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Some Refinements for Numerical Radius Inequalities of Operators

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Abstract. Several recent papers gave numerical radius inequalities for sums and products of operators which are defined on a complex separable Hilbert space H. In this paper, we prove a numerical radius inequality which generalizes and refines a recent inequality proved by Kittaneh.

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1. Introduction

Let H be a Hilbert space over the field of complex numbers with inner product $\langle .,. \rangle$. The set of all operators in H is denoted by $B(H)$. Upper case letters will denote elements of $B(H)$. If $A \in B(H)$ such that the operator A is equal to its conjugate transpose, then we say that the operator A is self-adjoint and in this case all its eigenvalues are real numbers. The set of all eigenvalues of $A \in B(H)$ is denoted by $\sigma(A)$. The conjugate transpose (adjoint) of A is denoted by A^*. A self-adjoint operator $A \in B(H)$ is called positive semi-definite if $\langle Ax, x \rangle \geq 0$ for all $x \in H$. For any operator $A \in B(H)$, the operator A^*A is positive semi-definite. The square root of A^*A, denoted by $|A|$, is defined as $|A| = (A^*A)^{\frac{1}{2}}$. The singular values of $A \in B(H)$ are ordered descendingly as follows, $s_1(A) \geq s_2(A) \geq \cdots$ and they are the eigenvalues of $|A|$. In fact $s_j(A) = \lambda_j(|A|) = s_j(|A|)$ for $j = 1, 2, ...$ For recent studies about singular values, we advise the readers to read [[1]-[4]], [[8]-[10]], [[13]] and [[14]-[18]]. We denote the identity operator on H by $I \in B(H)$ and we denote the zero operator on H by $O \in B(H)$. Each operator A can be written as a sum of its real and imaginary parts, as follows, $A = B + iC$, where $B = Re(A) = \frac{A+A^*}{2}$

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and $C = Im(A) = \frac{A-A^*}{2i}$. Note that $Re(A)$ and $Im(A)$ are self-adjoint operators. For $A \in B(H)$, the spectral radius, numerical radius and the usual operator norm of A are given, respectively, by

$$r(A) = \max_{\lambda \in \sigma(A)} \{|\lambda|\},$$

$$w(A) = \sup_{||x||=1} |\langle Ax, x\rangle|$$

and

$$\|A\| = \sup_{||x||=||y||=1} |\langle Ax, y\rangle| .$$

The computation process to reach the exact value of $w(A)$ is not always of that simplicity, this gave the authors the area to obtain lower and upper bounds of $w(A)$, these bounds are usually in terms of $||A||$. For recent studies of numerical radius inequalities of operaors, we refer the reader to [5]-[6], [11]-[12] and [7]. The most known famous lower and upper bounds for $w(A)$ is the following equivalence between $w(A)$ and $||A||$, see [22],

$$\frac{\|A\|}{2} \leq w(A) \leq \|A\| . \tag{1}$$

Several improvements and generalizations of inequality (1) has been given. For example, in [26], it is shown that

$$w(A) \leq \frac{1}{2}||\ |A| + |A^*|\ || \tag{2}$$

and

$$w(A) \leq \frac{1}{2}\left(||\ A\ || + ||\ A^2\ ||^{\frac{1}{2}}\right), \tag{3}$$

as a refinement of the second inequality in (1).
In [25], another refinement of the second inequality in (1) has been given as follows:

$$w^2(A) \leq \frac{1}{2}||\ A^*A + AA^*\ ||. \tag{4}$$

This bound for the numerical radius is given as one of the sharpest simple bounds in the literature.
The author in [20] refines inequality (3) as follows,

$$w(A) \leq \frac{1}{2}\left(||\ A\ || + \sqrt{r(|A||A^*|)}\right). \tag{5}$$

Recently, in [23], it is shown that:

$$w(AB^*) \leq \frac{1}{4}||\ |A| + |B|\ ||\ ||\ |A^*| + |B^*|\ ||. \tag{6}$$

In this paper, we give a remarkable refinement and a generalization of inequality (4). An attractive generalization of inequality (5) is also given. Moreover, we give an inequality that is equivalent, if A and B are self-adjoint, to inequality (6). A new proof of inequality (2) is obtained. Several numerical radius inequalities are included.

2. Numerical radius inequalities via inner product

The main result in this section is a numerical radius inequality for finite sums of operators. To prove this result and to make some comparisons between some of its special cases and recent inequalities proved by different authors, we need the following lemmas, which are proved in [28], and they are essential in our analysis.

Lemma 1. *Let $A \in B(H)$ be positive semidefinite and $x \in H$ such that $\|x\| \leq 1$. Then*

(i) $\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle$ for $r \geq 1$.
(ii) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ for $0 < r \leq 1$.

Lemma 2. *Let $A \in B(H)$ be self-adjoint operator and $x \in H$. Then*

$$|\langle Ax, x\rangle| \leq \langle |A|\, x, x\rangle . \tag{7}$$

Lemma 3. *Let $A \in B(H)$ and $x, y \in H$ be any vectors. If f and g are nonnegative continuous functions on $[0,\infty)$ satisfying the relation $f(a)g(a) = a$ $(a \in [0,\infty))$, then*

$$|\langle Ax, y\rangle|^2 \leq \langle |A|\, x, x\rangle \langle |A^*|\, y, y\rangle \tag{8}$$

and more general

$$|\langle Ax, y\rangle|^2 \leq \left\langle f^2\left(|A|\right) x, x\right\rangle \left\langle g^2\left(|A^*|\right) y, y\right\rangle . \tag{9}$$

The following lemma follows by Weyl's monotonocity principle (see, e.g., [5, p.63] or [5, p. 20]).

Lemma 4. *If $A, B \in B(H)$ are positive semidefinite such that $A \leq B$. Then*

$$||\ A\ || \leq ||\ B\ ||. \tag{10}$$

Lemma 5. *Let $A, B \geq O$. Then*

$$||\ A + B\ ||^2 \leq 2||\ A^2 + B^2\ ||. \tag{11}$$

Equality holds iff $A = B$.

Proof. It is well known that $||\ (A+B)^2\ || = ||\ A + B\ ||^2$ (since $A, B \geq 0$). To reach inequality (11), it is enough to prove that $(A+B)^2 \leq 2(A^2 + B^2)$ and then applying inequality (10).
Now,

$$\begin{aligned} & 2A^2 + 2B^2 - (A+B)^2 = 2A^2 + 2B^2 - (A^2 + B^2 + AB + BA) \\ = \ & A^2 + B^2 - AB - BA \\ = \ & (A-B)^2 \geq 0 \text{ (since A-B is Hermitian)}. \end{aligned}$$

This implies that $(A+B)^2 \leq 2(A^2 + B^2)$, and so we reach our claim.

Theorem 1. *Let $A_1, A_2, ..., A_n, B_1, B_2, ..., B_n$ be operators in $B(H)$. If f and g are non-negative continuous functions on $[0,\infty)$ satisfying the relation $f(a)g(a) = a$ $(a \in [0,\infty))$, then*

$$w\left(\sum_{i=1}^{n}(A_i + B_i)\right) \leq \frac{1}{2}\sum_{i=1}^{n}\left|\left| \; f^2(|A_i|) + g^2(|A_i^*|) + f^2(|B_i|) + g^2(|B_i^*|) \; \right|\right|. \tag{12}$$

Proof.

$$\begin{aligned}\left|\left\langle \sum_{i=1}^{n}(A_i + B_i)x, x \right\rangle\right| &= |\langle((A_1 + B_1) + (A_2 + B_2) + \cdots + (A_n + B_n))\, x, x\rangle| \\ &= |\langle(A_1 + B_1)x, x\rangle + \langle(A_2 + B_2)x, x\rangle + \cdots + \langle(A_n + B_n)x, x\rangle|\end{aligned}$$

$$\begin{aligned}&\leq (|\langle(A_1 + B_1)x, x\rangle| + |\langle(A_2 + B_2)x, x\rangle| + \cdots + |\langle(A_n + B_n)x, x\rangle|) \\ &\quad \text{(by using triangle inequality)} \\ &= \sum_{i=1}^{n}|\langle(A_i + B_i)x, x\rangle| \\ &= \sum_{i=1}^{n}|\langle A_i x, x\rangle + \langle B_i x, x\rangle| \\ &\leq \sum_{i=1}^{n}(|\langle A_i x, x\rangle| + |\langle B_i x, x\rangle|)\,\text{(by using triangle inequality)} \\ &\leq \sum_{i=1}^{n}\left(\left\langle f^2(|A_i|)x, x\right\rangle^{\frac{1}{2}}\left\langle g^2(|A_i^*|)x, x\right\rangle^{\frac{1}{2}} + \left\langle f^2(|B_i|)x, x\right\rangle^{\frac{1}{2}}\left\langle g^2(|B_i^*|)x, x\right\rangle^{\frac{1}{2}}\right) \\ &\quad \text{(by using inequality (9))} \\ &\leq \frac{1}{2}\sum_{i=1}^{n}\left(\left\langle f^2(|A_i|)x, x\right\rangle + \left\langle g^2(|A_i^*|)x, x\right\rangle + \left\langle f^2(|B_i|)x, x\right\rangle + \left\langle g^2(|B_i^*|)x, x\right\rangle\right) \\ &\quad \text{(by using arithmetic geometric mean inequality)} \\ &= \frac{1}{2}\sum_{i=1}^{n}\left\langle\left(f^2(|A_i|) + g^2(|A_i^*|) + f^2(|B_i|) + g^2(|B_i^*|)\right)x, x\right\rangle.\end{aligned}$$

Taking the supremum over all unit vectors $x \in H$, we obtain the inequality (12).

Corollary 1. *Let $A_1, A_2, ..., A_n, B_1, B_2, ..., B_n$ be operators in $B(H)$. Then*

$$w\left(\sum_{i=1}^{n}(A_i + B_i)\right) \leq \frac{1}{2}\sum_{i=1}^{n}\left|\left| \; |A_i| + |A_i^*| + |B_i| + |B_i^*| \; \right|\right|. \tag{13}$$

Proof. Letting $f(t) = g(t) = \sqrt{t}$ in inequality (12), we obtain the inequality (13).

Corollary 2. *Let A, B be operators in $B(H)$. Then*

$$w(A+B) \leq \frac{1}{2} \| \, |A| + |A^*| + |B| + |B^*| \, \| . \tag{14}$$

Proof. Letting $A_i = B_i = O$ for $i = 2, 3, 4, ..., n$ in inequality (13), we obtain the inequality (14).

Corollary 3. *Let A, B be operators in $B(H)$. Then*

$$w^2(A+B) \leq \frac{1}{4} \| \, |A| + |A^*| + |B| + |B^*| \, \|^2 . \tag{15}$$

Proof. By squaring both sides of inequality (14), we obtain the inequality (15).

Remark 1. *Letting $B = O$ in inequality (14), we derive inequality (2). This is considered as a new proof of inequality (2).*

Corollary 4. *Let $A \in B(H)$. Then*

$$w^2(A) \leq \frac{1}{4} \| \, |A| + |A^*| \, \|^2. \tag{16}$$

Proof. Letting $B = O$ in inequality (15), we obtain the inequality (16).

Remark 2. *Inequality (16) refines inequality (4). To show this, note that while the left sides of inequalities (4) and (16) are the same. The right side of inequality (16) is*

$$\frac{1}{4} \| \, |A| + |A^*| \, \|^2 \leq \frac{1}{2} \| \, |A|^2 + |A^*|^2 \, \| \text{ (by using inequality (11))}$$

$$= \frac{1}{2} \| \, A^*A + AA^* \, \|,$$

which is the right side of inequality (4).

The following lemma [27] is essential to prove the following theorem which is a generalization of inequality (5).

Lemma 6. *Let A, B be positive semidefinite operators in $B(H)$. Then*

$$\| \, A + B \, \| \leq max\{\| \, A \, \|, \| \, B \, \|\} + \| \, A^{\frac{1}{2}} B^{\frac{1}{2}} \, \| \tag{17}$$

Theorem 2. *Let A, B be operators in $B(H)$. Then*

$$w(A+B) \leq \frac{1}{2} \left(max\{\| \, |A| + |B| \, \|, \| \, |A^*| + |B^*| \, \|\} + \sqrt{r\left((|A| + |B|)(|A^*| + |B^*|)\right)} \right) . \tag{18}$$

Proof.

$$
\begin{aligned}
& w(A+B) \leq \frac{1}{2} \| \; |A| + |A^*| + |B| + |B^*| \; \| \ \text{(by inequality (14))} \\
= \ & \frac{1}{2} \| \; (|A| + |B|) + (|A^*| + |B^*|) \; \| \\
\leq \ & \frac{1}{2} \left(max\{ \| \; |A| + |B| \; \|, \| \; |A^*| + |B^*| \; \| \} + \| \; (|A| + |B|)^{\frac{1}{2}} (|A^*| + |B^*|)^{\frac{1}{2}} \; \| \right) \\
& \text{(by inequality (17)} \\
= \ & \frac{1}{2} \left(max\{ \| \; |A| + |B| \; \|, \| \; |A^*| + |B^*| \; \| \} + \sqrt{r\left((|A| + |B|)(|A^*| + |B^*|) \right)} \right) \\
& \text{(since } \| \; A^{\frac{1}{2}} B^{\frac{1}{2}} \; \|^2 = r(AB)).
\end{aligned}
$$

Remark 3. *Letting $B = O$ in inequality (18), we obtain the inequality (5).*

3. Numerical radius inequalities via block matrices

In this section, new numerical radius inequalities are proved by using block matrices. We begin by proving the following inequality which is a direct application of inequality (15).

Theorem 3. *Let X, Y, W, Z be operators in $B(H)$. Then*

$$
w^2 \begin{bmatrix} X & Z \\ W & Y \end{bmatrix} \leq \frac{1}{4} \left\| \begin{bmatrix} |X| + |X^*| + |W| + |Z^*| & O \\ O & |Y| + |Y^*| + |W^*| + |Z| \end{bmatrix} \right\|^2 . \tag{19}
$$

Proof. Letting $A = \begin{bmatrix} X & O \\ O & Y \end{bmatrix}$ and $B = \begin{bmatrix} O & Z \\ W & O \end{bmatrix}$ in inequality (15), we obtain the inequality (19).

Corollary 5. *Let X, Y be operators in $B(H)$. Then*

$$
w^2 \begin{bmatrix} X & O \\ O & Y \end{bmatrix} \leq \frac{1}{4} \left\| \begin{bmatrix} |X| + |X^*| & O \\ O & |Y| + |Y^*| \end{bmatrix} \right\|^2 . \tag{20}
$$

Proof. Letting $Z = W = O$ in inequality (19), we obtain the inequality (20).

Remark 4. *Letting $Y = O$ in inequality (20), we obtain the inequality (16). In that sense, inequality (20) is a generalization of inequality (16).*

Corollary 6. *Let W, Z be operators in $B(H)$. Then*

$$
w^2 \begin{bmatrix} O & Z \\ W & O \end{bmatrix} \leq \frac{1}{4} \left\| \begin{bmatrix} |W| + |Z^*| & O \\ O & |W^*| + |Z| \end{bmatrix} \right\|^2 . \tag{21}
$$

Proof. Letting $X = Y = O$ in inequality (19), we obtain the inequality (21).

Remark 5. *Replacing W by W^* in inequality (21), we obtain*

$$w^2 \begin{bmatrix} O & Z \\ W^* & O \end{bmatrix} \leq \frac{1}{4} \left\| \begin{bmatrix} |W^*| + |Z^*| & O \\ O & |W| + |Z| \end{bmatrix} \right\|^2 = \frac{1}{4} max\{ \| \, |W^*|+|Z^*| \, \|, \| \, |W|+|Z| \, \|\}^2. \tag{22}$$

But $max\{w(ZW^*), w(W^*Z)\} = w \begin{bmatrix} ZW^* & O \\ O & W^*Z \end{bmatrix} = w \begin{bmatrix} O & Z \\ W^* & O \end{bmatrix}^2 \leq w^2 \begin{bmatrix} O & Z \\ W^* & O \end{bmatrix}$.
Thus

$$max\{w(ZW^*), w(W^*Z)\} \leq w^2 \begin{bmatrix} O & Z \\ W^* & O \end{bmatrix}. \tag{23}$$

Combining inequalities (22) and (23), we obtain:

$$max\{w(ZW^*), w(W^*Z)\} \leq \frac{1}{4} max\{\| \, |W^*| + |Z^*| \, \|, \| \, |W| + |Z| \, \|\}^2. \tag{24}$$

Note that inequality (24) is equivalent to inequality (6) if W and Z are self-adjoint operators.
In that sense, inequality (19) is a generalization of inequality (6) when W and Z are self-adjoint operators.

Lemma 7. *[7] Let $A \in B(H)$ and $r \geq 2$. Then*

$$w^r(A) \leq 2^{r-3} \| \, |A|^r + |A^*|^r \, \|. \tag{25}$$

The following theorem is another generalization of inequality (4).

Theorem 4. *Let X, Y be operators in $B(H)$, $r \geq 2$. Then*

$$w^r \begin{bmatrix} O & X \\ Y^* & O \end{bmatrix} \leq 2^{r-3} max\{\| \, |X^*|^r + |Y^*|^r \, \|, \ \| \, |X|^r + |Y|^r \, \|\}. \tag{26}$$

In particular,

$$w^2 \begin{bmatrix} O & X \\ Y^* & O \end{bmatrix} \leq \frac{1}{2} max\{\| \, |X|^2 + |Y|^2 \, \|, \| \, |X^*|^2 + |Y^*|^2 \, \|\} \tag{27}$$

Proof. Letting $A = \begin{bmatrix} O & X \\ Y^* & O \end{bmatrix}$ in inequality (25), implies that

$$\begin{aligned} w^r \begin{bmatrix} O & X \\ Y^* & O \end{bmatrix} &\leq 2^{r-3} \left\| \begin{bmatrix} |Y^*|^r & O \\ O & |X|^r \end{bmatrix} + \begin{bmatrix} |X^*|^r & O \\ O & |Y|^r \end{bmatrix} \right\| \\ &= 2^{r-3} \left\| \begin{bmatrix} |X^*|^r + |Y^*|^r & O \\ O & |X|^r + |Y|^r \end{bmatrix} \right\| \\ &= 2^{r-3} max\{\| \, |X^*|^r + |Y^*|^r \, \|, \| \, |X|^r + |Y|^r \, \|\}. \end{aligned}$$

Remark 6. *Inequality (24) is sharper than inequality (29). To show this, note that*

$$max\{w(XY^*), w(Y^*X)\} = w\begin{bmatrix} XY^* & O \\ O & Y^*X \end{bmatrix} = w\begin{bmatrix} O & X \\ Y^* & O \end{bmatrix}^2 \leq w^2\begin{bmatrix} O & X \\ Y^* & O \end{bmatrix}. \quad (28)$$

Combining inequalities (27) and (28), it follows that

$$max\{\ w(XY^*), w(Y^*X)\ \} \leq max\{\frac{1}{2}||\ |X|^2 + |Y|^2\ ||, \frac{1}{2}||\ |X^*|^2 + |Y^*|^2\ ||\}. \quad (29)$$

By comparing inequality (24) and inequality (29), we note that while the left sides of both inequalities are the same, the right side in inequality (24) is

$$\frac{1}{4}max\{||\ |W| + |Z|\ ||, ||\ |W^*| + |Z^*|\ ||\}^2 \leq \frac{1}{2}max\{||\ |W|^2 + |Z|^2\ ||, ||\ |W^*|^2 + |Z^*|^2\ ||\}$$

(by inequality (11)), which is the right side of inequality (29). This implies that inequality (24) is sharper than inequality (29).

Corollary 7. *Let X be operators in $B(H)$, $r \geq 2$. Then*

$$w^r\begin{bmatrix} O & X \\ O & O \end{bmatrix} \leq 2^{r-3} max\{||\ |X^*|^r\ ||,\ ||\ |X|^r\ ||\}. \quad (30)$$

Proof. Letting $Y^* = O$ in inequality (26), we obtain the inequality (30).

Remark 7. *Letting $r = 2$ in inequality (30), we obtain*

$$w^2\left(\begin{bmatrix} O & X \\ O & O \end{bmatrix}\right) \leq \frac{1}{2}max\{||\ XX^*\ ||, ||\ X^*X\ ||\} = \frac{1}{2}||\ X^*X\ || = \frac{1}{2}||\ X\ ||^2. \quad (31)$$

Theorem 5. *Let X and Y be operators in $B(H)$, $r \geq 2$. Then*

$$w^r\begin{bmatrix} X & O \\ O & Y^* \end{bmatrix} \leq \frac{1}{2}\left|\left|\begin{bmatrix} |X|^r + |X^*|^r & O \\ O & |Y|^r + |Y^*|^r \end{bmatrix}\right|\right|. \quad (32)$$

In particular, if $r = 2$, we obtain

$$w^2\begin{bmatrix} X & O \\ O & Y^* \end{bmatrix} \leq \frac{1}{2}\left|\left|\begin{bmatrix} |X|^2 + |X^*|^2 & O \\ O & |Y|^2 + |Y^*|^2 \end{bmatrix}\right|\right|. \quad (33)$$

Proof. Let $A = \begin{bmatrix} X & O \\ O & Y^* \end{bmatrix}$ in inequality (25), we obtain the inequality (32).

Remark 8. *Letting $Y = O$ in inequality (33), we obtain the inequality (4). In that sense, inequality (33) is a generalization of inequality (4).*

Recall that the cartesian decomposition of the operator X is given by $X = A + iB$ where $A = Re(X)$ and $B = Im(X)$.
The author in [24] proved that if $X = A + iB$ and $0 < r \leq 2$, then

$$\frac{1}{2}|| \ |A|^r + |B|^r \ || \leq w^r(X) \leq || \ |A|^r + |B|^r \ ||. \tag{34}$$

This implies , if $r = 2$,

$$\frac{1}{2}|| \ |A|^2 + |B|^2 \ || \leq w^2(X) \leq || \ |A|^2 + |B|^2 \ || = || \ A^2 + B^2 \ ||. \tag{35}$$

In the following, we provide a new proof of the second inequality in (35).

Theorem 6. *Let $X \in B(H)$ with cartesian decomposition $X = A + iB$. Then*

$$w^2(X) \leq || \ A^2 + B^2 \ ||. \tag{36}$$

Proof.

$$\begin{aligned} & |\langle Xx, x\rangle|^2 = |\langle (A+iB)x, x\rangle|^2 \\ = \ & |\langle Ax, x\rangle + i\langle Bx, x\rangle|^2 \\ = \ & |\langle Ax, x\rangle|^2 + |\langle Bx, x\rangle|^2 \\ = \ & \langle Ax, x\rangle^2 + \langle Bx, x\rangle^2 \ \ (\text{since} < Ax, x > \in \mathbb{R}) \\ \leq \ & \left\langle A^2x, x\right\rangle + \left\langle B^2x, x\right\rangle \ \ (\text{by Lemma (1) (i)}) \\ = \ & \left\langle \left(A^2 + B^2\right)x, x\right\rangle. \end{aligned}$$

Taking the supremum over all unit vectors $x \in H$, we obtain the inequality (36).

4. Numerical radius inequalities via singular values and aluthge transform

In this section, we prove numerical radius inequalities using recent singular values inequalities and aluthge transform.
The author in [15] proves that if $A, B \in B(H)$, then

$$2s_j(AB^* + BA^*) \leq s_j^2 \begin{bmatrix} A & B \\ B & A \end{bmatrix}. \tag{37}$$

This inequality implies, since unitarily invariant norms and in particular the spectral norm, are increasing functions of singular values, that

$$||AB^* + BA^*|| \leq \frac{1}{2}\left|\left|\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right|\right|^2. \tag{38}$$

Theorem 7. *Let A and B be operators in $B(H)$. Then*

$$w(AB^*) \leq \frac{1}{4}|| \ |A| + |B| \ ||^2 \tag{39}$$

Proof. If A and B are operators in $B(H)$, then

$$\begin{aligned}
&||Re(e^{i\theta}AB^*)|| = \frac{1}{2}||e^{i\theta}AB^* + e^{-i\theta}BA^*|| \\
=\ & \frac{1}{2}||Ae^{i\theta}B^* + e^{-i\theta}BA^*|| \\
\leq\ & \frac{1}{4}\left|\left|\begin{bmatrix} A & e^{-i\theta}B \\ e^{-i\theta}B & A \end{bmatrix}\right|\right|^2 \ (\textit{by inequality } (38)) \\
=\ & \frac{1}{4}\left|\left|\ \left|\begin{bmatrix} A & e^{-i\theta}B \\ e^{-i\theta}B & A \end{bmatrix}\right|\ \right|\right|^2 (\text{since } ||A|| = ||\ |A|\ ||) \\
=\ & \frac{1}{4}\left|\left|\ \left|\begin{bmatrix} A & O \\ O & A \end{bmatrix} + \begin{bmatrix} O & e^{-i\theta}B \\ e^{-i\theta}B & O \end{bmatrix}\right|\ \right|\right|^2 \\
\leq\ & \frac{1}{4}\left|\left|\ \left|\begin{bmatrix} A & O \\ O & A \end{bmatrix}\right| + \left|\begin{bmatrix} O & e^{-i\theta}B \\ e^{-i\theta}B & O \end{bmatrix}\right|\ \right|\right|^2 \\
\leq\ & \frac{1}{4}\left|\left|\ \left|\begin{bmatrix} A & O \\ O & A \end{bmatrix}\right| + \left|\begin{bmatrix} |e^{-i\theta}B| & O \\ O & |e^{-i\theta}B| \end{bmatrix}\right|\ \right|\right|^2 \ (\textit{by triangle inequality}) \\
=\ & \frac{1}{4}\left|\left|\begin{bmatrix} |A| & O \\ O & |A| \end{bmatrix} + \begin{bmatrix} |B| & O \\ O & |B| \end{bmatrix}\right|\right|^2 \\
=\ & \frac{1}{4}\left|\left|\begin{bmatrix} |A|+|B| & O \\ O & |A|+|B| \end{bmatrix}\right|\right|^2 \\
=\ & \frac{1}{4}||\ |A| + |B|\ ||^2.
\end{aligned}$$

Taking the supremum over all $\theta \in \mathbb{R}$, we obtain the inequality (39).

It can be shown easily that inequalities (6) and (39), if A and B are self-adjoint operators, are the same.
It is obvious that in the general case if A and B are not self- adjoint operators, when inequality (6) is sharper than inequality (39) then replacing A by A^* and B by B^* in the same example will make inequality (39) is sharper than inequality (6).
The following example shows that inequality (39) is sharper than inequality (6).

Example 1. *Let* $A = \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$. *Then*

$$||\ |A| + |B|\ || = 11 \textit{ and } ||\ |A^*| + |B^*|\ || = 13.$$

This implies that inequality (39) is sharper than inequality (6), since

$$\| \, |A| + |B| \, \| < \| \, |A^*| + |B^*| \, \|.$$

This implies that

$$\| \, |A| + |B| \, \|^2 < \| \, |A| + |B| \, \| \; \| \, |A^*| + |B^*| \, \|.$$

Remark 9. *If we replace A by A^* and B by B^*, in example 1, we obtain*

$$\| \, |A^*| + |B^*| \, \| < \| \, |A| + |B| \, \|.$$

This implies that inequality (6) is sharper than inequality (39).

The Aluthge transform is used to study numerical radius bounds. Recall that if $A \in B(H)$, then the polar decomposition of A is given by $A = U|A|$, where U is a partial isometry. For $0 \leq t \leq 1$, the weighted Aluthge transform is defined by $\tilde{A}_t = |A|^{1-t}U|A|^t$. If $t = 2$, we write $\tilde{A}$ instead of $\tilde{A}_{\frac{1}{2}}$.
The following corollary is an application of Theorem 7, which inturns a generalization of the second inequality of (1).

Corollary 8. *Let $T \in B(H)$. Then*

$$w(T) \leq \frac{1}{4} \| \, |T|^t + |T|^{1-t} \, \|^2, \; 0 \leq t \leq 1. \tag{40}$$

In particular, if $t = \frac{1}{2}$, we obtain the second inequality of (1).

Proof. Letting $A = |T|^t$ and $B^* = |T|^{1-t}U^*$ in inequality (39), we obtain the inequality (40).

5. Conclusions

Several numerical radius inequalities of operators are proved. We compare these new inequalities with recent inequalities proved by Kittaneh. Our new inequalities refine and generalize Kittaneh inequalities. We use several techniques to reach our new bounds for numerical radius inequalities of operators. These techniques included inner products, block matrices, singular values and aluthge transform.

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References

[1] M. Al-Labadi, R. Al-Naimi and W. Audeh, Singular value inequalities for generalized anticommutators, J Inequal Appl, 15(2025).

[2] W. Audeh, H.R. Moradi and M. Sababheh, Commutator bounds via singular values with applications to the numerical radius, Mediterr. J. Math. 22,8(2025).

[3] W. Audeh, A. Al-Boustanji, M. Al-Labadi and R. Al-Naimi, Singular value inequalities of matrices via increasing functions, J Inequal Appl, 114(2024).

[4] W. Audeh, H.R. Moradi and M. Sababheh, Generalizations of recent singular value inequalities for sums of products of matrices, Filomat, 38,28(2024), $9905 - 9919$.

[5] W. Audeh, H.R. Moradi and M. Sababheh, Matrix Holder inequalities and numerical radius applications, Linear Algebra Appl., 696(2024), $68 - 84$.

[6] W. Audeh, M. Al-Labadi and R. Al-Naimi, Numerical radius inequalities via block matrices, Acta Sci. Math. (2024).

[7] W. Audeh and M. Al-Labadi, Numerical Radius Inequalities for Finite Sums of Operators, Complex Anal. Oper. Theory, **17** (2023).

[8] W. Audeh, Singular value inequalities for operators and matrices, Ann. Funct. Anal., 13,24(2022).

[9] W. Audeh, Singular value inequalities for accretive-dissipative normal operators, J. Math. Inequal., 16(2022), $729 - 737$.

[10] W.Audeh, Singular value inequalities with applications, J. Math. Computer Sci., 24(2022), $323 - 329$.

[11] W. Audeh and M. Al-Labadi, Some results about numerical radius inequalities, Int. J. Math. Compute. Sci., 17(2022), $33 - 39$.

[12] A. Al-Boustanji and W. Audeh, Applications of numerical radius inequalities, Int. J. Math. Compute. Sci., 17,3(2022), $1305 - 1312$.

[13] W. Audeh, Singular value and norm inequalities of Davidson-Power type, J. Math. Inequal., 15(2021), $1311 - 1320$.

[14] W. Audeh, Some generalizations for singular value inequalities of compact operators, Adv. Oper. Theory, **6** (2021).

[15] W. Audeh, Generalizations for singular value and arithmetic-geometric mean inequalities of operators, J.Math. Anal. appl., **489** (2020), $1 - 8$.

[16] W. Audeh, Generalizations for singular value inequalities of operators, Adv. Oper. Theory, **5** (2020), $371 - 381$.

[17] W. Audeh, Singular value inequalities and applications, Positivity, **25** (2020), 843-852.

[18] W. Audeh and F. kittaneh, Singular value inequalities for compact operators, Linear Algebra Appl, **437** (2012), 2516-2522.

[19] R. Bhatia, Matrix Analysis, GTM^{169}, Springer-Verlag, New York, (1997).

[20] P. Bhunia and K. Paul, Furtherance of numerical radius inequalities of Hilbert space operators. Arch. Math. (Basel)., **117** (5) (2021), $537 - 546$.

[21] L. C. Gohberg and M.G. Krein, Introduction to the theory of Linear Nonselfadjoint Operators. Amer. Math. Soc, Providence, RI (1969).

[22] K.E. Gustafson and D.K.M. Rao, Numerical range, the field of values of linear operators and matrices.Springer, New York (1997).

[23] F. Kittaneh, H. Moradi and M. Sababheh, Sharper bounds for the numerical radius. Linear Multilinear Algebra, (2023).

[24] F. Kittaneh, Numerical radius inequalities, associated with the cartesian decomposition. MIA., **18** (2015), $915-922$.

[25] F. Kittaneh, Numerical radius inequalities for Hilbert space operators. Stud. Math., **168** (2003), $73-80$.

[26] F. Kittaneh, Numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix. Stud Math., **158** (2003), $11-17$.

[27] F. Kittaneh, Norm inequalities for certain operators sums. J. Funct. Anal., **143** (1997), $337-348$.

[28] F. Kittaneh, Notes on some inequalities for Hilbert space operators, Publications of the Research Institute for Mathematical Sciences, 24(1988), $283-293$.

[29] H.R. Moradi, W. Audeh and M. Sababheh, Singular value inequalities via matrix monotone functions, Anal. Math. Phys. 13, 71(2023).