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An Inductive Product of Terms

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Abstract. Over the years, many binary operations on the set of all n-ary terms of type τ have been defined, derived from superpositions and forming semigroups. Later, an inductive composition of terms was introduced as a generalization of a superposition, extending its scope from variable replacement to subterm replacement. From this, a binary operation called an r-inductive product was introduced by fixing a specific subterm to be replaced. In this study, we define a new binary operation, called an rs-inductive product, which generalizes the r-inductive product by allowing the simultaneous replacement of two specific subterms. We construct a semigroup equipped with the new operation and investigate its algebraic properties, including regular elements, idempotent elements, and Green's relations.

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1. Introduction

In universal algebra, the concept of terms plays a crucial role as formal representations in equations and identities within algebraic structures. In addition to their foundational role in algebra, terms have found applications in other areas, particularly in computer science and formal languages. For further background and applications, the readers are referred to [1].

In the study of terms, various operations on the set of terms have been introduced over the past decades. Among these, superpositions have been extensively studied due to their satisfaction of the superassociative law, a generalization of associativity (see, e.g., [1]). One of the earliest forms is the superposition S^n_m, which maps an n-ary term and an n-tuple of m-ary terms to an m-ary term by substituting each variable in the n-ary term with the corresponding m-ary term. In 2001, Denecke and Leeratanavalee [2] extended this operation to a more general form, denoted by S^n_g, which is an $(n+1)$-ary operation defined on terms of arbitrary arity. A special case where $n = m$, denoted S^n, was later

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mentioned in [3]. Moreover, the idea of superpositions has also been extended to sets of terms, known as tree languages. This extension, denoted by $\hat{S}^n_m$, was introduced in [4].

Superpositions, which satisfy the superassociative law, have inspired many researchers to transform them into binary operations and study the semigroups equipped with these operations. Following this approach, Denecke and Jampachon [5] introduced four binary operations, denoted by $+, *, +_g$, and $*_g$, derived from the superposition S^n, and investigated idempotent and regular elements as well as Green's relations in the resulting semigroups. Similarly, in the context of tree languages, Denecke and Sarasit [6] defined a product of tree languages, denoted by $\cdot_{x_i}$, which is a binary operation induced from the superposition $\hat{S}^n_m$.

The binary operation $\cdot_{x_i}$ can also be restricted to the set of terms, as shown by Kumduang and Leeratanavalee in [7]. This restricted operation corresponds to substituting every occurrence of the variable x_i in a term with another term. In a more general approach, which is not limited to variable replacement but also extends to subterm replacement, Shtrakov [8] introduced an inductive composition, a ternary operation that maps a triple (t, r, q) to a term obtained by simultaneously replacing every occurrence of the subterm r in t with q. Building on this, Kritpratyakul and Pibaljommee [9] defined a binary operation, called the r-inductive product and denoted by $\cdot_r$, by fixing a specific term r in the inductive composition.

Unlike other binary operations derived from superpositions, the r-inductive product is not associative on the entire set of terms, as demonstrated by Kritpratyakul and Pibaljommee [9]. However, they showed that the operation $\cdot_r$ becomes associative and closed on a certain subset of the set of terms. They further investigated the algebraic structure of the resulting semigroup, examining properties such as idempotent and regular elements, Green's relations, ideals, and special substructures (see [9–11]).

In this paper, we extend the concept of the r-inductive product by allowing two specific subterms to be replaced simultaneously. The resulting binary operation is called the rs-inductive product and is denoted by $\cdot_{rs}$, where r and s are the two fixed terms. This operation generalizes the binary operation $\cdot_{ij}$, introduced by Boonsol *et al.* [12] on the set of tree languages, when restricted to the set of terms. We construct a semigroup of terms under this new operation and investigate its algebraic structure, focusing on the characterization of idempotent and regular elements, as well as all five types of Green's relations.

2. Preliminaries

We begin by recalling the definition of terms. Let $X_n = \{x_1, \ldots, x_n\}$ be a finite set of n elements, called the *alphabet*, whose elements are called *variables*. Let $\{f_i \mid i \in I\}$ be a set of operation symbols, where I is a non-empty index set, and assume that $\{f_i \mid i \in I\}$ and X_n are disjoint. For each f_i, we assign a positive integer n_i, called the *arity* of f_i. The sequence $\tau = (n_i)_{i\in I}$ is called the *type*. The *n-ary terms* of type τ are inductively defined as follows.

(i) Every variable x_i is an n-ary term of type τ.

(ii) If $t_1, \dots, t_{n_i}$ are n-ary terms of type τ and f_i is an operation symbol of arity n_i, then $f_i(t_1, \dots, t_{n_i})$ is an n-ary term of type τ.

(iii) The set $W_\tau(X_n)$ of all n-ary terms is the smallest set containing $x_1, \dots, x_n$, and closed under finite applications of (ii).

As subterm replacement plays a crucial role in our study, we now recall the formal definition of subterms. This notion appears in several works (see, e.g., [8, 9]). Let $t \in W_\tau(X_n)$. The set of *subterms* of t, denoted by $\mathrm{sub}(t)$, is defined inductively as follows:

(i) if $t \in X_n$, then $\mathrm{sub}(t) = \{t\}$;

(ii) if $t = f_i(t_1, \dots, t_{n_i})$, then $\mathrm{sub}(t) = \{t\} \cup \mathrm{sub}(t_1) \cup \cdots \cup \mathrm{sub}(t_{n_i})$.

For $r, t \in W_\tau(X_n)$, let $n_r(t)$ denote the number of occurrences of r in t. This notion was introduced in [9]. It is defined inductively as follows:

(i) $n_r(t) = 0$ if $r \notin \mathrm{sub}(t)$;

(ii) $n_r(t) = 1$ if $r = t$;

(iii) $n_r(t) = \sum_{j=1}^{n_i} n_r(t_j)$ if $t = f_i(t_1, \dots, t_{n_i})$ and $r \in \mathrm{sub}(t) \setminus \{t\}$.

To investigate the structure of terms, various measures have been introduced, one of which is the *operation-symbol count*. For $t \in W_\tau(X_n)$, the operation-symbol count of t, denoted by $op(t)$, is defined inductively as follows:

(i) $op(t) = 0$ if $t \in X_n$;

(ii) $op(t) = 1 + \sum\limits_{j=1}^{n_i} op(t_j)$ if $t = f_i(t_1, \dots, t_{n_i})$.

We refer the readers to [13] for further discussion on term complexity measures. It is easy to see that if a term $t \in \mathrm{sub}(q)$, then $op(t) \leq op(q)$. Moreover, if $t \neq q$, then $op(t) < op(q)$.

3. The rs-Inductive Product

We define a binary operation, called *rs-inductive product* on the set of terms $W_\tau(X_n)$, which extends the r-inductive product by allowing the simultaneous replacement of two fixed subterms within a given term.

Definition 1. *Let $r, s \in W_\tau(X_n)$ be fixed terms and let $t, q \in W_\tau(X_n)$. The binary operation rs-inductive product, denoted by $\cdot_{rs}$, is inductively defined as follows:*

(i) $t \cdot_{rs} q = t$ if $\{r, s\} \cap \mathrm{sub}(t) = \emptyset$;

(ii) $t \cdot_{rs} q = q$ if $t \in \{r, s\}$;

(iii) $t \cdot_{rs} q = f_i(t_1 \cdot_{rs} q, \dots, t_{n_i} \cdot_{rs} q)$ if $t = f_i(t_1, \dots, t_{n_i})$, $\{r, s\} \cap \mathrm{sub}(t) \neq \emptyset$, and $t \notin \{r, s\}$.

Example 1. *Let $\tau = (1, 2, 3)$ with a unary operation symbol g, a binary operation f, and a ternary operation symbol h. Fix the terms $r = g(x_2)$ and $s = f(x_2, x_1)$. Consider 3-ary terms of type τ given by $t = h(f(x_2, x_1), x_3, g(x_2))$ and $q = g(x_3)$. Then we have*

$$\begin{aligned} t \cdot_{rs} q &= h(f(x_2, x_1), x_3, g(x_2)) \cdot_{rs} g(x_3) \\ &= h(f(x_2, x_1) \cdot_{rs} g(x_3), x_3 \cdot_{rs} g(x_3), g(x_2) \cdot_{rs} g(x_3)) \\ &= h(g(x_3), x_3, g(x_3)). \end{aligned}$$

Unlike the operation $\cdot_r$, which has r as an identity element, the operation $\cdot_{rs}$ does not possess an identity element in general. In fact, when $r \neq s$, the operation has no identity element at all. Typically, the terms r and s serve only as left identities of $\cdot_{rs}$.

The next result establishes a formula for computing the operation-symbol count of a term under the binary operation $\cdot_{rs}$. This formula reduces to the one given for the r-inductive product $\cdot_r$ in [9] when $r = s$. Note that for any two terms r and s, at least one of them is not a proper subterm of the other.

Theorem 1. *Let $r, s \in W_\tau(X_n)$ be fixed terms such that $r \notin \text{sub}(s) \setminus \{s\}$, and let $t, q \in W_\tau(X_n)$. Then, the operation-symbol count of $t \cdot_{rs} q$ is given by*

$$op(t \cdot_{rs} q) = op(t) + n_r(t)(op(q) - op(r)) + (n_s(t) - n_s(r)n_r(t))(op(q) - op(s)).$$

Proof. We proceed by induction on the structure of t. If $\{r, s\} \cap \text{sub}(t) = \emptyset$, then $t \cdot_{rs} q = t$, and we have $n_r(t) = n_s(t) = 0$. The formula holds trivially. If $t = r$, then $t \cdot_{rs} q = q$, and we observe that $n_r(t) = 1$ and $n_s(t) = n_s(r)$. The formula follows by substituting these values. If $t = s$, then $t \cdot_{rs} q = q$ and $n_s(t) = 1$. By the assumption that $r \notin \text{sub}(s) \setminus \{s\}$, we have $t = r$ or $r \notin \text{sub}(t)$. The first case implies the formula by the argument discussed above. In the second case, we have $n_r(t) = 0$, and the formula follows immediately from the known values. For $t = f_i(t_1, \ldots, t_{n_i})$ with $\{r, s\} \cap \text{sub}(t) \neq \emptyset$ and $t \notin \{r, s\}$, we inductively assume that

$$op(t_j \cdot_{rs} q) = op(t_j) + n_r(t_j)(op(q) - op(r)) + (n_s(t_j) - n_s(r)n_r(t_j))(op(q) - op(s))$$

for all $1 \leq j \leq n_i$. Then

$$\begin{aligned} op(t \cdot_{rs} q) &= op(f_i(t_1 \cdot_{rs} q, \ldots, t_{n_i} \cdot_{rs} q)) \\ &= 1 + \sum_{j=1}^{n_i} op(t_j \cdot_{rs} q) \\ &= 1 + \sum_{j=1}^{n_i} \Big[op(t_j) + n_r(t_j)(op(q) - op(r)) \\ &\qquad + (n_s(t_j) - n_s(r)n_r(t_j))(op(q) - op(s)) \Big] \\ &= 1 + \sum_{j=1}^{n_i} op(t_j) + (op(q) - op(r)) \sum_{j=1}^{n_i} n_r(t_j) \end{aligned}$$

$$+ (op(q) - op(s))\Big(\sum_{j=1}^{n_i} n_s(t_j) - n_s(r)\sum_{j=1}^{n_i} n_r(t_j)\Big)$$
$$= op(t) + n_r(t)(op(q) - op(r)) + (n_s(t) - n_s(r)n_r(t))(op(q) - op(s)).$$

This completes the proof.

As a direct consequence of Theorem 1 , we obtain the following corollary.

Corollary 1. *Let $r, s \in W_\tau(X_n)$ be fixed terms such that $r \notin \mathrm{sub}(s) \setminus \{s\}$ and $s \notin \mathrm{sub}(r)\{r\}$. For each $t, q \in W_\tau(X_n)$, the following statements hold true.*

(i) If $r = s$, then $op(t \cdot_{rs} q) = op(t) + n_r(t)(op(q) - op(r))$.

(ii) If $r \neq s$, then $op(t \cdot_{rs} q) = op(t) + n_r(t)(op(q) - op(r)) + n_s(t)(op(q) - op(s))$.

We now establish some fundamental properties of the operation $\cdot_{rs}$.

Lemma 1. *Let $r, s \in W_\tau(X_n)$ be fixed terms and $t, q \in W_\tau(X_n)$. If $\{r, s\} \cap \mathrm{sub}(t) \neq \emptyset$, then $q \in \mathrm{sub}(t \cdot_{rs} q)$. If, in addition, $t \notin \{r, s\}$, then $q \in \mathrm{sub}(t \cdot_{rs} q) \setminus \{t \cdot_{rs} q\}$.*

Proof. We proceed by induction on the structure of t. If $t \in \{r, s\}$, then $t \cdot_{rs} q = q$, so $q \in \mathrm{sub}(t \cdot_{rs} q)$. For $t = f_i(t_1, \dots, t_{n_i})$ with $\{r, s\} \cap \mathrm{sub}(t) \neq \emptyset$ and $t \notin \{r, s\}$, we assume that $\{r, s\} \cap \mathrm{sub}(t_j) \neq \emptyset$ implies $q \in \mathrm{sub}(t_j \cdot_{rs} q)$ for all $1 \leq j \leq n_i$. Since $\{r, s\} \cap \mathrm{sub}(t) \neq \emptyset$ and $t \neq \{r, s\}$, there exists $1 \leq j \leq n_i$ such that $\{r, s\} \cap \mathrm{sub}(t_j) \neq \emptyset$. By inductive hypothesis, it follows that $q \in \mathrm{sub}(t_j \cdot_{rs} q)$. This completes the first part.

For the second part, assume that $\{r, s\} \cap \mathrm{sub}(t) \neq \emptyset$ and $t \neq \{r, s\}$. Then t cannot be a variable, so let $t = f_i(t_1, \dots, t_{n_i})$. We have $t \cdot_{rs} q = f_i(t_1 \cdot_{rs} q, \dots, t_{n_i} \cdot_{rs} q)$, and as above, there is $1 \leq j \leq$ such that $\{r, s\} \cap \mathrm{sub}(t_j) \neq \emptyset$. By the first part, $q \in \mathrm{sub}(t_j \cdot_{rs} q)$. Thus, $q \in \mathrm{sub}(t \cdot_{rs} q) \setminus \{t \cdot_{rs} q\}$.

Lemma 2. *Let $r, s \in W_\tau(X_n)$ be fixed terms and $t, q \in W_\tau(X_n)$. If $\{r, s\} \cap \mathrm{sub}(t \cdot_{rs} q) \neq \emptyset$, then $\{r, s\} \cap \mathrm{sub}(t) \neq \emptyset$.*

Proof. Assume that $\{r, s\} \cap \mathrm{sub}(t) = \emptyset$. Then $t \cdot_{rs} q = t$. Thus, $\{r, s\} \cap \mathrm{sub}(t \cdot_{rs} q) = \emptyset$.

Lemma 3. *Let $r, s \in W_\tau(X_n)$ be fixed terms and $t, q \in W_\tau(X_n)$. Then the following statements hold.*

(i) If $\{r, s\} \cap \mathrm{sub}(t) \neq \emptyset$ and $r \in \mathrm{sub}(q)$, then $r \in \mathrm{sub}(t \cdot_{rs} q)$.

(ii) If $\{r, s\} \cap \mathrm{sub}(t) \neq \emptyset$ and $s \in \mathrm{sub}(q)$, then $s \in \mathrm{sub}(t \cdot_{rs} q)$.

Proof. The second part follows by a similar argument, so we prove only the first part. Assume $\{r, s\} \cap \mathrm{sub}(t) \neq \emptyset$ and $r \in \mathrm{sub}(q)$. By Lemma 1, we have $q \in \mathrm{sub}(t \cdot_{rs} q)$. Thus, $r \in \mathrm{sub}(t \cdot_{rs} q)$.

Lemma 4. *Let $r, s \in W_\tau(X_n)$ be fixed terms, and let $t, q \in W_\tau(X_n)$. Then the following statements hold.*

(i) If $t \cdot_{rs} q = r$, then $\{r, s\} \cap \operatorname{sub}(t) \neq \emptyset$ and $q \in \operatorname{sub}(r)$.

(ii) If $t \cdot_{rs} q = s$, then $\{r, s\} \cap \operatorname{sub}(t) \neq \emptyset$ and $q \in \operatorname{sub}(s)$.

Proof. We prove only the first part, as the second follows by a similar argument. Assume that $t \cdot_{rs} q = r$. Then $r \in \operatorname{sub}(t \cdot_{rs} q)$. By Lemma 2, we obtain $\{r, s\} \cap \operatorname{sub}(t) \neq \emptyset$. Applying Lemma 1, it follows that $q \in \operatorname{sub}(t \cdot_{rs} q) = \operatorname{sub}(r)$.

As in the case of $\cdot_r$, the operation $\cdot_{rs}$ is not necessarily associative over $W_\tau(X_n)$. The following example illustrates this fact.

Example 2. *Let $\tau = (1, 2)$ with a unary operation symbol g and a binary operation symbol f. Fix the terms $r = f(x_1, x_1)$ and $s = g(x_2)$. Let $t = f(g(x_2), f(x_1, x_1))$, $q = x_1$, and $h = g(f(x_1, x_2))$ be 2-ary terms of type τ. We have*

$$\begin{aligned}(t \cdot_{rs} q) \cdot_{rs} h &= (f(g(x_2), f(x_1, x_1)) \cdot_{rs} x_1) \cdot_{rs} g(f(x_1, x_2)) \\ &= f(x_1, x_1) \cdot_{rs} g(f(x_1, x_2)) \\ &= g(f(x_1, x_2)).\end{aligned}$$

On the other hand,

$$\begin{aligned}t \cdot_{rs} (q \cdot_{rs} h) &= f(g(x_2), f(x_1, x_1)) \cdot_{rs} (x_1 \cdot_{rs} g(f(x_1, x_2))) \\ &= f(g(x_2), f(x_1, x_1)) \cdot_{rs} x_1 \\ &= f(x_1, x_1).\end{aligned}$$

Thus, $(t \cdot_{rs} q) \cdot_{rs} h \neq t \cdot_{rs} (q \cdot_{rs} h)$, and the operation $\cdot_{rs}$ is not associative on $W_\tau(X_2)$.

Next, we investigate the associativity of the operation $\cdot_{rs}$. A necessary and sufficient condition is first established to characterize when associativity holds on a given subset of $W_\tau(X_n)$. Using this criterion, we construct a subset of $W_\tau(X_n)$ on which $\cdot_{rs}$ is both associative and closed, which leads to a semigroup under the operation.

Theorem 2. *Let $r, s \in W_\tau(X_n)$ be fixed terms and A a non-empty subset of $W_\tau(X_n)$. The following statements are equivalent:*

(i) For all $t, q \in A$, if $t \notin \{r, s\}$, then $t \cdot_{rs} q \notin \{r, s\}$.

(ii) For all $t, q, u \in A$, $(t \cdot_{rs} q) \cdot_{rs} u = t \cdot_{rs} (q \cdot_{rs} u)$.

Proof. Assume (i) and let $t, q, u \in A$. We prove by induction on the structure of t. If $\{r, s\} \cap \operatorname{sub}(t) = \emptyset$, then

$$(t \cdot_{rs} q) \cdot_{rs} u = t \cdot_{rs} u = t = t \cdot_{rs} (q \cdot_{rs} u).$$

If $t \in \{r, s\}$, then

$$(t \cdot_{rs} q) \cdot_{rs} u = q \cdot_{rs} u = t \cdot_{rs} (q \cdot_{rs} u).$$

For $t = f_i(t_1, \ldots, t_{n_i})$ with $\{r, s\} \cap \mathrm{sub}(t) \neq \emptyset$ and $t \notin \{r, s\}$, assume inductively that $(t_j \cdot_{rs} q) \cdot_{rs} u = t_j \cdot_{rs} (q \cdot_{rs} u)$ for all j. We consider the following two cases:

Case 1: $\{r, s\} \cap \mathrm{sub}(t \cdot_{rs} q) = \emptyset$. Then $(t \cdot_{rs} q) \cdot_{rs} u = t \cdot_{rs} q$. Since $\{r, s\} \cap \mathrm{sub}(t) \neq \emptyset$, Lemma 1 implies that $q \in \mathrm{sub}(t \cdot_{rs} q)$. Consequently, $\{r, s\} \cap \mathrm{sub}(q) = \emptyset$. It follows that $t \cdot_{rs} (q \cdot_{rs} u) = t \cdot_{rs} q$. Thus, $(t \cdot_{rs} q) \cdot_{rs} u = t \cdot_{rs} (q \cdot_{rs} u)$.

Case 2: $\{r, s\} \cap \mathrm{sub}(t \cdot_{rs} q) \neq \emptyset$. By (i) and the fact that $t \notin \{r, s\}$, we obtain $t \cdot_{rs} q \notin \{r, s\}$. From the definition of $\cdot_{rs}$ and the inductive hypothesis, it follows that

$$\begin{aligned}
(t \cdot_{rs} q) \cdot_{rs} u &= (f_i(t_1, \ldots, t_{n_i}) \cdot_{rs} q) \cdot_{rs} u \\
&= f_i(t_1 \cdot_{rs} q, \ldots, t_{n_i} \cdot_{rs} q) \cdot_{rs} u \\
&= f_i((t_1 \cdot_{rs} q) \cdot_{rs} u, \ldots, (t_{n_i} \cdot_{rs} q) \cdot_{rs} u) \\
&= f_i(t_1 \cdot_{rs} (q \cdot_{rs} u), \ldots, t_{n_i} \cdot_{rs} (q \cdot_{rs} u)) \\
&= f_i(t_1, \ldots, t_{n_i}) \cdot_{rs} (q \cdot_{rs} u) \\
&= t \cdot_{rs} (q \cdot_{rs} u).
\end{aligned}$$

Conversely, assume (ii) and let $t, q \in A$ be such that $t \notin \{r, s\}$. Suppose $t \cdot_{rs} q \in \{r, s\}$, say $t \cdot_{rs} q = r$. By Lemma 4, we get $q \in \mathrm{sub}(r)$ and $\{r, s\} \cap \mathrm{sub}(t) \neq \emptyset$. We consider the following two cases:

Case 1: $q \in \{r, s\}$. By the associativity of $\cdot_{rs}$, we have

$$t \cdot_{rs} t = t \cdot_{rs} (q \cdot_{rs} t) = (t \cdot_{rs} q) \cdot_{rs} t = r \cdot_{rs} t = t.$$

Since $\{r, s\} \cap \mathrm{sub}(t) \neq \emptyset$ and $t \notin \{r, s\}$, Lemma 1 gives $t \in \mathrm{sub}(t \cdot_{rs} t) \setminus \{t \cdot_{rs} t\} = \mathrm{sub}(t) \setminus \{t\}$, a contradiction.

Case 2: $q \notin \{r, s\}$. This implies $q \in \mathrm{sub}(r) \setminus \{r\}$, so $op(q) < op(r)$. Hence, $r \notin \mathrm{sub}(q)$. If $s \in \mathrm{sub}(q)$, then Lemma 1 provides that $t \in \mathrm{sub}(q \cdot_{rs} t) \setminus \{q \cdot_{rs} t\}$ and $q \cdot_{rs} t \in \mathrm{sub}(t \cdot_{rs} (q \cdot_{rs} t))$. Thus

$$op(t) < op(q \cdot_{rs} t) \leq op(t \cdot_{rs} (q \cdot_{rs} t)) = op((t \cdot_{rs} q) \cdot_{rs} t) = op(r \cdot_{rs} t) = op(t),$$

which is impossible. It follows that $\{r, s\} \cap \mathrm{sub}(q) = \emptyset$, so $q \cdot_{rs} t = q$. As a result,

$$r = t \cdot_{rs} q = t \cdot_{rs} (q \cdot_{rs} t) = (t \cdot_{rs} q) \cdot_{rs} t = r \cdot_{rs} t = t,$$

contradicting $t \notin \{r, s\}$.

Theorem 3. *Let $r, s \in W_\tau(X_n)$ be fixed terms, and define*

$$W_\tau^{r,s}(X_n) = W_\tau(X_n) \setminus \big[(\mathrm{sub}(r) \setminus \{r\}) \cup (\mathrm{sub}(s) \setminus \{s\})\big].$$

Then $(W_\tau^{r,s}(X_n), \cdot_{rs})$ is a semigroup.

Proof. To show closure, let $t, q \in W_\tau^{r,s}(X_n)$. If $\{r, s\} \cap \mathrm{sub}(t) = \emptyset$, then $t \cdot_{rs} q = t \in W_\tau^{r,s}(X_n)$. Now, assume $\{r, s\} \cap \mathrm{sub}(t) \neq \emptyset$. By Lemma 1, we obtain $q \in \mathrm{sub}(t \cdot_{rs} q)$. Since q is not a proper subterm of either r or s, the same must hold for $t \cdot_{rs} q$. Thus, $t \cdot_{rs} q \in W_\tau^{r,s}(X_n)$.

Next, we establish associativity using Theorem 2. Let $t, q \in W_\tau^{r,s}(X_n)$ with $t \notin \{r, s\}$. Suppose that $t \cdot_{rs} q \in \{r, s\}$, say $t \cdot_{rs} q = r$. By Lemma 4, we have $\{r, s\} \cap \mathrm{sub}(t) \neq \emptyset$ and $q \in \mathrm{sub}(r)$. Since q is not a proper subterm of r, it follows that $q = r$. Applying Lemma 1, we deduce

$$r = q \in \mathrm{sub}(t \cdot_{rs} q) \setminus \{t \cdot_{rs} q\} = \mathrm{sub}(r) \setminus \{r\},$$

a contradiction. Therefore, $t \cdot_{rs} q \notin \{r, s\}$.

We further show that $W_\tau^{r,s}(X_n)$ is a maximal semigroup in $W_\tau(X_n)$ with respect to the operation $\cdot_{rs}$, assuming that neither r nor s is a proper subterm of the other.

Theorem 4. *Let $r, s \in W_\tau(X_n)$ be fixed terms such that $r \notin \mathrm{sub}(s) \setminus \{s\}$ and $s \notin \mathrm{sub}(r) \setminus \{r\}$. Then $W_\tau^{r,s}(X_n)$ is a maximal subset of $W_\tau(X_n)$ that forms a semigroup under the operation $\cdot_{rs}$.*

Proof. By Theorem 3, we know that $W_\tau^{r,s}(X_n)$ is a semigroup. We now proceed to show its maximality. Note that when $r, s \in X_n$, the set $W_\tau^{r,s}(X_n)$ becomes equal to $W_\tau(X_n)$, so the maximality clearly holds in this case. Now, assume that at least one of r or s is a compound term. In this case, we have $W_\tau^{r,s}(X_n) \subsetneq W_\tau(X_n)$. Let Q be a subset of $W_\tau(X_n)$ such that $W_\tau^{r,s}(X_n) \subsetneq Q$. This implies that there exists an element $a \in Q$ such that $a \in (\mathrm{sub}(r) \setminus \{r\}) \cup (\mathrm{sub}(s) \setminus \{s\})$. Without loss of generality, assume that $a \in \mathrm{sub}(r) \setminus \{r\}$. Let $r = f_i(t_1, \ldots, t_{n_i})$, and let $1 \leq l \leq n_i$ be such that $a \in \mathrm{sub}(t_l)$. We will demonstrate that Q does not satisfy associativity under $\cdot_{rs}$ by applying Theorem 2. We proceed with the following two steps.

Step 1. We show that there is $b \in W_\tau^{r,s}(X_n)$ such that $r \in \mathrm{sub}(b)$ and $b \cdot_{rs} a = t_l$. We prove by induction on the complexity of t_l. If $t_l \in X_n$, then $a \in \mathrm{sub}(t_l) = \{t_l\}$, which implies $a = t_l$. In this case, we set $b = r$. For $t_l = f_j(q_1, \ldots, q_{n_j})$, assume that for each $1 \leq m \leq n_j$, if $a \in \mathrm{sub}(q_m)$, then there exists $b_m \in W_\tau^{r,s}(X_n)$ such that $r \in \mathrm{sub}(b_m)$ and $b_m \cdot_{rs} a = q_m$. If $a = t_l$, then we set $b = r$ as in the base case. Next, consider the case where $a \in \mathrm{sub}(t_l) \setminus \{t_l\}$. We obtain that $a \in \mathrm{sub}(q_m)$ for some $1 \leq m \leq n_j$. By inductive hypothesis, there exists $b_m \in W_\tau^{r,s}(X_n)$ such that $r \in \mathrm{sub}(b_m)$ and $b_m \cdot_{rs} a = q_m$. We define

$$b = f_j(q_1, \ldots, q_{m-1}, b_m, q_{m+1}, \ldots, q_{n_j}).$$

Since $r \in \mathrm{sub}(b_m) \subseteq \mathrm{sub}(b) \setminus \{b\}$, $b \notin \mathrm{sub}(r)$. Additionally, since $r \notin \mathrm{sub}(s) \setminus \{s\}$, $b \notin \mathrm{sub}(s)$. Thus, $b \in W_\tau^{r,s}(X_n)$ and $b \notin \{r, s\}$. Note that $\{r, s\} \cap \mathrm{sub}(q_k) = \emptyset$ for all $1 \leq q_k \leq n_j$ because each q_k is a proper subterm of r and $s \notin \mathrm{sub}(r) \setminus \{r\}$. It follows that

$$\begin{aligned} b \cdot_{rs} a &= f_j(q_1, \ldots, q_{m-1}, b_m, q_{m+1}, \ldots, q_{n_j}) \cdot_{rs} a \\ &= f_j(q_1 \cdot_{rs} a, \ldots, q_{m-1} \cdot_{rs} a, b_m \cdot_{rs} a, q_{m+1} \cdot_{rs} a, \ldots, q_{n_j} \cdot_{rs} a) \\ &= f_j(q_1, \ldots, q_{m-1}, q_m, q_{m+1}, \ldots, q_{n_j}) \end{aligned}$$

$= t_l.$

Step 2. We establish the existence of $c \in W_\tau^{r,s}(X_n)$ *such that* $c \notin \{r, s\}$ *and* $c \cdot_{rs} a = r$. Let b be the term constructed in Step 1. We define

$$c = f_i(t_1, \ldots, t_{l-1}, b, t_{l+1}, \ldots, t_{n_i}).$$

Since $r \in \text{sub}(b) \subseteq \text{sub}(c) \setminus \{c\}$, $c \notin \text{sub}(r)$. Furthermore, since $r \notin \text{sub}(s) \setminus \{s\}$, $c \notin \text{sub}(s)$. Thus, $c \in W_\tau^{r,s}(X_n)$ and $c \notin \{r, s\}$. Observe that $\{r, s\} \cap \text{sub}(t_k) = \emptyset$ for all $1 \leq k \leq n_i$ as each t_k is a proper subterm of r and $s \notin \text{sub}(r) \setminus \{r\}$. Hence,

$$\begin{aligned} c \cdot_{rs} a &= f_i(t_1, \ldots, t_{l-1}, b, t_{l+1}, \ldots, t_{n_i}) \cdot_{rs} a \\ &= f_i(t_1 \cdot_{rs} a, \ldots, t_{l-1} \cdot_{rs} a, b \cdot_{rs} a, t_{l+1} \cdot_{rs} a, \ldots, t_{n_i} \cdot_{rs} a) \\ &= f_i(t_1, \ldots, t_{l-1}, t_l, t_{l+1}, \ldots, t_{n_i}) \\ &= r. \end{aligned}$$

By Theorem 2, Q is not associative under $\cdot_{rs}$.

We conclude this section with two lemmas, which show that the converses of Lemma 3 and Lemma 4 hold when restricted to the subset $W_\tau^{r,s}(X_n)$. These properties will play a key role in establishing the algebraic properties of the semigroup in the next two sections.

Lemma 5. *Let* $r, s \in W_\tau(X_n)$ *be fixed terms, and let* $t, q \in W_\tau^{r,s}(X_n)$. *Then the following statements hold true:*

(i) $r \in \text{sub}(t \cdot_{rs} q)$ *if and only if* $\{r, s\} \cap \text{sub}(t) \neq \emptyset$ *and* $r \in \text{sub}(q)$.

(ii) $s \in \text{sub}(t \cdot_{rs} q)$ *if and only if* $\{r, s\} \cap \text{sub}(t) \neq \emptyset$ *and* $s \in \text{sub}(q)$.

Proof. (i) The reverse direction follows directly from Lemma 3. Additionally, Lemma 2 shows that $r \in \text{sub}(t \cdot_{rs} q)$ implies $\{r, s\} \cap \text{sub}(t) \neq \emptyset$. It remains to demonstrate that $r \in \text{sub}(t \cdot_{rs} q)$ implies $r \in \text{sub}(q)$. We prove this by induction on the structure of t. If $\{r, s\} \cap \text{sub}(t) = \emptyset$, then $t \cdot_{rs} q = t$, which implies $r \notin \text{sub}(t \cdot_{rs} q)$. Thus, the claim holds trivially. If $t \in \{r, s\}$, then $t \cdot_{rs} q = q$, and therefore, $r \in \text{sub}(t \cdot_{rs} q)$ implies $r \in \text{sub}(q)$. For $t = f_i(t_1, \ldots, t_{n_i})$ with $\{r, s\} \cap \text{sub}(t) \neq \emptyset$ and $t \neq \{r, s\}$, assume that for each $1 \leq j \leq n_i$, if $r \in \text{sub}(t_j \cdot_{rs} q)$, then $r \in \text{sub}(q)$. In this case, we have $t \cdot_{rs} q = f_i(t_1 \cdot_{rs} q, \ldots, t_{n_i} \cdot_{rs} q)$. Now, assume $r \in \text{sub}(t \cdot_{rs} q)$. In the case where $r = t \cdot_{rs} q$, Lemma 4 and the fact that $q \in W_\tau^{r,s}(X_n)$ provides that $q = r$, and thus $r \in \text{sub}(q)$. On the other hand, if $r \in \text{sub}(t_j \cdot_{rs} q)$ for some $1 \leq j \leq n_i$, then the desired result follows directly from the inductive hypothesis. This complete the proof for part (i).

(ii) This follows by a similar reasoning.

Lemma 6. *Let* $r, s \in W_\tau(X_n)$ *be fixed terms, and let* $t, q \in W_\tau^{r,s}(X_n)$. *Then the following statements hold:*

(i) $t \cdot_{rs} q = r$ *if and only if* $t \in \{r, s\}$ *and* $q = r$.

(ii) $t \cdot_{rs} q = s$ *if and only if* $t \in \{r, s\}$ *and* $q = s$.

Proof. We will prove only (i), as (ii) follows by a similar argument. Assume $t \cdot_{rs} q = r$. By Lemma 4, it follows that $\{r, s\} \cap \mathrm{sub}(t) \neq \emptyset$ and $q \in \mathrm{sub}(r)$. Since q is not a proper subterm of r, $q = r$. Now, suppose $t \notin \{r, s\}$. By Lemma 1,

$$r = q \in \mathrm{sub}(t \cdot_{rs} q) \setminus \{t \cdot_{rs} q\} = \mathrm{sub}(r) \setminus \{r\},$$

which is a contradiction. Hence, $t \in \{r, s\}$. The converse is straightforward.

4. Idempotent and Regular Elements in the Semigroup $(W_\tau^{r,s}(X_n), \cdot_{rs})$

In this section, we investigate the idempotent and regular elements with respect to $\cdot_{rs}$ in $W_\tau^{r,s}(X_n)$. Recall that an element t in a semigroup S is called *idempotent* if $t = tt$, and is called *regular* if there exists $q \in S$ such that $t = tqt$. It is straightforward that every idempotent element is also regular.

Theorem 5. *Let* $r, s \in W_\tau(X_n)$ *be fixed terms and* $t \in W_\tau^{r,s}(X_n)$. *Then* t *is idempotent with respect to* $\cdot_{rs}$ *if and only if* $\{r, s\} \cap \mathrm{sub}(t) = \emptyset$ *or* $t \in \{r, s\}$.

Proof. Assume that $\{r, s\} \cap \mathrm{sub}(t) \neq \emptyset$ and $t \notin \{r, s\}$. By Lemma 1, $t \in \mathrm{sub}(t \cdot_{rs} t) \setminus \{t \cdot_{rs} t\}$. Thus, $t \neq t \cdot_{rs} t$, which implies that t is not idempotent. The converse follows directly from the definition of $\cdot_{rs}$.

Theorem 6. *Let* $r, s \in W_\tau(X_n)$ *be fixed terms and* $t \in W_\tau^{r,s}(X_n)$. *Then* t *is regular with respect to* $\cdot_{rs}$ *if and only if* t *is idempotent.*

Proof. Assume that t is regular but not idempotent. Then $t = t \cdot_{rs} q \cdot_{rs} t$ for some $q \in W_\tau^{r,s}(X_n)$. By Theorem 5, we obtain $\{r, s\} \cap \mathrm{sub}(t) \neq \emptyset$ and $t \notin \{r, s\}$. Now we have $\{r, s\} \cap \mathrm{sub}((t \cdot_{rs} q) \cdot_{rs} t) \neq \emptyset$. By Lemma 5, it follows that $\{r, s\} \cap \mathrm{sub}(t \cdot_{rs} q) \neq \emptyset$. By using Lemma 5 again, we have $\{r, s\} \cap \mathrm{sub}(q) \neq \emptyset$. Lemma 1 provides that $t \in \mathrm{sub}(q \cdot_{rs} t)$ and

$$q \cdot_{rs} t \in \mathrm{sub}(t \cdot_{rs} q \cdot_{rs} t) \setminus \{t \cdot_{rs} q \cdot_{rs} t\} = \mathrm{sub}(t) \setminus \{t\}.$$

Hence, $t \in \mathrm{sub}(t) \setminus \{t\}$, a contradiction.

Theorem 5 and Theorem 6 show that idempotent elements and regular elements are coincide in the semigroup $(W_\tau^{r,s}(X_n), \cdot_{rs})$. We now give an example of a term that is both regular and idempotent with respect to $\cdot_{rs}$ in order to better demonstrate this result.

Example 3. *Let* $\tau = (1, 2)$ *with a unary operation symbol* g *and a binary operation symbol* f. *Fix the terms* $r = f(x_1, x_1)$ *and* $s = g(x_2)$. *Let* $t = g(f(x_2, x_1))$ *and* $q = g(x_1)$ *be* 2*-ary terms of type* τ. *We see that* t *and* q *are not subterms of* r *and* s, *so* $t, q \in W_\tau^{r,s}(X_2)$. *Moreover,* $\{r, s\} \cap \mathrm{sub}(t) = \emptyset$. *By the definition of* $\cdot_{rs}$, *we obtain* $t = t \cdot_{rs} (q \cdot_{rs} t)$ *and* $t = t \cdot_{rs} t$. *Thus,* t *is both regular and idempotent under* $\cdot_{rs}$.

As in the case of $\cdot_r$, where it was shown that any term expressible as a product of several terms, with at least two of them equal to the term itself, is idempotent (see [9]), we find that a similar behavior holds in the $\cdot_{rs}$ setting. The next lemma plays a key role in verifying this implication.

Lemma 7. *Let $r, s \in W_\tau(X_n)$ be fixed terms. For each $t, q, u \in W_\tau^{r,s}(X_n)$, if $t = u \cdot_{rs} t \cdot_{rs} q$ and $\{r, s\} \cap \mathrm{sub}(t) \neq \emptyset$, then $t = u \cdot_{rs} t$ and $q \in \{r, s\}$.*

Proof. Assume that $t = u \cdot_{rs} t \cdot_{rs} q$ and $\{r, s\} \cap \mathrm{sub}(t) \neq \emptyset$. By Lemma 5, we have $\{r, s\} \cap \mathrm{sub}(u \cdot_{rs} t) \neq \emptyset$ and $\{r, s\} \cap \mathrm{sub}(u) \neq \emptyset$. By Lemma 1, we obtain

$$q \in \mathrm{sub}((u \cdot_{rs} t) \cdot_{rs} q) = \mathrm{sub}(t) \quad \text{and} \quad t \in \mathrm{sub}(u \cdot_{rs} t).$$

We claim the following statement:

$$r \in \mathrm{sub}(t) \Leftrightarrow r \in \mathrm{sub}(u \cdot_{rs} t) \Leftrightarrow r \in \mathrm{sub}(q). \tag{1}$$

To prove this, it suffices to show both $r \in \mathrm{sub}(t) \Leftrightarrow r \in \mathrm{sub}(u \cdot_{rs} t)$ and $r \in \mathrm{sub}(t) \Leftrightarrow r \in \mathrm{sub}(q)$. Assume $r \in \mathrm{sub}(t)$. Since $t \in \mathrm{sub}(u \cdot_{rs} t)$, $r \in \mathrm{sub}(u \cdot_{rs} t)$. Moreover, by Lemma 5 and condition $t = (u \cdot_{rs} t) \cdot_{rs} q$, we conclude that $r \in \mathrm{sub}(q)$. Conversely, assume that $r \in \mathrm{sub}(u \cdot_{rs} t)$. Lemma 5 provides that $r \in \mathrm{sub}(t)$. Finally, we assume $r \in \mathrm{sub}(q)$. Since $q \in \mathrm{sub}(t)$, $r \in \mathrm{sub}(t)$. This establishes the statement (1). A similar reasoning shows the corresponding statement for s.

Next, we show that

$$n_r(u \cdot_{rs} t)(op(q) - op(r)) \geq 0, \text{ and} \tag{2}$$

$$(n_s(u \cdot_{rs} t) - n_s(r) n_r(u \cdot_{rs} t))(op(q) - op(s)) \geq 0. \tag{3}$$

Note that $n_r(u \cdot_{rs} t) \geq 0$ and $n_s(u \cdot_{rs} t) - n_s(r) n_r(u \cdot_{rs} t) \geq 0$. If $r \in \mathrm{sub}(t)$, then by (1), we have $r \in \mathrm{sub}(q)$. Therefore, $op(q) - op(r) \geq 0$, and inequality (2) follows. On the other hand, if $r \notin \mathrm{sub}(t)$, then by (1), we also have $r \notin \mathrm{sub}(u \cdot_{rs} t)$. In this case, we have $n_r(u \cdot_{rs} t) = 0$, which implies inequality (2). Thus, inequality (2) always holds. A similar argument shows that inequality (3) also holds.

By applying operation-symbol count formula from Theorem 1, we have

$$\begin{aligned} op(t) &= op(u \cdot_{rs} t \cdot_{rs} q) \\ &= op(u \cdot_{rs} t) + n_r(u \cdot_{rs} t)(op(q) - op(r)) \\ &\quad + (n_s(u \cdot_{rs} t) - n_s(r) n_r(u \cdot_{rs} t))(op(q) - op(s)) \\ &\geq op(u \cdot_{rs} t) \end{aligned}$$

Since $t \in \mathrm{sub}(u \cdot_{rs} t)$, we have $op(t) \leq op(u \cdot_{rs} t)$. Therefore, $op(t) = op(u \cdot_{rs} t)$, which implies $t = u \cdot_{rs} t$. Substituting this into the equation above shows that

$$n_r(u \cdot_{rs} t)(op(q) - op(r)) = 0, \text{ and}$$
$$(n_s(u \cdot_{rs} t) - n_s(r) n_r(u \cdot_{rs} t))(op(q) - op(s)) = 0.$$

If $r \in \mathrm{sub}(t)$, then by (1), we have $n_r(u \cdot_{rs} t) = 1$ and $r \in \mathrm{sub}(q)$. Thus, $op(q) = op(r)$, which implies $q = r$. If $r \notin \mathrm{sub}(t)$, then by the assumption that $\{r, s\} \cap \mathrm{sub}(t) \neq \emptyset$, we have $s \in \mathrm{sub}(t)$. By a similar reasoning, we conclude $q = s$. This completes the proof.

The next theorem establishes that any term in $(W_\tau^{r,s}(X_n), \cdot_{rs})$ that can be written as a product of several terms, where the term itself appears at least twice, is necessarily idempotent.

Theorem 7. *Let $r, s \in W_\tau(X_n)$ be fixed terms and $t \in W_\tau^{r,s}(X_n)$. Then t is idempotent with respect to $\cdot_{rs}$ if and only if there exist $t_1, \ldots, t_m \in W_\tau^{r,s}(X_n)$ with at least two of them equal to t such that*

$$t = t_1 \cdot_{rs} t_2 \cdot_{rs} \cdots \cdot_{rs} t_m.$$

Proof. Assume that $t = t_1 \cdot_{rs} t_2 \cdot_{rs} \cdots \cdot_{rs} t_m$ for some $t_1, \ldots, t_m \in W_\tau^{r,s}(X_n)$ with at least two of them equal to t. Suppose that t is not idempotent. By Theorem 5, we obtain $\{r, s\} \cap \mathrm{sub}(t) \neq \emptyset$ and $t \notin \{r, s\}$. We first consider the case where $t_m = t$. Then we write

$$t = a \cdot_{rs} t \cdot_{rs} b \cdot_{rs} t,$$

where each of a and b is either a product of t_j's or the element r, which is the left identity in $W_\tau^{r,s}(X_n)$. By Lemma 5, we obtain $\{r, s\} \cap \mathrm{sub}(a) \neq \emptyset$ and $\{r, s\} \cap \mathrm{sub}(b) \neq \emptyset$. Applying Lemma 1, it follows that $t \in \mathrm{sub}(b \cdot_{rs} t)$, $b \cdot_{rs} t \in \mathrm{sub}(t \cdot_{rs} b \cdot_{rs} t) \setminus \{t \cdot_{rs} b \cdot_{rs} t\}$, and $t \cdot_{rs} b \cdot_{rs} t \in \mathrm{sub}(a \cdot_{rs} t \cdot_{rs} b \cdot_{rs} t)$. Hence,

$$op(t) \leq op(b \cdot_{rs} t) < op(t \cdot_{rs} b \cdot_{rs} t) \leq op(a \cdot_{rs} t \cdot_{rs} b \cdot_{rs} t) = op(t),$$

a contradiction. Now, assume $t_m \neq t$. Then t can be written as

$$t = a \cdot_{rs} t \cdot_{rs} b \cdot_{rs} t \cdot_{rs} c,$$

where c is a product of t_j's. By Lemma 7, we have $t = a \cdot_{rs} t \cdot_{rs} b \cdot_{rs} t$, and the same contradiction follows as in the previous case.

5. Green's Relations on the Semigroup $(W_\tau^{r,s}(X_n), \cdot_{rs})$

In this section, we provide explicit characterizations for all five types of Green's relations on the semigroup $(W_\tau^{r,s}(X_n), \cdot_{rs})$. Let S be a semigroup and S^1 denote the monoid obtained by adjoining an identity element 1 to S, if necessary. For elements $x, y \in S$, we say that $x \,\mathcal{L}\, y$ if and only if there exist $u, v \in S^1$ such that $ux = y$ and $vy = x$. Similarly, $x \,\mathcal{R}\, y$ if and only if there exist $u, v \in S^1$ such that $xu = y$ and $yv = x$. The relation $\mathcal{H}$ is defined as the intersection of $\mathcal{L}$ and $\mathcal{R}$. The relation $\mathcal{D}$ is defined as the composition $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$, where $\circ$ denotes the composition of binary relations. Finally, $x \,\mathcal{J}\, y$ if and only if there exist $u, v, a, b \in S^1$ such that $uxv = y$ and $ayb = x$. For further background on Green's relations, we refer the reader to [14].

Theorem 8. *Let $r, s \in W_\tau(X_n)$ be fixed terms, and let $t, q \in W_\tau^{r,s}(X_n)$. Then $t \,\mathcal{L}\, q$ if and only if $\{r, s\} \cap (\mathrm{sub}(t) \cup \mathrm{sub}(q)) = \emptyset$ or $t = q$.*

Proof. Assume that $t\,\mathcal{L}\,q$ and $\{r,s\}\cap(\mathrm{sub}(t)\cup\mathrm{sub}(q))\neq\emptyset$. Without loss of generality, assume $\{r,s\}\cap\mathrm{sub}(t)\neq\emptyset$. Since $t\,\mathcal{L}\,q$, there exist $u,v\in(W_\tau^{r,s}(X_n))^1$ such that

$$u\cdot_{rs}t=q \quad\text{and}\quad v\cdot_{rs}q=t.$$

If $u=1$ or $v=1$, then $t=q$, and we are done. Next, assume $u,v\in W_\tau^{r,s}(X_n)$. By Lemma 5, we have $\{r,s\}\cap\mathrm{sub}(v)\neq\emptyset$ and $\{r,s\}\cap\mathrm{sub}(u)\neq\emptyset$. Then, by Lemma 1, it follows that

$$t\in\mathrm{sub}(u\cdot_{rs}t)=\mathrm{sub}(q)\quad\text{and}\quad q\in\mathrm{sub}(v\cdot_{rs}q)=\mathrm{sub}(t).$$

Hence, $t=q$. The converse is clear.

Theorem 9. *Let $r,s\in W_\tau(X_n)$ be fixed terms, and let $t,q\in W_\tau^{r,s}(X_n)$. Then $t\,\mathcal{R}\,q$ if and only if $t=q$ or there exists $u,v\in\{r,s\}$ such that $t\cdot_{rs}u=q$ and $q\cdot_{rs}v=t$.*

Proof. Assume that $t\,\mathcal{R}\,q$. Then there exists $u,v\in(W_\tau^{r,s}(X_n))^1$ such that

$$t\cdot_{rs}u=q \quad\text{and}\quad q\cdot_{rs}v=t.$$

If $u=1$ or $v=1$, then $t=q$, and the result follows. Now, we assume that $u,v\in W_\tau^{r,s}(X_n)$. If $\{r,s\}\cap\mathrm{sub}(t)=\emptyset$, then $t=t\cdot_{rs}u=q$. If $\{r,s\}\cap\mathrm{sub}(t)\neq\emptyset$, we consider

$$t=q\cdot_{rs}v=t\cdot_{rs}u\cdot_{rs}v=r\cdot_{rs}t\cdot_{rs}u\cdot_{rs}v.$$

By Lemma 7, we obtain that $u\cdot_{rs}v\in\{r,s\}$. Then, by Lemma 6, it follows that $u,v\in\{r,s\}$. The converse is straightforward.

Theorem 10. *Let $r,s\in W_\tau(X_n)$ be fixed terms, and let $t,q\in W_\tau^{r,s}(X_n)$. Then $t\,\mathcal{H}\,q$ if and only if $t=q$.*

Proof. Assume $t\,\mathcal{H}\,q$ and suppose $t\neq q$. Then $t\,\mathcal{L}\,q$ and $t\,\mathcal{R}\,q$. Therefore, there is $u\in W_\tau^{r,s}(X_n)$ such that $t\cdot_{rs}u=q$. By Theorem 8, it follows that $\{r,s\}\cap\mathrm{sub}(t)=\emptyset$. Thus, $t=t\cdot_{rs}u=q$, a contradiction. The reverse implication holds by the reflexivity of $\mathcal{H}$.

Theorem 11. *Let $r,s\in W_\tau(X_n)$ be fixed terms, and let $t,q\in W_\tau^{r,s}(X_n)$. Then the following statements hold:*

(i) If $\{r,s\}\cap\mathrm{sub}(t)=\emptyset$, then $t\,\mathcal{D}\,q$ if and only if $t\,\mathcal{L}\,q$.

(ii) If $\{r,s\}\cap\mathrm{sub}(t)\neq\emptyset$, then $t\,\mathcal{D}\,q$ if and only if $t\,\mathcal{R}\,q$.

Proof. (i) Assume that $\{r,s\}\cap\mathrm{sub}(t)=\emptyset$ and $t\,\mathcal{D}\,q$. Then there exists $u\in W_\tau^{r,s}(X_n)$ such that $t\,\mathcal{L}\,u$ and $u\,\mathcal{R}\,q$. By Lemma 5, it follows that $\{r,s\}\cap\mathrm{sub}(u)=\emptyset$, and hence $\{r,s\}\cap\mathrm{sub}(q)=\emptyset$. Applying Theorem 8, we conclude that $t\,\mathcal{L}\,q$. The converse is obvious.

(ii) Assume that $\{r,s\}\cap\mathrm{sub}(t)\neq\emptyset$ and $t\,\mathcal{D}\,q$. Then there is $u\in W_\tau^{r,s}(X_n)$ such that $t\,\mathcal{L}\,u$ and $u\,\mathcal{R}\,q$. By Theorem 8, we have $t=u$, and thus $t\,\mathcal{R}\,q$. The converse is clear.

Theorem 12. *Let $r, s \in W_\tau(X_n)$ be fixed terms, and let $t, q \in W_\tau^{r,s}(X_n)$. Then the following statements hold:*

(i) If $\{r,s\} \cap \mathrm{sub}(t) = \emptyset$, then $t \,\mathcal{J}\, q$ if and only if $t \,\mathcal{L}\, q$.

(ii) If $\{r,s\} \cap \mathrm{sub}(t) \neq \emptyset$, then $t \,\mathcal{J}\, q$ if and only if $t \,\mathcal{R}\, q$.

Proof. Part (i) follows from the same reasoning as in part (i) of Theorem 11. For part (ii), assume $\{r,s\} \cap \mathrm{sub}(t) \neq \emptyset$ and $t \,\mathcal{J}\, q$. Then there are $u, v, a, b \in (W_\tau^{r,s}(X_n))^1$ such that

$$u \cdot_{rs} t \cdot_{rs} v = q \quad \text{and} \quad a \cdot_{rs} q \cdot_{rs} b = t. \tag{4}$$

Then $t = (a \cdot_{rs} u) \cdot_{rs} t \cdot_{rs} (v \cdot_{rs} b)$. We consider the following cases.

Case 1: $v = b = 1$. In this case, we have $t \,\mathcal{L}\, q$. By Theorem 8, it follows that $t = q$, which clearly implies $t \,\mathcal{R}\, q$.

Case 2: One of v or b belong to $W_\tau^{r,s}(X_n)$. Then $v \cdot_{rs} b \in W_\tau^{r,s}(X_n)$. If $u = 1$ or $a = 1$, we may substitute it with r, which is a left identity in $W_\tau^{r,s}(X_n)$, without affecting the equality in (4). Thus, we may assume $u, a \in W_\tau^{r,s}(X_n)$. By Lemma 7, we obtain $t = (a \cdot_{rs} u) \cdot_{rs} t$. Applying Lemma 5 with the condition $\{r,s\} \cap \mathrm{sub}((a \cdot_{rs} u) \cdot_{rs} (t \cdot_{rs} (v \cdot_{rs} b))) \neq \emptyset$, we have $\{r,s\} \cap \mathrm{sub}(a \cdot_{rs} u) \neq \emptyset$. If $a \cdot_{rs} u \notin \{r,s\}$, then Lemma 1 yields

$$t \in \mathrm{sub}((a \cdot_{rs} u) \cdot_{rs} t) \setminus \{(a \cdot_{rs} u) \cdot_{rs} t\} = \mathrm{sub}(t) \setminus \{t\},$$

a contradiction. Consequently, $a \cdot_{rs} u \in \{r,s\}$, and then Lemma 6 implies $a, u \in \{r,s\}$. As both r and s act as left identities, we conclude that

$$t \cdot_{rs} v = q \quad \text{and} \quad q \cdot_{rs} b = t,$$

which implies $t \,\mathcal{R}\, q$, as desired.

The characterizations of Green's relations for $(W_\tau^{r,s}(X_n), \cdot_{rs})$ exhibit notable differences from those in the case of $\cdot_r$. For the operation $\cdot_r$, the relations satisfy $\mathcal{L} = \mathcal{D} = \mathcal{J}$ and $\mathcal{R} = \mathcal{H}$ (see [9]). However, under $\cdot_{rs}$, we generally have only $\mathcal{D} = \mathcal{J}$, while $\mathcal{L} \neq \mathcal{J}$ and $\mathcal{R} \neq \mathcal{H}$ may occur. For instance, let $r = x_1$, $s = g(x_2)$, $t = f(x_1, x_1)$, and $q = f(g(x_2), g(x_2))$. It can be verified that $t \cdot_{rs} s = q$ and $q \cdot_{rs} r = t$, so $t \,\mathcal{R}\, q$. However, since $t \neq q$, it follows from Theorem 10 that $(t, q) \notin \mathcal{H}$. In the same example, we also have $r \cdot_{rs} t \cdot_{rs} s = q$ and $r \cdot_{rs} q \cdot_{rs} r = t$, which implies $t \,\mathcal{J}\, q$. Nevertheless, $(t, q) \notin \mathcal{L}$ by Theorem 8, as $t \neq q$ and both terms contain either r or s as subterms.

6. Conclusion

In this paper, we introduced a binary operation $\cdot_{rs}$ on terms, extending the r-inductive product by simultaneously replacing two fixed subterms. Since $\cdot_{rs}$ is not associative on the entire set of terms, we constructed the subset $W_\tau^{r,s}(X_n)$ by excluding all proper subterms of r and s, and showed that this set forms a semigroup under $\cdot_{rs}$. However, we were only able to prove that this semigroup is maximal under the condition that r and s are not proper subterms of each other, while the general case remains open. We showed that idempotent and regular elements coincide in this semigroup, and that any term expressible as a product in which it occurs at least twice is necessarily idempotent. Moreover, we provided complete characterizations for all five types of Green's relations on this semigroup, highlighting that $\mathcal{D} = \mathcal{J}$ still holds, whereas $\mathcal{R} \neq \mathcal{H}$ and $\mathcal{L} \neq \mathcal{J}$ can occur, unlike in the case of r-inductive product. Future research could further explore other algebraic properties of this semigroup, such as ideals and special subsemigroups. Additionally, one might investigate the behavior of semigroups under more general forms of subterm replacement or define new operations inspired by inductive compositions.

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References

[1] K Denecke and S L Wismath. *Universal Algebra and Applications in Theoretical Computer Science*. Chapman & Hall/CRC, Boca Raton, FL, 2002.

[2] K Denecke and S Leeratanavalee. Kernels of generalized hypersubstitutions. In *Discrete mathematics and applications (Bansko, 2001)*, volume 6 of *Res. Math. Comput. Sci.*, pages 87–96. South-West Univ., Blagoevgrad, 2002.

[3] K Denecke. Menger algebras and clones of terms. *East-West J. Math.*, 5(2):179–193, 2003.

[4] K Denecke, P Glubudom, and J Koppitz. Power clones and non-deterministic hypersubstitutions. *Asian-Eur. J. Math.*, 1(2):177–188, 2008.

[5] K Denecke and P Jampachon. Regular elements and Green's relations in Menger algebras of terms. *Discuss. Math. Gen. Algebra Appl.*, 26(1):85–109, 2006.

[6] K Denecke and N Sarasit. Products of tree languages. *Bull. Sect. Logic Univ. Łódź*, 40(1-2):13–36, 2011.

[7] T Kumduang and S Leeratanavalee. Semigroups of terms, tree languages, menger algebra of n-ary functions and their embedding theorems. *Symmetry*, 13:558, 2021.

[8] S Shtrakov. Multi-solid varieties and mh-transducers. *Algebra Discrete Math.*, (3):113–131, 2007.

[9] P Kitpratyakul and B Pibaljommee. Semigroups of an inductive composition of terms. *Asian-Eur. J. Math.*, 15(2):Paper No. 2250038, 16, 2022.

[10] P Kitpratyakul and B Pibaljommee. Ideal characterizations of semigroups of inductive terms. *International Journal of Innovative Computing, Information and Control*, 18:801–813, 2022.

[11] P Kitpratyakul and B Pibaljommee. On substructures of semigroups of inductive terms. *AIMS Math.*, 7(6):9835–9845, 2022.

[12] J Boonsol, P Kitpratyakul, T Changphas, and B Pibaljommee. A product of tree languages. *Int. J. Math. Comput. Sci.*, 19(2):279–288, 2024.

[13] K Denecke and S L Wismath. Complexity of terms, composition, and hypersubstitution. *Int. J. Math. Math. Sci.*, (15):959–969, 2003.

[14] J M Howie. *Fundamentals of Semigroup Theory*, volume 12 of *London Mathematical Society Monographs. New Series.* The Clarendon Press, Oxford University Press, New York, 1995. Oxford Science Publications.