \in U \cap V$, then

$$\begin{aligned}
L_\omega^+(\kappa+\xi) &= B\{J_\omega^+(\kappa+\xi), K_\omega^+(\kappa+\xi)\}\\
&\geq B\{A\{J_\omega^+(\kappa), J_\omega^+(\xi)\}, A\{K_\omega^+(\kappa), K_\omega^+(\xi)\}\}\\
&\geq A\{B\{J_\omega^+(\kappa), K_\omega^+(\kappa)\}, B\{J_\omega^+(\xi), K_\omega^+(\xi)\}\}\\
&= A\{L_\omega^+(\kappa), L_\omega^+(\xi)\},\\
L_\omega^-(\kappa+\xi) &= A\{J_\omega^-(\kappa+\xi), K_\omega^-(\kappa+\xi)\}\\
&\leq A\{B\{J_\omega^-(\kappa), J_\omega^-(\xi)\}, B\{K_\omega^-(\kappa), K_\omega^-(\xi)\}\}\\
&\leq B\{A\{J_\omega^-(\kappa), K_\omega^-(\kappa)\}, A\{J_\omega^-(\xi), K_\omega^-(\xi)\}\}\\
&= B\{L_\omega^-(\kappa), L_\omega^-(\xi)\},\\
L_\omega^+(\kappa\xi) &= B\{J_\omega^+(\kappa\xi), K_\omega^+(\kappa\xi)\}\\
&\geq B\{A\{J_\omega^+(\kappa), J_\omega^+(\xi)\}, A\{K_\omega^+(\kappa), K_\omega^+(\xi)\}\}\\
&\geq A\{B\{J_\omega^+(\kappa), K_\omega^+(\kappa)\}, B\{J_\omega^+(\xi), K_\omega^+(\xi)\}\}\\
&= A\{L_\omega^+(\kappa), L_\omega^+(\xi)\},\\
L_\omega^-(\kappa\xi) &= A\{J_\omega^-(\kappa\xi), K_\omega^-(\kappa\xi)\}\\
&\leq A\{B\{J_\omega^-(\kappa), J_\omega^-(\xi)\}, B\{K_\omega^-(\kappa), K_\omega^-(\xi)\}\}\\
&\leq B\{A\{J_\omega^-(\kappa), K_\omega^-(\kappa)\}, A\{J_\omega^-(\xi), K_\omega^-(\xi)\}\}\\
&= B\{L_\omega^-(\kappa), L_\omega^-(\xi)\}.
\end{aligned}$$

Therefore, $(L, W) = (J, U) \cup (K, V)$ is a BFSBR over $\Re$.

Example 3. *Let the BR be $\Re = \{0, \kappa, \xi, \tau\}$, and the table is provided in Example 1 above. Let the parameters be $E = \{e_1, e_2, e_3\}$ and $U = \{e_1, e_2\} \subseteq E$. Then (J, U) is a BFSS defined as, $(J, U) = \{J(e_1), J(e_2)\}$, where*

$$\begin{aligned}
J(e_1) &= \{(0, 0.9, -0.7), (\kappa, 0.5, -0.3), (\xi, 0.7, -0.2), (\tau, 0.5, -0.2)\},\\
J(e_2) &= \{(0, 0.8, -0.7), (\kappa, 0.4, -0.1), (\xi, 0.2, -0.5), (\tau, 0.2, -0.1)\}.
\end{aligned}$$

Let $V = \{e_2, e_3\} \subseteq E$. Then (K, V) is a BFSS defined as, $(K, V) = \{K(e_2), K(e_3)\}$, where

$$K(e_2) = \{(0, 0.7, -0.6), (\kappa, 0.6, -0.4), (\xi, 0.3, -0.2), (\tau, 0.3, -0.2)\},$$
$$K(e_3) = \{(0, 0.6, -0.9), (\kappa, 0.4, -0.5), (\xi, 0.3, -0.5), (\tau, 0.3, -0.5)\}.$$

It is seen that (J, U) and (K, V) are BFSBRs of $\Re$. Now, let $(J, U) \wedge (K, V) = (L, W)$, where

$$W = U \times V = \{(e_1, e_2), (e_1, e_3), (e_2, e_2), (e_2, e_3)\}.$$

Then $(J \wedge K, W) = \{(J \wedge K)(e_1, e_2),\ (J \wedge K)(e_1, e_3),\ (J \wedge K)(e_2, e_2),\ (J \wedge K)(e_2, e_3)\}$, where

$$(J \wedge K)(e_1, e_2) = \{(0, 0.7 - 0.6), (\kappa, 0.5, -0.3), (\xi, 0.3, -0.2), (\tau, 0.3, -0.2)\},$$
$$(J \wedge K)(e_1, e_3) = \{(0, 0.6 - 0.7), (\kappa, 0.4, -0.3), (\xi, 0.3, -0.2), (\tau, 0.3, -0.2)\},$$
$$(J \wedge K)(e_2, e_2) = \{(0, 0.7 - 0.6), (\kappa, 0.4, -0.1), (\xi, 0.2, -0.2), (\tau, 0.2, -0.1)\},$$
$$(J \wedge K)(e_2, e_3) = \{(0, 0.6 - 0.7), (\kappa, 0.4, -0.1), (\xi, 0.2, -0.5), (\tau, 0.2, -0.1)\}.$$

Obviously, for all $\kappa, \xi \in \Re$,

$$(J \wedge K)^+(\kappa + \xi) \geq (J \wedge K)^+(\kappa) \wedge (J \wedge K)^+(\xi),$$
$$(J \wedge K)^+(\kappa\xi) \geq (J \wedge K)^+(\kappa) \wedge (J \wedge K)^+(\xi),$$
$$(J \wedge K)^-(\kappa + \xi) \leq (J \wedge K)^-(\kappa) \vee (J \wedge K)^-(\xi),$$
$$(J \wedge K)^-(\kappa\xi) \leq (J \wedge K)^-(\kappa) \vee (J \wedge K)^-(\xi).$$

4. Bipolar Fuzzy Soft Ideals over Boolean Rings

Building upon the framework of BFSBRs, this section introduces the notion of BFSIs. These ideals adapt classical ring-theoretic concepts to the bipolar fuzzy soft context, allowing for the representation of parameterized uncertainty within ideal structures. We define BFSIs over BRs and explore their algebraic properties, focusing on their behavior under binary operations and their role in preserving the integrity of the extended structure.

Definition 15. *A BFSS (J, U) over $\Re$ is called a bipolar fuzzy soft ideal (BFSI) over $\Re$ if*
(i) $J^+(\kappa + \xi) \geq A\{J^+(\kappa), J^+(\xi)\}$
(ii) $J^-(\kappa + \xi) \leq B\{J^-(\kappa), J^-(\xi)\}$
(iii) $J^+(\kappa\xi) \geq J^+(\xi)$
(iv) $J^-(\kappa\xi) \leq J^-(\xi)$ for all $\kappa, \xi \in \Re$.

Example 4. *The non-empty set $R = \{0, \kappa, \xi, \tau\}$ can be subjected to the binary operations in the observing terms: Let $U = \{e_1, e_2, e_3\}$ be the group of parameters. Then now define a BFSS (J, U) on $\Re$ as follows:*

$$J(e_1) = \{(0, 0.1, -0.2), (\kappa, 0.5, -0.3), (\xi, 0.1, -0.4), (\tau, 0.7, -0.4)\},$$
$$J(e_2) = \{(0, 0.3, -0.3), (\kappa, 0.3, -0.3), (\xi, 0.7, -0.4), (\tau, 0.9, -0.4)\},$$
$$J(e_3) = \{(0, 0.9, -0.3), (\kappa, 0.7, -0.3), (\xi, 0.4, -0.4), (\tau, 0.4, -0.4)\}.$$

It is easy to verify that (J, U) is a BFSI over $\Re$. Hence, (J, U) is a BFSI over $\Re$.

$+$	0	κ	ξ	τ
0	0	κ	ξ	τ
κ	κ	0	τ	ξ
ξ	ξ	τ	0	κ
τ	τ	ξ	κ	0

$*$	0	κ	ξ	τ
0	0	0	0	0
κ	0	κ	0	κ
ξ	0	0	ξ	ξ
τ	0	κ	ξ	τ

Theorem 5. *If (J,U) and (K,V) are two BFSIs over $\Re$, then $(J,U) \wedge (K,V)$ is also a BFSI over $\Re$.*

Proof. Let (J,U) and (K,V) be two BFSIs over $\Re$. Then as defined $(J,U) \wedge (K,V)$, where $W = U \times V$ and $L(\rho,\tau) = J(\rho) \cap K(\tau)$, for all $(\rho,\tau) \in W = U \times V$, as (J,U) and (K,V) are BFSIs over $\Re$. Thus, for $\kappa, \xi \in \Re$,

$$\begin{aligned}
L^+_{(\rho,\tau)}(\kappa+\xi) &= A\{J^+_\rho(\kappa+\xi), K^+_\tau(\kappa+\xi)\} \\
&\geq A\{A\{J^+_\rho(\kappa), J^+_\rho(\xi)\}, A\{K^+_\tau(\kappa), K^+_\tau(\xi)\}\} \\
&= A\{A\{J^+_\rho(\kappa), K^+_\tau(\kappa)\}, A\{J^+_\rho(\xi), K^+_\tau(\xi)\}\} \\
&= A\{L^+_{(\rho,\tau)}(\kappa), L^+_{(\rho,\tau)}(\xi)\}, \\
L^-_{(\rho,\tau)}(\kappa+\xi) &= B\{J^-_\rho(\kappa+\xi), K^-_\tau(\kappa+\xi)\} \\
&\leq B\{B\{J^-_\rho(\kappa), J^-_\rho(\xi)\}, B\{K^-_\tau(\xi), K^-_\tau(\xi)\}\} \\
&= B\{B\{J^-_\rho(\kappa), K^-_\tau(\kappa)\}, B\{J^-_\rho(\xi), K^-_\tau(\kappa)\}\} \\
&= B\{L^-_{(\rho,\tau)}(\kappa), L^-_{(\rho,\tau)}(\xi)\}, \\
L^+_{(\rho,\tau)}(\kappa\xi) &= A\{J^+_\rho(\kappa\xi), K^+_\tau(\kappa\xi)\} \\
&\geq A\{J^+_\rho(\xi), K^+_\tau(\xi)\} \\
&= L^+_{(\rho,\tau)}(\xi), \\
L^-_{(\rho,\tau)}(\kappa\xi) &= B\{J^-_\rho(\kappa\xi), K^-_\tau(\kappa\xi)\} \\
&\leq B\{J^-_\rho(\xi), K^-_\tau(\xi)\} \\
&= L^-_{(\rho,\tau)}(\xi).
\end{aligned}$$

Therefore, $(L,W) = (J,U) \wedge (K,V)$ is a BFSI over $\Re$.

Theorem 6. *If (J,U) and (K,V) are two BFSIs over $\Re$, then $(J,U) \vee (K,V)$ is also a BFSI over $\Re$.*

Proof. Let (J,U) and (K,V) be two BFSIs over $\Re$. Then as defined $(J,U) \vee (K,V) = (L,W)$, where $W = U \times V$ and $L(\rho,\tau) = J(\rho) \cup K(\tau)$, for all $(\rho,\tau) \in W = U \times V$, as (J,U) and (K,V) are BFSIs over $\Re$. Thus, for $\kappa, \xi \in \Re$,

$$\begin{aligned}
L^+_{(\rho,\tau)}(\kappa+\xi) &= B\{J^+_\rho(\kappa+\xi), K^+_\tau(\kappa+\xi)\} \\
&\geq B\{A\{J^+_\rho(\kappa), J^+_\rho(\xi)\}, A\{K^+_\tau(\kappa), K^+_\tau(\xi)\}\}
\end{aligned}$$

$$
\begin{aligned}
&\geq A\{B\{J^+_\rho(\kappa), K^+_\tau(\kappa)\}, B\{J^+_\rho(\xi), K^+_\tau(\xi)\}\} \\
&= A\{L^+_{(\rho,\tau)}(\kappa), L^+_{(\rho,\tau)}(\xi)\}, \\
L^-_{(\rho,\tau)}(\kappa+\xi) &= A\{J^-_\rho(\kappa+\xi), K^-_\tau(\kappa+\xi)\} \\
&\leq A\{B\{J^-_\rho(\kappa), J^-_\rho(\xi)\}, B\{K^-_\tau(\kappa), K^-_\tau(\xi)\}\} \\
&\leq B\{A\{J^-_\rho(\kappa), K^-_\tau(\kappa)\}, A\{J^-_\rho(\xi), K^-_\tau(\xi)\}\} \\
&= B\{L^-_{(\rho,\tau)}(\kappa), L^-_{(\rho,\tau)}(\xi)\}, \\
L^+_{(\rho,\tau)}(\kappa\xi) &= B\{J^+_\rho(\kappa\xi), K^+_\tau(\kappa\xi)\} \\
&\geq B\{J^+_\rho(\xi), K^+_\tau(\xi)\} \\
&= L^+_{(\rho,\tau)}(\xi), \\
L^-_{(\rho,\tau)}(\kappa\xi) &= A\{J^-_\rho(\kappa\xi), K^-_\tau(\kappa\xi)\} \\
&\leq A\{J^-_\rho(\xi), K^-_\tau(\xi)\} \\
&= L^-_{(\rho,\tau)}(\xi).
\end{aligned}
$$

Therefore, $(L, W) = (J, U) \vee (K, V)$ is a BFSI over $\Re$.

Theorem 7. *If (J, U) and (K, V) are two BFSIs over $\Re$, then $(J, U) \cap (K, V)$ is also a BFSI over $\Re$.*

Proof. Let (J, U) and (K, V) be two BFSIs over $\Re$. Then $(J, U) \cap (K, V) = (L, W)$, where $W = U \cap V$ and $L(\omega) = J(\omega) \cap K(\omega)$, for all $\omega \in W$. Now,

$$
\begin{aligned}
L^+_\omega(\kappa+\xi) &= A\{J^+_\omega(\kappa+\xi), K^+_\omega(\kappa+\xi)\} \\
&\geq A\{A\{J^+_\omega(\kappa), J^+_\omega(\xi)\}, A\{K^+_\omega(\kappa), K^+_\omega(\xi)\}\} \\
&= A\{J^+_\omega(\kappa), K^+_\omega(\kappa)\}, A\{J^+_\omega(\xi), K^+_\omega(\xi)\}\} \\
&= A\{L^+_\omega(\kappa), L^+_\omega(\xi)\}, \\
L^-_\omega(\kappa+\xi) &= B\{J^-_\omega(\kappa+\xi), K^-_\omega(\kappa+\xi)\} \\
&\leq B\{B\{J^-_\omega(\kappa), J^-_\omega(\xi)\}, B\{K^-_\omega(\kappa), K^-_\omega(\xi)\}\} \\
&= B\{B\{J^-_\omega(\kappa), K^-_\omega(\kappa)\}, B\{J^-_\omega(\xi), K^-_\omega(\xi)\}\} \\
&= B\{L^-_\omega(\kappa), L^-_\omega(\xi)\}, \\
L^+_\omega(\kappa\xi) &= A\{J^+_\omega(\kappa\xi), K^+_\omega(\kappa\xi)\} \\
&\geq A\{J^+_\omega(\xi), K^+_\omega(\xi)\} \\
&= A\{J^+_\omega(\xi), K^+_\omega(\xi)\} \\
&= L^+_\omega(\xi), \\
L^-_\omega(\kappa\xi) &= B\{J^-_\omega(\kappa+\xi), K^-_\omega(\kappa+\xi)\} \\
&\leq B\{J^-_\omega(\xi), K^-_\omega(\xi)\} \\
&= B\{J^-_\omega(\xi), K^-_\omega(\xi)\} \\
&= L^-_\omega(\xi).
\end{aligned}
$$

Therefore, $(L, W) = (J, U) \cap (K, V)$ is a BFSI over $\Re$.

Theorem 8. *If (J, U) and (K, V) are two BFSIs over $\Re$, then $(J, U) \cup (K, V)$ is also a BFSI over $\Re$.*

Proof. Let (J, U) and (K, V) be two BFSIs over $\Re$. Then $(J, U) \cup (K, V) = (L, W)$, where $W = U \cup V$ and

$$L(\omega) = \begin{cases} J(\omega) & \text{if } \omega \in U - V \\ K(\omega) & \text{if } \omega \in V - U \\ J(\omega) \vee K(\omega) & \text{if } \omega \in U \cap V \end{cases} \quad \text{for all } \omega \in W.$$

Then we have the following cases:
Case 1: If $\omega \in U - V$, then

$$\begin{aligned} L^+_\omega(\kappa + \xi) &= J^+_\omega(\kappa + \xi) \\ &\geq A\{J^+_\omega(\kappa), J^+_\omega(\xi)\} \\ &= A\{L^+_\omega(\kappa), L^+_\omega(\xi)\}, \\ L^-_\omega(\kappa + \xi) &= J^-_\omega(\kappa + \xi) \\ &\leq B\{J^-_\omega(\kappa), J^-_\omega(\xi)\} \\ &= B\{L^-_\omega(\kappa), L^-_\omega(\xi)\}, \\ L^+_\omega(\kappa\xi) &= J^+_\omega(\kappa\xi) \\ &\geq J^+_\omega(\xi) \\ &= L^+_\omega(\xi), \\ L^-_\omega(\kappa\xi) &= J^-_\omega(\kappa + \xi) \\ &\leq J^-_\omega(\xi) \\ &= L^-_\omega(\xi). \end{aligned}$$

Case 2: If $\omega \in V - U$, then

$$\begin{aligned} L^+_\omega(\kappa + \xi) &= K^+_\omega(\kappa + \xi) \\ &\geq A\{K^+_\omega(\kappa), K^+_\omega(\xi)\} \\ &= A\{L^+_\omega(\kappa), L^+_\omega(\xi)\}, \\ L^-_\omega(\kappa + \xi) &= K^-_\omega(\kappa + \xi) \\ &\leq B\{K^-_\omega(\kappa), K^-_\omega(\xi)\} \\ &= B\{L^-_\omega(\kappa), L^-_\omega(\xi)\}, \\ L^+_\omega(\kappa\xi) &= K^+_\omega(\kappa\xi) \\ &\geq K^+_\omega(\xi) \\ &= L^+_\omega(\xi), \\ L^-_\omega(\kappa\xi) &= K^-_\omega(\kappa + \xi) \end{aligned}$$

$$\begin{aligned} &\leq K_\omega^-(\xi) \\ &= L_\omega^-(\xi). \end{aligned}$$

Case 3: If $\omega \in U \cap V$, then

$$\begin{aligned} L_\omega^+(\kappa+\xi) &= B\{J_\omega^+(\kappa+\xi), K_\omega^+(\kappa+\xi)\} \\ &\geq B\{A\{J_\omega^+(\kappa), J_\omega^+(\xi)\}, A\{K_\omega^+(\kappa), K_\omega^+(\xi)\}\} \\ &\geq A\{B\{J_\omega^+(\kappa), K_\omega^+(\kappa)\}, B\{J_\omega^+(\xi), K_\omega^+(\xi)\}\} \\ &= A\{L_\omega^+(\kappa), L_\omega^+(\xi)\}, \\ L_\omega^-(\kappa+\xi) &= A\{J_\omega^-(\kappa+\xi), K_\omega^-(\kappa+\xi)\} \\ &\leq A\{B\{J_\omega^-(\kappa), J_\omega^-(\xi)\}, B\{K_\omega^-(\kappa), K_\omega^-(\xi)\}\} \\ &\leq B\{A\{J_\omega^-(\kappa), K_\omega^-(\kappa)\}, A\{J_\omega^-(\xi), K_\omega^-(\xi)\}\} \\ &= B\{L_\omega^-(\kappa), L_\omega^-(\xi)\}, \\ L_\omega^+(\kappa\xi) &= B\{J_\omega^+(\kappa\xi), K_\omega^+(\kappa\xi)\} \\ &\geq B\{J_\omega^+(\xi), K_\omega^+(\xi)\} \\ &= B\{J_\omega^+(\xi), K_\omega^+(\xi)\} \\ &= L_\omega^+(\xi), \\ L_\omega^-(\kappa\xi) &= A\{J_\omega^-(\kappa\xi), K_\omega^-(\kappa\xi)\} \\ &\leq A\{J_\omega^-(\xi), K_\omega^-(\xi)\} \\ &= A\{J_\omega^-(\xi), K_\omega^-(\xi)\} \\ &= L_\omega^-(\xi). \end{aligned}$$

Therefore, $(L, W) = (J, U) \cup (K, V)$ is a BFSI over $\Re$.

5. Conclusion

This study examined the integration of BFSs and SSs within Boolean ring structures, resulting in the development of BFSBRs and their corresponding BFSIs. The algebraic properties and structural behaviors of these models were systematically examined. This framework not only advances mathematical modeling under uncertainty but also serves as a valuable foundation for students beginning research in abstract algebra. Supporting hands-on exploration and conceptual learning helps cultivate mathematical intuition and collaborative research practices in educational settings.

Acknowledgements

This research was supported by University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund 2025, Grant No. 5027/2567).

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