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Exact Solutions oF Nonlinear Delay Voltera Integro-Differential Equations Using Modified Homotopy Perturbation Method

Nidal Anakira[1,2,*], Ala Amourah[1], Adel Almalki[3], Abdullah Alsoboh[4,*], Tala Sasa[5]

[1] *Faculty of Education and Arts, Sohar University, Sohar 3111, Oman*
[2] *Jadara University Research Center, Jadara University, Jordan*
[3] *Al-Gunfudah University College, Umm Al- Qura University, Mecca 21955, Saudi Arabia*
[4] *College of Applied and Health Sciences, Al Sharqiyah University, Post Box No. 42, Post Code No. 400, Ibra, Sultanate of Oman*
[5] *Department of Mathematics, Faculty of Science, Private Applied Science University, Amman, Jordan*

Abstract. This paper presents a robust and efficient approach for solving delay Volterra integro-differential equations (DVIDEs), which model systems with memory and delay effects commonly encountered in fields such as biology, control theory, and epidemiology. Due to their complexity, these equations require accurate and efficient solution methods. The proposed method modifies the traditional homotopy perturbation method (HPM) to form a new version (MHPM), integrating it with the Laplace transformation and Padé approximants. The incorporation of the Laplace transformation simplifies the problem by converting integro-differential equations into algebraic equations, streamlining the solution process. To further enhance the accuracy and convergence of the solution series, Padé approximants are employed, enabling the method to overcome the limitations of standard perturbation techniques. This hybrid approach effectively combines the strengths of homotopy perturbation, Laplace transformation, and Padé approximants, yielding highly accurate solutions that closely approximate the exact ones for various nonlinear DVIDEs. Numerical experiments and illustrative examples confirm the method's efficiency and superior accuracy, even for equations with complex delay terms. The results highlight the potential of this combined approach as a powerful analytical tool for solving nonlinear delay integro-differential equations of the Volterra type in scientific and engineering applications.

2020 Mathematics Subject Classifications: 26A33, 11Cxx, 34A08

Key Words and Phrases: Homotopy Perturbation Method (HPM), Nonlinear Delay Volterra Integro-Differential Equations, Laplace Transform, Padé Approximants

*Corresponding author.
*Corresponding author.
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Email addresses: nanakira@su.edu.om (N. Anakira), AAmourah@su.edu.om (A. Amourah), aaamalki@uqu.edu.su (A. Almalki), abdullah.alsoboh@asu.edu.om (A. Alsoboh), t_sasa@asu.edu.jo (T. Sasa)

1. Introduction

Ordinary, partial, integral, fractional, fuzzy, and functional differential equations are vital tools for modeling complex phenomena in science and engineering. Each type captures distinct dependencies and dynamic behaviors, making them essential for accurately describing real-world systems. To solve these equations effectively, researchers have developed a wide range of analytical and numerical methods, including the homotopy perturbation method (HPM), the Adomian decomposition method (ADM), and the variational iteration method (VIM), along with various modifications. These techniques enhance the accuracy, convergence, and applicability of solutions across diverse problems. For instance, the optimal homotopy asymptotic Method (OHAM), originally introduced by Marinca et al. in 2008 [1, 2], has been successfully applied to different types of differential equations, such as Volterra integro-differential equations [3], delay differential equations [4], singular two-point boundary value problems [5], nonlinear anharmonic oscillators [6], and fuzzy heat equations [7]. Other widely used methods include HPM [8–11], ADM [12–15], homotopy analysis method (HAM) [16–19], the Collection Method [20, 21], and VIM [22–24], all of which have become indispensable in modern applications, especially with the support of powerful computational tools and software [25–35].

Numerical methods for solving Volterra integro-differential equations play a pivotal role in addressing complex problems in science and engineering [31–37]. These equations, characterized by their dependence on both the current state and past data, arise in diverse fields such as population dynamics, viscoelasticity, and electrical circuit analysis. A particular subclass, DVIDEs, incorporates both integral and delay terms in addition to differential components. These equations are especially useful for modeling systems where the present state depends on previous states, rates of change, and delayed responses. They are frequently used to describe processes involving hereditary effects and time lags—common in disciplines such as engineering, biology, economics, and applied sciences. As such, the development of efficient numerical solutions for DVIDEs is of significant interest. Among the semi-analytical methods, the HPM and its modifications—originally introduced by Ji-Huan He in 2003—are particularly notable for their simplicity and efficiency in solving a wide range of linear and nonlinear differential equations. Over time, several enhancements have been developed to improve its accuracy and applicability [38–43]. HPM combines concepts from topology and classical perturbation theory to construct a homotopy that smoothly transitions from a simple solvable problem to the original complex one. An embedding parameter is introduced, and the solution is expressed as a series expansion in terms of this parameter. Once the terms are obtained iteratively, the parameter is set to $P = 1$, yielding an approximate solution. A key advantage of HPM is that it does not require the presence of a small parameter in the original equation, making it applicable to a wider range of problems. Additionally, HPM often converges rapidly, providing accurate results with minimal computational effort. It has been successfully applied in fields such as fluid dynamics, heat and mass transfer, structural analysis, and quantum mechanics. However, the success of HPM relies on the proper construction of the homotopy, and convergence is not guaranteed for all problems. To overcome such limitations, a modified version of HPM (MHPM) has been proposed, integrating Laplace transforms and Padé approximants to improve solution accuracy and convergence behavior. The Modified HPM, when combined with the Laplace transform and Padé approximants, forms a powerful hybrid method for solving nonlinear differential equations. This approach leverages the strengths of each component: the Modified HPM reformulates the nonlinear problem into a more manageable form via homotopy; the Laplace transform simplifies the handling of derivatives and initial conditions by converting the problem into the Laplace domain; and Padé approximants enhance solution accuracy by converting truncated series into rational functions, which often converge more rapidly and represent the solution more effectively over a broader range. This hybrid technique is particularly effective for tackling strongly nonlinear problems, singularities, and boundary layers—situations where standard series solutions may diverge or converge slowly. By efficiently addressing initial and boundary conditions and improving convergence properties, the combined use of MHPM, Laplace transforms, and Padé approximants has shown great promise in solving problems across various scientific and engineering domains, including fluid mechanics, heat transfer, and nonlinear oscillations.

2. Methodology

2.1. Homotopy Perturbation Method

To explain the core idea of the Homotopy Perturbation Method (HPM), we will analyze the following equation according to, see [8–11].

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{1}$$

where A is the integral operator composed of the linear operator L and the nonlinear operator $N, f(r)$ is a dependent variable, and Γ is the boundary of the domain Ω. Eq. (1) can be

$$L(u) - N(u) - f(r) = 0. \tag{2}$$

A homotopy equation $v : \Omega[0,1] \rightarrow \mathrm{R}$ that satisfies

$$H(v;p) = L(v) - L(v_0) + pL(v_0) + p[N(v) - f(r)] = 0, \tag{3}$$

or

$$H(v;p) = (1-p)\left[L(v) - L(v_0)\right] + p\left[A(v_0) - F(r)\right] = 0, \tag{4}$$

is being constructed here, $r \in \Omega, p \in [0,1]$ is the homotopy parameter, and $v_0(x)$ is an initial approximation of Eq. (1). It is observed that

$$(v;0) = L(u) - L(v_0) = 0, \quad H(v;1) = A(v) - F(r) = 0. \tag{5}$$

The process of change p from zero to one includes the transformation of $H(v;p)$ from $L(u) - L(v_0)$ to $A(v) - F(r)$ is named deformation, Furthermore, $L(u) - L(v_0)$ and

$A(v) - F(r)$ are named homotopic. Observe that $0 \leq p \leq 1$ is considered as a small parameter, the solution of Eqs. (4) or (5) can be expressed as a series in p, as follows:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots \tag{6}$$

when $p \to 1$, Eq. (2.1.4) or Eq. (5) corresponds to Eq. (3) and becomes the approximate solution of Eq. (1). i.e.

$$u(x) = \lim_{p \to 1} v(x) = v_1 + v_2 + v_3 + \cdots \tag{7}$$

2.2 Padè approximation

For the function $u(x)$, the Padé approximation of order $\left[\frac{L}{M}\right]$, for more details, see [44, 45] , can be

$$\left[\frac{L}{M}\right] = \frac{P_L(x)}{Q_M(x)},$$

The Padé approximant of order $\left[\frac{L}{M}\right]$ to $u(x)$ is the rational function

$$u_{[L/M]}(x) = \frac{P_L(x)}{Q_M(x)} = \frac{b_0 + b_1 x + \cdots + b_m x^m}{1 + c_1 x + \cdots + c_n x^n}$$

where $P_L(x)$ and $Q_M(x)$, are two polynomials of the highest degree L and M. The power series

$$u(x) = \sum_{i=1}^{\infty} a_i x^i$$

The coefficients of the polynomials $P_L(x)$ and $Q_M(x)$, can be obtained from

$$u(x) - \frac{P_L(x)}{Q_M(x)} = O\left(x^{L+M+1}\right) \tag{8}$$

When the denominator and numerator functions $\frac{P_L(x)}{Q_M(x)}$ are multiplied by a constant that is not zero, the fractional values stay the same, so that we can set the normalization requirement as follows:

$$\mathrm{Q_M}(0) = 1 \tag{9}$$

The polynomial associated with the functions $P_L(x)$ and $Q_M(x)$ was found to have no public factors. The coefficients of the polynomial $Q_M(x)$ and $P_L(x)$ are given by

$$\begin{aligned} P_L(t) &= P_0 + P_1 t + P_2 t^2 + \cdots + P_L t^L. \\ Q_M(t) &= q_0 + q_1 t + q_2 t^2 + \cdots + q_M t^M. \end{aligned} \tag{10}$$

The following linear systems of coefficients can be obtained by multiplying Eq. (1) by $Q_M(x)$ considering Eq. (3).

$$\left\{\begin{array}{c} a_{L+1}+a_Lq_1+\cdots+a_{L-M+1}q_M=0 \\ a_{L+2}+a_{L+1}q_1+\cdots+a_{L-M+2}q_M=0 \\ . \\ . \\ a_{L+M}+a_{L+M-1}q_1+\cdots+a_Lq_M=0 \end{array}\right\}, \tag{11}$$

$$a_0=P_0$$

$$\left\{\begin{array}{c} a_0+a_0q_1=P_1 \\ a_2+a_1q_1+a_0q_2=P_2 \\ . \\ . \\ a_L+a_{L-1}q_1+\cdots+a_0q_L=P_L \end{array}\right\}. \tag{12}$$

These equations will be solved using Eq. (4), It is seen as a set of linear formulas for the unidentified q 's. When the q 's are identified, then (5) have an explicit formula for the unknown p's, this concludes the solution to the problem. If Eqs. (4) and (5) are non-singular, then we can solve them directly and get (6), where (6) possesses, if the lower index on a sum exceeds the upper, the sum is replaced by zero:

$$\left[\frac{L}{M}\right]=\frac{\det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ . & . & \cdots & . \\ . & & . & . & . \\ a_L & a_{L+1} & & a_{L+M} \\ \sum_{j=M}^{L}a_{j-M}X^j & \sum_{j=M-1}^{L}a_{j-M+1}X^j & \ldots & \sum_{j=0}^{L}a_jX^j \end{bmatrix}}{\det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ . & . & . & . \\ . & . & . & . \\ a_L & a_{L+1} & \ldots & a_{L+M} \\ X^M & X^{M-1} & \ldots & 1 \end{bmatrix}}.$$

3. Applications of HPM

In this section, we present two illustrative examples of nonlinear DVIDEs. These examples aim to demonstrate the effectiveness and reliability of the modified HPM procedure.

3.1. Example 1 Given the following nonlinear DVIDE [46],

$$u'(x)=u^2\left(\frac{x}{2}\right)+\int_0^t u^2\left(\frac{S}{2}\right)\mathrm{d}S-e^x+1,\quad u(x)=e^x,0\leq x, \tag{13}$$

with exact solution $u(x) = e^x$.
To find approximate solutions to this problem, we rewrite the above problem into the following linear and nonlinear operators:

$$L\left(u(x)\right) = u'(x),$$
$$N(u(x)) = -u^2\left(\frac{S}{2}\right) - \int_0^x u^2\left(\frac{s}{2}\right)ds + \sum_{n=0}^{5}\frac{x^n}{n!} - 1. \tag{14}$$

Then we construct the homotopy equation.

$$u'(x) - u_0(x) + p\left[-u^2\left(\frac{s}{2}\right) - \int_0^x u^2\left(\frac{s}{2}\right)ds + \sum_{n=0}^{5}\frac{x^n}{n!} - 1\right]. \tag{15}$$

Using HPM, we expand $u(x)$ as a perturbation series in p :

$$v(x) = v_0(x) + pv_1(x) + p^2v_2(x) + p^3v_3(x) + p^4v_4(x) + p^5v_5(x) + \cdots \tag{16}$$

The zeroth order approximation that is obtained by setting $p = 0$, is as follows

$$u_0'(x) = 0, \quad u_0(0) = 1, \tag{17}$$

which has the following solutions $u_0(x) = 1$.

Moreover, the first, second, and other orders of approximations will be obtained by substituting the homotopy equations and matching the terms of power p to get the following.

$$u_1'(x) = 1 - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{24} - \frac{x^5}{120} - \frac{x^6}{720}, \tag{18}$$

$$u_2'(x) = x + \frac{x^2}{2} - \frac{x^3}{24} - \frac{x^4}{64} - \frac{x^5}{640} - \frac{x^6}{7680} - \frac{x^7}{107520} - \frac{x^8}{2580480}, \tag{19}$$

$$u_3'(x) = 2x + \frac{3x^2}{2} + \frac{x^3}{8} - \frac{11x^4}{256} - \frac{53x^5}{5120} - \frac{23x^6}{40960} + \frac{3x^7}{32768} + \frac{575x^8}{22020096}$$
$$+\frac{297x^9}{73400320} + \frac{28031x^{10}}{59454259200} + \frac{71x^{11}}{1703116800} + \frac{43x^{12}}{14863564800}$$
$$+\frac{x^{13}}{6038323200} + \frac{x^{14}}{138726604800} + \frac{x^{15}}{6242697216000}, \tag{20}$$

Up to the $n'th$ order problem, and by solving these problems and substituting them in (??) , we have the HPM approximate solutions of order five

$$u_5(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} - \frac{42397x^7}{82575360} - \frac{103239x^8}{1174405120} - \frac{4027213x^9}{507343011840}$$

$$-\frac{16426909x^{10}}{81174881894400} + \frac{1159751881x^{11}}{42860337640243200} + \frac{3745787009x^{12}}{822918482692669440}$$

$$+\frac{29180882551x^{13}}{713196018333364684800} + \frac{41682262283x^{14}}{1996948851334211174400}$$

$$+\frac{2848076657x^{15}}{1797253966200790056960 0} - \frac{2620331837x^{16}}{4108009065601805844480 0}$$

$$-\frac{150638876327x^{17}}{2607216420301946109296640 0} - \frac{9022206041257x^{18}}{281579373392610179804037120 00}$$

$$-\frac{15360137046493x^{19}}{107000161889191868325534105 6000} - \frac{755701829501x^{20}}{142666882518922491100712140 8000}$$

$$-\frac{101051x^{21}}{629038888170276519936 0000} - \frac{28139x^{23}}{150969333160866364784 64000}$$

$$-\frac{19x^{24}}{347229466269992639004672 0000} - \frac{6079x^{22}}{166503640169427039682560 000}$$

$$-\frac{x^{25}}{126687552302824921497600 0000}. \tag{21}$$

This leads to $u = e^x$. as $\lim_{n\to\infty} \tilde{u}_n(x)$. Table 1 provides a comparative analysis of the absolute errors obtained using the HPM on various orders: third-order, fifth-order, and seventh-order. Absolute errors are assessed at different values of x, and presented in graphical form in Fig 1. For example, at $x = 0.2$, the third-order HPM shows an error of 7.40×10^{-8}, while the fifth order significantly reduces it to 9.83×10^{-8}, and the seventh order further refines the result to an absolute error of 9.83×10^{-12}. Similarly, for larger values of x, the error decreases consistently as the order increases. The substantial decrease in absolute error as the order increases highlights the effectiveness of HPM in approximating solutions. These results indicate that, for practical applications that need high precision, a seventh-order approximation or higher is recommended. The comparison clearly shows that increasing the order of HPM leads to a significant reduction in error. The third-order approximation provides a reasonable estimate, while the fifth-order greatly improves accuracy, and the seventh-order results in an almost negligible error. Therefore, for greater accuracy in solving non-linear problems, using a higher-order HPM is advantageous. To improve the accuracy of the HPM approximation, we employ MHPM. This method extends the original HPM to overcome its limitations, such as the number of terms and computational complexity. The techniques we will consider include the Padé approximation, Laplace transformation, and ultimately inverse Laplace, as follows.

$$L\left(\tilde{u}_1(t)\right) = -\frac{671}{512s^8} - \frac{263}{128s^7} - \frac{53}{32s^6} + \frac{1}{s^4} + \frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{s}, \tag{22}$$

The use of $s = \frac{1}{z}$, leads to

$$L\left(\tilde{u}_1(t)\right) = z + z^2 + z^3 + z^4 - \frac{53z^6}{32} - \frac{263z^7}{128} - \frac{671z^8}{512}, \tag{23}$$

The Pade approximation of order $\left[\frac{3}{3}\right]$ in terms of $x = \frac{1}{s}$, gives.

$$\left[\frac{3}{3}\right] = \frac{1}{\left(1 - \frac{1}{s}\right) s}, \tag{24}$$

The modified approximation solution $u_1(x) = e^x$ is obtained by applying the inverse Laplace transform to the $\left[\frac{3}{3}\right]$ Pade approximates.

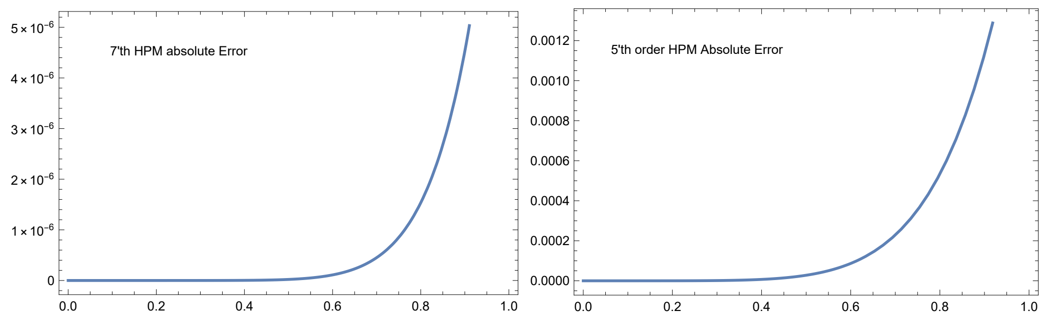

Figure 1: Absolute error resulted from the HPM procedure for example 1

3.2. Example 2 The second example considered in this study is the following non-linear DVIDE [46]

.

$$u(x) = u\left(\frac{x}{2}\right) - \frac{3}{2}\sin x - \frac{x}{2} - \cos\left(\frac{x}{2}\right) + \int_0^x u^2\left(\frac{s}{2}\right) ds, u(0) = 1. \tag{25}$$

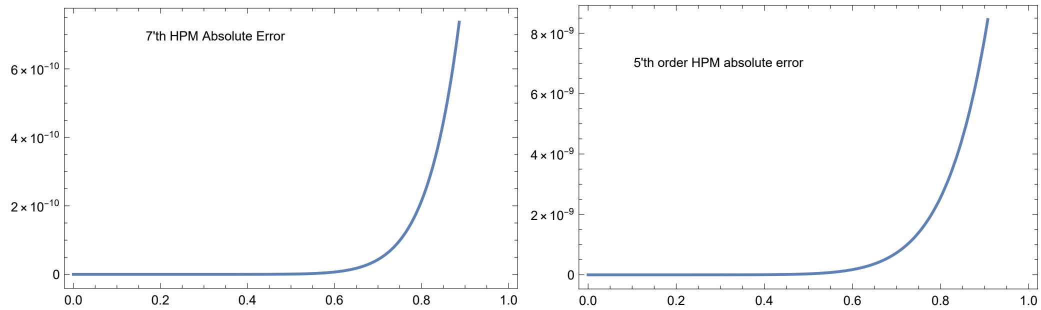

Figure 2: Absolute error resulted from the HPM procedure for example 2

Table 1: Numerical result of example 1

x	3th order HPM Absolute Error	5th order HPM Absolute Error	7th order HPM Absolute Error
0.0	0.0000000000	0.0000000000	0.00
0.2	7.40×10^{-8}	9.83×10^{-8}	5.14×10^{-12}
0.4	1.31×10^{-3}	6.93×10^{-6}	2.75×10^{-9}
0.6	7.33×10^{-3}	8.67×10^{-5}	1.10×10^{-7}
0.8	2.55×10^{-2}	5.34×10^{-4}	1.53×10^{-6}
1.0	6.85×10^{-2}	2.22×10^{-3}	1.19×10^{-5}

Following the same process in example one, we have the 5^{th} -order HPM approximate solution.

$$L\{u(x)\} = u'(x),$$

$$N(u) = -u\left(\frac{x}{2}\right) + \frac{3}{2}\sum_{k=0}^{6}(-1)^k\frac{x^{2k+1}}{(2k+1)!} + \frac{x}{2} + \sum_{k=0}^{6}(-1)^k\frac{\left(\frac{x}{2}\right)^{2k}}{(2k)!} - \int_0^x u^2\left(\frac{s}{2}\right)ds. \quad (26)$$

Then we construct the homotopy equation.

$$u'(x) - u_0(x) + p\left[-u\left(\frac{x}{2}\right) + \frac{3}{2}\sum_{k=0}^{6}(-1)^k\frac{x^{2k+1}}{(2k+1)!} + \frac{x}{2} + \sum_{k=0}^{6}(-1)^k\frac{\left(\frac{x}{2}\right)^{2k}}{(2k)!} - \int_0^x u^2\left(\frac{s}{2}\right)ds\right]. \quad (27)$$

Using HPM, we expand $u(x)$ as a perturbation series in p

$$u(x) = u_0(x) + pu_1(x) + p^2u_2(x) + p^3u_3(x) + p^4u_4(x) + p^5u_5(x) + \cdots \quad (28)$$

By substituting this series into the provided equation, we arrange the terms in accordance with the powers of p, and we have: The zeroth order approximation that is obtained by setting $p = 0$, is as follows

$$u_0'(x) = 0, \quad u_0(0) = 1, \quad (29)$$

which has the following solutions $u_0(x) = 1$. Moreover, the first, second and other orders of approximations will be obtained by substituting into the homotopy equations and matching the terms of power p to get the following.

$$u_1'(x) = \frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{384}+\frac{x^6}{46080}-\frac{x^8}{10321920}+\frac{x^{10}}{3715891200}-\frac{3}{2}\left(x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}+\frac{x^9}{362880}\right), \tag{30}$$

$$u_2'(x) = -\frac{x^2}{8}-\frac{5x^3}{64}+\frac{5x^4}{768}+\frac{19x^5}{12288}-\frac{7x^6}{184320}-\frac{383x^7}{41287680}+\frac{5x^8}{33030144} +\frac{307x^9}{9512681472}-\frac{97x^{10}}{237817036800}-\frac{6143x^{11}}{83711596953600}+\frac{x^{12}}{502269581721600}, \tag{31}$$

$$u_3'(x) = -\frac{x^3}{192}-\frac{47x^4}{12288}+\frac{329x^5}{122880}-\frac{941x^6}{4718592}-\frac{22217x^7}{165150720}+\frac{471809x^8}{84557168640}+\frac{1963291x^9}{761014517760} -\frac{22998401x^{10}}{487049291366400}-\frac{47100961x^{11}}{1785847401676800}+\frac{37961641x^{12}}{164583696538533888}+\frac{9442463747x^{13}}{5348970137502351360 0} -\frac{25965661x^{14}}{340389008750149632 00}-\frac{1419107x^{15}}{1687625794584576000}+\frac{24349x^{16}}{1800134180890214400 0} +\frac{639917x^{17}}{229517108063502336000}-\frac{17x^{18}}{115722911628656640 00}-\frac{2085431x^{19}}{337725745297071538176000} +\frac{51622011x^{21}}{5184386440963817472 0000}+\frac{x^{22}}{20903446129966112047104 0000} -\frac{1}{228113003402407968768 0000}+\frac{x^{23}}{16117552368400537441271808 0000}, \tag{32}$$

Up to the $n'th$ order problem, and by solving these problems and substituting them in (??) , we have the HPM approximate solutions of order five

$$\tilde{u}(x) = 1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\frac{x^7}{5284823040}+\frac{133153x^8}{5368709120}-\frac{5419361x^9}{389639433093120} -\frac{876413917x^{10}}{3117115464744960}+\frac{6295085773x^{11}}{8777797148721807360}-\frac{1814058456809x^{12}}{21066713156932337664 00} -\frac{130118761434241x^{13}}{70110021386270819745792 00}+\frac{455479147538587x^{14}}{13087203992103886352547840 0}$$

Table 2: Numerical result of example 2

x	3th order HPM Absolute Error	5th order HPM Absolute Error	7th order HPM Absolute Error
0.0	0.0000000000	0.0000000000	0.00
0.2	5.41×10^{-9}	5.11×10^{-15}	5.14×10^{-12}
0.4	1.52×10^{-7}	3.88×10^{-12}	5.26×10^{-14}
0.6	7.59×10^{-7}	1.71×10^{-10}	6.81×10^{-12}
0.8	5.213×10^{-7}	2.55×10^{-9}	2.15×10^{-10}
1.0	9.88×10^{-6}	2.15×10^{-8}	3.12×10^{-9}

$$+\frac{19978465659899683x^{15}}{1370587181718516098012282880000}-\frac{716086078759136753x^{16}}{964893375929835333000647147520000} \tag{33}$$

This converges to the exact solution $\tilde{u}(x) = \cos x$, as in $\lim_{n\to\infty} \tilde{u}_i(t)$. Table 2 provides a comparative analysis of the absolute errors obtained using the HPM on various orders: third-order, fifth-order, and seventh-order. Absolute errors are assessed at different values of x and plotted in Fig 2. For example, at $x = 0.2$, the third-order HPM shows an error of 5.41×10^{-9}, whereas the fifth order reduces it significantly to 5.11×10^{-15}, and the seventh order further refines the result to an absolute error of zero. Similarly, for larger values of x, the error decreases consistently as the order increases. The substantial decrease in absolute error as the order increases highlights the effectiveness of HPM in approximating solutions. These results indicate that, for practical applications that need high precision, a seventh-order approximation or higher is recommended. The comparison clearly shows that increasing the order of HPM leads to a significant reduction in error. The third-order approximation provides a reasonable estimate, while the fifth-order greatly improves accuracy, and the seventh-order results in an almost negligible error. Therefore, for greater accuracy in solving non-linear problems, using a higher-order HPM is advantageous. Therefore, to enhance the precision of the HPM procedure, we will begin by applying the Laplace transformation to the initial terms in the HPM series solutions. Next, we will utilize the Pade approximants and, finally, we will conclude by implementing the inverse Laplace transformation. The process is described below.

$$L\left(\tilde{u}_1(t)\right) = \frac{1}{34359738368s^{10}} + \frac{1}{s^9} - \frac{1}{s^7} + \frac{1}{s^5} - \frac{1}{s^3} + \frac{1}{s}, \tag{34}$$

Use $s = \frac{1}{z}$, leads to

$$L\left(\tilde{u}_2(t)\right) = z - z^3 + z^5 - z^7 + z^9 + \frac{z^{10}}{34359738368}, \tag{35}$$

The Pade approximates of order $\left[\frac{4}{4}\right]$ in term of $x = \frac{1}{s}$, gives

$$\left[\frac{3}{3}\right] = \frac{1}{\left(1 + \frac{1}{s^2}\right) s}. \tag{36}$$

The exact solutions $\tilde{u}(x) = \cos x$ are obtained by applying the inverse Laplace transform to the $\left[\frac{3}{3}\right]$ Pade approximate.

4. Conclusions

The present study holds considerable importance as it proposes a new and effective analytical method for addressing Delay Volterra Integro-Differential Equations (DVIDEs), which are critical for modeling systems influenced by memory and time-delay effects in real-world scenarios. By enhancing the Homotopy Perturbation Method (HPM) and incorporating both the Laplace transformation and Padé approximants, the suggested approach successfully overcomes common challenges associated with nonlinearity and limited convergence in conventional methods. This combined technique not only streamlines the solution process but also improves accuracy, minimizes computational workload, and achieves rapid convergence with fewer terms. Consequently, it provides a dependable and practical solution for researchers and engineers dealing with mathematical models across various domains such as biological systems, control theory, population models, and engineering problems involving time delays and hereditary behavior. This work advances the field of analytical techniques for integro-differential equations and opens new avenues for tackling more complex or multidimensional problems.

References

[1] Vasile Marinca, Nicolae Herişanu, and Iacob Nemeş. Optimal homotopy asymptotic method with application to thin film flow. *Open Physics*, 6(3):648–653, 2008.

[2] Vasile Marinca and Nicolae Herişanu. Application of optimal homotopy asymptotic method for solving nonlinear equations arising in heat transfer. *International communications in heat and mass transfer*, 35(6):710–715, 2008.

[3] Praveen Agarwal, Muhammad Akbar, Rashid Nawaz, and Mohamed Jleli. Solutions of system of volterra integro-differential equations using optimal homotopy asymptotic method. *Mathematical Methods in the Applied Sciences*, 44(3):2671–2681, 2021.

[4] N Ratib Anakira, AK Alomari, and Ishak Hashim. Application of optimal homotopy asymptotic method for solving linear delay differential equations. In *AIP Conference Proceedings*, volume 1571, pages 1013–1019. American Institute of Physics, 2013.

[5] N Ratib Anakira, AK Alomari, and Ishak Hashim. Numerical scheme for solving singular two-point boundary value problems. *Journal of Applied Mathematics*, 2013(1):468909, 2013.

[6] Remus-Daniel Ene and Nicolina Pop. Optimal homotopy asymptotic method for an anharmonic oscillator: application to the chen system. *Mathematics*, 11(5):1124, 2023.

[7] Ali F Jameel, Akram H Shather, NR Anakira, AK Alomari, and Azizan Saaban. Comparison for the approximate solution of the second-order fuzzy nonlinear differential equation with fuzzy initial conditions. *Mathematics and Statistics*, 8(5):527–534, 2020.

[8] Taiye Oyedepo Ayinde, Matthew Olanrewaju Oluwayemi, Muhammed Abdullahi, Johnson Adekunle Osilagun, and Lukman Olalekan Ahmed. Homotopy perturbation technique for fractional volterra and fredholm integro differential equations. In *2023 International Conference on Science, Engineering and Business for Sustainable Development Goals (SEB-SDG)*, volume 1, pages 1–6. IEEE, 2023.

[9] Samad Noeiaghdam, Aliona Dreglea, Jihuan He, Zakieh Avazzadeh, Muhammad Suleman, Mohammad Ali Fariborzi Araghi, Denis N Sidorov, and Nikolai Sidorov. Error estimation of the homotopy perturbation method to solve second kind volterra integral equations with piecewise smooth kernels: Application of the cadna library. *Symmetry*, 12(10):1730, 2020.

[10] Moustafa El-Shahed. Application of he's homotopy perturbation method to volterra's integro-differential equation. *International Journal of Nonlinear Sciences and Numerical Simulation*, 6(2):163–168, 2005.

[11] Nidal Anakira, Adel Almalki, MJ Mohammed, Safwat Hamad, Osoma Oqilat, and Ala Amourah. Analytical approaches for computing exact solutions to system of volterra integro-differential equations. *WSEAS Trans. Math*, 23:400–407, 2024.

[12] S Alao, FS Akinboro, FO Akinpelu, and R Oderinu. Numerical solution of integro-differential equation using adomian decomposition and variational iteration methods. *IOSR Journal of Mathematics*, 10(4):18–22, 2014.

[13] Mehdi Dehghan, Mohammad Shakourifar, and Asgar Hamidi. The solution of linear and nonlinear systems of volterra functional equations using adomian–pade technique. *Chaos, Solitons & Fractals*, 39(5):2509–2521, 2009.

[14] Nidal Anakira, Gada Bani-Hani, Osama Ababneh, Ali Jameel, and Khamis Al-Kalbani. Modified adomian decomposition method for solving volterra integro-differential equations. In *The International Arab Conference on Mathematics and Computations*, pages 335–341. Springer, 2023.

[15] MA Fariborzi Araghi and Sh Sadigh Behzadi. Solving nonlinear volterra— fredholm integro-differential equations using the modified adomian decomposition method. *Computational Methods in Applied Mathematics*, 9(4):321–331, 2009.

[16] Xindong Zhang, Bo Tang, and Yinnian He. Homotopy analysis method for higher-order fractional integro-differential equations. *Computers & mathematics with applications*, 62(8):3194–3203, 2011.

[17] AF Jameel, Azizan Saaban, SA Altaie, NR Anakira, AK Alomari, and N Ahmad. Solving first order nonlinear fuzzy differential equations using optimal homotopy asymptotic method. *International Journal of Pure and Applied Mathematics*, 118(1):49–64, 2018.

[18] Ali F Jameel, Akram H Shather, NR Anakira, AK Alomari, and Azizan Saaban. Comparison for the approximate solution of the second-order fuzzy nonlinear differential equation with fuzzy initial conditions. *Mathematics and Statistics*, 8(5):527–534,

2020.

[19] Nidal Ratib Anakira, AF Jameel, Abedel-Karrem Alomari, Azizan Saaban, Mohammad Almahameed, and IHM Hashim. Approximate solutions of multi-pantograph type delay differential equations using multistage optimal homotopy asymptotic method. *Journal of Mathematical and Fundamental Sciences*, 2018.

[20] FM Alharbi. Numerical solutions of an integro-differential equation with smooth and singular kernels. *International Journal of Mathematical Analysis*, 13(12):573–586, 2019.

[21] Ivo Babuška, Fabio Nobile, and Raúl Tempone. A stochastic collocation method for elliptic partial differential equations with random input data. *SIAM Journal on Numerical Analysis*, 45(3):1005–1034, 2007.

[22] H Ghaneai, MM Hosseini, and Syed Tauseef Mohyud-Din. Modified variational iteration method for solving a neutral functional-differential equation with proportional delays. *International Journal of Numerical Methods for Heat & Fluid Flow*, 22(8):1086–1095, 2012.

[23] TL Yookesh, ED Boobalan, and TP Latchoumi. Variational iteration method to deal with time delay differential equations under uncertainty conditions. In *2020 International Conference on Emerging Smart Computing and Informatics (ESCI)*, pages 252–256. IEEE, 2020.

[24] Ahmet Yıldırım. Applying heś variational iteration method for solving differential-difference equation. *Mathematical Problems in Engineering*, 2008(1):869614, 2008.

[25] Ali Fareed Jameel, Nidal Ratib Anakira, AK Alomari, M Al-Mahameed, and Azizan Saaban. A new approximate solution of the fuzzy delay differential equations. *International Journal of Mathematical Modelling and Numerical Optimisation*, 9(3):221–240, 2019.

[26] Brijendra Kumar Chaurasiya and Avadhesh Kumar. Analysis of positive solutions for the fractional derivative with delay and integral boundary conditions. *Gulf Journal of Mathematics*, 19(2):315–327, 2025.

[27] Zhor Mellah, El Bekkaye Mermri, and Mohammed Bouchlaghem. A numerical approach for solving a semilinear obstacle problem on the boundary. *Gulf Journal of Mathematics*, 19(2):20–35, 2025.

[28] Mouhssine Zakaria and Abdelaziz Moujahid. Computational spectral method for solving two-dimensional riesz multi-term time-fractional diffusion equation. *Gulf Journal of Mathematics*, 19(2):181–199, 2025.

[29] Ali Jameel, NR Anakira, AK Alomari, Ishak Hashim, and MA Shakhatreh. Numerical solution of n'th order fuzzy initial value problems by six stages. *Journal of nonlinear science and applications*, 2016.

[30] N Rabit Anakira, AK Alomari, AF Jameel, and Ishak Hashim. Multistage optimal homotopy asymptotic method for solving initial-value problems. *J. Nonlinear Sci. Appl*, 9(4):1826–1843, 2016.

[31] AF Jameel, NR Anakira, MM Rashidi, AK Alomari, A Saaban, and MA Shakhatreh. Differential transformation method for solving high order fuzzy initial value problems. *Italian Journal of Pure and Applied Mathematics*, 39:194–208, 2018.

[32] Erkan Cimen and Sabahattin Yatar. Numerical solution of volterra integro-differential equation with delay. *J. Math. Comput. Sci*, 20(3):255–263, 2020.

[33] Rohul Amin, Kamal Shah, Muhammad Asif, and Imran Khan. Efficient numerical technique for solution of delay volterra-fredholm integral equations using haar wavelet. *Heliyon*, 6(10), 2020.

[34] Erkan Cimen and Sabahattin Yatar. Numerical solution of volterra integro-differential equation with delay. *J. Math. Comput. Sci*, 20(3):255–263, 2020.

[35] Emiko Ishiwata and Yoshiaki Muroya. On collocation methods for delay differential and volterra integral equations with proportional delay. *Frontiers of mathematics in China*, 4(1):89–111, 2009.

[36] Aidouni Yamina. *Numerical Solution of Integro-Delay Differential Equations on a Half Line.* PhD thesis, University BBA, 2023.

[37] Ghada H Ibraheem, Mustafa Turkyilmazoglu, and MA Al-Jawary. Novel approximate solution for fractional differential equations by the optimal variational iteration method. *Journal of Computational Science*, 64:101841, 2022.

[38] Adedapo Ismaila Alaje, Morufu Oyedunsi Olayiwola, Kamilu Adewale Adedokun, Joseph Adeleke Adedeji, and Asimiyu Olamilekan Oladapo. Modified homotopy perturbation method and its application to analytical solitons of fractional-order korteweg–de vries equation. *Beni-Suef University Journal of Basic and Applied Sciences*, 11(1):139, 2022.

[39] Adedapo Ismaila Alaje and Morufu Oyedunsi Olayiwola. A fractional-order mathematical model for examining the spatiotemporal spread of covid-19 in the presence of vaccine distribution. *Healthcare Analytics*, 4:100230, 2023.

[40] Morufu Oyedunsi Olayiwola and Adedapo Ismaila Alaje. Mathematical analysis of intrahost spread and control of dengue virus: unraveling the crucial role of antigenic immunity. *Franklin Open*, 7:100117, 2024.

[41] Mutairu Kayode Kolawole, Morufu Oyedunsi Olayiwola, Adedapo Ismaila Alaje, Hammed Ololade Adekunle, and Kazeem Abidoye Odeyemi. Conceptual analysis of the combined effects of vaccination, therapeutic actions, and human subjection to physical constraint in reducing the prevalence of covid-19 using the homotopy perturbation method. *Beni-Suef University Journal of Basic and Applied Sciences*, 12(1):10, 2023.

[42] M34108671396 Turkyilmazoglu. Is homotopy perturbation method the traditional taylor series expansion. *Hacettepe journal of mathematics and statistics*, 44(3):651–657, 2015.

[43] Mustafa Turkyilmazoglu. Optimization by the convergence control parameter in iterative methods. *Journal of Applied Mathematics and Computational Mechanics*, 23(2), 2024.

[44] Shadi Al-Ahmad, Mustafa Mamat, Nidal Anakira, and Rami Alahmad. Modified differential transformation method for solving classes of non-linear differential equations. *TWMS Journal of Applied and Engineering Mathematics*, 2022.

[45] Shadi Al-Ahmad, Nidal Ratib Anakira, Mustafa Mamat, Ali Fareed Jameel, Rami Alahmad, and Abedel-Karrem Alomari. Accurate approximate solution of classes

of boundary value problems using modified differential transform method. *TWMS Journal of Applied and Engineering Mathematics*, 2022.

[46] Fazli Hadi, Rohul Amin, Ilyas Khan, J Alzahrani, KS Nisar, Amnah S Al-Johani, and Elsayed Tag Eldin. Numerical solutions of nonlinear delay integro-differential equations using haar wavelet collocation method. *Fractals*, 31(02):2340039, 2023.