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Stability Results for Stochastic Delay Differential Equations in the Framework of Conformable Fractional Derivatives

Muhammad Imran Liaqat[1], Dumitru Baleanu[2], Majeed Ahmad Yousif[3], Pshtiwan Othman Mohammed[4,5,6,*]

[1] *Abdus Salam School of Mathematical Sciences, Government College University, 68-B, New Muslim Town, Lahore 54600, Pakistan*
[2] *Department of Computer Science and Mathematics, Lebanese American University, Beirut 11022801, Lebanon*
[3] *Department of Mathematics, College of Education, University of Zakho, Zakho 42002, Iraq*
[4] *Research and Development Center, University of Sulaimani, Sulaymaniyah 46001, Iraq*
[5] *Research Center, University of Halabja, Halabja 46018, Iraq*
[6] *Associate Member, Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio, Italy*

Abstract. Well-Posedness is essential in various scientific and engineering fields, including physics, engineering, biological sciences, economics, and environmental science. Well-posedness ensures that the mathematical model corresponds to a physically meaningful situation. Without well-posedness, the solution might not make sense in the context of the problem being modeled. The concept of well-posedness refers to certain desirable properties that a differential equation must satisfy, which are existence, uniqueness, and continuous dependency. Regularization is an additional feature of the solution of the differential equation, such as the smoothness of the solution. Stability theory is one of the indispensable qualitative concepts of dynamical systems. We established results about the well-posedness, regularity, and Ulam-Hyers stability of solutions to conformable fractional stochastic delay differential equations. First, we discussed the results about the existence and uniqueness of solutions when the global and local Lipschitz conditions of the coefficients are satisfied. Second, we demonstrated the results about continuously depending on the fractional order ξ and initial values under the global Lipschitz condition of coefficients. Thirdly, we constructed results regarding regularity and Ulam-Hyers stability, and two examples that demonstrate our results are presented. The main part of the proof made use of the truncation process, the Banach fixed point theorem, Itô isometry, temporally weighted norm, Grönwall's inequality, and Hölder's inequality.
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[*]Corresponding author.
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Email addresses: imran_liaqat_22@sms.edu.pk (M. I. Liaqat), dumitru.baleanu@lau.edu.lb (D. Baleanu), majeed.yousif@uoz.edu.krd (M. A. Yousif), pshtiwansangawi@gmail.com (P. O. Mohammed)

1. Introduction

Fractional calculus (FC) is a branch of mathematical analysis that extends the concepts of differentiation and integration to non-integer orders. Instead of dealing exclusively with integer-order derivatives and integrals, FC involves operations of differentiation and integration of fractional orders.

The history of FC dates back to the 17th century, with the concept of fractional derivatives and integrals emerging alongside the development of traditional calculus. Here is a brief overview of the history of FC [1–3]:

i. Beginnings (17th Century)

- Leibniz and L'Hôpital: Leibniz introduced the idea of non-integer order derivatives in correspondence with L'Hôpital, pondering the concept of a half-order derivative.

ii. Exploration and Early Development (18th and 19th Centuries)

- Euler: Leonhard Euler made early contributions to the theory of FC, investigating the properties of the gamma function and its connection to FC.
- Liouville: Joseph Liouville expanded upon Euler's work, developing integral representations for fractional derivatives and integrals.
- Grnwald and Letnikov: Independently, Grünwald and Letnikov formulated discrete versions of fractional derivatives, laying down the groundwork for numerical methods in FC.
- Riemann-Liouville FC: Bernhard Riemann and Joseph Liouville developed the Riemann-Liouville approach to FC, providing a rigorous mathematical framework for fractional derivatives and integrals.

iii. Consolidation and Formalization (20th Century)

- Caputo FC: Michele Caputo modified the Riemann-Liouville approach known as the Caputo fractional derivative, which is widely used due to its compatibility with initial value problems.
- Modern Developments: Throughout the 20th century, mathematicians continued to explore and develop the theory of FC, advancing its applications in various scientific disciplines.

iv. Resurgence and Application (Late 20th Century to Present):

- Resurgence of Interest: FC experienced a resurgence of interest in the late 20th century and continues to be an active area of research, driven by its applications in physics, engineering, biology, finance, and signal processing.
- Applications: FC finds applications in modeling anomalous transport phenomena, viscoelastic materials, biological processes, financial markets, signal processing, and many other fields.

FC offers several advantages over traditional integer-order calculus, particularly when dealing with complex, real-world systems. Here are some of the key benefits [4–7]:

i. Modeling Complex Phenomena: FC provides a powerful framework for modeling and analyzing complex phenomena that exhibit non-local interactions, memory effects, and fractal properties. Systems with long-range dependencies, anomalous diffusion, and hereditary behaviors can be accurately described using fractional differential equations.

ii. Memory and Non-local Effects: FC captures memory effects and non-local interactions that are not adequately addressed by classical calculus. Fractional derivatives and integrals allow for the incorporation of memory kernels and long-range dependencies, leading to more realistic models of dynamic systems.

iii. Flexibility in Representation: FC offers a flexible and versatile approach to representing and analyzing dynamic systems. The fractional order provides a continuous spectrum of differentiation and integration, allowing for a finer-grained characterization of system dynamics compared to integer-order models.

iv. Fractional Dynamics: FC enables the study of fractional-order dynamical systems, which exhibit rich and diverse behaviors. Fractional dynamics encompasses phenomena such as fractional-order chaos, fractal patterns, and anomalous transport processes, providing insights into the underlying mechanisms governing complex systems.

v. Improved Accuracy: In many cases, FC provides more accurate descriptions of real-world phenomena compared to classical integer-order models. By capturing memory effects and long-range dependencies, FC can better match experimental observations and empirical data, leading to more reliable predictions and interpretations.

vi. Generalization of Classical Calculus: FC generalizes classical calculus by extending differentiation and integration to non-integer orders. This allows for a seamless transition between classical and FC, providing a unified framework for addressing a wide range of problems in mathematics, science, and engineering.

vii. Applications in Various Fields: FC has diverse applications across numerous fields, including physics, engineering, biology, finance, and signal processing. It is used to model and analyze phenomena such as diffusion processes, viscoelasticity, population dynamics, financial time series, and medical imaging, among others.

viii. Novel Mathematical Properties: FC possesses unique mathematical properties that distinguish it from classical calculus. It introduces fractional-order operators, fractional Taylor series, and fractional boundary value problems, enriching the theoretical toolkit of mathematicians and researchers.

Fractional derivatives serve as a foundational element of FC, providing a robust mechanism for modeling and analyzing systems that exhibit non-locality or memory-dependent behavior. In contrast to classical integer-order derivatives, which capture only the instantaneous rate of change, fractional derivatives account for a function's history over an interval, effectively incorporating memory into the system's dynamics. There are several formulations of fractional-order derivatives, each with distinct definitions and characteristics. Commonly utilized forms include the Riemann-Liouville, Caputo, Grünwald-Letnikov, Caputo-Fabrizio, and conformable operators [8–15]. The choice of a particular definition depends on the nature of the application and the mathematical properties required for accurate modeling.

Conformable fractional derivatives (Con-FrD) offer a practical and intuitive framework for embedding memory effects into differential equations. They preserve several key characteristics of classical derivatives, which simplifies their application, while still providing the adaptability inherent in FC. This balance makes Con-FrD especially useful for modeling real-world processes across diverse disciplines such as physics, engineering, biology, and control theory.

The Con-FrD of order ξ for a function $\Psi(\tau)$ is expressed as [16]:

$$\mathrm{D}_{\tau}^{\xi}\Psi(\tau) = \lim_{\epsilon\to 0}\frac{\Psi^{\lceil\xi\rceil-1}(\tau+\epsilon\tau^{\lceil\xi\rceil-\xi}) - \Psi^{\lceil\xi\rceil-1}(\tau)}{\epsilon}, \tag{1}$$

here, the Con-FrD with respect to time is represented by D_{τ}^{ξ}; $\lceil\xi\rceil$ denotes the smallest integer not less than ξ, where $\varkappa \in \mathscr{N}$ and satisfies $\varkappa - 1 < \xi \leq \varkappa$ for $\tau > 0$.

In a specific case, if $0 < \xi \leq 1$, we then obtain

$$\mathrm{D}_{\tau}^{\xi}\Psi(\tau) = \lim_{\epsilon\to 0}\frac{\Psi(\tau+\epsilon\tau^{1-\xi}) - \Psi(\tau)}{\epsilon},\ \tau > 0. \tag{2}$$

If $\Psi(\tau)$ is ξ-differentiable in some $(0, \Lambda)$, $\Lambda > 0$ and $\lim_{\tau\to 0^{+}} \Psi^{\xi}(\tau)$ exists, then define $\Psi^{(\xi)}(0) = \lim_{\tau\to 0^{+}} \Psi^{(\xi)}(\tau)$. An integral of a function $\Psi(\tau)$ that is conformable fractional $\Psi(\tau)$ starting from $\tilde{\alpha} \geq 0$ is defined as:

$$\mathfrak{I}_{\xi}^{\tilde{\alpha}}(\Psi)(\tau) = \int_{\tilde{\alpha}}^{\tau} \frac{\Psi(\mathrm{s})}{\left(\mathrm{s}-\tilde{\alpha}\right)^{1-\xi}} d\mathrm{s},\ \xi \in (0,\ 1]. \tag{3}$$

The Con-FrD offers a simplified approach to fractional differentiation, easing computational complexity while retaining essential characteristics of traditional derivatives. The Con-FrD offers several advantages compared to other types of fractional derivatives. Here are some of the key benefits [17–20]:

i. Compatibility with Stochastic Processes: Con-FrD maintains compatibility with stochastic processes commonly encountered in real-life applications, such as Brownian motion and Lévy processes. This compatibility ensures that models incorporating Con-FrD accurately capture the stochastic nature of the underlying phenomena.

ii. Conformity with Classical Calculus: The properties of Con-FrD closely resemble those of classical derivatives, making them more intuitive and easier to apply in the context of stochastic calculus. This conformity facilitates the development and analysis of fractional stochastic differential equations (FSDEs), as it allows researchers to leverage their existing knowledge and techniques from classical calculus.

iii. Simplicity and Computational Efficiency: Con-FrD is simpler to work with and manipulate compared to other fractional derivatives, which can lead to more efficient computational algorithms for solving FSDEs. This simplicity can reduce the computational cost associated with simulating stochastic processes involving FC.

iv. Mathematical Properties: Con-FrD possesses certain mathematical properties, such as the fractional Itô formula, which can simplify the analysis of FSDEs and facilitate the derivation of important results. These properties provide a solid theoretical foundation for studying the behavior of stochastic processes involving FC.

v. Generalization: Con-FrD offers a generalized framework for modeling stochastic processes with non-integer orders of differentiation. This generalization allows researchers to explore a broader range of stochastic systems, including those with anomalous diffusion or long-range dependencies. Conformable derivatives provide a versatile tool for capturing complex dynamics in various fields.

FSDEs constitute a class of differential equations that combine stochastic elements with FC. They serve to describe systems demonstrating randomness, generally represented by stochastic processes, alongside non-local or memory-dependent characteristics embodied by fractional derivatives. FSDEs play a crucial role across diverse scientific and engineering domains for modeling complex phenomena involving both randomness and memory effects. Delay fractional stochastic differential equations (DF-SDEs) are a class of differential equations that incorporate both delay effects and FC, along with stochastic components, to model systems influenced by randomness. DF-SDEs combine the advantages of FC, delay differential equations, and stochastic differential equations. This combination is crucial for accurately capturing the dynamics of systems that exhibit memory effects, delays, and randomness. By using DF-SDEs, researchers and engineers can develop more realistic models, leading to better predictions, control strategies, and understanding of complex phenomena across various fields.

DF-SDEs offer several advantages in understanding various phenomena across different fields due to their ability to capture memory effects, delays, and stochastic influences. Here are some key advantages [21, 22]:

i. Modeling Memory Effects

- DF-SDEs can effectively model systems where future states depend not only on the current state but also on past states. This is particularly useful for systems with long memory or hereditary properties. For example, in finance, asset prices often exhibit long-range dependence and volatility clustering. DF-SDEs can model these phenomena more accurately than traditional models, capturing the impact of past market events on current price dynamics.

ii. Incorporating Time Delays

- Time delays are common in many real-world systems, such as biological systems where there are time lags in reaction processes or in engineering where control actions have delayed effects. DF-SDEs explicitly incorporate these delays, leading to more realistic models. For instance, in population dynamics, the birth rate may depend on the population size at some previous time due to gestation periods. DF-SDEs can model this by including a delay term, providing a more accurate representation of population changes over time.

iii. Handling Stochastic Fluctuations

- Many systems are subject to random perturbations or noise. DF-SDEs incorporate stochastic processes, allowing for the modeling of randomness alongside deterministic trends and memory effects. For instance, in climate modeling, weather conditions exhibit both deterministic seasonal trends and stochastic variations due to unpredictable environmental factors. DF-SDEs can capture both aspects, improving climate prediction models.

iv. Improved Accuracy in Complex Systems

- By integrating fractional derivatives, delays, and stochastic components, DF-SDEs offer a more comprehensive and accurate description of complex systems than traditional differential equations. For example, in epidemiology, the spread of diseases can be influenced by past infection rates (memory), delays in the incubation period, and random contact rates. DF-SDEs can simultaneously account for these factors, leading to better predictions of disease outbreaks and control strategies.

v. Flexibility in Model Formulation

- DF-SDEs provide a flexible framework for model formulation. The orders of fractional derivatives and the lengths of delays can be adjusted to fit empirical data more closely, allowing for tailored models that match specific system behaviors. For example, in mechanical systems, different materials exhibit varying degrees of viscoelasticity and response times. DF-SDEs can be customized to model the specific viscoelastic properties and response delays of different materials, improving the design and analysis of mechanical structures.

vi. Capturing Anomalous Diffusion

- Anomalous diffusion, where the rate of diffusion deviates from classical behavior, can be effectively modeled using DF-SDEs. This is crucial in fields like materials science and biology, where such phenomena are common. For instance, in porous media, the diffusion of fluids can be anomalous due to the complex structure of the medium. DF-SDEs can model anomalous diffusion more accurately than standard models, aiding in the understanding and prediction of fluid flow in such media.

vii. Enhanced Predictive Power

- The combined effects of memory, delays, and randomness in DF-SDEs often result in models with superior predictive power compared to traditional models. This is particularly valuable in fields where accurate forecasting is critical. For example, in financial markets, accurate prediction of asset prices and risk management are crucial. DF-SDEs can provide better forecasts by incorporating historical data, delays in market reactions, and stochastic volatility.

A central concept in mathematics and scientific computing is ensuring the well-posedness of differential equations. Existence, uniqueness, and continuous dependency are three desirable properties that a differential equation must meet in order to be considered well-posed. Differential equations are excellent for practical usage in a variety of domains, including physics, engineering, and biological sciences, since well-posedness guarantees that the equations are solvable, unique, and stable. Differential equations must be well-posed in order to have meaningful solutions; otherwise, the solution may not make sense in relation to the situation being described. If there is at least one function that can satisfy the given differential equation, the differential equation is said to have the existence property. The relationship between variables and time or other factors is described by differential equations. We can obtain functions that forecast the behavior of dynamic systems by solving these equations. Differential equations, for example, are used to simulate electrical circuits, motion, and heat transport in physics. We are unable to forecast these phenomena if there are no solutions to differential equations. In scientific modeling and engineering applications, uniqueness is crucial for consistency and predictability. It is guaranteed that a well-posed problem has a unique solution. Ambiguity can result from a differential equation that is not well-posed, as it may have several solutions or none at all. The solution must continuously depend on initial conditions, boundary conditions, and additional data to be well-posed. Modest modifications to the input data ought to yield modest modifications to the solution. For stability and robustness in real-world applications, this attribute is essential. Regularization is incorporating extra features, like solution smoothness, into the differential equation solution.

Stability theory stands out as a significant qualitative concept in dynamical systems. Consequently, it has garnered heightened attention across various fields of study and practical applications. Understanding the behavior of solutions under small perturbations requires an understanding of the Ulam-Hyers (UH) stability of the differential equations.

Numerous researchers have established various results for FSDEs. For example, Kahouli et al. [23] proved the stability of FSDEs. Ali et al. [24] demonstrated the EU of solutions using Picard's iteration method for coupled systems of FSDEs. Raheem et al. [25] investigated the Existence and uniqueness (EU) of mild solutions and controllability by employing sectorial operators for FSDEs involving the Ψ-Hilfer fractional derivative. Ramkumar et al. [26] studied mild solutions and optimal control approaches for FSDEs in Hilbert spaces. Djaouti and Liaqat [27] presented several important results regarding the solutions of FSDEs under the Caputo derivative. Ali et al. [28] demonstrated the EU for Caputo–Fabrizio FSDEs driven by multiplicative white noise. They also verified the convergence of the Euler–Maruyama approach. Lavanya and Vadivoo [29] derived results concerning the controllability of Caputo–Hadamard FSDEs. The EU was further confirmed using the Banach contraction principle. Moualkia and Xu [30] investigated the EU of solutions to FSDEs with variable order. They additionally obtained solutions via Picard iterations and introduced new sufficient conditions. Liping et al. [31] examined a novel financial chaotic model formulated as an FSDE involving the Atangana–Baleanu operator. The authors numerically solved the model and provided graphical results under various scenarios. Abouagwa et al. [32] explored the EU for FSDEs with impulses. Asadzade and Mahmudov [33] addressed the existence of a unique solution and analyzed the finite-time stability of a class of FSDEs.

In this study, we have established the EU, continuous dependence (Con-D), regularity, and UH stability of the solutions to DF-SDEs in the framework of Con-FrD. More specifically, there have been two steps to proving EU: first, we have demonstrated the EU via the global Lipschitz condition (GLC) of the coefficients. In the second stage, using a truncation approach, we have demonstrated the EU according to the local Lipschitz condition (LLC) of the coefficients. We have also shown that, under the GLC of the coefficients, solutions exhibit Con-D on the initial value and on the fractional exponent ξ of Con-FrD. Next, we have established the result of regularity in time for the solutions to the DF-SDEs. Using the generalized Grönwall inequality, we have presented the result of the UH stability for the solutions of the DF-SDEs. Lastly, we have provided two examples to help illustrate the results we have established. The main part of the proof of our established results has involved the use of the truncation process, Itô isometry (I-Is), Grönwall's inequality (G-I), temporally weighted norm, and Hölder's inequality (Höld-I).

The contributions of this research are outlined as follows:

i. In contrast to the findings in [23–33], our research has yielded results concerning the EU, regularity, Con-D on both the initial condition and fractional component, and UH stability in the context of Con-FrD.

ii. In contrast to [23–33], we established the result of EU by using a truncation approach based on the LLC of the coefficients.

We examined the following DF-SDEs of order $\frac{1}{2} < \xi < 1$ in this research driven by Brownian motions, which serves as the generalization of the classical stochastic differential equation.

$$\mathrm{D}_{\tau}^{\xi}\mathrm{x}(\tau) = \mathrm{V}(\tau, \mathrm{x}(\tau), \mathrm{x}(\tau - \mathrm{m})) + F(\tau, \mathrm{x}(\tau), \mathrm{x}(\tau - \mathrm{m}))\frac{d\mathrm{w}_{\tau}}{d\tau}, \ \forall \tau \in [0, \hbar], \tag{4}$$

Here, V and F are measurable functions from $[0,\ \hbar] \times (\mathbb{R}^{\mathrm{r}})^2$ into $\mathbb{R}^{\mathrm{r}}$. The process w_{τ}, indexed over $\tau \in [0, \infty)$, denotes a standard scalar Wiener process on the complete filtered probability space $(\Omega, \mathbf{F} = \mathbb{F}_{\tau \in [0,\infty)}, \mathbb{P})$. The delay is governed by a non-negative real constant $\mathrm{m} \in \mathbb{R}$, and $\Phi(\tau)$ denotes the history-dependent function over the interval $\tau \in [-\mathrm{m}, 0]$. Further suppose, $\mathrm{x}_{\tau} = \{\mathrm{x}(\tau + \varpi), -\mathrm{m} \leq \varpi \leq 0\}$ and $\tau \in [0, \hbar]$. We examine Eq. (4) using the initial value listed below:

$$\mathrm{x}_0 = \Phi = \Phi(\tau) : \tau \in [-\mathrm{m}, 0], \tag{5}$$

which is an $\mathbb{F}_0-$measurable $\widetilde{\mathrm{C}}([-\mathrm{m},0],\mathbb{R}^{\mathrm{r}})-$ valued random variable in a manner that $\mathbf{E}\|\Phi\|^2<\infty$.

The format of the study is as follows: in the following part, we employ some important points, assumptions, and definitions that will serve as foundations to establish our results regarding DF-SDEs. In Section 3, we prove the well-posedness, regularity, and UH stability of solutions to DF-SDEs. In Section 4, we present two examples that demonstrate our results. Finally, in Section 5, we present conclusions.

2. Preliminaries

We go over key definitions, assumptions, and annotations in this part that will be used throughout the work.

The paper uses the following annotations throughout: When $\mathrm{m}>0$, suppose $\widetilde{\mathrm{C}}([-\mathrm{m},0],\mathbb{R}^{\mathrm{r}})$ indicate the continuous function space $\mathcal{Y}$ from $[-\mathrm{m},0]$ to $\mathbb{R}^{\mathrm{r}}$ via the norm $\sup_{\mathrm{m}\leq\varpi\leq 0}\|\mathcal{Y}(\varpi)\|$ when $\|\cdot\|$ represents in $\mathbb{R}^{\mathrm{r}}$ the Euclidean norm. $\tilde{\mathrm{p}}\geq 1$, suppose $\mathrm{L}^{\tilde{\mathrm{p}}}\big([0,\hbar],\mathbb{R}^{\mathrm{r}}\big)$ indicate the space where all $\mathbb{R}^{\mathrm{r}}-$valued $\mathbb{F}_\tau-$ adapted stochastic process $(\mathrm{x}(\tau))_{0\leq\tau\leq\hbar}$ on $(\Omega,\mathbb{F},\mathbb{P})$, where $\int_0^\hbar\|\mathrm{x}(\tau)\|^{\tilde{\mathrm{p}}}d\tau<\infty$ is quite certainly achieved using a filtration $\mathbf{F}=(\mathbb{F}_\tau)_{\tau\in[0,\hbar]}$. Suppose $\mathrm{L}^2(\Omega,\mathbb{F},\mathbb{P})$ represent all $\mathbb{F}_\tau-$measurable, mean square$-$integrable functions $\mathbb{F}:\Omega\to\mathbb{R}^{\mathrm{r}}$ with $\|\mathbb{F}\|_{ms}^2=\mathbf{E}\|\mathbb{F}\|^2$ in the space.

Definition 1. *If the following conditions are met, an $\mathbb{R}^{\mathrm{r}}-$valued stochastic process $\mathrm{x}(\tau)$ on the interval $[-\mathrm{m},\hbar]$ is referred to as the solution of Eq.* (4) *with the initial value Eq.* (5)*:*

(i) It is a continuous process and $(\mathrm{x}(\tau))_{0\leq\tau\leq\hbar}$ is $\mathbb{F}_\tau-$adapted,

(ii) $\mathrm{V}(\tau,\mathrm{x}(\tau),\mathrm{x}(\tau-\mathrm{m}))\in\mathrm{L}^2([0,\hbar],\mathbb{R}^{\mathrm{r}})$ *and* $\digamma(\tau,\mathrm{x}(\tau),\mathrm{x}(\tau-\mathrm{m}))\in\mathrm{L}^2([0,\hbar],\mathbb{R}^{\mathrm{r}})$

(iii) Probability one gives us $\mathrm{x}_0=\Phi$, and the subsequent equality holds $\tau\in[0,\hbar]$

$$\mathrm{x}(\tau)=\Phi+\int_0^\tau \mathrm{s}^{\xi-1}\mathrm{V}\big(\mathrm{s},\mathrm{x}(\mathrm{s}),\mathrm{x}(\mathrm{s}-\mathrm{m})\big)d\mathrm{s}+\int_0^\tau \mathrm{s}^{\xi-1}\digamma\big(\mathrm{s},\mathrm{x}(\mathrm{s}),\mathrm{x}(\mathrm{s}-\mathrm{m})\big)d\mathrm{w}_{\mathrm{s}}. \tag{6}$$

Definition 2. *The EU of solutions to Eq.* (4) *when the coefficients* V *and* $\digamma$ *meet the following requirements is our first major finding in our paper.*

(i) $(\mathbf{A}_1)$: *With each integer $\upsilon\geq 1$, there is $\mathrm{L}_\upsilon>0$ that corresponds to $\forall\in[0,\hbar]$ and all $\mathrm{y}_1,\ \mathrm{y}_2,\ \mathrm{z}_1,\ \mathrm{z}_2\in\mathbb{R}^{\mathrm{r}}$ with $\|\mathrm{y}_1\|\vee\|\mathrm{y}_1\|_2\vee\|\mathrm{z}_1\|\vee\|\mathrm{z}_2\|\leq\upsilon$,*

$$\|\mathrm{V}(\tau,\mathrm{y}_1,\mathrm{y}_2)-\mathrm{V}(\tau,\mathrm{z}_1,\mathrm{z}_2)\|^2\leq\mathrm{L}_\upsilon\big(\|\mathrm{y}_1-\mathrm{z}_1\|^2+\|\mathrm{y}_2-\mathrm{z}_2\|^2\big),$$
$$\|\digamma(\tau,\mathrm{y}_1,\mathrm{y}_2)-\digamma(\tau,\mathrm{z}_1,\mathrm{z}_2)\|^2\leq\mathrm{L}_\upsilon\big(\|\mathrm{y}_1-\mathrm{z}_1\|^2+\|\mathrm{y}_2-\mathrm{z}_2\|^2\big),$$

where $\mathrm{y}\vee\mathrm{z}=\max(\mathrm{y},\mathrm{z}),\mathrm{y},\mathrm{z}\in\mathbb{R}$.

Local Lipschitz continuity is a property of a function that ensures it behaves "nicely" (i.e., does not change too abruptly) in a neighborhood around every point of its domain, but not necessarily globally. It generalizes the concept of global Lipschitz continuity by relaxing the requirement of a uniform bound on the function's rate of change.

(ii) $(\mathbf{A}_2) : \mathrm{V}(.,0,0)$ *and* $\digamma(.,0,0)$ *are bounded, i.e*

$$\sup_{\tau\in[0,\hbar]}\|\mathrm{V}(\tau,0,0)\| < \mathbb{K}, \quad \sup_{\tau\in[0,\hbar]}\|\digamma(\tau,0,0)\| < \mathbb{K}.$$

The LLC $(\mathbf{A}_1)$ *is substituted with the following GLC in order to examine the Con-D on the initial value, the fractional order* ξ*, and the regularity of the results of DF-SDEs:*

(iii) $(\mathbf{A}_3) : \forall \mathrm{y}_1,\ \mathrm{y}_2,\ \mathrm{z}_1,\ \mathrm{z}_2 \in \mathbb{R}^{\mathrm{r}}$ *there exists* $\mathrm{L} > 0$ *such that*

$$\|\mathrm{V}(\tau,\mathrm{y}_1,\mathrm{y}_2) - \mathrm{V}(\tau,\mathrm{z}_1,\mathrm{z}_2)\|^2 \le \mathrm{L}^2\big(\|\mathrm{y}_1-\mathrm{z}_1\|^2 + \|\mathrm{y}_2-\mathrm{z}_2\|^2\big),$$
$$\|\digamma(\tau,\mathrm{y}_1,\mathrm{y}_2) - \digamma(\tau,\mathrm{z}_1,\mathrm{z}_2)\|^2 \le \mathrm{L}^2\big(\|\mathrm{y}_1-\mathrm{z}_1\|^2 + \|\mathrm{y}_2-\mathrm{z}_2\|^2\big).$$

In the next section, we will prove the well-posedness, regularity, and UH stability of solutions to DF-SDEs.

3. The main results

In this section, we established the EU, Con-D, and regularity of the solutions to DF-SDEs. We utilized the concept introduced in [34] to demonstrate the EU. Specifically, the process to prove EU consists of two steps: Initially, we establish the EU through the GLC of coefficients. In the subsequent phase, we illustrate the EU according to the LLC of coefficients by employing a truncation method. To establish the Con-D of solutions, we demonstrate that solutions are contingent upon the initial value of the fractional order ξ.

3.1. Existence, uniqueness, continuous dependence, and regularity result

First, we must construct a sufficient Banach space covering the solution Eq. (4) in order to demonstrate the subsequent result. Assume $\widetilde{\mathbb{H}}^2(0,\hbar)$ is the space where all processes exist $\mathrm{x}(\tau) : [0,\hbar] \to \mathrm{L}^2(\Omega,\mathbb{F},\mathbb{P})$ which are measurable $\mathbf{F}_\hbar$−adapted, with $\mathbf{F}_\hbar = (\mathbb{F}_\tau)_{\tau\in[0,\hbar]}$ and satisfy the following:

$$\|\mathrm{x}(\tau)\|_{\widetilde{\mathbb{H}}^2} = \sup_{\tau\in[0,\hbar]}\|\mathrm{x}(\tau)\|_{ms} < \infty. \tag{7}$$

$\big(\widetilde{\mathbb{H}}^2([0,\hbar]), \|\cdot\|_{\widetilde{\mathbb{H}}^2}\big)$ is surely a Banach space. We construct an operator $\mathbb{G}_\Phi : \widetilde{\mathbb{H}}^2(0,\hbar) \to \widetilde{\mathbb{H}}^2(0,\hbar)$ by $\mathbb{G}_\Phi(\mathrm{x}(0)) = \Phi$ for any $\Phi \in \widetilde{\mathrm{C}}([-\mathrm{m},0],\mathbb{R}^{\mathrm{r}})$ and $\forall\tau \in [0,\hbar]$, the subsequent equality is valid.

$$\mathbb{G}_\Phi(\mathrm{x}(\tau)) = \Phi + \int_0^\tau \mathrm{s}^{\xi-1}\mathrm{V}\big(\mathrm{s},\mathrm{x}(\mathrm{s}),\mathrm{x}(\mathrm{s}-\mathrm{m})\big)ds + \int_0^\tau \mathrm{s}^{\xi-1}\digamma\big(\mathrm{s},\mathrm{x}(\mathrm{s}),\mathrm{x}(\mathrm{s}-\mathrm{m})\big)d\mathrm{w}_\mathrm{s}. \tag{8}$$

The subsequent lemma verifies that the operator is well-defined. The fundamental inequality provided below supports the proof of this statement and other later results.

$$\big\|\mathrm{x}_1 + \mathrm{x}_2 + \cdots + \mathrm{x}_\upsilon\|^2 \le \upsilon\big(\|\mathrm{x}_1\|^2 + \|\mathrm{x}_2\|^2 + \cdots + \|\mathrm{x}_\upsilon\|^2\big),\ \forall \mathrm{x}_1,\mathrm{x}_2,\cdots,\mathrm{x}_\upsilon \in \mathbb{R}^{\mathrm{r}}. \tag{9}$$

Lemma 1. *Assume that* $(\mathbf{A}_1)$ *and* $(\mathbf{A}_2)$ *are valid. The operator* $\mathbb{G}_\Phi$ *is then well defined for any* $\Phi \in \widetilde{\mathrm{C}}([-\mathrm{m}, 0], \mathbb{R}^{\mathrm{r}})\big)$.

Proof. Assume that $\mathrm{x}(\tau) \in \widetilde{\mathbb{H}}^2[0, \hbar]$, where $\mathrm{x}(\tau)$ is considered to be arbitrary. Based on the formulation of $\mathbb{G}_\Phi(\mathrm{x}(\tau))$ given in Eq. (8) and the inequality in Eq. (9), we derive the following.

$$\begin{aligned}\mathbf{E}\big\|\mathbb{G}_\Phi(\mathrm{x}(\tau))\big\|^2 \leq & 3\mathbf{E}\big\|\Phi\big\|^2 + 3\mathbf{E}\bigg\|\int_0^\tau \mathrm{s}^{\xi-1}\mathrm{V}\big(\mathrm{s}, \mathrm{x}(\mathrm{s}), \mathrm{x}(\mathrm{s}-\mathrm{m})\big)d\mathrm{s}\bigg\|^2 \\ & +3\mathbf{E}\bigg\|\int_0^\tau \mathrm{s}^{\xi-1}\digamma\big(\mathrm{s}, \mathrm{x}(\mathrm{s}), \mathrm{x}(\mathrm{s}-\mathrm{m})\big)d\mathrm{w}_\mathrm{s}\bigg\|^2. \end{aligned} \tag{10}$$

Now we will simplify Eq. (10) in two parts. The Höld-I gives us the result that

$$\begin{aligned} & \mathbf{E}\left[\bigg\|\int_0^\tau \mathrm{s}^{\xi-1}\mathrm{V}\big(\mathrm{s}, \mathrm{x}(\mathrm{s}), \mathrm{x}(\mathrm{s}-\mathrm{m})\big)d\mathrm{s}\bigg\|^2\right] \\ & \leq \left(\left(\int_0^\tau \mathrm{s}^{(2\xi-2)}d\mathrm{s}\right)\mathbf{E}\left[\int_0^\tau \big\|\mathrm{V}\big(\mathrm{s}, \mathrm{x}(\mathrm{s}), \mathrm{x}(\mathrm{s}-\mathrm{m})\big)\big\|^2 d\mathrm{s}\right]\right) \\ & \leq \frac{\tau^{(2\xi-1)}}{2\xi-1}\mathbf{E}\left[\int_0^\tau \big\|\big(\mathrm{s}, \mathrm{x}(\mathrm{s}), \mathrm{x}(\mathrm{s}-\mathrm{m})\big)\big\|^2 d\mathrm{s}\right] \\ & \leq \frac{\hbar^{(2\xi-1)}}{2\xi-1}\mathbf{E}\big\|\big(\mathrm{V}(\mathrm{s}, \mathrm{x}(\mathrm{s}), \mathrm{x}(\mathrm{s}-\mathrm{m})\big)\big\|^2\hbar. \end{aligned} \tag{11}$$

Because of $(\mathbf{A}_2)$ and $(\mathbf{A}_3)$, we acquire the following:

$$\big\|\big(\mathrm{V}(\mathrm{s}, \mathrm{x}(\mathrm{s}), \mathrm{x}(\mathrm{s}-\mathrm{m})\big)\big\|^2 \leq 2\mathrm{L}^2\big(\|\mathrm{x}(\mathrm{s})\|^2 + \big(\|\mathrm{x}(\mathrm{s}-\mathrm{m})\|^2\big) + 2\mathbb{K}^2. \tag{12}$$

Consequently, we employ Eq. (12) in Eq. (11) to obtain the following:

$$\begin{aligned} & \mathbf{E}\left[\bigg\|\int_0^\tau \mathrm{s}^{\xi-1}\mathrm{V}\big(\mathrm{s}, \mathrm{x}(\mathrm{s}),\mathrm{x}(\mathrm{s}-\mathrm{m})\big)d\mathrm{s}\bigg\|^2\right] \\ & \qquad \leq \frac{\hbar^{(2\xi-1)}}{2\xi-1}\mathbf{E}\big[2\mathrm{L}^2\big(\|\mathrm{x}(\mathrm{s})\|^2 + \big(\|\mathrm{x}(\mathrm{s}-\mathrm{m})\|^2\big) + 2\mathbb{K}^2\big]\hbar \\ & \qquad = \frac{\hbar^{(2\xi-1)}}{2\xi-1}\left(2\mathrm{L}^2\big(\|\mathrm{x}(\mathrm{s})\|^2_{\widetilde{\mathbb{H}}^2} + \big(\|\mathrm{x}(\mathrm{s}-\mathrm{m})\|^2_{\widetilde{\mathbb{H}}^2}\big) + 2\mathbb{K}^2\right)\hbar. \end{aligned} \tag{13}$$

Now, we will simplify the following result: For this, use I-Is.

$$\begin{aligned} \mathbf{E}\left[\bigg\|\int_0^\tau \mathrm{s}^{\xi-1}\digamma\big(\mathrm{s}, \mathrm{x}(\mathrm{s}), \mathrm{x}(\mathrm{s}-\mathrm{m})\big)d\mathrm{w}_\mathrm{s}\bigg\|^2\right] & = \mathbf{E}\left[\left(\int_0^\tau \mathrm{s}^{\xi-1}\big\|\digamma\big(\mathrm{s}, \mathrm{x}(\mathrm{s}), \mathrm{x}(\mathrm{s}-\mathrm{m})\big)\big\|d\mathrm{s}\right)^2\right] \\ & = \mathbf{E}\left[\int_0^\tau \mathrm{s}^{2\xi-2}\big\|\digamma\big(\mathrm{s}, \mathrm{x}(\mathrm{s}), \mathrm{x}(\mathrm{s}-\mathrm{m})\big)\big\|^2 d\mathrm{s}\right] \\ & = \mathbf{E}\left[\frac{\mathrm{s}^{2\xi-1}}{2\xi-1}\big\|\digamma\big(\mathrm{s}, \mathrm{x}(\mathrm{s}), \mathrm{x}(\mathrm{s}-\mathrm{m})\big)\big\|^2\right] \end{aligned}$$

$$\leq \mathbf{E}\left[\frac{\hbar^{2\xi-1}}{2\xi-1}\left\|\digamma\big(\mathrm{s},\mathrm{x(s)},\mathrm{x(s-m)}\big)\right\|^2\right]. \tag{14}$$

Because of $(\mathbf{A}_2)$ and $(\mathbf{A}_3)$, we acquire the following

$$\left\|\digamma\big(\mathrm{s},\mathrm{x(s)},\mathrm{x(s-m)}\big)\right\|^2 \leq 2\mathrm{L}^2\bigg(\|\mathrm{x(s)}\|^2 + \big(\|\mathrm{x(s-m)}\|^2\big) + 2\mathbb{K}\bigg). \tag{15}$$

Implementing Eq. (15) to Eq. (14) yields the following results:

$$\mathbf{E}\left[\left\|\int_0^\tau \mathrm{s}^{\xi-1}\digamma\big(\mathrm{s},\mathrm{x(s)},\mathrm{x(s-m)}\big)d\mathrm{w_s}\right\|^2\right] \leq \frac{\hbar^{2\xi-1}}{2\xi-1}$$
$$\bigg(2\mathrm{L}^2\big(\|\mathrm{x(s)}\|^2_{\widetilde{\mathbb{H}}^2} + \big(\|\mathrm{x(s-m)}\|^2\big) + 2\mathbb{K}^2\bigg). \tag{16}$$

Combining Eqs. (16) and (13) in (10), we obtain the subsequent outcome:

$$\mathbf{E}\left\|\mathbb{G}_\Phi(\mathrm{x}(\tau))\right\|^2 \leq \frac{6\hbar^{2\xi-1}}{2\xi-1}(\hbar+1)$$
$$\bigg(2\mathrm{L}^2\big(\|\mathrm{x(s)}\|^2_{\widetilde{\mathbb{H}}^2} + \big(\|\mathrm{x(s-m)}\|^2_{\widetilde{\mathbb{H}}^2}\big) + 2\mathbb{K}^2\bigg) + 3\|\Phi\|^2. \tag{17}$$

It follows that $\|\mathbb{G}_\Phi(\mathrm{x}(\tau))\|_{\widetilde{\mathbb{H}}^2} < \infty$. Therefore, the operator $\mathbb{G}_\Phi$ is properly defined. In order to confirm EU, we proceed to prove the following lemma:

Lemma 2. *In the occurrence where $\tau > 0$ and $\xi > \frac{1}{2}$, the subsequent inequality is true:*

$$\vartheta\int_0^\tau \mathrm{s}^{2\xi-2}\mathbb{E}_{2\xi-1}(\vartheta \mathrm{s}^{2\xi-1})d\mathrm{s} \leq \mathbb{E}_{2\xi-1}(\vartheta\tau^{2\xi-1}), \tag{18}$$

here the Mittag-Leffler function $\mathbb{E}_{2\xi-1}(.)$ is defined as

$$\mathbb{E}_{(2\xi-1)}(\tau) = \sum\nolimits_{\upsilon=0}^{\infty}\frac{\tau^\upsilon}{\Gamma((2\xi-1)\upsilon+1)}. \tag{19}$$

Proof. Let $\vartheta > 0$ be fixed arbitrarily. We begin by replacing the integral with a corresponding sum, and subsequently, we employ the following identity.

$$\int_0^\tau \mathrm{s}^{2\xi-2}\mathrm{s}^{\upsilon(2\xi-1)}d\mathrm{s} = \tau^{(\upsilon+1)(2\xi-1)}\mathbb{B}\big(2\xi-1,\upsilon(2\xi-1)+1\big),\ \upsilon = 0,\ 1,\ 2,\cdots.$$

So, we get

$$\vartheta\int_0^\tau \mathrm{s}^{2\xi-2}\mathbb{E}_{2\xi-1}(\vartheta\mathrm{s}^{2\xi-1})ds = \vartheta\sum\nolimits_{\upsilon=0}^{\infty}\frac{\vartheta^\upsilon}{\Gamma(\upsilon(2\xi-1)+1)}\int_0^\tau \mathrm{s}^{2\xi-2}\mathrm{s}^{\upsilon(2\xi-1)}d\mathrm{s}$$
$$= \sum\nolimits_{\upsilon=0}^{\infty}\frac{\vartheta^{\upsilon+1}\tau^{(\upsilon+1)(2\xi-1)}}{\Gamma(2\xi-1)\Gamma(\upsilon(2\xi-1)+1)}$$

$$
\begin{aligned}
&=\sum_{\upsilon=1}^{\infty}\frac{\vartheta^{\upsilon}\tau^{\upsilon(2\xi-1)}}{\Gamma(\upsilon(2\xi-1)+1)}\\
&=\mathbb{E}_{2\xi-1}(\vartheta\tau^{2\xi-1})-1\\
&\leq\mathbb{E}_{2\xi-1}(\vartheta\tau^{2\xi-1}),
\end{aligned}
$$

here, $\mathbb{B}$ is a beta function. Hence, completes the proof.

To demonstrate the EU of solutions, we aim to prove that the operator $\mathbb{G}_\Phi$ is a contraction in the space $\widetilde{\mathbb{H}}^2([0,\hbar])$ with respect to an appropriately chosen weighted norm ([], [Remark 2.1]). In this context, the Mittag-Leffler function $\mathbb{E}_{(2\xi-1)}(\tau)$, defined in Eq. (19), acts as the weighting function. To demonstrate the EU of solutions, we aim to prove that the operator $\mathbb{G}_\Phi$ is a contraction in the space $\widetilde{\mathbb{H}}^2([0,\hbar])$ with respect to an appropriately chosen weighted norm ([35], [Remark 2.1]). In this context, the Mittag-Leffler function $\mathbb{E}_{(2\xi-1)}(\tau)$, defined in Eq. (19), acts as the weighting function. To demonstrate the EU of solutions, we aim to prove that the operator $\mathbb{G}_\Phi$ is a contraction in the space $\widetilde{\mathbb{H}}^2([0,\hbar])$ with respect to an appropriately chosen weighted norm ([], [Remark 2.1]). In this context, the Mittag-Leffler function $\mathbb{E}_{(2\xi-1)}(\tau)$, defined in Eq. (19), acts as the weighting function.

Theorem 1. *If* $(\mathbf{A}_2)$ *and* $(\mathbf{A}_3)$ *are valid, then the Problem Eq.* (4) *with* $\mathrm{x}(0)=\Phi$ *has unique solution on* $[-\mathrm{m},0]$ *for any* $\Phi\in\widetilde{\mathrm{C}}([-\mathrm{m},0],\mathbb{R}^{\mathrm{r}})$.

Proof. First of all, choose a fix positive constant ϑ as follows:

$$\vartheta > 8\mathrm{L}^2(\hbar+1)\Gamma(2\xi-1). \tag{20}$$

We construct a weighted norm $\|\cdot\|_\vartheta$ over the space $\widetilde{\mathbb{H}}^2([o,\hbar])$ as:

$$\|\mathrm{x}(\tau)\|_\vartheta = \underset{\tau\in[0,\hbar]}{Sup}\left(\frac{\mathbf{E}(\|\mathrm{x}(\tau)\|^2)}{\mathbb{E}_{2\xi-1}\big(\vartheta(\tau+\mathrm{m})^{2\xi-1}\big)}\right)^{\frac{1}{2}},\ \forall\ \mathrm{x}(\tau)\in\widetilde{\mathbb{H}}^2([0,\hbar]). \tag{21}$$

Let us now fix an arbitrary $\Phi\in\widetilde{\mathrm{C}}([-\mathrm{m},0],\mathbb{R}^{\mathrm{r}})$. The operator $\mathbb{G}_\Phi$ is well-posed by virtue of the previously stated lemma. We shall now establish the contractive property of the mapping $\mathbb{G}_\Phi$ with respect to the norm $\|\cdot\|_\vartheta$. To this end, consider arbitrary elements $\mathrm{x}(\tau),\widetilde{\mathrm{x}}(\tau)\in\widetilde{\mathbb{H}}^2([0,\hbar])$. Then, for every $\tau\in[0,\hbar]$, applying Eqs. (7) and (8), we obtain the following:

$$
\begin{aligned}
&\mathbf{E}\big(\|\mathbb{G}_\Phi\big(\mathrm{x}(\tau)\big)-\mathbb{G}_\Phi\big(\widetilde{\mathrm{x}}(\tau)\big)\|^2\big)\\
\leq&2\mathbf{E}\left[\left\|\int_0^\tau \mathrm{s}^{\xi-1}\Big(\mathrm{V}(\mathrm{s},\mathrm{x}(\mathrm{s}),\mathrm{x}(\mathrm{s}-\mathrm{m})-\mathrm{V}(\mathrm{s},\tilde{\mathrm{x}}(\mathrm{s}),\tilde{\mathrm{x}}(\mathrm{s}-\mathrm{m}))\Big)d\mathrm{s}\right\|^2\right]\\
+&2\mathbf{E}\left[\left\|\int_0^\tau \mathrm{s}^{\xi-1}\Big(\digamma(\mathrm{s},\mathrm{x}(\mathrm{s}),\mathrm{x}(\mathrm{s}-\mathrm{m})-\digamma(\mathrm{s},\tilde{\mathrm{x}}(\mathrm{s}),\tilde{\mathrm{x}}(\mathrm{s}-\mathrm{m}))\Big)d\mathrm{w}_\mathrm{s}\right\|^2\right].
\end{aligned}
\tag{22}
$$

Now we will simplify Eq. (22) in two parts. The Höld-I and $(\mathbf{A}_3)$ gives us the result that

$$\mathbf{E}\left[\left\|\int_0^\tau \mathrm{s}^{\xi-1}\Big(\mathrm{V}(\mathrm{s},\mathrm{x}(\mathrm{s}),\mathrm{x}(\mathrm{s}-\mathrm{m})-\mathrm{V}(\mathrm{s},\tilde{\mathrm{x}}(\mathrm{s}),\tilde{\mathrm{x}}(\mathrm{s}-\mathrm{m}))\Big)d\mathrm{s}\right\|^2\right]\leq 2\hbar$$

$$\int_0^\tau s^{2\xi-2}L^2\Big(\mathbf{E}\big[\|x(s)-\tilde{x}(s)\|^2\big]+\mathbf{E}\big[\|x(s-m)-\tilde{x}(s-m))\|^2\big]\Big)ds. \quad (23)$$

Now, we will simplify the following result: For this, use I-Is and $(\mathbf{A}_3)$

$$\mathbf{E}\Big[\Big\|\int_0^\tau s^{\xi-1}\Big(\digamma(s,x(s),x(s-m)-\digamma(s,\tilde{x}(s),\tilde{x}(s-m))\Big)dw_s\Big\|^2\Big]\leq 2$$
$$\int_0^\tau s^{2\xi-2}L^2\Big(\mathbf{E}\big[\|x(s)-\tilde{x}(s)\|^2\big]+\mathbf{E}\big[\|x(s-m)-\tilde{x}(s-m))\|^2\big]\Big)ds. \quad (24)$$

Utilizing Eqs. (24) and (23) in Eq. (22), we acquire the subsequent outcome.

$$\begin{aligned}\mathbf{E}\big(\|\mathbb{G}_\Phi\big(x(\tau)\big)-\mathbb{G}_\Phi\big(\widetilde{x}(\tau)\big)\|^2\big)\leq &2(\tau+1)\int_0^\tau s^{2\xi-2}L^2\big(\mathbf{E}\|x(s)-\widetilde{x}(s)\|^2+\\ &\mathbf{E}\|x(s-m)-\widetilde{x}(s-m)\|^2\big)ds. \end{aligned} \quad (25)$$

However, in light of Lemma 2 and the monotone increasing of function $\mathbb{E}_{2\xi-1}$ on $\mathbb{R}^+$, we have $\forall\tau\in[0,\hbar]$.

$$\begin{aligned}&\frac{\mathbf{E}\big(\|\mathbb{G}_\Phi\big(x(\tau)\big)-\mathbb{G}_\Phi\big(\widetilde{x}(\tau)\big)\|^2\big)}{\mathbf{E}_{2\xi-1}(\vartheta(\tau+m)^{2\xi-1})}\\ &\leq\Gamma(2\xi-1)\frac{4L^2(\hbar+1)}{\vartheta}\sup_{\tau\in[0,\hbar]}\frac{\mathbf{E}\big(\|\mathbb{G}_\Phi\big(x(\tau)\big)-\mathbb{G}_\Phi\big(\widetilde{x}(\tau)\big)\|^2\big)}{\mathbf{E}_{2\xi-1}(\vartheta(\tau+m)^{2\xi-1})}.\end{aligned} \quad (26)$$

By the definition of $\|\cdot\|_\vartheta$ in Eq. (21), consequently

$$\|\mathbb{G}_\Phi\big(x(\tau)\big)-\mathbb{G}_\Phi\big(\widetilde{x}(\tau)\big)\|_\vartheta\leq y\|x(\tau)-\widetilde{x}(\tau)\|, \quad (27)$$

where $y=\Big(\Gamma(2\xi-1)\frac{4L^2(\hbar+1)}{\vartheta}\Big)^{\frac{1}{2}}$.

In light of Eq. (20), we derive $y<1$ and therefore the operator $\mathbb{G}_\Phi$ is a contractive map on $\big(\widetilde{\mathbb{H}}^2[0,\hbar],\|\cdot\|_\vartheta\big)$. In accordance with the Banach fixed point theorem, there is only one fixed point $x(\tau)$ on this map in $\big(\widetilde{\mathbb{H}}^2[0,\hbar],\|\cdot\|_\vartheta\big)$. Additionally, this fixed point is the sole solution to Eq. (4) with the initial value $x(0)=\Phi$. Therefore, we proved the required result.

In the subsequent theorem, we establish the EU according to the LLC of coefficients by employing a truncation method.

Theorem 2. *Suppose that* $(\mathbf{A}_1)$ *and* $(\mathbf{A}_2)$ *are valid. Then, the Problem Eq.* (4) *involving the initial value* $x(0)=\Phi$ *has a unique solution on* $[-m,\hbar]$ *for any* $\Phi\in\widetilde{C}([-m,0],\mathbb{R}^r)$*.*

Proof. **Step 1:** Existence: We define the truncation functions V_υ and $\digamma_\upsilon$ for each $\upsilon\leq 1$.

$$V_\upsilon\big(s,x(s),x(s-m)\big)=\begin{cases}V\big(s,x(s),x(s-m)\big), & \text{if } \|x_\tau\|\leq\upsilon,\\ V\Big(s,\frac{\upsilon x(s)}{\|x(\tau)\|},\frac{\upsilon x(s-m)}{\|x(\tau-m)\|}\Big), & \text{if } \|x_\tau\|>\upsilon.\end{cases}$$

$$\mathcal{F}_{\upsilon}\big(\mathrm{s},\mathrm{x}(\mathrm{s}),\mathrm{x}(\mathrm{s}-\mathrm{m})\big)=\begin{cases}\mathcal{F}\big(\mathrm{s},\mathrm{x}(\mathrm{s}),\mathrm{x}(\mathrm{s}-\mathrm{m})\big), & \text{if } \|\mathrm{x}_\tau\|\leq\upsilon,\\ \mathcal{F}\left(\mathrm{s},\dfrac{\upsilon\mathrm{x}(\mathrm{s})}{\|\mathrm{x}(\tau)\|},\dfrac{\upsilon\mathrm{x}(\mathrm{s}-\mathrm{m})}{\|\mathrm{x}(\tau-\mathrm{m})\|}\right), & \text{if } \|\mathrm{x}_\tau\|>\upsilon.\end{cases}$$

Notice that $\mathrm{x}_\tau=\{\mathrm{x}(\tau+\delta),-\mathrm{m}\leq\delta\leq 0\}$ and $\tau\in[0,\hbar]$. Then, V_υ and $\mathcal{F}_\upsilon$ satisfy the GLC ($\mathbf{A}_3$) and ($\mathbf{A}_2$) and by Lemma 2, there is a unique solution $\mathrm{x}^\upsilon(\tau)$ to the following:

$$\begin{aligned}\mathrm{x}^\upsilon(\tau)=&\Phi+\int_0^\tau \mathrm{s}^{\xi-1}\mathrm{V}^\upsilon\big(\mathrm{s},\mathrm{x}^\upsilon(\mathrm{s}),\mathrm{x}^\upsilon(\mathrm{s}-\mathrm{m})\big)d\mathrm{s}\\&+\int_0^\tau \mathrm{s}^{\xi-1}\mathcal{F}\big(\mathrm{s},\mathrm{x}^\upsilon(\mathrm{s}),\mathrm{x}^\upsilon(\mathrm{s}-\mathrm{m})\big)d\mathrm{w}_\mathrm{s}.\end{aligned}\tag{28}$$

Define the stopping time $\lambda_\upsilon=\hbar\wedge\inf\{\tau\in[0,\hbar:\|\mathrm{x}^\upsilon_\tau\|\geq\upsilon]\}$. Now, we shall prove

$$\mathrm{x}^\upsilon(\tau)=\mathrm{x}^{\upsilon+1}(\tau),\ \forall\tau\in[0,\lambda_\upsilon].\tag{29}$$

By Eq. (28) and using the Höld-I and I-Is, we obtain

$$\begin{aligned}&\mathbf{E}\big[\big\|\mathrm{x}^{\upsilon+1}(\tau)-\mathrm{x}^\upsilon(\tau)\big\|^2\big]\\&\leq 4\tau\mathbf{E}\int_0^\tau \mathrm{s}^{2\xi-2}\big\|\mathrm{V}_{\upsilon+1}\big(\mathrm{s},\mathrm{x}^{\upsilon+1}(\mathrm{s}),\mathrm{x}^{\upsilon+1}(\mathrm{s}-\mathrm{m})-\mathrm{V}_{\upsilon+1}\big(\mathrm{s},\mathrm{x}^\upsilon(\mathrm{s}),\mathrm{x}^\upsilon(\mathrm{s}-\mathrm{m})\big)\big\|^2d\mathrm{s}\\&+4\tau\mathbf{E}\int_0^\tau \mathrm{s}^{2\xi-2}\big\|\mathrm{V}_{\upsilon+1}\big(\mathrm{s},\mathrm{x}^{\upsilon+1}(\mathrm{s}),\mathrm{x}^{\upsilon+1}(\mathrm{s}-\mathrm{m})-\mathrm{V}_{\upsilon}\big(\mathrm{s},\mathrm{x}^\upsilon(\mathrm{s}),\mathrm{x}^\upsilon(\mathrm{s}-\mathrm{m})\big)\big\|^2d\mathrm{s}\\&+4\tau\mathbf{E}\int_0^\tau \mathrm{s}^{2\xi-2}\big\|\mathcal{F}_{\upsilon+1}\big(\mathrm{s},\mathrm{x}^{\upsilon+1}(\mathrm{s}),\mathrm{x}^{\upsilon+1}(\mathrm{s}-\mathrm{m})-\mathcal{F}_{\upsilon+1}\big(\mathrm{s},\mathrm{x}^\upsilon(\mathrm{s}),\mathrm{x}^\upsilon(\mathrm{s}-\mathrm{m})\big)\big\|^2d\mathrm{s}\\&+4\tau\mathbf{E}\int_0^\tau \mathrm{s}^{2\xi-2}\big\|\mathcal{F}_{\upsilon+1}\big(\mathrm{s},\mathrm{x}^{\upsilon+1}(\mathrm{s}),\mathrm{x}^{\upsilon+1}(\mathrm{s}-\mathrm{m})-\mathcal{F}_{\upsilon}\big(\mathrm{s},\mathrm{x}^\upsilon(\mathrm{s}),\mathrm{x}^\upsilon(\mathrm{s}-\mathrm{m})\big)\big\|^2d\mathrm{s}.\end{aligned}\tag{30}$$

From ($\mathbf{A}_1$) and noting that $\mathrm{x}^{\upsilon+1}(\mathrm{s})=\mathrm{x}^\upsilon(\mathrm{s})=\Phi(\mathrm{s}),\ \mathrm{s}\in[-\mathrm{m},0]$, we derive

$$\begin{aligned}&\mathbf{E}\big[\big\|\mathrm{x}^{\upsilon+1}(\tau)-\mathrm{x}^\upsilon(\tau)\big\|^2\big]\\&\leq 4(\tau+1)\mathrm{L}_\upsilon\int_0^\tau \mathrm{s}^{2\xi-2}\Big(\mathbf{E}\big[\|\mathrm{x}^{\upsilon+1}(\mathrm{s})-\mathrm{x}^\upsilon(\mathrm{s})\big\|^2\big]+\mathbf{E}\big[\big\|\mathrm{x}^{\upsilon+1}(\mathrm{s}-\mathrm{m})-\mathrm{x}^\upsilon(\mathrm{s}-\mathrm{m})\big\|^2\big]\Big)\\&\leq 8\tau\mathrm{L}_\upsilon\int_0^\tau \mathrm{s}^{2\mathrm{s}-2}\underset{\leq\delta\leq\mathrm{s}}{sup}\,\mathbf{E}\big[\big\|\mathrm{x}^{\upsilon+1}(\delta)-\mathrm{x}^\upsilon(\delta)\big\|^2\big]d\mathrm{s}.\end{aligned}\tag{31}$$

When we use the G-I, we get

$$\underset{0\leq\eta\leq\tau}{sup}\ \mathbf{E}[\|\mathrm{x}^{\upsilon+1}(\eta)-\mathrm{x}^\upsilon(\eta)\|^2]=0,\ \forall 0\leq\tau\leq\lambda_\upsilon,\tag{32}$$

which demonstrates Eq. (29). As a result, λ_υ is increasing. However, V_υ and $\mathcal{F}_\upsilon$ meet the linear growth requirement since V_υ and $\mathcal{F}_\upsilon$ satisfy conditions ($\mathbf{A}_2$) and ($\mathbf{A}_3$) . As a result, there exists an integer $\upsilon_0=\upsilon_0(\varpi)$ that corresponds to $\lambda_\upsilon=\hbar$ whenever $\upsilon\geq\upsilon_0$ for almost every value $\varpi\in\Omega$. Next, let $\mathrm{x}(\tau)$ be defined by $\mathrm{x}(\tau)=\mathrm{x}^{(\upsilon_0)}(\tau),\ \tau\in[0,\hbar]$.

Therefore, to complete the proof, it is sufficient to show that $\mathrm{x}(\tau)$ is the solution of Problem Eq. (4). Indeed, by virtue of Eq. (29), we get that

$$\mathrm{x}(\tau \wedge \lambda_\upsilon) = \mathrm{x}^{(\upsilon)}(\tau \wedge \lambda_\upsilon) = \mathrm{x}^{(\upsilon_0)}(\tau \wedge \lambda_\upsilon). \tag{33}$$

Using Eq. (28), we yield that

$$\begin{aligned} \mathrm{x}(\tau \wedge \hbar_\upsilon) =& \Phi + \int_0^{\tau\wedge\hbar_\upsilon} \mathrm{s}^{\xi-1} \mathrm{V}\big(\mathrm{s}, \mathrm{x}(\mathrm{s}), \mathrm{x}(\mathrm{s}-\mathrm{m})\big) ds \\ &+ \int_0^{\tau\wedge\hbar_\upsilon} \mathrm{s}^{\xi-1} F\big(\mathrm{s}, \mathrm{x}(\mathrm{s}), \mathrm{x}(\mathrm{s}-\mathrm{m})\big) d\mathrm{w}_\mathrm{s}. \end{aligned} \tag{34}$$

Let $\upsilon \to \infty$ and by $\lambda_\upsilon \to \hbar$ as $\upsilon \to \infty$, As a result, we can conclude that the solution to Eq. (4) is $\mathrm{x}(\tau)$. Hence, it proved the desired result.

Step 2: Uniqueness: Assume that, with respect to the same Brownian motion w_τ and the same initial value Φ, x and $\widetilde{\mathrm{x}}$ are the solutions stated for all $\tau \in [-\mathrm{m}, \hbar]$ of the Eq. (4). We determine the stopping timings associated with every $\upsilon \geq 1$.

$$\lambda_\upsilon = \hbar \wedge inf\{\tau \in [0, \hbar] : \|\mathrm{x}_\tau\| \geq \upsilon\},\ \widetilde{\lambda}_\upsilon = \hbar \wedge inf\{\tau \in [0, \hbar] : \|\widetilde{\mathrm{x}}_\tau\| \geq \upsilon\}$$

and we put $\mathbb{U}_\upsilon = \lambda_\upsilon \wedge \widetilde{\lambda}_\upsilon$. Obviously, $\lim_{\upsilon\to\infty} \mathbb{U}_\upsilon = \hbar$ $\mathbb{P}$-almost surely and

$$\begin{aligned} \mathrm{x}(\tau \wedge \nu_\upsilon) - & \widetilde{\mathrm{x}}(\tau \wedge \nu_\upsilon) \\ &= \int_0^{\tau\wedge\nu_\upsilon} \mathrm{s}^{\xi-1} \Big(\mathrm{V}\big(\mathrm{s}, \mathrm{x}(\mathrm{s}), \mathrm{x}(\mathrm{s}-\mathrm{m} - \mathrm{V}\big(\mathrm{s}, \widetilde{\mathrm{x}}(\mathrm{s}), \widetilde{\mathrm{x}}(\mathrm{s}-\mathrm{m})\big) \Big) ds \\ &+ \int_0^{\tau\wedge\nu_\upsilon} \mathrm{s}^{\xi-1} \Big(F\big(\mathrm{s}, \mathrm{x}(\mathrm{s}), \mathrm{x}(\mathrm{s}-\mathrm{m})\big) - F\big(\mathrm{s}, \widetilde{\mathrm{x}}(\mathrm{s}), \widetilde{\mathrm{x}}(\mathrm{s}-\mathrm{m})\big) \Big) d\mathrm{w}_\mathrm{s}. \end{aligned} \tag{35}$$

Utilizing the I-Is, Höld-I, and $(\mathbf{A}_1)$, we accomplished

$$\begin{aligned} \mathbf{E}\big[\|\mathrm{x}(\tau) - \widetilde{\mathrm{x}}(\tau)\|^2 \aleph\{\tau \leq \mathbb{U}_\upsilon\}\big] &\leq 2(\tau+1)\mathrm{L}_\upsilon \\ \mathbf{E}\Big[\int_0^\tau \mathrm{s}^{2\xi-2} & \Big(\|\mathrm{x}(\tau) - \widetilde{\mathrm{x}}(\tau)\|^2 + \|\mathrm{x}(\tau-\mathrm{m}) - \widetilde{\mathrm{x}}(\tau-\mathrm{m})\|^2 \Big) \aleph\{\tau \leq \mathbb{U}_\upsilon\} \Big] \\ &\leq 4(\hbar+1)\mathrm{L}_\upsilon \int_0^\tau \sup_{0\leq\delta\leq\mathrm{s}} \mathbf{E}\big[\|\mathrm{x}(\delta) - \widetilde{\mathrm{x}}(\delta)\|^2 \aleph\{\tau \leq \mathbb{U}_\upsilon\}\big] \mathrm{s}^{2\xi-2} d\mathrm{s}. \end{aligned} \tag{36}$$

Utilizing the G-I, we achieved

$$\underset{0\leq\tau\leq\hbar}{sup}\ \mathbf{E}\big[\|\mathrm{x}(\delta) - \widetilde{\mathrm{x}}(\delta)\|^2 \aleph\{\tau \leq \mathbb{U}_\upsilon\}\big] = 0. \tag{37}$$

However, we achieve the following:

$$\begin{aligned} \sup_{0\leq\tau\leq\hbar} \mathbf{E}\big[\|\mathrm{x}(\delta) - \widetilde{\mathrm{x}}(\delta)\|^2\big] \leq \big[\|\mathrm{x}(\delta) - \widetilde{\mathrm{x}}(\delta)\|^2 \\ \aleph(\tau \leq \mathbb{U}_\upsilon\big] + \big[\|\mathrm{x}(\delta) - \widetilde{\mathrm{x}}(\delta)\|^2 \aleph\{\tau > \mathbb{U}_\upsilon\}\big]. \end{aligned} \tag{38}$$

Let $\upsilon \to \infty$ and note that $\mathrm{x}(\tau) = \widetilde{\mathrm{x}}(\tau) = \Phi(\tau)$ for $\tau \in [-\mathrm{m}, 0]$, it is clear that

$$\{\mathrm{x}(\tau),\ -\mathrm{m} \leq \tau \leq \hbar\},$$

and

$$\{\widetilde{\mathrm{x}}(\tau),\ -\mathrm{m} \leq \tau \leq \hbar\},$$

are the modification of each other and thus are indistinguishable. Hence, it proved the required result.

In the following theorem, we will illustrate the Con-D of solutions based on the initial value Φ and fractional exponent ξ for DF-SDEs.

Theorem 3. $(\mathbf{A}_2)$ *and* $(\mathbf{A}_3)$ *are assumed to be valid. Thus, for every* $\Phi, \beta \in \widetilde{\mathrm{C}}([-\mathrm{m}, 0], \mathbb{R}^{\mathrm{r}})$, *the solution* $\mho_\xi(., \Phi)$ *of Eq.* (4) *continues to be relying on* Φ, *i.e.*

$$\lim_{\Phi \to \beta} \sup_{\tau \in [-\mathrm{m}, \hbar]} \|\mho_\xi(\tau, \Phi) - \mho_\xi(\tau, \beta)\|_{ms} = 0. \tag{39}$$

Proof. Choose and fix $\hbar > 0$ and $\Phi, \beta \in \widetilde{\mathrm{C}}([-\mathrm{m}, 0], \mathbb{R}^{\mathrm{r}})$. Because $\mho_\xi(., \beta)$ and $\mho_\xi(., \Phi)$ are solutions of Eq. (4), utilizing the Eq. (9), the Höld-I, I-Is, $(\mathbf{A}_3)$, and Lemma 2, we yield

$$\begin{aligned} \mathop{sup}_{\tau \in [0, \hbar]} \frac{E\left[\|\mho_\xi(\tau, \Phi) - \mho_\xi(\tau, \beta)\|^2\right]}{\mathbf{E}_{2\Phi-1}(\vartheta(\tau + \mathrm{m})^{2\Phi-1})} \leq & 3\mathbf{E}\left[\|\Phi - \beta\|^2\right] + \frac{\Gamma(2\xi - 1)}{\vartheta} 6\mathrm{L}^2(\tau + 1) \\ & \times \mathop{sup}_{\mathrm{s} \in [0, \hbar]} \frac{\mathbb{E}\left[\|\mho_\xi(\mathrm{s}, \Phi) - \mho_\xi(\mathrm{s}, \beta)\|^2\right]}{\mathbf{E}_{2\Phi-1}(\vartheta(\mathrm{s} + \mathrm{m})^{2\Phi-1})}. \end{aligned} \tag{40}$$

By definition of $\| \cdot \|_\vartheta$, gives us

$$\left(1 - \frac{\Gamma(2\xi - 1)}{\vartheta} 6\mathrm{L}^2(\tau + 1)\right) \|\mho_\xi(\tau, \Phi) - \mho_\xi(\tau, \beta)\|_\vartheta^2 \leq 3\|\Phi - \beta\|_{ms}^2, \tag{41}$$

which together with Eq. (20) and $\mho_\xi(\tau, \Phi) = \Phi(\tau)$ with $\tau \in [-\mathrm{m}, 0]$, we derive

$$\lim_{\Phi \to \beta} \sup_{\tau \in [-\mathrm{m}, \hbar]} \|\mho_\xi(\tau, \Phi) - \mho_\xi(\tau, \beta)\|_\vartheta^2 = 0. \tag{42}$$

The Con-D of solutions on the fractional exponents ξ and $\tilde{\xi}$ for DF-SDEs will be demonstrated in the ensuing theorem.

Theorem 4. *Assume that* $(\mathbb{H}_2)$ *and* $(\mathbb{H}_3)$ *are both valid. Following this, the solution* $\mho_\xi(., \Phi)$ *to Eq.* (4) *exhibits Con-D on* ξ, *i.e.*

$$\lim_{\tilde{\xi} \to \xi} \sup_{\tau \in [-\mathrm{m}, \hbar]} \|\mho_\xi(\tau, \Phi) - \mho_{\tilde{\xi}}(\tau, \Phi)\|_{ms} = 0. \tag{43}$$

Proof. Let $\xi, \tilde{\xi} \in (\frac{1}{2}, 1)$ be arbitrary but fixed. Select and adjust $\Phi \in \widetilde{\mathrm{C}}([-\mathrm{m}, 0], \mathbb{R}^{\mathrm{r}})$. As $\mho_\xi(., \Phi)$ and $\mho_{\tilde{\xi}}(., \Phi)$ are solutions of Eq. (4) and by utilizing Höld-I, I-Is, Eq. (9), and $(\mathbf{A}_3)$, we achieve

$$\begin{aligned}\frac{\mathbf{E}\big[\|\mho_\xi(\tau,\Phi)-\mho_{\tilde{\xi}}(\tau,\beta)\|^2\big]}{\mathbb{E}_{2\xi-1}(\vartheta(\tau+\mathrm{m})^{2\Phi-1})} \leq & \sup_{\mathrm{s}\in[0,\hbar]} \frac{\mathbf{E}\big[\|\mho_\xi(\mathrm{s},\Phi)-\mho_{\tilde{\xi}}(\mathrm{s},\beta)\|^2\big]}{\mathbb{E}_{2\xi-1}(\vartheta(\mathrm{s}+\mathrm{m})^{2\Phi-1})} 8(\tau+1)\mathrm{L}^2 \\ & \frac{\int_0^\tau \mathrm{s}^{2\xi-2}\mathbb{E}_{2\xi-1}(\vartheta(\mathrm{s}+\mathrm{m})^{2\Phi-1})d\mathrm{s}}{\mathbb{E}_{2\xi-1}(\vartheta(\mathrm{s}+\mathrm{m})^{2\Phi-1})} \\ & +4(\tau+1)\big(4\mathrm{L}^2 \sup_{\mathrm{s}\in[0,\hbar]}\|\mho_{\tilde{\xi}}(\mathrm{s},\Phi)\|^2+2\mathrm{L}^2\|\Phi\|_{ms}^2+2\mathbb{K}^2\big) \\ & \times\int_0^\tau \big(\lambda(\tau,\mathrm{s},\Phi,\tilde{\Phi})\big)^2, \end{aligned} \tag{44}$$

where

$$\lambda(\tau,\mathrm{s},\Phi,\tilde{\Phi}) = |\tau^{\Phi-1}-\tau^{\tilde{\Phi}-1}|. \tag{45}$$

By virtue of Lemma 2 and by the definition of $\|\cdot\|_\vartheta$, we attain

$$\begin{aligned}&\left(1-\frac{8(\hbar+1)\mathrm{L}^2\Gamma(2\xi-1)}{\vartheta}\right)\|\mho_\xi(\tau,\Phi)-\mho_{\tilde{\xi}}(\tau,\beta)\|_\vartheta^2 \\ &\leq 4(\hbar+1)4(\tau+1)\big(4\mathrm{L}^2 \sup_{\mathrm{s}\in[0,\hbar]}\|\mho_{\tilde{\xi}}(\mathrm{s},\Phi)\|^2+2\mathrm{L}^2\|\Phi\|_{ms}^2+2\mathbb{K}^2\big) \\ &\times\int_0^\tau \big(\lambda(\tau,\mathrm{s},\Phi,\tilde{\Phi})\big)^2 d\mathrm{s}.\end{aligned}$$

However, we yield

$$\begin{aligned}\int_0^\tau \big(\lambda(\tau,\mathrm{s},\Phi,\tilde{\Phi})\big)^2 d\mathrm{s} &= \int_0^\tau \mathrm{s}^{2\xi-2}d\mathrm{s}+\int_0^\tau \mathrm{s}^{2\tilde{\xi}-2}d\mathrm{s}+\int_0^\tau \mathrm{s}^{\xi+\tilde{\xi}-2}d\mathrm{s} \\ &= \frac{\tau^{2\xi-1}}{2\xi-1}+\frac{\tau^{2\tilde{\xi}-1}}{2\tilde{\xi}-1}+\frac{\tau^{\xi+\tilde{\xi}-1}}{\xi+\tilde{\xi}-1}.\end{aligned}$$

As a result,

$$\lim_{\tilde{\xi}\to\xi}\sup_{\tau\in[0,\hbar]}\int_0^\tau \big(\lambda(\tau,\mathrm{s},\Phi,\tilde{\Phi})\big)^2 d\mathrm{s} = 0.$$

When associated with Eq. (4), this suggests the completion of the proof.

We shall demonstrate the regularity of solutions to DF-SDEs in the following subsection.

3.2. The regularity of solutions to DF-SDEs

In the following theorem, we established the result for regularity of the solutions of DF-SDEs.

Theorem 5. *Assume that* $(\mathbf{A}_2)$ *and* $(\mathbf{A}_3)$ *are true. Assume further that* Φ *is* $\xi-\frac{1}{2}$*-Hölder continuous (Höld-Con) i.e.,* $\forall \tau, \tilde{\alpha} \in [-\mathrm{m}, 0]$ *there is a constant* $\rho_1 > 0$

$$\|\Phi(\tau) - \Phi(\tilde{\alpha})\| \leq \rho_1 |\tau - \tilde{\alpha}|^{\xi-\frac{1}{2}}.$$

Next, based on $\xi, \hbar, \mathbb{K}, \mathrm{L}, \rho_1$*, there is a constant* $\rho > 0$ *that corresponds to*

$$\|\mho_\xi(\tau, \Phi) - \mho_\xi(\tilde{\alpha}, \Phi)\|_{ms} \leq \rho |\tau - \tilde{\alpha}|^{\xi-\frac{1}{2}}, \ \forall \tau, \tilde{\alpha} \in [-\mathrm{m}, \hbar].$$

Proof. The proof has been separated into three parts for explanation.
Step 1: $\tau, \ \tilde{\alpha} \in [0, \hbar]$: Select and fix $\tau, \ \tilde{\alpha} \in [0, \hbar], \tau > \tilde{\alpha}$. By utilizing Eq. (9) and I-Is, we attained the following:

$$\frac{1}{4}\mathbf{E}\big(\|\mho_\xi(\tau, \Phi) - \mho_\xi(\tilde{\alpha}, \Phi)\|^2\big) \leq \mathbf{E}\Big(\Big\| \int_c^\tau \mathrm{s}^{\xi-1} \mathrm{V}(\mathrm{s}, \mho_\xi(\mathrm{s}, \Phi), \mho_\xi(\mathrm{s}-\mathrm{m}, \Phi)) d\mathrm{s} \Big\|^2\Big) + \int_c^\tau \mathrm{s}^{2\xi-2} \mathbf{E}\|F(\mathrm{s}, \mho_\xi(\mathrm{s}, \Phi), \mho_\xi(\mathrm{s}-\mathrm{m}, \Phi))\|^2 d\mathrm{s}.$$

Additionally, considering $\mho_\xi(., \Phi) \in \widetilde{\mathbb{H}}\big([-\mathrm{m}, \hbar]\big)$, there is $\mathbb{K}_1 > 0$, which means

$$\sup_{\tau \in [-\mathrm{m}, \hbar]} \mathbf{E}\|\mho_\xi(\tau, \Phi)\|^2 \leq \mathbb{K}_1. \tag{42}$$

By utilizing $(\mathbf{A}_2)$, $(\mathbf{A}_3)$, Höld-I and Eq. (3.2), we have the following:

$$\frac{1}{4}\mathbf{E}\big(\|\mho_\xi(\tau, \Phi) - \mho_\xi(\tilde{\alpha}, \Phi)\|^2\big) \leq \frac{2\mathbb{K}^2 + 4\mathrm{L}^2\mathbb{K}_1 + 2\mathrm{L}^2\mathbf{E}\|\Phi\|^2}{2\xi - 1}\big((\tau - c)^{2\xi} + (\tau - c)^{2\xi-1}\big).$$

Consequently, we attained the following:

$$\|\mho_\xi(\tau, \Phi) - \mho_\xi(c, \Phi)\|_{ms} \leq \rho_2 (\tau - c)^{\xi-\frac{1}{2}},$$

where

$$\rho_2^2 = 4\frac{2\mathbb{K}^2 + 4\mathrm{L}^2\mathbb{K}_1 + 2\mathrm{L}^2\mathbf{E}\|\Phi\|^2(3\hbar + 1)}{2\xi - 1}$$

As a result, we get the following result:

$$\lim_{c \to \tau} \|\mho_\xi(\tau, \Phi) - \mho_\xi(c, \Phi)\|_{ms} = 0.$$

Step 2:$\tau, c \in [-\mathrm{m}, 0]$, since Φ is $\xi - \frac{1}{2}$-Höld-Con and $\mho_\xi(\tau, \Phi) = \Phi(\tau)$ with $\tau \in [-\mathrm{m}, 0]$, Eqs. (5) and (6), we yield the following:

$$\|\mho_\xi(\tau, \Phi) - \mho_\xi(c, \Phi)\|_{ms} \leq \rho_1 (\tau - c)^{\xi-\frac{1}{2}}.$$

Step 3: $-\mathrm{m} < c \leq 0 \leq \tau \leq \hbar$: We address the outcomes in **Step 1** and **Step 2** and achieve

$$\|\mho_\xi(\tau, \Phi) - \mho_\xi(c, \Phi)\|_{ms} \leq \|\mho_\xi(\tau, \Phi) - \mho_\xi(0, \Phi)\|_{ms} + \|\mho_\xi(0, \Phi) - \mho_\xi(c, \Phi)\|_{ms}$$

$$\rho_2\ |\tau|^{\xi-\frac{1}{2}} + \rho_1\ |-c|^{\xi-\frac{1}{2}}.$$

In addition, if $\theta_1, \theta_2 > 0$ and $0 < \xi < 1$, then there is a constant $\mu \in (0, \frac{1}{2}]$ that

$$\theta_1^{\xi} + \theta_2^{\xi} \leq \mu^{\xi-1}(\theta_1 + \theta_2)^{\xi}.$$

As a result, we get

$$\|\mho_{\xi}(\tau, \Phi) - \mho_{\xi}(c, \Phi)\|_{ms} \leq \rho|\tau - c|^{\xi-\frac{1}{2}},$$

where, $\rho = \max(\rho_1, \rho_2)\mu^{\xi-\frac{3}{2}}$. The required inequality is established.

3.3. The UH stability of DF-SDEs

We now provide an explanation of UH stability.

Definition 3. *The Eq.* (4) *is UH stability with respect to* ϵ *if there is a constant* $\chi > 0$ *such that for each* $\epsilon > 0$ *and* $\mathbb{J}(\tau) \in \widetilde{\mathbb{H}}^2([0, \hbar])$ *of the following inequality:* $\forall \tau \in [0, \hbar]$

$$\mathbf{E}\left[\left\|\mathbb{J}(\tau) - \mathbb{J}(0) - \left(\int_0^{\tau} s^{\xi-1}\Big(V\big(s, \mathbb{J}(s), \mathbb{J}(s-m)\big)ds + \digamma\big(s, \mathbb{J}(s), \mathbb{J}(s-m)\big)dw(s)\Big)\right)\right\|^2\right]$$
$$\leq \epsilon,$$

there exists a solution $x(\tau) \in \widetilde{\mathbb{H}}^2([0, \hbar])$ *of Eq.* (4)*, with* $x_0 = \Phi = \Phi(\tau) : \tau \in [-m, 0]$*, satisfies* $\mathbf{E}\big[\|\mathbb{J}(\tau) - x(\tau)\|^2\big] \leq \chi\epsilon$*,* $\forall \tau \in [0, \hbar]$*.*

Theorem 6. *Under* ($\mathbf{A}_2$) *and* ($\mathbf{A}_3$)*, the DF-SDEs Eq.* (4) *is UH stability on* $[0, \hbar]$*.*

Proof. Set $\epsilon > 0$ and $\mathbb{J}(\tau) \in \widetilde{\mathbb{H}}^2([0, \hbar])$ satisfies Eq. (3). Denote by $x(\tau) \in \widetilde{\mathbb{H}}^2([0, \hbar])$ be the unique solution of Eq. (4) with initial value $x(\tau) = \mathbb{J}(\tau)$ for $\tau \in [-m, 0]$, then

$$x(\tau) = \mathbb{J}(0) + \left(\int_0^{\tau} s^{\xi-1} V\big(s, x(s), x(s-m)\big)ds \right.$$
$$\left. + \int_0^{\tau} s^{\xi-1} \digamma\big(s, x(s), x(s-m)\big)dw(s)\right).$$

Thus,

$$\mathbf{E}\big[\|\mathbb{J}(\tau) - x(\tau)\|^2\big] \leq 2\mathbf{E}\left[\left\|\mathbb{J}(\tau) - \mathbb{J}(0) - \left(\int_0^{\tau} s^{\xi-1} V\big(s, \mathbb{J}(s), \mathbb{J}(s-m)\big)ds \right.\right.\right.$$
$$\left.\left.\left. + \int_0^{\tau} s^{\xi-1} \digamma\big(s, \mathbb{J}(s), \mathbb{J}(s-m)\big)dw(s)\right)\right\|^2\right]$$
$$+ 2\mathbf{E}\left[\left\|\left(\int_0^{\tau} s^{\xi-1}\Big(V\big(s, \mathbb{J}(s), \mathbb{J}(s-m)\big) - V\big(s, x(s), x(s-m)\big)\Big)ds\right.\right.\right.$$

$$+\int_0^{\tau} \mathrm{s}^{\xi-1}\Big(\digamma\big(\mathrm{s},\mathbb{J}(\mathrm{s}),\mathbb{J}(\mathrm{s}-\mathrm{m})\big)-\digamma\big(\mathrm{s},\mathrm{x}(\mathrm{s}),\mathrm{x}(\mathrm{s}-\mathrm{m})\big)\Big)d\mathrm{w}(\mathrm{s})\Big)\Big\|^2\Bigg].$$

Thus, using Cauchy-Schwarz inequality, $(\mathbf{A}_2)$ and $(\mathbf{A}_3)$, we can derive that

$$\begin{aligned}
&\mathbf{E}\big[\|\mathbb{J}(\tau)-\mathrm{x}(\tau)\|^2\big] \leq 2\epsilon \\
&+4\mathbf{E}\Bigg[\Big\|\int_0^{\tau} \mathrm{s}^{\xi-1}\Big(\mathrm{V}\big(\mathrm{s},\mathbb{J}(\mathrm{s}),\mathbb{J}(\mathrm{s}-\mathrm{m})\big)-\mathrm{V}\big(\mathrm{s},\mathrm{x}(\mathrm{s}),\mathrm{x}(\mathrm{s}-\mathrm{m})\big)\Big)d\mathrm{s}\Big\|^2\Bigg] \\
&+4\mathbf{E}\Big\|\int_0^{\tau} \mathrm{s}^{\xi-1}\Big(\digamma\big(\mathrm{s},\mathbb{J}(\mathrm{s}),\mathbb{J}(\mathrm{s}-\mathrm{m})\big)-\digamma\big(\mathrm{s},\mathrm{x}(\mathrm{s}),\mathrm{x}(\mathrm{s}-\mathrm{m})\big)\Big)d\mathrm{w}(\mathrm{s})\Big\|^2\Bigg] \\
&\leq 2\epsilon+\frac{8\mathrm{L}^2\hbar^{2\xi-1}}{(2\xi-1)}\mathbf{E}\Bigg[\int_0^{\tau}\Big(\|\mathbb{J}(\mathrm{s})-\mathrm{x}(\mathrm{s})\|^2+\|\mathbb{J}(\mathrm{s}-\mathrm{m})-\mathrm{x}(\mathrm{s}-\mathrm{m})\|^2\Big)d\mathrm{s}\Bigg] \\
&+8\mathrm{L}^2\mathbf{E}\Bigg[\int_0^{\tau}\mathrm{s}^{2\xi-2}\Big(\|\mathbb{J}(\mathrm{s})-\mathrm{x}(\mathrm{s})\|^2+\|\mathbb{J}(\mathrm{s}-\mathrm{m})-\mathrm{x}(\mathrm{s}-\mathrm{m})\|^2\Big)d\mathrm{s}\Bigg].
\end{aligned}$$

Let $\phi(\tau)=\sup_{\mathrm{s}\in[0,\hbar]}\mathbf{E}\big[\|\mathbb{J}(\mathrm{s})-\mathrm{x}(\mathrm{s})\|^2\big]$ for $\tau\in[0,\hbar]$.
We have

$$\mathbf{E}\big[\|\mathbb{J}(\mathrm{s})-\mathrm{x}(\mathrm{s})\|^2\big]\leq\phi(\mathrm{s}),$$

and

$$\mathbf{E}\big[\|\mathbb{J}(\mathrm{s}-\mathrm{m})-\mathrm{x}(\mathrm{s}-\mathrm{m})\|^2\big]\leq\phi(\mathrm{s}),\ \forall\mathrm{s}\in[0,\hbar].$$

Then, for $\tau\in[0,\hbar]$, we obtain

$$\begin{aligned}
\mathbf{E}\big[\|\mathbb{J}(\tau)-\mathrm{x}(\tau)\|^2\big] \leq & 2\epsilon+\frac{16\mathrm{L}^2\hbar^{2\xi-1}}{(2\xi-1)}\int_0^{\tau}\phi(\mathrm{s})d\mathrm{s} \\
&+16\mathrm{L}^2\int_0^{\tau}\mathrm{s}^{2\xi-2}\phi(\mathrm{s})d\mathrm{s}.
\end{aligned}$$

Hence, $\forall\tau\in[0,\hbar]$,

$$\begin{aligned}
\mathbf{E}\big[\|\mathbb{J}(\mathrm{s})-\mathrm{x}(\mathrm{s})\|^2\big] \leq & 2\epsilon+\frac{16\mathrm{L}^2\hbar^{2\xi-1}}{(2\xi-1)}\int_0^{\tau}\phi(\mathrm{s})d\mathrm{s} \\
&+16\mathrm{L}^2\int_0^{\tau}\mathrm{s}^{2\xi-2}\phi(\mathrm{s})d\mathrm{s}.
\end{aligned}$$

Then,

$$\phi(\tau)\leq 2\epsilon+\gamma_1\int_0^{\tau}\phi(\mathrm{s})d\mathrm{s}+\gamma_2\int_0^{\tau}\mathrm{s}^{2\xi-2}\phi(\mathrm{s})d\mathrm{s},$$

for all $\tau\in[0,\hbar]$, where $\gamma_1=\frac{16\mathrm{L}^2\hbar^{2\xi-1}}{(2\xi-1)}$ and $\gamma_2=16\mathrm{L}^2$.
Using G-I, we get

$$\phi(\tau)\leq\left(2\epsilon+\gamma_1\int_0^{\tau}\phi(\mathrm{s})d\mathrm{s}\right)\mathbb{E}_{2\xi-1}\left(\gamma_2\Gamma(2\xi-1)\,(\xi)^{2\xi-1}\right)$$

$$\leq \gamma_3\epsilon + \gamma_4 \int_0^\tau \phi(\mathrm{s})d\mathrm{s},$$

where $\gamma_3 = 2\mathbb{E}_{2\xi-1}\left(\gamma_2\Gamma(2\xi-1)\hbar^{2\xi-1}\right)$ and $\gamma_4 = \gamma_1\mathbb{E}_{2\xi-1}\left(\gamma_2\Gamma(2\xi-1)\mathrm{x}^{2\xi-1}\right)$.
Using G-I, we get

$$\phi(\tau) \leq \gamma_3\epsilon e^{\gamma_4(\xi)}.$$

Hence,

$$\mathbf{E}[\|\mathbb{J}(\tau) - \mathrm{x}(\tau)\|^2] \leq \aleph\epsilon,\ \forall\tau \in [0, \hbar],$$

where $\chi = \gamma_3 e^{\gamma_4\hbar}$. Thus, Eq. (4) is UH stability.

4. Examples

This section provides examples to demonstrate our results.
Example 1. The following is the Langevin DF-SDE:

$$\mathrm{D}_\tau^\xi\mathrm{x}(\tau) = -\Upsilon\mathrm{x}(\tau - \mathrm{m}) + \omega\frac{d\mathrm{w}_\tau}{d\tau},\ \frac{1}{2} < \xi < 1,\ 0 < \tau < \hbar, \tag{28}$$

$$\mathrm{x}(\tau) = 1,\ -\mathrm{m} \leq \tau \leq 0, \tag{28}$$

where, m is delay factor and $\Upsilon, \omega \in \mathbb{R}^+$, and $\mathrm{V} = -\Upsilon\mathrm{x}(\tau - \mathrm{m})$, $\digamma = \omega$ are drift and diffusion factors. In the earlier discussed model Eqs. (4) and (4), which was studied in [36] with $\xi = 1$ represented the statistical physics of auto traffic. We take $\mathrm{m} = 0.1$, $\Upsilon = 0.5, \omega = 1$. We can easily see that GLC is fulfilled by the $-\Upsilon\mathrm{x}(\tau - \mathrm{m})$ and ω coefficients in Eq. (4). As a consequence, Theorem 1 asserts the existence of a unique solution under the historical functions $\mathrm{x}(\tau) = 1$, $\mathrm{m} \leq \tau \leq 0$ in the interval $-\mathrm{m} \leq \tau \leq \hbar$.
Example 2. Consider the Mackey-Glass DF-SDE.

$$\mathrm{D}_\tau^\xi\mathrm{x}(\tau) = \frac{\kappa\mathrm{x}(\tau - \mathrm{m})}{1 + \mathrm{x}^{10}(\tau - \mathrm{m})} - \Lambda\mathrm{x}(\tau) + \ell\mathrm{x}\frac{d\mathrm{w}_\tau}{d\tau},\ \frac{1}{2} < \xi < 1,\ 0 < \tau < \hbar, \tag{28}$$

$$\mathrm{x}(\tau) = 0.5,\ -\mathrm{m} \leq \tau \leq 0, \tag{28}$$

where, m is delay factor and $\kappa, \Lambda, \ell \in \mathbb{R}^+$, with $\mathrm{V} = \frac{\ell\mathrm{x}(\tau-\mathrm{m})}{1+\mathrm{x}^{10}(\tau-\mathrm{m})}$ and $\digamma = -\kappa\mathrm{x}(\tau) + \Lambda\mathrm{x}$. $\xi = 1$ was used in [37] to analyze the model Eqs. (3.2) and (3.2), this model explains the stochastic increase in blood cell density. We consider $\mathrm{m} = 5$, $\kappa = 1, \Lambda = 2$, $\ell = 2$. The GLC is satisfied by the $\frac{\ell\mathrm{x}(\tau-\mathrm{m})}{1+\mathrm{x}^{10}(\tau-\mathrm{m})}$ and diffusion $-\kappa\mathrm{x}(\tau) + \lambda\mathrm{x}$ coefficients in Eq. (3.2). As a result, the solution exists with historical function $\mathrm{x} = 0.5$, $\mathrm{m} \leq \tau \leq 0$ in the interval $-\mathrm{m} \leq \tau \leq \hbar$, and Theorem 1 states that it is unique.
Example 3. We consider a population model governed by a DF-SDE, defined as:

$$\mathrm{D}_\tau^\xi\mathrm{P}(\tau) = \mathrm{V}(\tau, \mathrm{P}(\tau), \mathrm{P}(\tau - \mathrm{m})) + \digamma(\tau, \mathrm{P}(\tau), \mathrm{P}(\tau - \mathrm{m}))\frac{d\mathrm{w}_\tau}{d\tau}.$$

The deterministic part of the model is given by:

$$\mathrm{V}(\tau, \mathrm{P}(\tau), \mathrm{P}(\tau - \mathrm{m})) = \tilde{\mathrm{r}}\mathrm{P}(\tau)\left(1 - \frac{\mathrm{P}(\tau) + 0.2\mathrm{P}(\tau - \mathrm{m})}{\mathrm{K}}\right)$$

where:

- $\tilde{\mathrm{r}} > 0$ is the intrinsic growth rate,
- $\mathrm{K} > 0$ is the carrying capacity,
- The delayed feedback $0.2\mathrm{P}(\tau - \mathrm{m})$ introduces a historical influence on growth.

This represents a logistic-type growth with delay, where the population's current growth is affected by both its present and past sizes. Such delayed effects may arise due to reproduction lags, environmental responses, or maturation delays.

The stochastic part of the model is:

$$\digamma(\tau, \mathrm{P}(\tau), \mathrm{P}(\tau - \mathrm{m})) = 1.5\sqrt{\mathrm{P}(\tau)}.$$

This term captures the influence of random environmental fluctuations. The noise is multiplicative and population-dependent:

- The coefficient 1.5 determines the intensity of randomness.
- The use of $\sqrt{\mathrm{P}(\tau)}$ ensures that noise increases with population size and becomes negligible near extinction.

By combining the deterministic and stochastic components, the population model is given by:

$$\mathrm{D}_{\tau}^{\xi}\mathrm{P}(\tau) = \tilde{\mathrm{r}}\mathrm{P}(\tau)\left(1 - \frac{\mathrm{P}(\tau) + 0.2\mathrm{P}(\tau - \mathrm{m})}{\mathrm{K}}\right) + 1.5\sqrt{\mathrm{P}(\tau)}\frac{d\mathrm{w}_{\tau}}{d\tau}.$$

This equation captures

- Nonlinear logistic growth with delay,
- Memory effects through conformable fractional calculus,
- Environmental uncertainty via stochastic forcing.

To obtain an approximate solution of the DF-SDE for population growth, we employ the Euler–Maruyama method. This approach allows us to simulate the evolution of population size under the influence of random fluctuations, delayed feedback mechanisms, and intrinsic growth dynamics. Multiple simulation paths are generated to capture the system's inherent stochasticity, and the average of these trajectories provides an estimate of the expected population behavior over time. The resulting population dynamics are illustrated in the figure below.

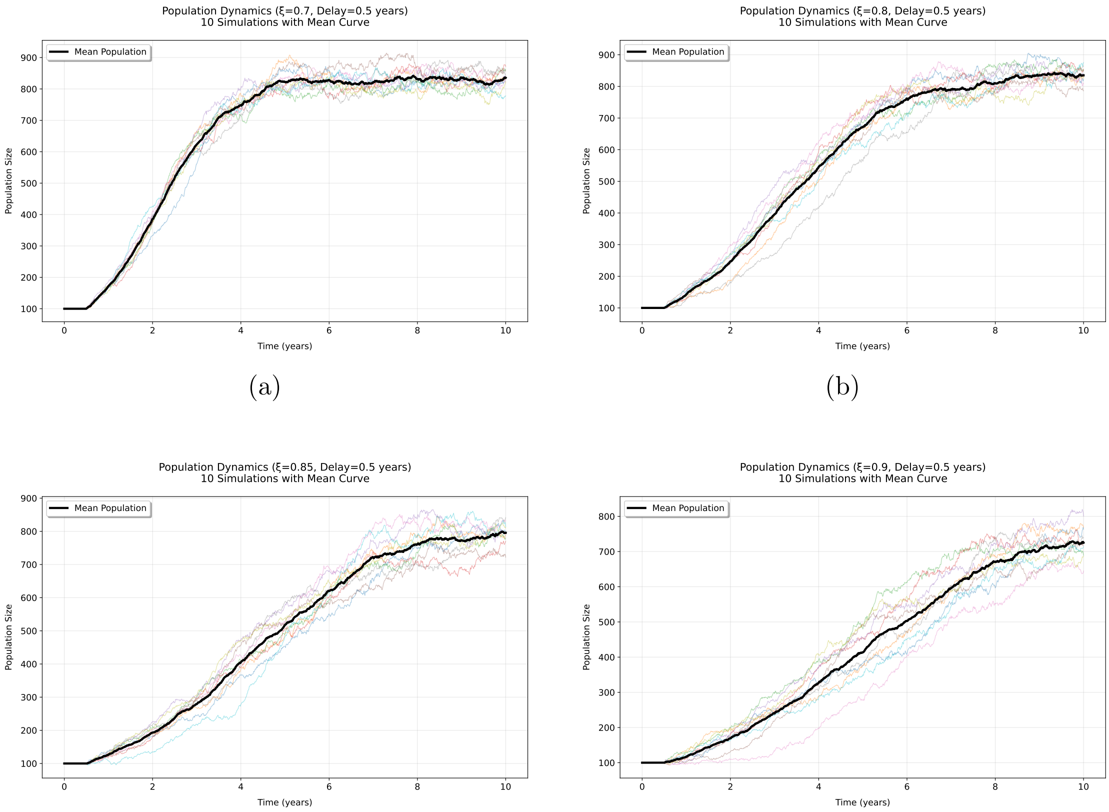


Figure 1: Population dynamics under different conformable fractional orders: (a) $\xi = 0.7$ showing strong memory effects, (b) $\xi = 0.8$ with moderate memory, (c) $\xi = 0.85$ demonstrating weak memory, and (d) $\xi = 0.9$ (near-classical case). All simulations use $\tilde{\mathrm{r}} = 0.3$, $\mathrm{K} = 1000$, $\sigma = 1.5$, $\mathrm{m} = 0.5$ years, and $\mathrm{P}(0) = 100$. Light curves represent individual realizations; bold curves show the mean.

Example 4. We consider the DF-SDE for modeling temperature dynamics as follows:

$$\mathrm{D}_{\tau}^{\xi}\mathrm{T}(\tau) = -\alpha\left(\mathrm{T}(\tau) - \mathrm{T}_{\mathrm{env}}(\tau)\right) + \beta\left(\mathrm{T}(\tau - \mathrm{m}) - \mathrm{T}_{\mathrm{env}}(\tau)\right) + \sigma(\mathrm{T}(\tau))\,\frac{d\mathrm{w}(\tau)}{d\tau},$$

where:

The drift component is given below:

$$\mathrm{V}(\tau, \mathrm{T}(\tau), \mathrm{T}(\tau - \mathrm{m})) = -\alpha\left(\mathrm{T}(\tau) - \mathrm{T}_{\mathrm{env}}(\tau)\right) + \beta\left(\mathrm{T}(\tau - \mathrm{m}) - \mathrm{T}_{\mathrm{env}}(\tau)\right),$$

Here, $\mathrm{T}(\tau)$ denotes the indoor temperature at time τ, $\mathrm{T}(\tau - \mathrm{m})$ represents the delayed temperature with a delay $\mathrm{m} > 0$, and $\mathrm{T}_{\mathrm{env}}(\tau)$ is the ambient (environmental) temperature. The parameter $\alpha = 0.5$ represents the rate of heat exchange, while $\beta = 0.2$ denotes the strength of the delayed feedback. The environmental temperature is modeled as follows:

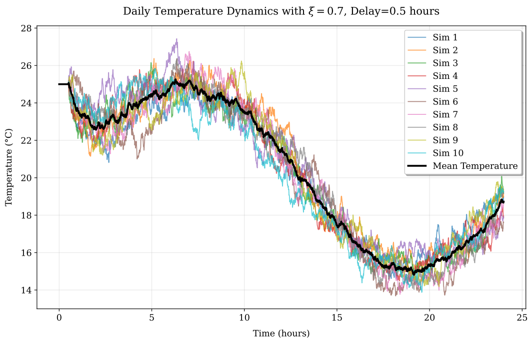
$$\mathrm{T}_{\mathrm{env}}(\tau) = 20 + 5\sin\left(\frac{2\pi\tau}{24}\right).$$

The diffusion component is given below:

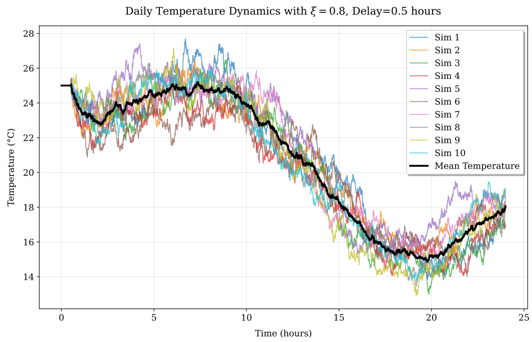
$$\mathcal{F}(\tau, \mathrm{T}(\tau), \mathrm{T}(\tau - \mathrm{m})) = 0.3\sqrt{|\mathrm{T}(\tau)|},$$

which introduces state-dependent stochastic fluctuations.

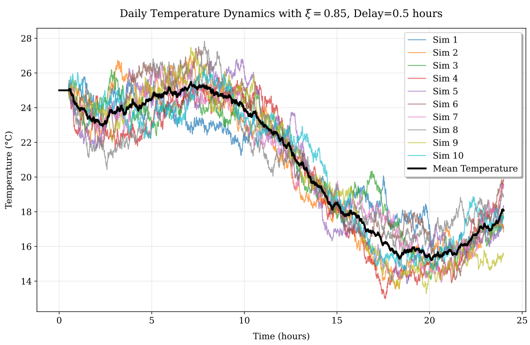
We numerically solve the DF-SDE governing temperature evolution using an adapted Euler-Maruyama method. The simulated thermal dynamics, including stochastic fluctuations and delayed feedback effects, are shown in Figure 2.



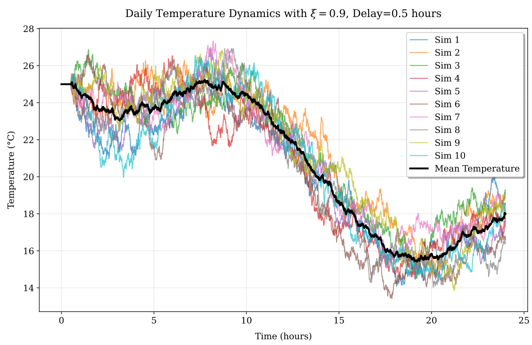
(a)



(b)



(c)



(d)

Figure 2: Population dynamics under different conformable fractional orders: (a) $\xi = 0.7$ showing strong memory effects, (b) $\xi = 0.8$ with moderate memory, (c) $\xi = 0.85$ demonstrating weak memory, and (d) $\xi = 0.9$ (near-classical case). All simulations use $\tilde{\mathrm{r}} = 0.3$, $\mathrm{K} = 1000$, $\sigma = 1.5$, $\mathrm{m} = 0.5$ years, and $\mathrm{P}(0) = 100$. Light curves represent individual realizations; bold curves show the mean.

5. Conclusions

In this study, we have established the EU, Con-D, regularity, and UH stability of the solutions to DF-SDEs in the framework of Con-FrD. We established the results of the EU in two ways: first, through the GLC of the coefficients. In the second stage, using a truncation approach, we have demonstrated the EU according to the LLC of the coefficients. We have also demonstrated that the solutions of DF-SDEs exhibit Con-D on the fractional order ξ and initial values under the GLC of the coefficients. Additionally, we have derived results concerning regularity and UH stability and presented two examples to illustrate our findings. The main part of the proof utilized the truncation process, the Banach fixed point approach, I-Is, temporally weighted norm, G-I, and Höld-I. In the future, we will focus on developing numerical methods for solving DF-SDEs within the framework of Con-FrD.

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