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A Novel of Ω-Proportional Fractional Integrals of a Function with Respect to Another Function

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Abstract. This paper explores a key topic in fractional calculus, which is the sophisticated idea of proportional fractional integrals with regard to another function. Our focus is on synchronous, monotonic, and bounded functions. We investigate the mathematical features and theoretical underpinnings of these integrals. The paper sheds fresh information on the behavior and uses of fractional integrals by concentrating on these particular types of functions, underscoring their potential for modeling intricate systems and processes. The findings provide new approaches for future study and useful applications, expanding our grasp of fractional calculus.

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1. Introduction

Integral inequalities are fundamental tools in mathematical analysis, as they provide valuable insights into the behavior of a function's integral—especially when exact evaluation is difficult or impossible. Common examples include Holder's and Minkowski's inequalities, both of which are closely related to L^p-spaces and norms. These inequalities play a key role in the study of function sequences and the stability of solutions to differential equations across various fields. By establishing upper and lower bounds, integral inequalities are also vital in solving optimization problems, see ([1]-[8]). The study of differential equations, functional analysis, and probability theory all depend on integral

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inequalities, which are basic tools in mathematical analysis that provide integrals boundaries. These inequalities, which frequently involve requirements on monotonicity, convexity, or other functional features, establish links between integrals of functions. Hölder's, Minkowski's, and Gronwall's inequalities are classical examples that are essential for estimating solutions and demonstrating the existence and uniqueness of a variety of mathematical problems. In addition to helping with theoretical research, integral inequalities have many uses in fields like economics, engineering, and physics where integral expressions naturally occur in modeling and analysis. For more details, see ([9]-[17]).

Differentiation and integration are extended to non-integer (fractional) orders in fractional calculus, a generalization of classical calculus. A more flexible and precise modeling of complex systems with memory and hereditary qualities is made possible by fractional calculus, which permits operations of arbitrary order in contrast to classical calculus, which works with integer-order derivatives and integrals. This discipline has received a lot of interest lately because of its applicability in a number of fields, including biological systems, control theory, viscoelasticity, anomalous diffusion, and signal processing. The Riemann–Liouville, Caputo, and Grunwald–Letnikov derivatives are among the concepts that form the mathematical basis of fractional calculus, and each is appropriate for a particular kind of issue. With the use of these instruments, fractional calculus offers a strong foundation for explaining dynamic phenomena that traditional models are unable to effectively represent. Mathematicians including Leibniz, Liouville, Riemann, and others investigated the idea of extending fractional calculus. The fractional derivative, which has multiple definitions (Riemann-Liouville, Caputo), is appropriate for a certain set of features and applications see ([18]-[21]).

Definition 1. *[22] Consider* $\mathfrak{f} \in \mathcal{L}[\mathfrak{a}, \mathfrak{b}]$*. The left-right-sided Riemann-Liouville (R–L) fractional integrals of order* $\xi > 0$ *are defined by*

$$ {}_{\mathfrak{a}}\mathfrak{J}^{\xi}\mathfrak{f}(\tau) = \frac{1}{\Gamma(\xi)} \int_{\mathfrak{a}}^{\tau} (\tau - \mu)^{\xi-1} \mathfrak{f}(\mu) d\mu, \qquad \mathfrak{a} < \tau \tag{1} $$

and

$$ \mathfrak{J}_{\mathfrak{b}}^{\xi}\mathfrak{f}(\tau) = \frac{1}{\Gamma(\xi)} \int_{\tau}^{\mathfrak{b}} (\mu - \tau)^{\xi-1} \mathfrak{f}(\mu) d\mu, \qquad \tau < \mathfrak{b}, \tag{2} $$

where the Gamma function is defined as $\Gamma(\xi) = \int_0^\infty e^{-u} u^{\xi-1} du$.

This integral is motivated by the reputed and well known Cauchy formula as follows:

$$ \int_{\mathfrak{a}}^{x} d\tau_1 \int_{\mathfrak{a}}^{\tau_1} d\tau_2 ... \int_{\mathfrak{a}}^{\tau_{n-1}} \mathfrak{f}\left(tau_n\right) d\tau_n = \frac{1}{\Gamma\left(n\right)} \int_{\mathfrak{a}}^{x} \left(-\tau\right)^{n-1} \mathfrak{f}\left(\tau\right) d\tau. \tag{3} $$

Definition 2. *[23, 24] Suppose* $(\mathfrak{a}, \mathfrak{b})$ *is a finite interval of real line* $\Re$ *and* $\Re(\xi) > 0$*. Also that suppose* $\Omega(x)$ *is an increasing and positive monotone function on* $(\mathfrak{a}, \mathfrak{b})$*, having a continuous derivative* $\Omega'(x)$ *on* $(\mathfrak{a}, \mathfrak{b})$*. The left-right sided fractional integrals of a function* $\mathfrak{f}$ *with respect to another function* Ω *on* $[\mathfrak{a}, \mathfrak{b}]$ *are defined by*

$$ (\mathfrak{J}_{\mathfrak{a}+,\Omega}^{\xi}\mathfrak{f})(\tau) = \frac{1}{\Gamma(\xi)} \int_{\mathfrak{a}}^{\tau} (\Omega(\tau) - \Omega(\mu))^{\xi-1} \Omega'(\mu) \mathfrak{f}(\mu) d\mu, \qquad \mathfrak{a} < \tau \tag{4} $$

and

$$(\mathfrak{J}^{\xi}_{\mathfrak{b}-,\Omega}\mathfrak{f})(\tau)=\frac{1}{\Gamma(\xi)}\int_{\tau}^{\mathfrak{b}}(\Omega(\mu)-\Omega(\tau))^{\xi-1}\Omega'(\mu)\mathfrak{f}(\mu)d\mu, \qquad \tau<\mathfrak{b}. \tag{5}$$

From (4) and (5),

$$(\mathfrak{J}^{\xi}_{\mathfrak{a}+,\Omega}\mathfrak{f})(\tau)=(\mathfrak{J}^{\xi}_{\mathfrak{b}-,\Omega}\mathfrak{f})(\tau)=0 \tag{6}$$

If we choose $\Omega(x)=x$ in the integral formulas (4) and (5), we have

$$\mathfrak{J}^{\xi}_{\mathfrak{a}+,\Omega}=\mathfrak{J}^{\xi}_{\mathfrak{a}+} \quad \textit{and} \quad \mathfrak{J}^{\xi}_{\mathfrak{b}-,\Omega}=\mathfrak{J}^{\xi}_{\mathfrak{b}-}. \tag{7}$$

If $\mathfrak{a}=0$ in (4), we can write

$$(\mathfrak{J}^{\xi}_{0+,\Omega}\mathfrak{f})(\tau)=\frac{1}{\Gamma(\xi)}\int_{0}^{\tau}(\Omega(\tau)-\Omega(\mu))^{\xi-1}\Omega'(\mu)\mathfrak{f}(\mu)d\mu, \qquad 0<\tau \tag{8}$$

$$(\mathfrak{J}^{\xi}_{0+,\Omega}\mathfrak{f})(\tau)=\mathfrak{f}(\tau).$$

For the convenience of establishing the results, we give the semi-group property:

$$\mathfrak{J}^{\xi}_{\mathfrak{a}+,\Omega}\mathfrak{J}^{\beta}_{\mathfrak{a}+,\Omega}\mathfrak{f}(\tau)=\mathfrak{J}^{\xi+\beta}_{\mathfrak{a}+,\Omega}\mathfrak{f}(\tau), \xi\geq 0, \beta\geq 0,$$

which gives the commutative property holding as

$$\mathfrak{J}^{\xi}_{\mathfrak{a}+,\Omega}\mathfrak{J}^{\beta}_{\mathfrak{a}+,\Omega}\mathfrak{f}(\tau)=J^{\beta}_{\mathfrak{a}+,\Omega}\mathfrak{J}^{\xi}_{\mathfrak{a}+,\Omega}\mathfrak{f}(\tau).$$

Definition 3. *(Modified conformable derivatives) For* $\gamma\in[0,1]$, *let the functions* $x_0, x_1 : [0,1]\times\Re\to[0,+\infty)$ *be continuous such that for all* $t\in\Re$, *we have*

$$\lim_{\varrho\to 0^+} x_1(\gamma,\kappa)=1, \ \lim_{\gamma\to 0^+} x_0(\gamma,\kappa)=0, \ \lim_{\gamma\to 1^-} x_1(\gamma,\kappa)=0, \ \lim_{\gamma\to 1^-} x_0(\gamma,\kappa)=1,$$

and

$$x_1(\gamma,\kappa)\neq 0, \quad \gamma\in[0,1], \quad x_0(\gamma,\kappa)\neq 0, \quad \gamma\in(0,1].$$

Then the modified conformable differential operator of order γ *is defined by*

$$D^{\gamma}\mathfrak{f}(\kappa)=x_1(\gamma,\kappa)\,\mathfrak{f}(\kappa)+x_0(\gamma,\kappa)\,\mathfrak{f}'(\kappa). \tag{9}$$

The derivative given in (9) *is said to be proportional derivative. For more details, see the literature [25]-[26].*

Definition 4. *[27] For $\gamma > 0$ and $\xi \in \mathfrak{C}, Re(\xi) > 0$, the left and right proportional fractional integrals of $\mathfrak{f}$ are respectively defined as*

$$\left({}_{\mathfrak{a}}I^{\xi,\gamma}\mathfrak{f}\right)(\kappa) = \frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{\kappa} e^{\frac{\gamma-1}{\gamma}(\kappa-\tau)}(\kappa-\tau)^{\xi-1}\mathfrak{f}(\tau)\,d\tau \tag{10}$$

and

$$\left(I_{\mathfrak{b}}^{\xi,\gamma}\mathfrak{f}\right)(\kappa) = \frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{\kappa} e^{\frac{\gamma-1}{\gamma}(\tau-\kappa)}(\tau-\kappa)^{\xi-1}\mathfrak{f}(\tau)\,d\tau. \tag{11}$$

Remark 1. *If we choose $\gamma = 1$ in the integral formulas* (10) *and* (11)*, we find* (1) *and* (2)*.*

The fractional proportional derivative of a function with respect to another function is as follows:

Definition 5. *[28] For $\gamma \in (0, 1]$, $\xi \in \mathfrak{C}$, such that for all $\kappa \in \Re$, $\Re(\xi) > 0, \Omega \in \mathfrak{C}[\mathfrak{a}, \mathfrak{b}]$ where $\Omega' > 0$, we define left and right fractional integrals of $\mathfrak{f}$ with respect to Ω by*

$$\left({}_{\mathfrak{a}}\mathfrak{I}^{\xi,\gamma,\Omega}\mathfrak{f}\right)(\kappa) = \frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{\kappa} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\mu))}(\Omega(\kappa)-\Omega(\mu))^{\xi-1}\Omega'(\mu)\mathfrak{f}(\mu)\,d\mu \tag{12}$$

and

$$\left(\mathfrak{I}_{\mathfrak{b}}^{\xi,\gamma,\Omega}\mathfrak{f}\right)(\kappa) = \frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\kappa}^{\mathfrak{b}} e^{\frac{\gamma-1}{\gamma}(\Omega(\mu)-\Omega(\kappa))}(\Omega(\mu)-\Omega(\kappa))^{\xi-1}\Omega'(\mu)\mathfrak{f}(\mu)\,d\mu. \tag{13}$$

Remark 2. *If we choose $\Omega(y) = y$ in the integral formulas* (12) *and* (13)*, we find* (10) *and* (11)*.*

Remark 3. *If we choose $\Omega(y) = y$ and $\gamma = 1$ in the integral formulas* (12) *and* (13)*, we find* (1) *and* (2)*.*

Remark 4. *If we choose $\gamma = 1$ in the integral formulas* (12) *and* (13)*, we find* (4) *and* (5)*.*

In fractional calculus, the proportional fractional integrals with respect to another function are an advanced topic. With a specific emphasis on synchronous, monotonic and bounded functions, it entails integrating a function using a fractional order that is proportionate to another function. In contrast to monotonic functions, which either continuously rise or decrease, synchronous functions change jointly in a predictable way.

2. Main Results

The development of new integral inequalities involving the Ω-proportional fractional integral of a function with respect to another function marks a significant advancement in the theory of fractional calculus and its applications. These inequalities are formulated within a generalized integral framework where the integration process is governed by a proportionality function Ω, and the integration is carried out with respect to another function rather than the independent variable. Such an approach allows for a more flexible and context-sensitive analysis of functions, especially those exhibiting memory effects, scaling behavior, or singularities.

In this section, we prove some $\Omega-$proportional fractional integrals of a function with respect to another function by synchronous and monotonic functions on $[0,+\infty)$.

Theorem 1. *Suppose that $\mathfrak{f}$ and $\mathfrak{g}$ are two synchronous functions on $[0,+\infty)$, then for $\kappa>\mathfrak{a},\xi\in\mathfrak{C}$, $\Re(\xi)>0$, $\gamma\in(0,1]$, the following $\Omega-$proportional holds:*

$$[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\mathfrak{g}\right)(\kappa)]\ .\ [{}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}(1)]\ \geq\ [\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\right)(\kappa)]\ .\ [\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{g}\right)(\kappa)]. \tag{14}$$

Proof. If $\mathfrak{f}$ and $\mathfrak{g}$ are synchronous functions, we have

$$[\mathfrak{f}(\tau)-\mathfrak{f}(\varrho)][\mathfrak{g}(\tau)-\mathfrak{g}(\varrho)]\geq 0. \tag{15}$$

From (15), it can be written as

$$\mathfrak{f}(\tau)\,\mathfrak{g}(\tau)+\mathfrak{f}(\varrho)\,\mathfrak{g}(\varrho)\ \geq\ \mathfrak{f}(\tau)\,\mathfrak{g}(\varrho)+\mathfrak{f}(\varrho)\,\mathfrak{g}(\tau)\,. \tag{16}$$

Multiplying both sides of (16) with $\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}\left(\Omega(\kappa)-\Omega(\tau)\right)^{\xi-1}\Omega'(\tau)$, $\tau\in(\mathfrak{a},\kappa)$ with respect to τ, we obtain

$$\begin{aligned}
&\mathfrak{f}(\tau)\,\mathfrak{g}(\tau)\,\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}\left(\Omega(\kappa)-\Omega(\tau)\right)^{\xi-1}\Omega'(\tau)+\\
&\mathfrak{f}(\varrho)\,\mathfrak{g}(\varrho)\,\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}\left(\Omega(\kappa)-\Omega(\tau)\right)^{\xi-1}\Omega'(\tau)\\
&\geq\ \mathfrak{f}(\tau)\,g(\varrho)\,\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}\left(\Omega(\kappa)-\Omega(\tau)\right)^{\xi-1}\Omega'(\tau)\\
&+\mathfrak{f}(\varrho)\,\mathfrak{g}(\tau)\,\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}\left(\Omega(\kappa)-\Omega(\tau)\right)^{\xi-1}\Omega'(\tau)\,.
\end{aligned} \tag{17}$$

Integrating the inequality (17) at $(\mathfrak{a},\kappa)$ with respect to τ, we have

$$\frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{t}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}\left(\Omega(\kappa)-\Omega(\tau)\right)^{\xi-1}\mathfrak{f}(\tau)\,\mathfrak{g}(\tau)\ \ \Omega'(\tau)\,d\tau+$$

$$\begin{aligned}
&\frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{t} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}\left(\Omega(\kappa)-\Omega(\tau)\right)^{\xi-1}\mathfrak{f}(\varrho)\,\mathfrak{g}(\varrho)\,\Omega'(\tau)\,d\tau\\
&\geq \frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{\kappa} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}\left(\Omega(\kappa)-\Omega(\tau)\right)^{\xi-1}\mathfrak{f}(\tau)\,\mathfrak{g}(\varrho)\,\Omega'(\tau)\,d\tau\\
&+\frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{\kappa} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}\left(\Omega(\kappa)-\Omega(\tau)\right)^{\xi-1}\mathfrak{f}(\varrho)\,\mathfrak{g}(\tau)\,\Omega'(\tau)\,d\tau.
\end{aligned}$$

$$[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\mathfrak{g}\right)(\kappa)] + \mathfrak{f}(\varrho)\,\mathfrak{g}(\varrho)\,[{}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}(1)] \geq \mathfrak{g}(\varrho)\,[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\right)(\kappa)] + \mathfrak{f}(\varrho)\,[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{g}\right)(\kappa)]. \tag{18}$$

Multiplying both sides of (18) with $\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\xi-1}\Omega'(\varrho)$, $\varrho\in(\mathfrak{a},\kappa)$ with respect to ϱ, we obtain

$$\begin{aligned}
&[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\mathfrak{g}\right)(\kappa)]\ \frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\xi-1}\Omega'(\varrho)+\\
&\mathfrak{f}(\varrho)\,\mathfrak{g}(\varrho)\,[{}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}(1)]\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\xi-1}\Omega'(\varrho)\\
&\geq \mathfrak{g}(\varrho)\,[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\right)(\kappa)]\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\xi-1}\Omega'(\varrho)\\
&+\mathfrak{f}(\varrho)\,[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{g}\right)(\kappa)]\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\xi-1}\Omega'(\varrho).
\end{aligned}\tag{19}$$

Integrating inequality (19) at $(\mathfrak{a},\kappa)$ with respect to ϱ, we have

$$\begin{aligned}
&[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\mathfrak{g}\right)(\kappa)]\ \frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{\kappa} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\xi-1}\Omega'(\varrho)\,d\varrho+\\
&[{}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}(1)]\frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{t} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\xi-1}\mathfrak{f}(\varrho)\,\mathfrak{g}(\varrho)\,\Omega'(\varrho)\,d\varrho\\
&\geq [\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\right)(\kappa)]\frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{\kappa} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\xi-1}\mathfrak{g}(\varrho)\,\Omega'(\varrho)\,d\varrho\\
&+[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{g}\right)(\kappa)]\frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{\kappa} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\xi-1}\mathfrak{f}(\varrho)\,\Omega'(\varrho)\,d\varrho.
\end{aligned}$$

Therefore, the inequality can be written as

$$[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\mathfrak{g}\right)(\kappa)]\cdot[{}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}(1)] \geq [\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\right)(\kappa)]\cdot[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{g}\right)(\kappa)].$$

This completes the proof.

Theorem 2. *Suppose that* $\mathfrak{f}$ *and* $\mathfrak{g}$ *are two synchronous functions on* $[0,+\infty)$*, then for* $\kappa > \mathfrak{a}, \xi \in \mathfrak{C}$*,* $\xi > 0, \beta > 0$*,* $\gamma \in (0,1]$*, the following* $\Omega-$*proportional holds:*

$$\begin{aligned}&[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\mathfrak{g}\right)(\kappa)] \;.\; [{}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}(1)] + [\left({}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}\mathfrak{f}\mathfrak{g}\right)(\kappa)][{}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}(1)] \\ &\geq\; [\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\right)(\kappa)] \;.[\left({}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}\mathfrak{g}\right)(\kappa) + [\left({}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}\mathfrak{f}\right)(\kappa)].[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{g}\right)(\kappa)].\end{aligned} \tag{20}$$

Proof. Let $\mathfrak{f}$ and $\mathfrak{g}$ be synchronous functions on $[0,+\infty)$. For all $\tau, \varrho \geq 0$, multiplying both sides of (18) with $\frac{1}{\gamma^{\beta}\Gamma(\beta)} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))} (\Omega(\kappa) - \Omega(\varrho))^{\beta-1} \Omega'(\varrho)$, $\varrho \in (\mathfrak{a}, \kappa)$ with respect to ϱ, we obtain

$$\begin{aligned}&[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\mathfrak{g}\right)(\kappa)] \;\frac{1}{\gamma^{\beta}\Gamma(\beta)} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))} (\Omega(\kappa) - \Omega(\varrho))^{\beta-1} \Omega'(\varrho) \\ &+ \mathfrak{f}(\varrho)\,\mathfrak{g}(\varrho)\,[{}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}(1)].\frac{1}{\gamma^{\beta}\Gamma(\beta)} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))} (\Omega(\kappa) - \Omega(\varrho))^{\beta-1} \Omega'(\varrho) \\ &\geq\; \mathfrak{g}(\varrho)\,[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\right)(\kappa)]\frac{1}{\gamma^{\beta}\Gamma(\beta)} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))} (\Omega(\kappa) - \Omega(\varrho))^{\beta-1} \Omega'(\varrho) \\ &+ \mathfrak{f}(\varrho)\,[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{g}\right)(\kappa)].\frac{1}{\gamma^{\beta}\Gamma(\beta)} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))} (\Omega(\kappa) - \Omega(\varrho))^{\beta-1} \Omega'(\varrho).\end{aligned} \tag{21}$$

Integrating the inequality (21) at $(\mathfrak{a}, \kappa)$ with respect to ϱ, then we have

$$\begin{aligned}&[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\mathfrak{g}\right)(\kappa)] \;\frac{1}{\gamma^{\beta}\Gamma(\beta)} \int_{\mathfrak{a}}^{\kappa} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))} (\Omega(\kappa) - \Omega(\varrho))^{\beta-1} \Omega'(\varrho)\, d\varrho \\ &+ [{}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}(1)].\frac{1}{\gamma^{\beta}\Gamma(\beta)} \int_{\mathfrak{a}}^{\kappa} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))} (\Omega(\kappa) - \Omega(\varrho))^{\beta-1} \mathfrak{f}(\varrho)\,\mathfrak{g}(\varrho)\,\Omega'(\varrho)\, d\varrho \\ &\geq\; [\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\right)(\kappa)]\frac{1}{\gamma^{\beta}\Gamma(\beta)} \int_{\mathfrak{a}}^{\kappa} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))} (\Omega(\kappa) - \Omega(\varrho))^{\beta-1} \mathfrak{g}(\varrho)\,\Omega'(\varrho)\, d\varrho \\ &+ [\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{g}\right)(\kappa)].\frac{1}{\gamma^{\beta}\Gamma(\beta)} \int_{\mathfrak{a}}^{\kappa} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))} (\Omega(\kappa) - \Omega(\varrho))^{\beta-1} \mathfrak{f}(\varrho)\,\Omega'(\varrho)\, d\varrho.\end{aligned} \tag{22}$$

Then we have

$$\begin{aligned}&[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\mathfrak{g}\right)(\kappa)] \;.\; [{}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}(1)] + [\left({}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}\mathfrak{f}\mathfrak{g}\right)(\kappa)][{}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}(1)] \\ &\geq\; [\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\right)(\kappa)] \;.[\left({}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}\mathfrak{g}\right)(\kappa) + [\left({}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}\mathfrak{f}\right)(\kappa)].[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{g}\right)(\kappa)].\end{aligned}$$

This completes the proof.

Remark 5. *It is obvious that if we let* $\xi = \beta$ *in Theorem 2, it reduces to Theorem 1.*

Theorem 3. *Suppose that* $\mathfrak{f}$*,* $\mathfrak{g}$ *and* θ *are three monotone functions defined on* $[0,+\infty)$*, satisfying the following inequality*

$$[\mathfrak{f}(\tau) - \mathfrak{f}(\varrho)][\mathfrak{g}(\tau) - \mathfrak{g}(\varrho)][\theta(\tau) - \theta(\varrho)] \geq 0,$$

then for all $\tau, \varrho \in [\mathfrak{a}, \kappa]$ $\kappa > \mathfrak{a}, \xi > 0, \beta > 0,$ $\gamma \in (0, 1]$, *the* $\Omega-$*proportional holds:*

$$\begin{aligned}
&[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\mathfrak{g}\theta\right)(\kappa)]\ [{}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}(1)]\ -[{}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}(1)][\left({}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}\mathfrak{f}\mathfrak{g}\theta\right)(\kappa)]\\
&\geq\ [\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\theta\right)(\kappa)]\ .\ [\left({}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}\mathfrak{g}\right)(\kappa)]+[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{g}\theta\right)(\kappa)]\ .\ [\left({}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}\mathfrak{f}\right)(\kappa)]\\
&-[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\theta\right)(\kappa)]\ .\ [\left({}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}\mathfrak{f}\mathfrak{g}\right)(\kappa)]+[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\mathfrak{g}\right)(\kappa)]\ .\ [\left({}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}\theta\right)(\kappa)]\\
&+[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\right)(\kappa)]\ .\ [\left({}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}\mathfrak{g}\theta\right)(\kappa)]-[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{g}\right)(\kappa)]\ .\ [\left({}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}\mathfrak{f}\theta\right)(\kappa)]. \qquad (23)
\end{aligned}$$

Proof. Since $\mathfrak{f}$, $\mathfrak{g}$ and θ are three monotonic functions defined on $[0, +\infty)$, then for all $\tau, \varrho \geq 0$, we have

$$[\mathfrak{f}(\tau)-\mathfrak{f}(\varrho)][\mathfrak{g}(\tau)-\mathfrak{g}(\varrho)][\theta(\tau)-\theta(\varrho)] \geq 0. \qquad (24)$$

From (24), it can be written as

$$\begin{aligned}
&\mathfrak{f}(\tau)\,\mathfrak{g}(\tau)\,\theta(\tau)-\mathfrak{f}(\varrho)\,\mathfrak{g}(\varrho)\,\theta(\varrho)-\mathfrak{f}(\tau)\,\mathfrak{g}(\varrho)\,\theta(\tau)-\mathfrak{f}(\varrho)\,\mathfrak{g}(\tau)\,\theta(\tau)\\
&+\mathfrak{f}(\varrho)\,\mathfrak{g}(\varrho)\,\theta(\tau)-\mathfrak{f}(\tau)\,\mathfrak{g}(\tau)\,\theta(\varrho)-\mathfrak{f}(\tau)\,\mathfrak{g}(\varrho)\,\theta(\varrho)+\mathfrak{f}(\varrho)\,\mathfrak{g}(\tau)\,\theta(\varrho) \geq\ 0. \qquad (25)
\end{aligned}$$

Multiplying both sides of (25) with $\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}(\Omega(\kappa)-\Omega(\tau))^{\xi-1}\Omega'(\tau)$, $\tau \in (\mathfrak{a}, \kappa)$ with respect to τ, we obtain

$$\begin{aligned}
&\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}(\Omega(\kappa)-\Omega(\tau))^{\xi-1}\Omega'(\tau)\,\mathfrak{f}(\tau)\,\mathfrak{g}(\tau)\,\theta(\tau)\\
&-\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}(\Omega(\kappa)-\Omega(\tau))^{\xi-1}\Omega'(\tau)\,\mathfrak{f}(\varrho)\,\mathfrak{g}(\varrho)\,\theta(\varrho)\\
&-\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}(\Omega(\kappa)-\Omega(\tau))^{\xi-1}\Omega'(\tau)\,\mathfrak{f}(\tau)\,\mathfrak{g}(\varrho)\,\theta(\tau)\\
&-\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}(\Omega(\kappa)-\Omega(\tau))^{\xi-1}\Omega'(\tau)\,\mathfrak{f}(\varrho)\,\mathfrak{g}(\tau)\,\theta(\tau)\\
&+\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}(\Omega(\kappa)-\Omega(\tau))^{\xi-1}\Omega'(\tau)\,\mathfrak{f}(\varrho)\,\mathfrak{g}(\varrho)\,\theta(\tau)\\
&-\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}(\Omega(\kappa)-\Omega(\tau))^{\xi-1}\Omega'(\tau)\,\mathfrak{f}(\tau)\,\mathfrak{g}(\tau)\,\theta(\varrho)\\
&-\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}(\Omega(\kappa)-\Omega(\tau))^{\xi-1}\Omega'(\tau)\,\mathfrak{f}(\tau)\,\mathfrak{g}(\varrho)\,\theta(\varrho)\\
&+\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}(\Omega(\kappa)-\Omega(\tau))^{\xi-1}\Omega'(\tau)\,\mathfrak{f}(\varrho)\,\mathfrak{g}(\tau)\,\theta(\varrho) \geq\ 0. \qquad (26)
\end{aligned}$$

Integrating the inequality (26) at $(\mathfrak{a}, \kappa)$ with respect to τ, we have

$$\frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{\kappa}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}(\Omega(\kappa)-\Omega(\tau))^{\xi-1}\Omega'(\tau)\,\mathfrak{f}(\tau)\,\mathfrak{g}(\tau)\,\theta(\tau)\,d\tau$$

$$
\begin{aligned}
&-\frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{\kappa} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}\left(\Omega(\kappa)-\Omega(\tau)\right)^{\xi-1}\Omega'(\tau)\,\mathfrak{f}(\varrho)\,\mathfrak{g}(\varrho)\,\theta(\varrho)\,d\tau\\
&-\frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{\kappa} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}\left(\Omega(\kappa)-\Omega(\tau)\right)^{\xi-1}\Omega'(\tau)\,\mathfrak{f}(\tau)\,\mathfrak{g}(\varrho)\,\theta(\tau)\,d\tau\\
&-\frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{\kappa} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}\left(\Omega(\kappa)-\Omega(\tau)\right)^{\xi-1}\Omega'(\tau)\,\mathfrak{f}(\varrho)\,\mathfrak{g}(\tau)\,\theta(\tau)\,d\tau\\
&+\frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{\kappa} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}\left(\Omega(\kappa)-\Omega(\tau)\right)^{\xi-1}\Omega'(\tau)\,\mathfrak{f}(\varrho)\,\mathfrak{g}(\varrho)\,\theta(\tau)\,d\tau\\
&-\frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{\kappa} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}\left(\Omega(\kappa)-\Omega(\tau)\right)^{\xi-1}\Omega'(\tau)\,\mathfrak{f}(\tau)\,\mathfrak{g}(\tau)\,\theta(\varrho)\,d\tau\\
&-\frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{\kappa} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}\left(\Omega(\kappa)-\Omega(\tau)\right)^{\xi-1}\Omega'(\tau)\,\mathfrak{f}(\tau)\,\mathfrak{g}(\varrho)\,\theta(\varrho)\,d\tau\\
&+\frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{\kappa} e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}\left(\Omega(\kappa)-\Omega(\tau)\right)^{\xi-1}\Omega'(\tau)\,\mathfrak{f}(\varrho)\,\mathfrak{g}(\tau)\,\theta(\varrho)\,d\tau \geq 0. \qquad (27)
\end{aligned}
$$

That is,

$$
\begin{aligned}
&[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\mathfrak{g}\theta\right)(\kappa)] - \mathfrak{f}(\varrho)\,\mathfrak{g}(\varrho)\,\theta(\varrho)\,.[{}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}(1)] \geq \mathfrak{g}(\varrho)\,[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\theta\right)(\kappa)]\\
&+\mathfrak{f}(\varrho)\,[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{g}\theta\right)(\kappa)] - \mathfrak{f}(\varrho)\,\mathfrak{g}(\varrho)\,[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\theta\right)(\kappa)] + \theta(\varrho)\,[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\mathfrak{g}\right)(\kappa)]\\
&+\mathfrak{g}(\varrho)\,\theta(\varrho)\,[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\right)(\kappa)] - \mathfrak{f}(\varrho)\,\theta(\varrho)\,[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{g}\right)(\kappa)]. \qquad (28)
\end{aligned}
$$

Multiplying both sides of (28) with $\frac{1}{\gamma^{\beta}\Gamma(\beta)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\beta-1}\Omega'(\varrho)$, $\varrho \in (\mathfrak{a},\kappa)$ with respect to ϱ, we obtain

$$
\begin{aligned}
&[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\mathfrak{g}\theta\right)(\kappa)]\int_{\mathfrak{a}}^{\kappa}\frac{1}{\gamma^{\beta}\Gamma(\beta)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\beta-1}\Omega'(\varrho)\\
&-[{}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}(1)]\int_{\mathfrak{a}}^{\kappa}\frac{1}{\gamma^{\beta}\Gamma(\beta)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\beta-1}\Omega'(\varrho)\,\mathfrak{f}(\varrho)\,\mathfrak{g}(\varrho)\,\theta(\varrho)\,d\varrho\\
&\geq [\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\theta\right)(\kappa)]\int_{\mathfrak{a}}^{\kappa}\frac{1}{\gamma^{\beta}\Gamma(\beta)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\beta-1}\Omega'(\varrho)\,\mathfrak{g}(\varrho)\,d\varrho\\
&+[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{g}\theta\right)(\kappa)]\int_{\mathfrak{a}}^{\kappa}\frac{1}{\gamma^{\beta}\Gamma(\beta)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\beta-1}\Omega'(\varrho)\,\mathfrak{f}(\varrho)\,d\varrho\\
&-[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\theta\right)(\kappa)]\int_{\mathfrak{a}}^{\kappa}\frac{1}{\gamma^{\beta}\Gamma(\beta)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\beta-1}\Omega'(\varrho)\,\mathfrak{f}(\varrho)\,\mathfrak{g}(\varrho)\,d\varrho\\
&+[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\mathfrak{g}\right)(\kappa)]\int_{\mathfrak{a}}^{\kappa}\frac{1}{\gamma^{\beta}\Gamma(\beta)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\beta-1}\Omega'(\varrho)\,\theta(\varrho)\,d\varrho\\
&+[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\right)(\kappa)]\int_{\mathfrak{a}}^{\kappa}\frac{1}{\gamma^{\beta}\Gamma(\beta)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\beta-1}\Omega'(\varrho)\,\mathfrak{g}(\varrho)\,\theta(\varrho)\,d\varrho\\
&-[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{g}\right)(\kappa)]\int_{\mathfrak{a}}^{\kappa}\frac{1}{\gamma^{\beta}\Gamma(\beta)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\beta-1}\Omega'(\varrho)\,\mathfrak{f}(\varrho)\,\theta(\varrho)\,d\varrho. \qquad (29)
\end{aligned}
$$

That is,

$$\begin{aligned}
&[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\mathfrak{g}\theta\right)(\kappa)]\ [{}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}(1)]\ -[{}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}(1)][\left({}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}\mathfrak{f}\mathfrak{g}\theta\right)(\kappa)]\\
&\geq\ [\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\theta\right)(\kappa)]\cdot[\left({}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}\mathfrak{g}\right)(\kappa)]+[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{g}\theta\right)(\kappa)]\cdot[\left({}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}\mathfrak{f}\right)(\kappa)]\\
&-[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\theta\right)(\kappa)]\cdot[\left({}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}\mathfrak{f}\mathfrak{g}\right)(\kappa)]+[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\mathfrak{g}\right)(\kappa)]\cdot[\left({}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}\theta\right)(\kappa)]\\
&+[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\right)(\kappa)]\cdot[\left({}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}\mathfrak{g}\theta\right)(\kappa)]-[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{g}\right)(\kappa)]\cdot[\left({}_{\mathfrak{a}}I^{\beta,\gamma,\Omega}\mathfrak{f}\theta\right)(\kappa)]. \qquad (30)
\end{aligned}$$

This completes the proof.

3. Inequalities involving $\Omega-$proportional fractional integrals of a function with respect to another function by bounded functions

In this section, we prove some $\Omega-$proportional fractional integrals of a function with respect to another function by bounded functions.

Theorem 4. *Suppose that $\mathfrak{f}$ is an integrable function on $[\mathfrak{a},\mathfrak{b}]$, then for $\kappa>\mathfrak{a},\xi\in\mathfrak{C}$, $\Re(\xi)>0$, $\gamma\in(0,1],\Phi_1,\Phi_2\in[\mathfrak{a},\mathfrak{b}]$ and $\Phi_1\leq\mathfrak{f}\leq\Phi_2$, the following $\Omega-$proportional holds:*

$$\begin{aligned}
&\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\Phi_2\right)(\kappa)]\ [\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}\mathfrak{f}\right)(\kappa)]+\ [\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}\mathfrak{f}\right)(\kappa)]\ [\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}\Phi_1\right)(\kappa)]\\
&\geq\ [\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}\Phi_2\right)(\kappa)][\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}\Phi_1\right)(\kappa)]+[\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}\mathfrak{f}\right)(\kappa)]^2. \qquad (31)
\end{aligned}$$

Proof. If $\tau,\varrho\in(\mathfrak{a},\kappa)$, then we have

$$[\Phi_2(\tau)-\mathfrak{f}(\tau)][\mathfrak{f}(\varrho)-\Phi_1(\varrho)]\geq 0. \qquad (32)$$

From (32), it can be written as

$$\Phi_2(\tau)\,\mathfrak{f}(\varrho)+\mathfrak{f}(\tau)\,\Phi_1(\varrho)\ \geq\ \Phi_2(\tau)\,\Phi_1(\varrho)+\mathfrak{f}(\varrho)\,\mathfrak{f}(\tau)\,. \qquad (33)$$

Multiplying both sides of (16) with $\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}(\Omega(\kappa)-\Omega(\tau))^{\xi-1}\Omega'(\tau)$, $\tau\in(\mathfrak{a},\kappa)$ with respect to τ, we obtain

$$\begin{aligned}
&\Phi_2(\tau)\,\mathfrak{f}(\varrho)\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}(\Omega(\kappa)-\Omega(\tau))^{\xi-1}\Omega'(\tau)+\\
&\mathfrak{f}(\tau)\,\Phi_1(\varrho)\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}(\Omega(\kappa)-\Omega(\tau))^{\xi-1}\Omega'(\tau)\\
&\geq\ \Phi_2(\tau)\,\Phi_1(\varrho))\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}(\Omega(\kappa)-\Omega(\tau))^{\xi-1}\Omega'(\tau)
\end{aligned}$$

$$+\mathfrak{f}(\varrho)\,\mathfrak{f}(\tau)\,\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}\left(\Omega(\kappa)-\Omega(\tau)\right)^{\xi-1}\Omega'(\tau)\,. \tag{34}$$

Integrating the inequality (34) at $(\mathfrak{a},\kappa)$ with respect to τ, we have

$$\begin{aligned}
&\mathfrak{f}(\varrho)\,\frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{\kappa}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}\left(\Omega(\kappa)-\Omega(\tau)\right)^{\xi-1}\Omega'(\tau)\,\Phi_2(\tau)\,d\tau+\\
&\Phi_1(\varrho)\,\frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{t}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}\left(\Omega(\kappa)-\Omega(\tau)\right)^{\xi-1}\Omega'(\tau)\,\mathfrak{f}(\tau)\,d\tau\\
&\geq\ \Phi_1(\varrho))\frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{\kappa}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}\left(\Omega(\kappa)-\Omega(\tau)\right)^{\xi-1}\Omega'(\tau)\,\Phi_2(\tau)\,d\tau\\
&+\mathfrak{f}(\varrho)\,\frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{t}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}\left(\Omega(\kappa)-\Omega(\tau)\right)^{\xi-1}\Omega'(\tau)\,\mathfrak{f}(\tau)\,d\tau.
\end{aligned}$$

That is,

$$\begin{aligned}
&\mathfrak{f}(\varrho)[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\Phi_2\right)(\kappa)]\ +\Phi_1(\varrho)\,[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\right)(\kappa)]\\
&\geq\ \Phi_1(\varrho)\,[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\Phi_2\right)(\kappa)]\ +\mathfrak{f}(\varrho)\,[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\right)(\kappa)].
\end{aligned} \tag{35}$$

Multiplying both sides of (35) with $\frac{1}{\gamma^{\xi}\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\xi-1}\Omega'(\varrho)$, $\varrho\in(\mathfrak{a},\kappa)$ with respect to ϱ and integrating inequality at $(\mathfrak{a},\kappa)$ with respect to ϱ, then we obtain

$$\begin{aligned}
&[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\Phi_2\right)(\kappa)]\frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{\kappa}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\xi-1}\Omega'(\varrho)\,\mathfrak{f}(\varrho)d\varrho\ +\\
&[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\right)(\kappa)]\frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{\kappa}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\xi-1}\Omega'(\varrho)\,\Phi_1(\varrho)\,d\varrho\\
&\geq\ [\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\Phi_2\right)(\kappa)]\frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{\kappa}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\xi-1}\Omega'(\varrho)\,\Phi_1(\varrho)\,d\varrho\ +\\
&[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\right)(\kappa)]\frac{1}{\gamma^{\xi}\Gamma(\xi)}\int_{\mathfrak{a}}^{\kappa}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\xi-1}\Omega'(\varrho)\,\mathfrak{f}(\varrho)\,d\varrho.
\end{aligned} \tag{36}$$

Thus, we get the inequality (31).

Theorem 5. *Suppose that* $\mathfrak{f}$ *is an integrable function on* $[\mathfrak{a},\mathfrak{b}]$*, then for* $\kappa>\mathfrak{a},\xi\in\mathfrak{C}$, $\Re(\xi)>0$, $\gamma\in(0,1],\Phi_1,\Phi_2\in[\mathfrak{a},\mathfrak{b}]$ *and* $\Phi_1\leq\mathfrak{f}\leq\Phi_2,\lambda_1,\lambda_2>0$ *and* $\frac{1}{\lambda_1}+\frac{1}{\lambda_2}=1$*, the following* $\Omega-$*proportional holds:*

$$\begin{aligned}
&\frac{1}{\lambda_1}\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\left(\Phi_2-\mathfrak{f}\right)^{\lambda_1}\right)(\kappa)\right]\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}(1)\right)\right]\\
&+\frac{1}{\lambda_2}\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}(1)\right)\right]\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\left(\mathfrak{f}-\Phi_1\right)^{\lambda_2}\right)(\kappa)\right]
\end{aligned}$$

$$+\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}\Phi_2\right)(\kappa)\right]\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}\Phi_1\right)(\kappa)\right]+\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}\mathfrak{f}\right)(\kappa)\right]^2\geq$$

$$\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}\Phi_2\right)(\kappa)\right]\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}\mathfrak{f}\right)(\kappa)\right]+\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}\mathfrak{f}\right)(\kappa)\right]\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}\Phi_1\right)(\kappa)\right] \tag{37}$$

Proof. With the help of well-known Young's inequality (see [16]), one has

$$\frac{1}{\lambda_1}u^{\lambda_1}+\frac{1}{\lambda_2}v^{\lambda_2}\geq uv; \qquad where \;\; u,v\geq 0. \tag{38}$$

By setting the requirement $u=\Phi_2-\mathfrak{f}$ and $v=\mathfrak{f}-\Phi_1$, we have

$$\frac{1}{\lambda_1}[\Phi_2-\mathfrak{f}]^{\lambda_1}+\frac{1}{\lambda_2}[\mathfrak{f}-\Phi_1]^{\lambda_2}\geq[\Phi_2-\mathfrak{f}][\mathfrak{f}-\Phi_1]; \qquad where \;\; u,v\geq 0. \tag{39}$$

Multiplying both sides of (39) with $\frac{1}{\gamma^\xi\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\tau))}\left(\Omega(\kappa)-\Omega(\tau)\right)^{\xi-1}\Omega'(\tau)$, $\tau\in(\mathfrak{a},\kappa)$ with respect to τ, and integrating with respect to $\tau\in(\mathfrak{a},\kappa)$, we obtain

$$\begin{aligned}&\frac{1}{\lambda_1}\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\left(\Phi_2-\mathfrak{f}\right)^{\lambda_1}\right)(\kappa)\right]+\frac{1}{\lambda_2}\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}(1)\right)\right]\left[\left(\left(\mathfrak{f}(\varrho)-\Phi_1(\varrho)\right)^{\lambda_2}\right)(\kappa)\right]\\&+\Phi_1(\varrho)\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\Phi_2\right)(\kappa)\right]+\mathfrak{f}(\varrho)\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\right)(\kappa)\right]\\&\geq\ \mathfrak{f}(\varrho)\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\Phi_2\right)(\kappa)\right]+\Phi_1(\varrho)\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\mathfrak{f}\right)(\kappa)\right]\end{aligned} \tag{40}$$

Multiplying both sides of (40) with $\frac{1}{\gamma^\xi\Gamma(\xi)}e^{\frac{\gamma-1}{\gamma}(\Omega(\kappa)-\Omega(\varrho))}\left(\Omega(\kappa)-\Omega(\varrho)\right)^{\xi-1}\Omega'(\varrho)$, $\varrho\in(\mathfrak{a},\kappa)$ with respect to ϱ and integrating inequality at $(\mathfrak{a},\kappa)$ with respect to ϱ, then after getting the simplification, we get

$$\begin{aligned}&\frac{1}{\lambda_1}\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\left(\Phi_2-\mathfrak{f}\right)^{\lambda_1}\right)(\kappa)\right]\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}(1)\right)\right]\\&+\frac{1}{\lambda_2}\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}(1)\right)\right]\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,\Omega}\left(\mathfrak{f}-\Phi_1\right)^{\lambda_2}\right)(\kappa)\right]\\&+\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}\Phi_2\right)(\kappa)\right]\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}\Phi_1\right)(\kappa)\right]+\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}\mathfrak{f}\right)(\kappa)\right]^2\geq\\&\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}\Phi_2\right)(\kappa)\right]\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}\mathfrak{f}\right)(\kappa)\right]+\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}\mathfrak{f}\right)(\kappa)\right]\left[\left({}_{\mathfrak{a}}I^{\xi,\gamma,h}\Phi_1\right)(\kappa)\right].\end{aligned}$$

This completes the proof.

4. Special cases

Here, we aim at present some new generalizations via proportional fractional integrals with respect to another function, which are the new estimates of the main consequences.

Corollary 1. *Under the assumptions of Theorem 1, the following inequality holds:*

$$[\left({}_{\mathfrak{a}}I^{\xi,\gamma}\mathfrak{f}\mathfrak{g}\right)(\kappa)]\ .\ [{}_{\mathfrak{a}}I^{\xi,\gamma}(1)]\ \geq\ [\left({}_{\mathfrak{a}}I^{\xi,\gamma}\mathfrak{f}\right)(\kappa)]\ .\ [\left({}_{\mathfrak{a}}I^{\xi,\gamma}\mathfrak{g}\right)(\kappa)].$$

Proof. Letting $\Omega(x) = x$ in Theorem 1 yields the proof of Corollary 1.

Corollary 2. *Under the assumptions of Theorem 2, the following inequality holds:*

$$\begin{aligned}&[\left({}_{\mathfrak{a}}I^{\xi,\gamma}\mathfrak{f}\mathfrak{g}\right)(\kappa)]\ .\ [{}_{\mathfrak{a}}I^{\beta,\gamma}(1)] + [\left({}_{\mathfrak{a}}I^{\beta,\gamma}\mathfrak{f}\mathfrak{g}\right)(\kappa)][{}_{\mathfrak{a}}I^{\xi,\gamma}(1)]\\ &\geq\ [\left({}_{\mathfrak{a}}I^{\xi,\gamma}\mathfrak{f}\right)(\kappa)]\ .[\left({}_{\mathfrak{a}}I^{\beta,\gamma}\mathfrak{g}\right)(\kappa) + [\left({}_{\mathfrak{a}}I^{\beta,\gamma}\mathfrak{f}\right)(\kappa)].[\left({}_{\mathfrak{a}}I^{\xi,\gamma}\mathfrak{g}\right)(\kappa)].\end{aligned}$$

Proof. Letting $\Omega(x) = x$ in Theorem 2 yields the proof of Corollary 2.

Corollary 3. *Under the assumptions of Theorem 2, the following inequality holds:*

$$\begin{aligned}&[\left({}_{\mathfrak{a}}I^{\gamma,\Omega}\mathfrak{f}\mathfrak{g}\right)(\kappa)]\ .\ [{}_{\mathfrak{a}}I^{\gamma,\Omega}(1)] + [\left({}_{\mathfrak{a}}I^{\gamma,\Omega}\mathfrak{f}\mathfrak{g}\right)(\kappa)][{}_{\mathfrak{a}}I^{\gamma,\Omega}(1)]\\ &\geq\ [\left({}_{\mathfrak{a}}I^{\gamma,\Omega}\mathfrak{f}\right)(\kappa)]\ .[\left({}_{\mathfrak{a}}I^{\gamma,\Omega}\mathfrak{g}\right)(\kappa) + [\left({}_{\mathfrak{a}}I^{\gamma,\Omega}\mathfrak{f}\right)(\kappa)].[\left({}_{\mathfrak{a}}I^{\gamma,\Omega}\mathfrak{g}\right)(\kappa)].\end{aligned}$$

Proof. Letting $\xi = 1 = \beta$ in Theorem 2 yields the proof of Corollary 3.

5. Conclusion

The Ω-proportional fractional integral of a function with respect to another function, in conclusion, provides a strong and cohesive framework that greatly expands the current and classical fractional integral operators. This innovative method expands the versatility and usefulness of fractional calculus in simulating intricate, nonlocal, and memory-dependent events by introducing the proportionality function Ω and permitting integration with respect to another function. In addition to generalizing well-known theorems, the recently developed integral inequalities within this framework provide new opportunities for theoretical investigation and real-world applications. These contributions provide the foundation for future study and advancement in the field by offering strong mathematical tools for the analysis of fractional differential equations and the creation of more precise models in a variety of scientific and technical fields.

Authors' Contributions

All authors contribute equally in this paper.

Conflict of interest

The authors declare that they have no conflict of interest.

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