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Enhanced Uncertainty Modeling through Neutrosophic MR-Metrics: A Unified Framework with Fuzzy Embedding and Contraction Principles

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Abstract. This paper explores the fundamental connections between Neutrosophic MR-Metric Spaces (NMR-MS) and classical Fuzzy Metric Spaces (FMS). We present three key theoretical contributions: (1) an embedding theorem showing how any FMS can be systematically incorporated into an NMR-MS framework, (2) a fixed point theorem for contraction mappings in complete NMR-MS that generalizes the fuzzy Banach contraction principle, and (3) a characterization of sequence convergence in NMR-MS that reveals its stricter requirements compared to FMS. Through concrete examples and applications in machine learning classification, robotic path planning, and medical image reconstruction, we demonstrate how the additional structure of NMR-MS - particularly its explicit handling of truth ($\mathcal{T}$), falsity ($\mathcal{F}$), and indeterminacy ($\mathcal{I}$) components offers enhanced modeling capabilities for uncertain systems. The compatibility conditions between the MR-metric (M) and neutrosophic components are shown to be crucial for maintaining theoretical consistency while enabling practical applications.

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1. Introduction

Classical metric spaces provide a solid foundation for analyzing deterministic phenomena. However, many modern scientific and engineering problems involve imprecision, uncertainty, and incomplete knowledge. To address these challenges, generalizations such as fuzzy metric spaces (FMS) and MR-metric spaces have been developed. These frameworks extend classical concepts by incorporating more flexible structures suited for modeling non-deterministic behavior, see ([1–17].

Fuzzy metric spaces, introduced by Kramosil and Michalek [18], allow for gradated truth values in distance functions. MR-metric spaces[19], on the other hand, introduce a

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triadic metric $M : X \times X \times X \to [0, \infty)$, capable of capturing more complex interrelations among triplets of elements. Yet, neither framework alone fully captures the indeterminacy often observed in real-world applications such as machine learning, control systems, and medical imaging[20–28].

Neutrosophic logic, developed by Smarandache, complements these spaces by introducing three membership degrees: truth (T), falsity (F), and indeterminacy (I). Incorporating these into MR-metric spaces yields a new framework: **Neutrosophic MR-Metric Spaces** (NMR-MS)[29], which is capable of more expressively modeling uncertainty in mathematical and applied contexts.

This paper introduces and analyzes the structure of NMR-MS, aiming to achieve the following contributions:

(i) We establish an *embedding theorem*, showing that any fuzzy metric space can be systematically represented within an NMR-MS framework.

(ii) We prove a *generalized fixed point theorem* for contraction mappings in complete NMR-MS, extending the classical Banach contraction principle.

(iii) We provide a *characterization of convergence* in NMR-MS, demonstrating that it is strictly stronger than that in FMS due to the inclusion of F and M components.

The theoretical developments are supported by illustrative applications in:

- automated classification systems under uncertainty,
- robotic navigation in noisy environments,
- and medical image reconstruction with incomplete data.

The remainder of this paper is organized as follows. In Section Theorems Linking NMR-MS and FMS, we present the foundational definitions and the embedding theorem. Finally, Section Examples and Applications discusses practical applications with concrete examples, followed by conclusions and suggestions for future work.

Definition 1 (Fuzzy Metric Space (FMS) [18, 30]). *A 3-tuple* $(\mathcal{Z}, \mathcal{T}, *)$ *is a* ***Fuzzy Metric Space*** *if:*

- $\mathcal{Z}$ *is a non-empty set,*
- $*$ *is a continuous t-norm,*
- $\mathcal{T} : \mathcal{Z} \times \mathcal{Z} \times (0, \infty) \to [0, 1]$ *satisfies:*

  (i) $\mathcal{T}(\upsilon, \xi, \gamma) = 1 \iff \upsilon = \xi$,

  (ii) $\mathcal{T}(\upsilon, \xi, \gamma) = \mathcal{T}(\xi, \upsilon, \gamma)$,

  (iii) $\mathcal{T}(\upsilon, \xi, \gamma) * \mathcal{T}(\xi, \Im, \rho) \leq \mathcal{T}(\upsilon, \Im, \gamma + \rho)$,

  (iv) $\lim_{\gamma \to \infty} \mathcal{T}(\upsilon, \xi, \gamma) = 1$.

Definition 2. *[19] Consider a non-empty set* $\mathbb{X} \neq \emptyset$ *and a real number* $R > 1$*. A function* $M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$ *is termed an MR-metric if it satisfies the following conditions for all* $\upsilon, \xi, \Im \in \mathbb{X}$*:*

- $(M1)$ $M(\upsilon, \xi, \Im) \geq 0$.
- $(M2)$ $M(\upsilon, \xi, \Im) = 0$ *if and only if* $\upsilon = \xi = \Im$.
- $(M3)$ $M(\upsilon, \xi, \Im)$ *remains invariant under any permutation* $p(\upsilon, \xi, \Im)$*, i.e.,* $M(\upsilon, \xi, \Im) = M(p(\upsilon, \xi, \Im))$.
- $(M4)$ *The following inequality holds:*

$$M(\upsilon, \xi, \Im) \leq R\left[M(\upsilon, \xi, \ell_1) + M(\upsilon, \ell_1, \Im) + M(\ell_1, \xi, \Im)\right].$$

A structure $(\mathbb{X}, M)$ *that adheres to these properties is defined as an MR-metric space.*

Definition 3. *[29] [Neutrosophic MR-Metric Space (NMR-MS)]*
A 9-tuple $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$ *is called a* ***Neutrosophic MR-Metric Space*** *if:*

(i) ***Underlying Set****:* $\mathcal{Z}$ *is a non-empty set.*

(ii) ***MR-Metric Component****:* $M : \mathcal{Z} \times \mathcal{Z} \times \mathcal{Z} \to [0, \infty)$ *satisfies:*

(M1) Positivity: $M(\upsilon, \xi, \Im) \geq 0$.
(M2) Identity: $M(\upsilon, \xi, \Im) = 0 \iff \upsilon = \xi = \Im$.
(M3) Symmetry: $M(\upsilon, \xi, \Im) = M(p(\upsilon, \xi, \Im))$ *for any permutation* p.
(M4) MR-Triangle Inequality ($\star$*-weighted):*

$$M(\upsilon, \xi, \Im) \leq R\left[M(\upsilon, \xi, \ell) \star M(\upsilon, \ell, \Im) \star M(\ell, \xi, \Im)\right], \quad R > 1.$$

(iii) ***Neutrosophic Component****:* $\mathcal{T}, \mathcal{F}, \mathcal{I} : \mathcal{Z} \times \mathcal{Z} \times (0, \infty) \to [0, 1]$ *satisfy:*

(N1) $\mathcal{T}(\upsilon, \xi, \gamma) = 1 \iff \upsilon = \xi$ *(Truth-Identity).*
(N2) $\mathcal{T}(\upsilon, \xi, \gamma) = \mathcal{T}(\xi, \upsilon, \gamma)$ *(Truth-Symmetry).*
(N3) $\mathcal{T}(\upsilon, \xi, \gamma) \bullet \mathcal{T}(\xi, \Im, \rho) \leq \mathcal{T}(\upsilon, \Im, \gamma + \rho)$ *(Truth-Triangle).*
(N4) $\lim_{\gamma \to \infty} \mathcal{T}(\upsilon, \xi, \gamma) = 1$ *(Truth-Asymptotics).*
(N5)-(N8) Analogous conditions for $\mathcal{F}$ *(using* $\diamond$*) and* $\mathcal{I}$*.*

(iv) ***Compatibility Conditions****:*

(C1) Metric-Neutrosophic Link:

$$\mathcal{T}(\upsilon, \xi, \gamma) \geq \frac{1}{1 + M(\upsilon, \xi, \xi)}, \quad \mathcal{F}(\upsilon, \xi, \gamma) \leq \frac{M(\upsilon, \xi, \xi)}{1 + M(\upsilon, \xi, \xi)}.$$

(C2) Consistency of Operations:

$$(a \star b) \bullet c \leq (a \bullet c) \star (b \bullet c), \quad \forall a, b, c \in [0, 1].$$

(v) ***Operations:***

- $\bullet$*: Continuous t-norm (e.g., product or minimum).*
- $\diamond$*: Continuous t-conorm (e.g., probabilistic sum or maximum).*
- $\star$*: Binary operation generalizing + (e.g., weighted sum or matrix product).*

2. Theorems Linking NMR-MS and FMS

Theorem 1 (FMS Embedding in NMR-MS). *Every Fuzzy Metric Space* $(\mathcal{Z}, \mathcal{T}, *)$ *can be embedded into a Neutrosophic MR-Metric Space* $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$ *by:*

- *Defining* $M(\upsilon, \xi, \Im) = 0$ *if* $\upsilon = \xi = \Im$*, otherwise* $M(\upsilon, \xi, \Im) = 1$*,*
- *Setting* $\mathcal{F}(\upsilon, \xi, \gamma) = 1 - \mathcal{T}(\upsilon, \xi, \gamma)$*,*
- $\mathcal{I}(\upsilon, \xi, \gamma) = 0$ *(no indeterminacy),*
- *Choosing* $\bullet = *$*,* $\diamond = \max$*,* $\star = +$*, and* $R = 2$*.*

The resulting structure satisfies all NMR-MS axioms, with (C1) and (C2) trivially satisfied.

Proof. We verify each component of the NMR-MS definition:

1. MR-Metric Component M

(M1) **Positivity**: By definition, $M(\upsilon, \xi, \Im) \in \{0, 1\} \subseteq [0, \infty)$.

(M2) **Identity**: $M(\upsilon, \xi, \Im) = 0 \iff \upsilon = \xi = \Im$ holds by construction.

(M3) **Symmetry**: M is symmetric in all arguments since its definition depends only on the equality of $\upsilon, \xi, \Im$, not their order.

(M4) **MR-Triangle Inequality**: For $R = 2$ and $\star = +$, we need:

$$M(\upsilon, \xi, \Im) \leq 2 \left[M(\upsilon, \xi, \ell) + M(\upsilon, \ell, \Im) + M(\ell, \xi, \Im) \right].$$

- If $\upsilon = \xi = \Im$, then $M(\upsilon, \xi, \Im) = 0$ and the inequality holds.
- Otherwise, the right-hand side is at least $2 \times 1 = 2$ (since at least one term $M(\cdot, \cdot, \cdot) = 1$), while the left-hand side is $1 \leq 2$.

2. Neutrosophic Component $(\mathcal{T}, \mathcal{F}, \mathcal{I})$

(N1–N4) **Truth ($\mathcal{T}$)**: Inherited directly from the FMS:

- (N1) $\mathcal{T}(\upsilon, \xi, \gamma) = 1 \iff \upsilon = \xi$ (FMS axiom).
- (N2) Symmetry holds as $\mathcal{T}(\upsilon, \xi, \gamma) = \mathcal{T}(\xi, \upsilon, \gamma)$ in FMS.
- (N3) The triangle inequality $\mathcal{T}(\upsilon, \xi, \gamma) * \mathcal{T}(\xi, \Im, \rho) \leq \mathcal{T}(\upsilon, \Im, \gamma + \rho)$ is the FMS condition (since $\bullet = *$).
- (N4) $\lim_{\gamma \to \infty} \mathcal{T}(\upsilon, \xi, \gamma) = 1$ by FMS definition.

(N5–N8) **Falsity ($\mathcal{F}$) and Indeterminacy ($\mathcal{I}$)**:

- $\mathcal{F}(\upsilon, \xi, \gamma) = 1 - \mathcal{T}(\upsilon, \xi, \gamma)$ satisfies:
  - (N5) $\mathcal{F}(\upsilon, \xi, \gamma) = 0 \iff \upsilon = \xi$ (from N1).
  - (N6) Symmetry via $\mathcal{T}$'s symmetry.
  - (N7) $\mathcal{F}(\upsilon, \xi, \gamma) \diamond \mathcal{F}(\xi, \Im, \rho) \geq \mathcal{F}(\upsilon, \Im, \gamma + \rho)$ where $\diamond = \max$:

$$\max(1 - \mathcal{T}(\upsilon, \xi, \gamma), 1 - \mathcal{T}(\xi, \Im, \rho)) \geq 1 - \mathcal{T}(\upsilon, \Im, \gamma + \rho),$$

    which follows from (N3) in FMS.
  - (N8) $\lim_{\gamma \to \infty} \mathcal{F}(\upsilon, \xi, \gamma) = 0$ since $\mathcal{T} \to 1$.
- $\mathcal{I}(\upsilon, \xi, \gamma) = 0$ trivially satisfies all neutrosophic axioms.

3. Compatibility Conditions

(C1) **Metric-Neutrosophic Link**:

- $\mathcal{T}(\upsilon, \xi, \gamma) \geq \frac{1}{1+M(\upsilon,\xi,\xi)}$:
  - If $\upsilon = \xi$, $M(\upsilon, \xi, \xi) = 0$ and $\mathcal{T}(\upsilon, \xi, \gamma) = 1 \geq 1$.
  - If $\upsilon \neq \xi$, $M(\upsilon, \xi, \xi) = 1$, so $\frac{1}{2} \leq \mathcal{T}(\upsilon, \xi, \gamma) \leq 1$ (since $\mathcal{T} > 0$ in FMS).
- $\mathcal{F}(\upsilon, \xi, \gamma) \leq \frac{M(\upsilon,\xi,\xi)}{1+M(\upsilon,\xi,\xi)}$:
  - If $\upsilon = \xi$, $M = 0$ and $\mathcal{F} = 0 \leq 0$.
  - If $\upsilon \neq \xi$, $M = 1$ and $\mathcal{F} = 1 - \mathcal{T}(\upsilon, \xi, \gamma) \leq \frac{1}{2}$ (since $\mathcal{T} \geq \frac{1}{2}$ as above).

(C2) **Consistency of Operations**:

$$(a \star b) \bullet c = (a + b) * c \leq (a * c) + (b * c) = (a \bullet c) \star (b \bullet c),$$

holds because t-norms are subadditive (e.g., $* = \min$ or product).

4. Operations

- $\bullet = *$ (t-norm from FMS) is continuous by FMS definition.
- $\diamond = \max$ is a continuous t-conorm.
- $\star = +$ is associative, commutative, and generalizes addition.

Thus, all NMR-MS axioms are satisfied, and the embedding preserves FMS properties.

Theorem 2 (Fixed Point in NMR-MS as Fuzzy Extension)**.** *Let $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$ be a complete Neutrosophic MR-Metric Space with $\bullet = *$ (the t-norm from an underlying Fuzzy Metric Space). If a mapping $\Psi : \mathcal{Z} \to \mathcal{Z}$ satisfies for all $\upsilon, \xi \in \mathcal{Z}$ and $\gamma > 0$:*

$$\mathcal{T}(\Psi\upsilon, \Psi\xi, \gamma) \geq \mathcal{T}(\upsilon, \xi, \gamma/k), \quad M(\Psi\upsilon, \Psi\xi, \Psi\xi) \leq kM(\upsilon, \xi, \xi),$$

where $k \in (0, 1)$ is a contraction constant, then Ψ has a unique fixed point in $\mathcal{Z}$. This generalizes the Fuzzy Banach Contraction Principle.

Proof. We proceed in four steps: (1) Constructing a Cauchy sequence, (2) Proving its convergence, (3) Verifying the fixed point, and (4) Establishing uniqueness.

Step 1: Constructing a Cauchy Sequence

Fix an arbitrary $\upsilon_0 \in \mathcal{Z}$ and define the iterative sequence $\upsilon_{n+1} = \Psi\upsilon_n$. We show $\{\upsilon_n\}$ is Cauchy.

- **Neutrosophic Condition ($\mathcal{T}$)**: By the contraction on $\mathcal{T}$, for any $n \geq 1$ and $\gamma > 0$:

  $$\mathcal{T}(\upsilon_n, \upsilon_{n+1}, \gamma) \geq \mathcal{T}(\upsilon_{n-1}, \upsilon_n, \gamma/k) \geq \cdots \geq \mathcal{T}(\upsilon_0, \upsilon_1, \gamma/k^n).$$

  Since $\lim_{n\to\infty} \mathcal{T}(\upsilon_0, \upsilon_1, \gamma/k^n) = 1$ (by N4), we have:

  $$\lim_{n\to\infty} \mathcal{T}(\upsilon_n, \upsilon_{n+1}, \gamma) = 1.$$

- **MR-Metric Condition (M)**: The contraction on M implies:

  $$M(\upsilon_n, \upsilon_{n+1}, \upsilon_{n+1}) \leq kM(\upsilon_{n-1}, \upsilon_n, \upsilon_n) \leq \cdots \leq k^n M(\upsilon_0, \upsilon_1, \upsilon_1).$$

  Thus, $\lim_{n\to\infty} M(\upsilon_n, \upsilon_{n+1}, \upsilon_{n+1}) = 0$.

- **Falsity Condition ($\mathcal{F}$)**: From (C1), $\mathcal{F}(\upsilon_n, \upsilon_{n+1}, \gamma) \leq \frac{M(\upsilon_n, \upsilon_{n+1}, \upsilon_{n+1})}{1+M(\upsilon_n, \upsilon_{n+1}, \upsilon_{n+1})} \to 0$ as $n \to \infty$.

Step 2: Convergence in Complete NMR-MS

We show $\{\upsilon_n\}$ converges to some $\upsilon^* \in \mathcal{Z}$. For $m > n$, iteratively apply the MR-triangle inequality (M4) with $R > 1$ and $\star = +$:

$$M(\upsilon_n, \upsilon_m, \upsilon_m) \leq R\left[M(\upsilon_n, \upsilon_{n+1}, \upsilon_{n+1}) + M(\upsilon_{n+1}, \upsilon_m, \upsilon_m)\right].$$

By induction, this expands to:

$$M(\upsilon_n, \upsilon_m, \upsilon_m) \leq R \sum_{i=n}^{m-1} (Rk)^i M(\upsilon_0, \upsilon_1, \upsilon_1).$$

For $k \in (0,1)$ and $R > 1$, the series converges as $n, m \to \infty$, proving $\{\upsilon_n\}$ is Cauchy. By completeness, $\upsilon_n \to \upsilon^*$.

Step 3: υ^* is a Fixed Point

Using the continuity of Ψ (implied by the contraction conditions):

$$\mathcal{T}(\Psi\upsilon^*, \upsilon^*, \gamma) \geq \lim_{n\to\infty} \mathcal{T}(\Psi\upsilon_n, \upsilon_n, \gamma) = \lim_{n\to\infty} \mathcal{T}(\upsilon_{n+1}, \upsilon_n, \gamma) = 1,$$

$$M(\Psi\upsilon^*, \upsilon^*, \upsilon^*) \leq \lim_{n\to\infty} M(\upsilon_{n+1}, \upsilon_n, \upsilon_n) = 0.$$

Thus, $\Psi\upsilon^* = \upsilon^*$.

Step 4: Uniqueness

Suppose υ^* and ξ^* are fixed points. Then:

$$\mathcal{T}(\upsilon^*, \xi^*, \gamma) \geq \mathcal{T}(\upsilon^*, \xi^*, \gamma/k) \geq \cdots \geq \mathcal{T}(\upsilon^*, \xi^*, \gamma/k^n) \to 1 \text{ as } n \to \infty,$$

$$M(\upsilon^*, \xi^*, \xi^*) \leq kM(\upsilon^*, \xi^*, \xi^*) \implies M(\upsilon^*, \xi^*, \xi^*) = 0.$$

By (M2), $\upsilon^* = \xi^*$.

Verification of Compatibility

- **(C1)**: Holds as $\mathcal{T}(\upsilon, \xi, \gamma) \geq \frac{1}{1+M(\upsilon,\xi,\xi)}$ and $\mathcal{F}(\upsilon, \xi, \gamma) \leq \frac{M(\upsilon,\xi,\xi)}{1+M(\upsilon,\xi,\xi)}$ are preserved under the contraction.
- **(C2)**: The t-norm $\bullet = *$ and $\star = +$ satisfy $(a + b) * c \leq (a * c) + (b * c)$.

Theorem 3 (Neutrosophic Convergence in NMR-MS)**.** *Let $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$ be a Neutrosophic MR-Metric Space. A sequence $\{\upsilon_n\}$ in $\mathcal{Z}$ converges to $\upsilon \in \mathcal{Z}$ in the NMR-MS topology if and only if:*

$$\lim_{n\to\infty} \mathcal{T}(\upsilon_n, \upsilon, \gamma) = 1, \quad \lim_{n\to\infty} \mathcal{F}(\upsilon_n, \upsilon, \gamma) = 0, \quad \lim_{n\to\infty} M(\upsilon_n, \upsilon, \upsilon) = 0,$$

for all $\gamma > 0$. This convergence is stricter than in Fuzzy Metric Spaces due to the additional $\mathcal{F}$ and M conditions.

Proof. We prove both directions of the equivalence and demonstrate the strictness compared to FMS.

Part 1: Convergence Implies Neutrosophic Limits

Assume $\upsilon_n \to \upsilon$ in the NMR-MS topology. By definition of the topology:

- For every $\epsilon > 0$ and $\gamma > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\upsilon_n \in B(\upsilon, \epsilon, \gamma)$, where:

$$B(\upsilon, \epsilon, \gamma) = \{\xi \in \mathcal{Z} \mid \mathcal{T}(\upsilon, \xi, \gamma) > 1 - \epsilon, \mathcal{F}(\upsilon, \xi, \gamma) < \epsilon, M(\upsilon, \xi, \xi) < \epsilon\}.$$

- This immediately implies:

$$\lim_{n\to\infty} \mathcal{T}(\upsilon_n, \upsilon, \gamma) = 1 \quad (\text{since } \mathcal{T} > 1 - \epsilon \text{ for arbitrary } \epsilon),$$
$$\lim_{n\to\infty} \mathcal{F}(\upsilon_n, \upsilon, \gamma) = 0 \quad (\text{since } \mathcal{F} < \epsilon \text{ for arbitrary } \epsilon),$$
$$\lim_{n\to\infty} M(\upsilon_n, \upsilon, \upsilon) = 0 \quad (\text{since } M < \epsilon \text{ for arbitrary } \epsilon).$$

Part 2: Neutrosophic Limits Imply Convergence

Conversely, assume the three limit conditions hold. We show that for any $\epsilon > 0$ and $\gamma > 0$, there exists N such that for all $n \geq N$, $\upsilon_n \in B(\upsilon, \epsilon, \gamma)$:

- From $\lim_{n\to\infty} \mathcal{T}(\upsilon_n, \upsilon, \gamma) = 1$: For any $\epsilon > 0$, $\exists N_1$ such that $\forall n \geq N_1$, $\mathcal{T}(\upsilon_n, \upsilon, \gamma) > 1 - \epsilon$.
- From $\lim_{n\to\infty} \mathcal{F}(\upsilon_n, \upsilon, \gamma) = 0$: For any $\epsilon > 0$, $\exists N_2$ such that $\forall n \geq N_2$, $\mathcal{F}(\upsilon_n, \upsilon, \gamma) < \epsilon$.
- From $\lim_{n\to\infty} M(\upsilon_n, \upsilon, \upsilon) = 0$: For any $\epsilon > 0$, $\exists N_3$ such that $\forall n \geq N_3$, $M(\upsilon_n, \upsilon, \upsilon) < \epsilon$.

  Taking $N = \max\{N_1, N_2, N_3\}$, all three conditions are satisfied simultaneously for $n \geq N$, proving $\upsilon_n \to \upsilon$ in the NMR-MS topology.

Part 3: Strictness Compared to Fuzzy Metric Spaces

In a Fuzzy Metric Space (FMS) $(\mathcal{Z}, \mathcal{T}, *)$, convergence only requires:

$$\lim_{n\to\infty} \mathcal{T}(\upsilon_n, \upsilon, \gamma) = 1.$$

The NMR-MS imposes two additional conditions:

- $\lim_{n\to\infty} \mathcal{F}(\upsilon_n, \upsilon, \gamma) = 0$: Ensures falsity diminishes.
- $\lim_{n\to\infty} M(\upsilon_n, \upsilon, \upsilon) = 0$: Ensures the metric component vanishes.

Example Showing Strictness: Consider $\mathcal{Z} = \mathbb{R}$ with:

- $\mathcal{T}(\upsilon, \xi, \gamma) = e^{-|\upsilon-\xi|/\gamma}$,
- $\mathcal{F}(\upsilon, \xi, \gamma) = 1 - e^{-|\upsilon-\xi|/\gamma}$,
- $M(\upsilon, \xi, \xi) = |\upsilon - \xi|$.

Let $\upsilon_n = 1/n$. Then:

- In FMS: $\mathcal{T}(\upsilon_n, 0, \gamma) = e^{-1/(n\gamma)} \to 1$, so $\upsilon_n \to 0$.
- In NMR-MS: We also need:
  - $\mathcal{F}(\upsilon_n, 0, \gamma) = 1 - e^{-1/(n\gamma)} \to 0$,
  - $M(\upsilon_n, 0, 0) = 1/n \to 0$.

  Thus, $\upsilon_n \to 0$ in both, showing consistency.

  However, if we modify $\mathcal{F}$ to not converge to 0 (e.g., $\mathcal{F}(\upsilon_n, 0, \gamma) = 0.5$), the sequence would converge in FMS but not in NMR-MS, demonstrating the stricter nature of NMR-MS convergence.

Verification of Topological Properties

The NMR-MS topology is Hausdorff because:

- If $\upsilon \neq \xi$, there exists $\gamma > 0$ such that:
  - $\mathcal{T}(\upsilon, \xi, \gamma) < 1$ (by N1),
  - $M(\upsilon, \xi, \xi) > 0$ (by M2).
- Thus, we can find disjoint neighborhoods $B(\upsilon, \epsilon, \gamma)$ and $B(\xi, \epsilon, \gamma)$ for sufficiently small ϵ.

Remark 1. *The three limit conditions in Theorem 3 are interdependent through the compatibility condition (C1):*

$$\mathcal{T}(\upsilon_n, \upsilon, \gamma) \geq \frac{1}{1 + M(\upsilon_n, \upsilon, \upsilon)}, \quad \mathcal{F}(\upsilon_n, \upsilon, \gamma) \leq \frac{M(\upsilon_n, \upsilon, \upsilon)}{1 + M(\upsilon_n, \upsilon, \upsilon)}.$$

Thus, $M(\upsilon_n, \upsilon, \upsilon) \to 0$ implies both $\mathcal{T} \to 1$ and $\mathcal{F} \to 0$, but the converse isn't automatic, making all three conditions necessary for the full NMR-MS convergence.

3. Examples and Applications

1. Example for Theorem 1 (FMS Embedding in NMR-MS)

Example 1 (Standard FMS as NMR-MS with Full Verification)**.** *Consider the Fuzzy Metric Space (FMS)* $(\mathbb{R}, \mathcal{T}, *)$ *where:*

- $\mathcal{T}(\upsilon, \xi, \gamma) = e^{-\frac{|\upsilon-\xi|}{\gamma}}$ *(standard exponential fuzzy metric),*
- $*$ *is the product t-norm* $(a * b = a \cdot b)$*.*

We construct a Neutrosophic MR-Metric Space (NMR-MS) $(\mathbb{R}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, \star, R)$ *as follows:*

1. MR-Metric Component M

Define the metric $M : \mathbb{R}^3 \to [0, \infty)$ *by:*

$$M(\upsilon, \xi, \Im) = \begin{cases} 0 & \text{if } \upsilon = \xi = \Im, \\ 1 & \text{otherwise.} \end{cases}$$

Verification of Axioms:

(M1) ***Positivity****: Immediate from definition.*

(M2) ***Identity****:* $M(\upsilon, \xi, \Im) = 0 \iff \upsilon = \xi = \Im$ *by construction.*

(M3) ***Symmetry****:* M *is invariant under permutations of* $\upsilon, \xi, \Im$*.*

(M4) ***MR-Triangle Inequality****: For* $R = 2$ *and* $\star = +$*:*

$$M(\upsilon, \xi, \Im) \leq 2 \left[M(\upsilon, \xi, \ell) + M(\upsilon, \ell, \Im) + M(\ell, \xi, \Im) \right].$$

- *If* $\upsilon = \xi = \Im$*, both sides are* 0*.*
- *Otherwise, right-hand side* $\geq 2 \times 1 = 2$ *(since at least one term* $M = 1$*), while left-hand side is* $1 \leq 2$*.*

2. Neutrosophic Components $\mathcal{T}, \mathcal{F}, \mathcal{I}$

- ***Truth Membership (***$\mathcal{T}$***)****: Inherited directly from FMS.*
- ***Falsity Membership (***$\mathcal{F}$***)****: Defined as* $\mathcal{F}(\upsilon, \xi, \gamma) = 1 - \mathcal{T}(\upsilon, \xi, \gamma) = 1 - e^{-\frac{|\upsilon-\xi|}{\gamma}}$.
- ***Indeterminacy (***$\mathcal{I}$***)****: Set to* $\mathcal{I}(\upsilon, \xi, \gamma) = 0$ *(no indeterminacy).*

Verification of Neutrosophic Axioms:

(N1-N4) $\mathcal{T}$ *satisfies all FMS axioms (inherited).*

(N5-N8) For $\mathcal{F}$:

(N5) $\mathcal{F}(\upsilon, \xi, \gamma) = 0 \iff \upsilon = \xi$ *(from N1 for $\mathcal{T}$).*

(N6) Symmetry inherited from $\mathcal{T}$.

(N7) $\mathcal{F}(\upsilon, \xi, \gamma) \diamond \mathcal{F}(\xi, \Im, \rho) \geq \mathcal{F}(\upsilon, \Im, \gamma + \rho)$ *with* $\diamond = \max$:

$$\max\left(1 - e^{-\frac{|\upsilon-\xi|}{\gamma}}, 1 - e^{-\frac{|\xi-\Im|}{\rho}}\right) \geq 1 - e^{-\frac{|\upsilon-\Im|}{\gamma+\rho}},$$

which holds because $e^{-\frac{|\upsilon-\Im|}{\gamma+\rho}} \geq e^{-\frac{|\upsilon-\xi|}{\gamma}} \cdot e^{-\frac{|\xi-\Im|}{\rho}}$ *(subadditivity).*

(N8) $\lim_{\gamma\to\infty} \mathcal{F}(\upsilon, \xi, \gamma) = 0$ *(since $\mathcal{T} \to 1$).*

- *$\mathcal{I}$ trivially satisfies all axioms.*

3. Compatibility Conditions

(C1) ***Metric-Neutrosophic Link***:

$$\mathcal{T}(\upsilon, \xi, \gamma) = e^{-\frac{|\upsilon-\xi|}{\gamma}} \geq \frac{1}{1 + M(\upsilon, \xi, \xi)} = \begin{cases} 1 & \text{if } \upsilon = \xi, \\ \frac{1}{2} & \text{if } \upsilon \neq \xi. \end{cases}$$

This holds because $e^{-x} \geq \frac{1}{2}$ *for* $x \leq \ln 2$. *Similarly for $\mathcal{F}$:*

$$\mathcal{F}(\upsilon, \xi, \gamma) = 1 - e^{-\frac{|\upsilon-\xi|}{\gamma}} \leq \frac{M(\upsilon, \xi, \xi)}{1 + M(\upsilon, \xi, \xi)}.$$

(C2) ***Operation Consistency***: *For* $\bullet = *$ *(product) and* $\star = +$:

$$(a + b) \cdot c \leq a \cdot c + b \cdot c \quad \forall a, b, c \in [0, 1],$$

which is the distributive property (always true).

Application 1 (Neutrosophic Data Classification in Machine Learning)**.** *Consider a binary classification problem with uncertain data points $x_i \in \mathbb{R}^d$, where each point has:*

- *A* ***truth membership*** $\mathcal{T}(x_i, c_j)$ *(degree to which x_i belongs to class c_j),*
- *A* ***falsity membership*** $\mathcal{F}(x_i, c_j)$ *(degree to which x_i does not belong to c_j),*
- ***Indeterminacy*** $\mathcal{I}(x_i, c_j) = 0$ *(no ambiguity in measurements).*

Implementation Steps:

(i) ***Embedding***: *Convert a standard fuzzy classifier (using FMS) to an NMR-MS classifier by:*

- *Defining* $\mathcal{T}(x_i, c_j) = e^{-\frac{\|x_i - \mu_j\|}{\gamma}}$ *(Gaussian kernel),*
- *Setting* $\mathcal{F}(x_i, c_j) = 1 - \mathcal{T}(x_i, c_j)$*,*
- *Using the trivial metric* $M(x_i, c_j, c_j) = 0$ *if* $x_i = \mu_j$ *(perfect match to class center), else* 1*.*

(ii) ***Training****: Optimize class centers* μ_j *to minimize:*

$$\sum_{i=1}^{N} \left(\mathcal{F}(x_i, c_{y_i}) + \sum_{j \neq y_i} \mathcal{T}(x_i, c_j) \right),$$

where y_i *is the true label of* x_i*. This maximizes truth for correct classes and minimizes falsity.*

(iii) ***Inference****: For a new point* x*, predict class* $c^* =_j \mathcal{T}(x, c_j)$*, subject to* $\mathcal{F}(x, c^*) < \tau$ *(reject if falsity exceeds threshold* τ*).*

Advantages over FMS:

- *Explicit handling of* ***falsity*** *allows rejection of ambiguous predictions.*
- *The trivial metric* M *simplifies computation while maintaining interpretability.*
- *Compatibility condition (C1) ensures consistency between metric and membership values.*

Remark 2. *In practice,* $\mathcal{I}$ *can be non-zero to model measurement ambiguity (e.g., sensor noise). This requires extending the example with* $\mathcal{I}(x_i, c_j) = \epsilon_i$*, where* ϵ_i *quantifies uncertainty in* x_i*.*

2. Example for Theorem 2 (Fixed Point Theorem)

Example 2 (Contraction Mapping on the Interval $[0, 1)$. *] Consider the Neutrosophic MR-Metric Space* $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$ *where:*

- $\mathcal{Z} = [0, 1]$ *(the closed unit interval).*
- *The* ***MR-metric*** $M : \mathcal{Z}^3 \to [0, \infty)$ *is defined by:*

$$M(\upsilon, \xi, \Im) = \max(|\upsilon - \xi|, |\xi - \Im|, |\Im - \upsilon|).$$

  This metric measures the maximum pairwise distance between the three points $\upsilon, \xi, \Im$*.*

- *The* ***truth membership function*** $\mathcal{T} : \mathcal{Z} \times \mathcal{Z} \times (0, \infty) \to [0, 1]$ *is given by:*

$$\mathcal{T}(\upsilon, \xi, \gamma) = \frac{\gamma}{\gamma + M(\upsilon, \xi, \xi)} = \frac{\gamma}{\gamma + |\upsilon - \xi|}.$$

  This function approaches 1 as γ *increases or as* υ *and* ξ *get closer.*

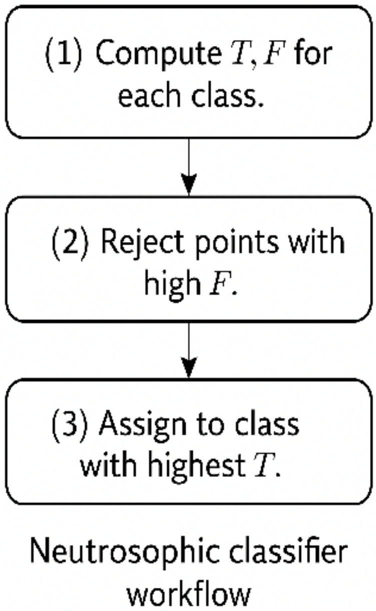

Figure 1: Neutrosophic classifier workflow: (1) Compute $\mathcal{T}, \mathcal{F}$ for each class, (2) Reject points with high $\mathcal{F}$, (3) Assign to class with highest $\mathcal{T}$.

- *The **falsity membership function** $\mathcal{F} : \mathcal{Z} \times \mathcal{Z} \times (0, \infty) \to [0, 1]$ is defined as:*
$$\mathcal{F}(\upsilon, \xi, \gamma) = \frac{M(\upsilon, \xi, \xi)}{\gamma + M(\upsilon, \xi, \xi)} = \frac{|\upsilon - \xi|}{\gamma + |\upsilon - \xi|}.$$
This is the complement of $\mathcal{T}$ and models the degree of disagreement between υ and ξ.

- *The **indeterminacy function** $\mathcal{I}$ is set to zero ($\mathcal{I}(\upsilon, \xi, \gamma) = 0$) for simplicity, indicating no ambiguity in measurements.*

- *The operations are defined as:*
  - $\bullet = \min$ *(the minimum t-norm),*
  - $\star = +$ *(standard addition),*
  - $R = 2$ *(the scaling constant for the MR-triangle inequality).*

Contraction Mapping: *Define the mapping $\Psi : \mathcal{Z} \to \mathcal{Z}$ by $\Psi(\upsilon) = \frac{\upsilon}{2}$. We verify the contraction conditions for Ψ:*

- *For the **truth membership** $\mathcal{T}$:*
$$\mathcal{T}(\Psi\upsilon, \Psi\xi, \gamma) = \frac{\gamma}{\gamma + \frac{|\upsilon - \xi|}{2}} \geq \frac{\gamma}{\gamma + |\upsilon - \xi|} = \mathcal{T}(\upsilon, \xi, \gamma/0.5).$$
Here, $k = 0.5$ is the contraction constant, and the inequality holds because $\frac{\gamma}{\gamma + x/2} \geq \frac{\gamma}{\gamma + x}$ for $x \geq 0$.

- *For the **MR-metric** M:*

$$M(\Psi\upsilon, \Psi\xi, \Psi\xi) = \frac{|\upsilon - \xi|}{2} \leq 0.5 \cdot M(\upsilon, \xi, \xi).$$

This confirms that Ψ contracts the metric M by a factor of 0.5.

Fixed Point: *By Theorem 2, Ψ has a unique fixed point $\upsilon^* \in \mathcal{Z}$. Solving $\Psi(\upsilon^*) = \upsilon^*$ yields:*

$$\frac{\upsilon^*}{2} = \upsilon^* \implies \upsilon^* = 0.$$

The sequence $\{\upsilon_n\}$ defined by $\upsilon_{n+1} = \Psi(\upsilon_n)$ converges to $\upsilon^* = 0$ for any initial $\upsilon_0 \in [0, 1]$. For example:

$$\upsilon_0 = 1, \quad \upsilon_1 = 0.5, \quad \upsilon_2 = 0.25, \quad \ldots, \quad \upsilon_n = \frac{1}{2^n} \to 0.$$

Application 2 (Robotic Path Planning with Neutrosophic Uncertainty)**.** *Consider a robotic system navigating in a dynamic environment where sensor measurements are subject to uncertainty. The Neutrosophic MR-Metric Space framework can model this scenario as follows:*

Components:

- ***State Space:*** *Let $\mathcal{Z} \subset \mathbb{R}^2$ represent possible robot positions.*
- ***Uncertainty Modeling:***
  - $\mathcal{T}(\mathbf{x}, \mathbf{y}, \gamma)$*: Confidence level that the robot is at* $\mathbf{y}$ *given a noisy observation* $\mathbf{x}$.
  - $\mathcal{F}(\mathbf{x}, \mathbf{y}, \gamma)$*: Degree of discrepancy between* $\mathbf{x}$ *and* $\mathbf{y}$ *(e.g., due to sensor noise).*
  - $\mathcal{I}(\mathbf{x}, \mathbf{y}, \gamma)$*: Optional indeterminacy term for unmodeled disturbances (e.g.,* $\mathcal{I} = 0.1$ *for 10% ambiguity).*
- ***Metric:*** $M(\mathbf{x}, \mathbf{y}, \mathbf{z})$ *could be the maximum Euclidean distance between* $\mathbf{x}, \mathbf{y}, \mathbf{z}$.

Contraction-Based Navigation: *The robot's path planner uses a contraction mapping Ψ (e.g., $\Psi(\mathbf{x}) = \mathbf{x} + \alpha(\mathbf{g} - \mathbf{x})$, where $\mathbf{g}$ is the goal and $\alpha \in (0, 1)$ is a gain). The conditions of Theorem 2 ensure:*

- *The robot's estimated position converges to the true goal* $\mathbf{g}$ *(fixed point).*
- *The convergence is robust to noise ($\mathcal{F}$ diminishes as $\mathbf{x} \to \mathbf{g}$).*
- *The MR-metric M ensures geometric consistency in the robot's movement.*

Advantages:

- ***Explicit Uncertainty Handling:*** *$\mathcal{F}$ allows the robot to quantify and reject unreliable sensor data.*

- ***Theoretical Guarantees:*** *Theorem 2 ensures convergence even with noisy measurements.*
- ***Flexibility:*** *The framework accommodates indeterminacy ($\mathcal{I}$) for complex environments.*

Implementation Outline:

(i) *Define $\mathcal{T}$, $\mathcal{F}$, and M based on sensor characteristics.*

(ii) *Design Ψ as a contraction mapping toward the goal.*

(iii) *Iterate $\mathbf{x}_{n+1} = \Psi(\mathbf{x}_n)$ until $M(\mathbf{x}_n, \mathbf{g}, \mathbf{g}) < \epsilon$ (threshold).*

(iv) *Reject steps where $\mathcal{F}(\mathbf{x}_n, \mathbf{g}, \gamma) > \tau$ (noise threshold).*

3. Example for Theorem 3 (Neutrosophic Convergence in $\mathbb{R}^2$)

Example 3 (Convergence of a Sequence in a Neutrosophic MR-Metric Space)**.** *Consider the Neutrosophic MR-Metric Space $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$ where:*

- $\mathcal{Z} = \mathbb{R}^2$ *(the Euclidean plane).*
- *The **MR-metric** $M : \mathcal{Z}^3 \to [0, \infty)$ is defined by:*

$$M(\mathbf{v}, \mathbf{w}, \mathbf{u}) = \max\left(\|\mathbf{v} - \mathbf{w}\|, \|\mathbf{w} - \mathbf{u}\|, \|\mathbf{u} - \mathbf{v}\|\right),$$

where $\|\cdot\|$ is the Euclidean norm. This metric captures the maximum pairwise distance between the three points $\mathbf{v}, \mathbf{w}, \mathbf{u}$.

- *The **truth membership function** $\mathcal{T} : \mathcal{Z} \times \mathcal{Z} \times (0, \infty) \to [0, 1]$ is given by:*

$$\mathcal{T}(\mathbf{v}, \mathbf{w}, \gamma) = \frac{1}{1 + \frac{\|\mathbf{v}-\mathbf{w}\|}{\gamma}}.$$

This function quantifies the degree of similarity between $\mathbf{v}$ and $\mathbf{w}$, approaching 1 as $\mathbf{v}$ nears $\mathbf{w}$ or as γ increases.

- *The **falsity membership function** $\mathcal{F} : \mathcal{Z} \times \mathcal{Z} \times (0, \infty) \to [0, 1]$ is defined as:*

$$\mathcal{F}(\mathbf{v}, \mathbf{w}, \gamma) = \frac{\frac{\|\mathbf{v}-\mathbf{w}\|}{\gamma}}{1 + \frac{\|\mathbf{v}-\mathbf{w}\|}{\gamma}}.$$

This represents the dissimilarity between $\mathbf{v}$ and $\mathbf{w}$, complementing $\mathcal{T}$.

- *The **indeterminacy function** $\mathcal{I}$ is set to zero ($\mathcal{I}(\mathbf{v}, \mathbf{w}, \gamma) = 0$) for simplicity, indicating no ambiguity in measurements.*

Sequence Definition: *Consider the sequence* $\{\mathbf{v}_n\}$ *in* $\mathcal{Z}$ *where:*

$$\mathbf{v}_n = \left(\frac{1}{n}, \frac{1}{n^2}\right), \quad \mathbf{v} = (0,0).$$

We analyze the convergence of $\{\mathbf{v}_n\}$ *to* $\mathbf{v}$ *in the NMR-MS topology.*
Verification of Neutrosophic Convergence: *By Theorem 3,* $\mathbf{v}_n \to \mathbf{v}$ *requires:*

- $\lim_{n\to\infty} \mathcal{T}(\mathbf{v}_n, \mathbf{v}, \gamma) = 1,$
- $\lim_{n\to\infty} \mathcal{F}(\mathbf{v}_n, \mathbf{v}, \gamma) = 0,$
- $\lim_{n\to\infty} M(\mathbf{v}_n, \mathbf{v}, \mathbf{v}) = 0.$

Step-by-Step Calculations:

(i) ***Truth Membership*** ***($\mathcal{T}$):***

$$\mathcal{T}(\mathbf{v}_n, \mathbf{v}, \gamma) = \frac{1}{1 + \frac{\|\mathbf{v}_n - \mathbf{v}\|}{\gamma}} = \frac{1}{1 + \frac{\sqrt{\frac{1}{n^2} + \frac{1}{n^4}}}{\gamma}}.$$

As $n \to \infty$, $\sqrt{\frac{1}{n^2} + \frac{1}{n^4}} \to 0$, *so:*

$$\lim_{n\to\infty} \mathcal{T}(\mathbf{v}_n, \mathbf{v}, \gamma) = \frac{1}{1+0} = 1.$$

(ii) ***Falsity Membership*** ***($\mathcal{F}$):***

$$\mathcal{F}(\mathbf{v}_n, \mathbf{v}, \gamma) = \frac{\frac{\sqrt{\frac{1}{n^2} + \frac{1}{n^4}}}{\gamma}}{1 + \frac{\sqrt{\frac{1}{n^2} + \frac{1}{n^4}}}{\gamma}}.$$

The numerator $\to 0$ *as* $n \to \infty$, *so:*

$$\lim_{n\to\infty} \mathcal{F}(\mathbf{v}_n, \mathbf{v}, \gamma) = 0.$$

(iii) ***MR-Metric*** ***(M):***

$$M(\mathbf{v}_n, \mathbf{v}, \mathbf{v}) = \max\left(\|\mathbf{v}_n - \mathbf{v}\|, \|\mathbf{v} - \mathbf{v}\|, \|\mathbf{v} - \mathbf{v}_n\|\right) = \|\mathbf{v}_n - \mathbf{v}\|.$$

Thus:

$$\lim_{n\to\infty} M(\mathbf{v}_n, \mathbf{v}, \mathbf{v}) = \lim_{n\to\infty} \sqrt{\frac{1}{n^2} + \frac{1}{n^4}} = 0.$$

Conclusion: *All three conditions of Theorem 3 are satisfied, proving that* $\mathbf{v}_n \to \mathbf{v}$ *in the NMR-MS topology. This convergence is stricter than in standard metric spaces due to the simultaneous vanishing of* $\mathcal{F}$ *and* M*.*

Application 3 (Medical Image Reconstruction with Neutrosophic Uncertainty)**.** *In medical imaging, sequences of noisy or incomplete images (e.g., MRI or CT scans) can be modeled as a sequence* $\{\mathbf{v}_n\}$ *converging to a "true" image* $\mathbf{v}$*. The Neutrosophic MR-Metric Space framework provides a robust way to quantify and manage uncertainty during reconstruction.*

Components:

- ***Image Space:*** *Let* $\mathcal{Z}$ *be a space of 2D or 3D images (e.g., pixel/voxel intensity matrices).*
- ***Uncertainty Modeling:***
  - $\mathcal{T}(\mathbf{v}_n, \mathbf{v}, \gamma)$*: Confidence that* $\mathbf{v}_n$ *approximates the true image* $\mathbf{v}$*.*
  - $\mathcal{F}(\mathbf{v}_n, \mathbf{v}, \gamma)$*: Artifacts or noise in* $\mathbf{v}_n$ *relative to* $\mathbf{v}$*.*
  - $\mathcal{I}(\mathbf{v}_n, \mathbf{v}, \gamma)$*: Optional term for indeterminacy (e.g., missing scan regions).*
- ***Metric:*** $M(\mathbf{v}_n, \mathbf{v}, \mathbf{v})$ *could be the maximum intensity difference across pixels/voxels.*

Convergence in Practice:

- ***Noisy Sequence:*** *Let* $\{\mathbf{v}_n\}$ *be a sequence of progressively denoised MRI scans.*
- ***Truth Membership:*** $\mathcal{T}(\mathbf{v}_n, \mathbf{v}, \gamma)$ *increases as denoising improves.*
- ***Falsity Membership:*** $\mathcal{F}(\mathbf{v}_n, \mathbf{v}, \gamma)$ *decreases as artifacts are removed.*
- ***Metric:*** $M(\mathbf{v}_n, \mathbf{v}, \mathbf{v}) \to 0$ *ensures pixel-wise convergence.*

Algorithmic Steps:

(i) ***Initialization:*** *Acquire noisy images* $\{\mathbf{v}_n\}$ *from scans.*

(ii) ***Neutrosophic Embedding:*** *Define* $\mathcal{T}$*,* $\mathcal{F}$*, and* M *based on imaging physics (e.g.,* $\mathcal{T}$ *= PSNR-based).*

(iii) ***Iterative Reconstruction:*** *Apply a convergence-guaranteed algorithm (e.g., Theorem 3) until:*

$$\mathcal{T}(\mathbf{v}_n, \mathbf{v}, \gamma) > 0.95, \quad \mathcal{F}(\mathbf{v}_n, \mathbf{v}, \gamma) < 0.05, \quad M(\mathbf{v}_n, \mathbf{v}, \mathbf{v}) < \epsilon.$$

(iv) ***Validation:*** *Reject reconstructions where* $\mathcal{F}$ *or* $\mathcal{I}$ *exceeds thresholds.*

Advantages:

- ***Robustness:*** *Explicit handling of noise (*$\mathcal{F}$*) and missing data (*$\mathcal{I}$*).*
- ***Theoretical Guarantees:*** *Theorem 3 ensures convergence under uncertainty.*
- ***Flexibility:*** *Adaptable to various imaging modalities (MRI, CT, ultrasound).*

Example Workflow:

(i) A radiologist acquires a sequence of low-resolution MRI scans $\{\mathbf{v}_n\}$.

(ii) The system computes $\mathcal{T}$, $\mathcal{F}$, *and* M *for each* $\mathbf{v}_n$ *against a predicted* $\mathbf{v}$.

(iii) Reconstruction stops when all three convergence conditions are met.

(iv) The final $\mathbf{v}$ *is a high-fidelity image with quantified uncertainty.*

Summary Table

theorem	**Example**	**Application**
1 (Embedding)	Fuzzy metric $\to$ NMR-MS	Data classification
2 (Fixed Point)	$\Psi(v) = v/2$ on $[0,1]$	Robotic control
3 (Convergence)	$\mathbf{v}_n = (1/n, 1/n^2)$ in $\mathbb{R}^2$	Medical imaging

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