EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6479
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

A Degree-Based Exponential Fuzzy Graph for Pollution Impact Analysis

Wadei Faris AL-Omeri[1,*], M.Kaviyarasu[2,*], R. Venitha[2]

[1] *Department of Mathematics, Faculty of Science and Information Technology, Jadara University, Irbid, Jordan*
[2] *Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology Avadi, Chennai, Tamil Nadu-600 062, India*

Abstract. Exponential Fuzzy Graphs (EFGs) are a new family of fuzzy graphs in which the vertices and edges' membership functions exhibit exponential decay. An EFG, which is characterised as a pair $G = (V, E)$, captures the uncertainty and degradation of influence in complex systems by giving each edge $(r, w) \in E$ a membership value $\vartheta_E(r, w) = \alpha_E(r, w) \cdot e^{-\lambda\alpha_E(r,w)}$ and each vertex $v \in V$ a membership value $\vartheta_V(w) = \alpha_V(w) \cdot e^{-\lambda\alpha_V(w)}$, where $\lambda > 0$ is a decay parameter. To maintain consistency inside the fuzzy structure, the edge membership values are limited by $\vartheta_E(r, w) \leq \min\{\vartheta_V(r), \vartheta_V(w)\}$. Some fundamental aspects such as vertex degree, order and size are explored in relation to the idea of exponential fuzzy graphs (EFGs). EFGs are characterised as complete and complement. Some basic operations like semi-strong product, union, join, composition and cartesian product are defined with graphical representing examples. The vertex degree of the generated vertices is examined for each operation and associated theorems are demonstrated. The theoretical findings are shown using examples. The use of EFGs in modelling real-life imprecise and uncertain data is explained in an application related to environmental contamination.

2020 Mathematics Subject Classifications: 03E72, 03B52, 28E10

Key Words and Phrases: Exponential Fuzzy Graphs, Degree of Vertices, Type of Product, Operations, Decision Making

---

1. Introduction

Zadeh [1] defined a fuzzy set as a class of objects having a range of membership grades in 1965. The function of membership that produces an element of membership grade from 0 to 1 provides such a collection. In certain fundamental operations, fuzzy sets are created.

---

*Corresponding author.
*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6479

Email addresses: wadeimoon1@hotmail.com (W. Al-Omeri), drkaviyarasum@veltech.edu.in (K. M.), vtd1606@veltech.edu.in (R. Venitha)

Specifically, it is possible to establish a separation theorem for convex fuzzy sets without the fuzzy sets having to be disjoint.

Fuzzy graphs, introduced by Rosenfeld (1975), expand on traditional graph theory by adding the concept of fuzziness into vertices and edges to model uncertainty and vagueness in relationships between objects [2]. Every vertex as well as edge in a fuzzy network has a membership level in the interval [0, 1], which indicates the level of connectedness or existence, respectively. Numerous fuzzy graph topics have been examined in [3].

Mordeson and Peng, fuzzy graph theory has emerged as a powerful tool for representing imprecise information in areas include network assessment, image processing, the process of decision-making and networking sites [4]. In [5] Allister demonstrates how effective numerical methods for fixing a network's failing node or link are provided by a novel definition. Akram presented the idea of strong bipolar fuzzy graphs and looked at a number of their basic characteristics in [6] and additionally investigates a number of statements regarding these graphs self-weak-complementary forms. Recent research has explored operations, connectivity and fuzzy subgraph properties, leading to the development of specialized structures of fuzzy graphs such as interval-valued, bipolar and exponential [7] [8]. These advancements enhance the capability of fuzzy graphs to handle diverse and complex real-world applications.

The fuzzification of graph structures allows for more realistic modeling of systems where binary connections are inadequate. Jun and colleagues (2014) further generalized fuzzy graphs to intuitionistic and interval-valued fuzzy environments, capturing additional uncertainty dimensions [9]. In [10], Thakur, Priya and Pawan Kumar presented a method that enhances optimal balanced histogram thresholding for converting grayscale images to binary. Their approach further incorporates Graphical abstraction through fuzzy graph theory and applies Widgerson's Color rendering algorithm for Image depiction. The ideas of efficient fuzzy graphs and dominance integrity in fuzzy graphs are presented and shown using instances in 2020 [11]. Muhiuddin et al. presented the original idea of node integrity in multi-parameter fuzzy graphs in 2023 [12] and thoroughly investigated a number of its associated features. Along with a number of significant findings, the paper also offers a thorough analysis of the many kinds of integrity in mPFG, such as edge, dominating and node integrity. Ji et al. [13] presented a thorough classification and new taxonomies of knowledge graph embedding in 2022. These were arranged according to four main components: auxiliary data, encoding designs, grading operations and illustration environment. A novel structure to support a rule-based sampling technique applied to fuzzy knowledge graphs was presented by Tan et al. [14] in 2025.

Ramot et al. 2002 [15] offered a mathematical approach that uses a complex fuzzy number to describe membership in a set. In [16] 2016, Thirunavukarasu et al. expand on the idea based on certain energies and their intricate systems. Shoaib et al, applied the complex fuzzy set in graph theory and varies operations are discussed in 2022 [17]. A sophisticated hesitant fuzzy graph model was presented by AbuHijleh (2023) to depict the influencing elements and cooperative ties amongst ministries. Vertex degrees in two of these graphs were also examined and a case study assessing intra-ministerial and inter-ministerial cooperation to enhance the representation of dual-variable systems was used to illustrate the model's usefulness in [18].

In 2025 [19], Kaviyarasu et al. developed the concept of exponential fuzzy sets, subsequently developing foundational definitions, illustrative examples, key properties and related theorems. The study concluded with an application in AI-powered investment decision-making, employing a Weighted average values to approach.AL-Omeri et al. [20]-[21] presented a novel class of sets, known as e-I-open sets, has been presented in the context of ideal topology and simple extension topology. It has been demonstrated that these sets are a less powerful variant of m-open sets. Numerous topological characteristics of the idea have been investigated, and it has been further expanded. Al-shami et al. [22] explores soft closed graphs and soft continuous mappings, characterizing soft continuity via soft points and establishing conditions for soft equalizers to be soft closed. It also shows the convergence equivalence between soft nets and soft filters, and proves that in soft Hausdorff and compact co-domains, soft continuity is equivalent to having a soft closed graph. In order to improve uncertainty modelling, Ibrahim et al. [23] presents complex nth power root fuzzy sets , which combine complex fuzzy logic with nth power root fuzzy sets. It uses the newly defined comparison tools and aggregation operators to decision-making issues such as venue and caterer selection. The n, m-rung picture fuzzy set , an improved model that goes beyond q-rung picture fuzzy sets by accounting for greater uncertainty in multi-attribute decision-making, is presented in this work. To evaluate expat living standards, a new aggregation operator, n, m-RPFWPA, is put forth and used. Through comparisons with current fuzzy decision-making operators, the model's efficacy is confirmed Ibrahim et al. [24].

Notation table:

Symbol	Meaning
$\mathcal{G}$	Exponential fuzzy graph
$\mathcal{E}_V$	Exponential fuzzy vertex set
$\mathcal{E}_E$	Exponential fuzzy edge set
$\vartheta_V(w)$	Vertex membership
$\vartheta_E(r, w)$	Edge membership
$\alpha_V(w)$	Vertex base-membership
$\alpha_E(r, w)$	Edge base-membership
λ	Parameter
$\deg_{\mathcal{G}}(w)$	Degree of a Vertex

Table 1: Notation Table

Research Gap and Contribution of this Study:
The extension of fuzzy graph theory's notions into the exponential fuzzy framework has not yet been investigated in the literature, despite the fact that the topic is well-established and extensively studied.

In this study, we introduce the innovative idea of exponential fuzzy graphs. The study explores vertex degrees, various types of graph products and provides illustrative examples accompanied by noteworthy properties. Furthermore, we present an application in the do-main of multi-criteria decision-making (MCDM). This work's vital contribution is to the improvement of exponential fuzzy graph structures and their specialized forms, designed to address limitations found in existing models within the literature. To support practical implementation, we propose an algorithm aimed at determining tolerance levels among real-world entities, with a focused application in solving MCDM problems.

Novelty

- An exponential decay function is directly incorporated into the vertex and edge membership values in EFGs.
- EFGs dynamically model how impact or relevance diminishes over time or through intricate interactions, in contrast to standard fuzzy graphs with members that are fixed or static.
- EFGs are more in line with real-world situations since they more accurately depict the impact's natural degradation. In network-based systems, this dynamic behavior improves the precision and dependability of uncertainty representation.
- EFGs' adaptability and time sensitivity make them appropriate for a variety of applications, such as flood prediction, where circumstances can change suddenly and without warning.

Motivation

- Static membership values, which are used in traditional fuzzy graphs, are insufficient to depict systems with interactions that change or degrade over time.
- Dynamic modeling is necessary for many real-world systems that show gradually deteriorating linkages, such as the spread of environmental pollution, biological network interactions and the development of social influence.
- EFGs are created to account for the time-sensitive and interaction-sensitive decay of relationships in order to overcome this constraint.
- The decay parameter λ, which is modifiable in EFGs, allows for flexible tweaking of the rate at which influence or connection strength declines. For complex and unpredictable systems, this dynamic structure offers deeper analytical insights and more accurate simulation.

Structure of this Paper:

This section is structured as follows: Section 2 provides Fundamental information of exponential fuzzy graphs. The basic concept of exponential fuzzy graphs and related functions is presented in Section 3, supported by examples and theorems. Section 4 discusses an application pertaining to environmental pollution. Lastly, the conclusion and future research prospects are discussed in Section 5.

2. Preliminaries

Definition 1. *[1] A function of membership ϑ_V which assumes values in unit interval $[0,1]$ characterises a fuzzy set V in a universal set $\mathcal{G}$.*

$$\vartheta_V(w) = \mathcal{G} \to [0,1].$$

The grade membership of $\mathcal{G}$ in V is represented by the value of $\vartheta_V(w)$, which is a point in $[0,1]$.

Definition 2. *A non-empty collection of vertices is denoted by V. Definition of fuzzy graph $\mathcal{G} = (\mathcal{E}_V, \mathcal{E}_E)$ is: Every vertex $w \in V$ was allotted a degree $\mathcal{E}_V(w)$ in a fuzzy subset $\mathcal{E}_V : V \to [0,1]$. Every edge $(r,w) \in V \times V$ was allotted a degree $\vartheta(r,w)$ by a fuzzy relation $\vartheta : V \times V \to [0,1]$, so that:*

$$\vartheta(r,w) \leq \min\{\mathcal{E}_V(r), \mathcal{E}_V(w)\} \quad \forall r, w \in V.$$

According to this condition, the edge membership cannot be more than the incident vertices membership.

Definition 3. *[19] If $\mathcal{G}$ is an universal set and w be any specific element of $\mathcal{G}$. The Exponential fuzzy set $\mathcal{E}_{\mathcal{A}}$ defined on $\mathcal{G}$ is a gathering of ordered pairs, $\mathcal{E}_{\mathcal{A}} = \left\{ \left(w, \vartheta_V(w)\mathfrak{e}^{-\lambda\vartheta_V(w)}\right) | w \in \mathcal{G}, \lambda > 0 \right\}$, where $\vartheta_V(w)\mathfrak{e}^{-\lambda\vartheta_{\mathcal{A}}(\mathfrak{y})} : \mathcal{G} \to [0,1]$ is called the membership function. The degree of membership function $0 \leq \vartheta_V(w)\mathfrak{e}^{-\lambda\vartheta_V(w)} \leq 1$.*

Example 1. *[19] Using $\mathcal{G} = \{1,2,3,4,5\}$ as the universal set, the decay parameter $\lambda = 0.02$ and the fuzzy membership values of $\mathcal{G}$ are $\vartheta_A(w) = \{0.9, 0.7, 0.5, 0.4, 0.3\}$. $\mathcal{E}_{\mathcal{A}}(w) = \vartheta_V(w)\mathfrak{e}^{-\lambda\vartheta_V(w)}$ is the exponential fuzzy membership function. The fuzzy membership values that are exponential $\mathcal{E}_{\mathcal{A}}(w) = \{(1, 0.8839), (2, 0.6903), (3, 0.4950), (4, 0.3968), (5, 0.2982)\}$.*

Definition 4. *[19] Consider two Exponential fuzzy sets on $\mathcal{G}$ with grade values $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$. Their values are determined by:*

$$\vartheta_{\mathcal{E}_{\mathcal{A}}}(w) = \vartheta_{V_1}(w)\mathfrak{e}^{-\lambda\vartheta_{V_1}(w)},\ \vartheta_{\mathcal{E}_{\mathcal{B}}}(w) = \vartheta_{V_2}(w)\mathfrak{e}^{-\lambda\vartheta_{V_2}(w)}$$

The following is the definition of the intersection of $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$:

$$\begin{aligned}\vartheta_{\mathcal{E}_{\mathcal{A}} \cap \mathcal{E}_{\mathcal{B}}}(w) &= min\left\{\vartheta_{\mathcal{E}_{\mathcal{A}}}(w), \vartheta_{\mathcal{E}_{\mathcal{B}}}(w)\right\} \\ &= min\left\{\vartheta_{V_1}(w)\mathfrak{e}^{-\lambda\vartheta_{V_1}(w)}, \vartheta_{V_2}(w)\mathfrak{e}^{-\lambda\vartheta_{V_2}(w)}\right\}\end{aligned}$$

Likewise, $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{B}}$ have an union that is defined as follows:

$$\begin{aligned}\vartheta_{\mathcal{E}_{\mathcal{A}} \cup \mathcal{E}_{\mathcal{B}}}(w) &= max\left\{\vartheta_{\mathcal{E}_{\mathcal{A}}}(w), \vartheta_{\mathcal{E}_{\mathcal{B}}}(w)\right\} \\ &= max\left\{\vartheta_{V_1}(w)\mathfrak{e}^{-\lambda\vartheta_{V_1}(w)}, \vartheta_{V_2}(w)\mathfrak{e}^{-\lambda\vartheta_{V_2}(w)}\right\}\end{aligned}$$

3. Main Results

Definition 5 (Exponential Fuzzy Graph). *Let V be a non-empty finite set. The exponential fuzzy graph (EFG) is denoted as $\mathcal{G} = (\mathcal{E}_V, \mathcal{E}_E)$, where: $\mathcal{E}_V = \{(w, \vartheta_V(w)) \mid w \in V\}$ is the exponential fuzzy vertex set, with $\vartheta_V(w) = \alpha_V(w) \cdot e^{-\lambda\alpha_V(w)}$, where $\alpha_V(w) \in [0,1]$ and $\lambda > 0$. $\mathcal{E}_E = \{((r, w), \vartheta_E(r, w)) \mid r, w \in V\}$ is the exponential fuzzy edge set, with $\vartheta_E(r, w) = \alpha_E(r, w) \cdot e^{-\lambda\alpha_E(r,w)}$, where $\alpha_E(r, w) \in [0, 1]$. The edge membership values must satisfy the condition:*

$$\vartheta_E(r, w) \leq \min\{\vartheta_V(r), \vartheta_V(w)\}, \ \forall r, w \in V.$$

Example 2. *Consider the vertex set $V = \{w_1, w_2, w_3\}$ with $\alpha_V(w_1) = 0.8$, $\alpha_V(w_2) = 0.5$, $\alpha_V(w_3) = 0.7$, and parameter $\lambda = 1$. Then the vertex membership values are: $\vartheta_V(w_1) = 0.3594, \vartheta_V(w_2) = 0.3032, \vartheta_V(w_3) = 0.3476$. Suppose the edge base membership values are: $\alpha_E(w_1, w_2) = 0.5$, $\alpha_E(w_2, w_3) = 0.4$, $\alpha_E(w_1, w_3) = 0.5$. Edge memberships are: $\vartheta_E(w_1, w_2) = 0.3032, \vartheta_E(w_2, w_3) = 0.2681, \vartheta_E(w_1, w_3) = 0.3032$.*

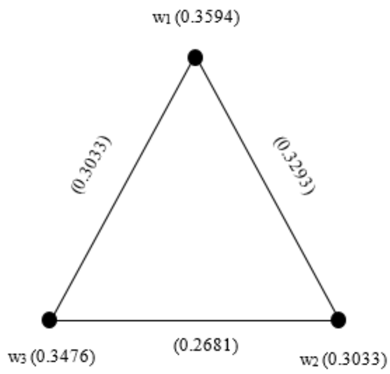

Figure 1: Exponential Fuzzy Graph

Definition 6 (Degree of a Vertex). *Let $\mathcal{G} = (\mathcal{E}_V, \mathcal{E}_E)$ is a EFG, the definition for degree of a vertex is defined as:*

$$\deg_{\mathcal{G}}(w) = \sum_{\substack{u \in V \\ u \neq v}} \vartheta_E(v, u) = \sum_{\substack{u \in V \\ u \neq v}} \alpha_E(v, u) \cdot e^{-\lambda\alpha_E(v,u)}.$$

Example 3. *Let $V = \{w_1, w_2, w_3\}$ be a set of vertices and let $\lambda = 2$. Let the vertex strengths $\alpha_V(w_i) \in [0, 1]$ be given as: $\alpha_V(w_1) = 0.8$, $\alpha_V(w_2) = 0.6$, $\alpha_V(w_3) = 0.9$. Then, the exponential fuzzy membership values are: $\vartheta_V(w_1) = 0.1615, \vartheta_V(w_2) = 0.1807, \vartheta_V(w_3) = 0.1488$. Let the edge strengths $\alpha_E(r, w) \in [0, 1]$ be: $\alpha_E(w_1, w_2) = 0.5$, $\alpha_E(w_1, w_3) = 0.6$, $\alpha_E(w_2, w_3) = 0.4$. Then the exponential fuzzy edge memberships are:*

$\vartheta_E(w_1,w_2) = 0.1839, \vartheta_E(w_1,w_3) = 0.1807, \vartheta_E(w_2,w_3) = 0.1797$. *Now, we compute the degree of each vertex:*

$$\deg_{\mathcal{G}}(w_1) = \vartheta_E(w_1,w_2) + \vartheta_E(w_1,w_3) = 0.1839 + 0.1807 = 0.3646,$$

$$\deg_{\mathcal{G}}(w_2) = \vartheta_E(w_2,w_1) + \vartheta_E(w_2,w_3) = 0.1839 + 0.1797 = 0.3636,$$

$$\deg_{\mathcal{G}}(w_3) = \vartheta_E(w_3,w_1) + \vartheta_E(w_3,w_2) = 0.1807 + 0.1797 = 0.3604.$$

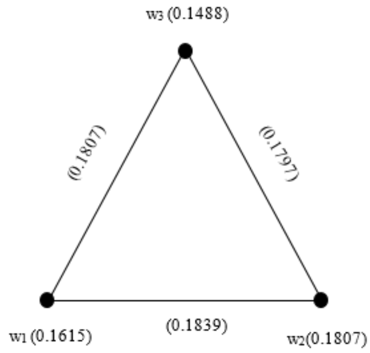

Figure 2: Exponential Fuzzy Graph

Definition 7 (Order)**.** *Let $\mathcal{G} = (\mathcal{E}_V, \mathcal{E}_E)$ be an exponential fuzzy graph with vertex membership function $\vartheta_V : V \to [0,1]$ defined by $\vartheta_V(w) = \alpha_V(w)e^{-\lambda\alpha_V(w)}$, where $\alpha_V(w) \in [0,1]$ is the base membership of vertex $w \in V$ and $\lambda > 0$ is a fixed parameter. The order of $\mathcal{G}$, denoted $|\mathcal{G}|$, is given by*

$$|\mathcal{G}| = \sum_{w\in V} \vartheta_V(w) = \sum_{w\in V} \alpha_V(w)e^{-\lambda\alpha_V(w)}.$$

Definition 8 (Size)**.** *Let $\vartheta_E : E \to [0,1]$ be the edge membership function defined by $\vartheta_E(r,w) = \alpha_E(r,w)e^{-\lambda\alpha_E(r,w)}$, where $\alpha_E(r,w) \in [0,1]$ is the base membership of edge $(r,w) \in E$. The size of $\mathcal{G}$, denoted $\|\mathcal{G}\|$, is given by*

$$\|\mathcal{G}\| = \sum_{(r,w)\in E} \vartheta_E(r,w) = \sum_{(r,w)\in E} \alpha_E(r,w)e^{-\lambda\alpha_E(r,w)}.$$

Example 4. *Consider the exponential fuzzy graph $\mathcal{G} = (\mathcal{E}_V, \mathcal{E}_E)$ where $V = \{w_1, w_2, w_3\}$, $\quad E = \{(w_1,w_2),(w_2,w_3)\}$, with $\lambda = 1$. The base vertex memberships are: $\alpha_V(w_1) = 0.7, \quad \alpha_V(w_2) = 0.5, \quad \alpha_V(w_3) = 0.8$, and the base edge memberships are:*

$$\alpha_E(w_1,w_2) = 0.5, \quad \alpha_E(w_2,w_3) = 0.4.$$

Compute vertex memberships: $\vartheta_V(w_1) = 0.3476, \vartheta_V(w_2) = 0.3032, \vartheta_V(w_3) = 0.3594$. Order of $\mathcal{G}$:

$$|\mathcal{G}| = 0.3476 + 0.3032 + 0.3594 = 1.0103.$$

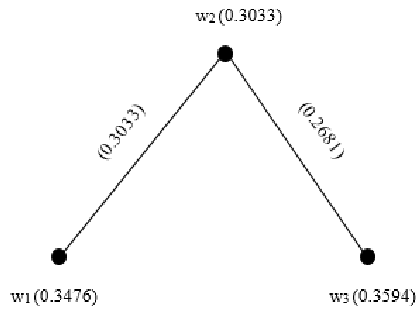

Figure 3: Order and Size of Exponential Fuzzy Graph

Compute edge memberships: $\vartheta_E(w_1, w_2) = 0.3032, \vartheta_E(w_2, w_3) = 0.2681$. *Size of* $\mathcal{G}$*:*

$$\|\mathcal{G}\| = 0.3032 + 0.2681 = 0.5714.$$

Definition 9 (Complete)**.** *Let* $V = \{w_1, \ldots, w_n\}$ *be a finite vertex set and let* $\vartheta_V(w) = \alpha_V(w)\, e^{-\lambda\, \alpha_V(w)}, \quad \alpha_V(w) \in [0,1],\ \lambda > 0$. *The complete exponential fuzzy graph on* V *is* $\mathcal{G}_k = (\mathcal{E}_V, \mathcal{E}_E, \vartheta_V, \vartheta_E)$, *where* $E = \{\, (r, w) \mid r, w \in V,\ r \neq w\}$. *The edge membership values must satisfy the condition:* $\vartheta_E(r, w) = \min\{\vartheta_V(r), \vartheta_V(w)\}\,,\ \forall r, w \in V$.

Example 5. *Take* $V = \{w_1, w_2, w_3, w_4\}$, $\lambda = 1$ *and base vertex-memberships* $\alpha_V(w_1) = 0.5,\ \alpha_V(w_2) = 0.6,\ \alpha_V(w_3) = 0.4,\ \alpha_V(w_4) = 0.7$. *Then* $\vartheta_V(w_1) = 0.5e^{-0.5} \approx 0.3032, \quad \vartheta_V(w_2) = 0.6e^{-0.6} \approx 0.3293, \vartheta_V(w_3) = 0.4e^{-0.4} \approx 0.2681, \quad \vartheta_V(w_4) = 0.7e^{-0.7} \approx 0.3476$. *For the complete graph,* $\vartheta_E(r, w) = \min\{\vartheta_V(r), \vartheta_V(w)\}\,,\ \forall r, w \in V$.
$\alpha_E(w_1, w_2) = 0.5$, $\alpha_E(w_2, w_3) = 0.4$, $\alpha_E(w_3, w_4) = 0.4$. *Then* $\vartheta_E(w_1, w_2) = 0.3032, \vartheta_E(w_2, w_3) = 0.2681, \vartheta_E(w_3, w_4) = 0.2681$.

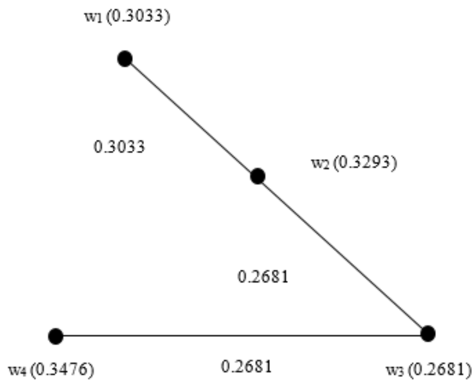

Figure 4: Exponential Fuzzy Graph

Definition 10 (Complement). *Let $\mathcal{G} = (\mathcal{E}_V, \mathcal{E}_E, \vartheta_V, \vartheta_E)$ be an exponential fuzzy graph with vertex membership function $\vartheta_V : V \to [0,1]$, $\vartheta_V(w) = \alpha_V(w)e^{-\lambda\alpha_V(w)}$, and edge membership function $\vartheta_E : E \to [0,1]$, $\vartheta_E(r,w) = \alpha_E(r,w)e^{-\lambda\alpha_E(r,w)}$, where $\alpha_V(w), \alpha_E(r,w) \in [0,1]$ are base membership degrees and $\lambda > 0$. The complement of $\mathcal{G}$, denoted $\mathcal{G}^c = (\mathcal{E}_{V^c}, \mathcal{E}_{E^c}, \vartheta_{V^c}, \vartheta_{E^c})$, is defined by:*

$$\vartheta_{V^c}(w) = 1 - \vartheta_V(w),$$

$$\vartheta_{E^c}(r,w) = \begin{cases} 1 - \vartheta_E(r,w), & \text{if } (r,w) \in E, \\ \vartheta_E(r,w), & \text{if } (r,w) \notin E. \end{cases}$$

Example 6. *Consider the exponential fuzzy graph $\mathcal{G} = (\mathcal{E}_V, \mathcal{E}_E)$ where $V = \{w_1, w_2, w_3, w_4\}$, $E = \{(w_1,w_2), (w_2,w_3), (w_3,w_4)\}$, with $\lambda = 1$. The base vertex memberships are: $\alpha_V(w_1) = 0.5$, $\alpha_V(w_2) = 0.7$, $\alpha_V(w_3) = 0.8$, $\alpha_V(w_4) = 0.4$ and the base edge memberships are: $\alpha_E(w_1,w_2) = 0.5$, $\alpha_E(w_2,w_3) = 0.6$, $\alpha_E(w_3,w_4) = 0.4$.*
Compute vertex memberships:

$$\vartheta_V(w_1) = 0.3032, \vartheta_V(w_2) = 0.3476, \vartheta_V(w_3) = 0.3594, \vartheta_V(w_4) = 0.2681.$$

Compute edge memberships:

$$\vartheta_E(w_1,w_2) = 0.3032, \vartheta_E(w_2,w_3) = 0.3292, \vartheta_E(w_3,w_4) = 0.2681.$$

The complement of the EFG is $\mathcal{G}^c = (\mathcal{E}_{V^c}, \mathcal{E}_{E^c})$, then
vertex memberships:

$$\vartheta_{V^c}(w_1) = 0.6968, \vartheta_{V^c}(w_2) = 0.6524, \vartheta_{V^c}(w_3) = 0.6404, \vartheta_{V^c}(w_4) = 0.7319.$$

edge memberships:

$$\vartheta_{E^c}(w_1,w_2) = 0.6968, \vartheta_{E^c}(w_2,w_3) = 0.6708, \vartheta_{E^c}(w_3,w_4) = 0.7319.$$

Definition 11 (Cartesian Product of EFGs). *Let $\mathcal{G}^1 = (\mathcal{E}_{V^1}, \mathcal{E}_{E^1}, \vartheta_{V^1}, \vartheta_{E^1})$, $\mathcal{G}^2 = (\mathcal{E}_{V^2}, \mathcal{E}_{E^2}, \vartheta_{V^2}, \vartheta_{E^2})$ be two exponential fuzzy graphs with $\vartheta_{V^i}(w) = \alpha_{V^i}(w)\,e^{-\lambda\,\alpha_{V^i}(w)}$, $\vartheta_{E^i}(r,w) = \alpha_{E^i}(r,w)\,e^{-\lambda\,\alpha_{E^i}(r,w)}$, $i = 1,2$. Their Cartesian product is $\mathcal{G} = \mathcal{G}^1 \times \mathcal{G}^2 = (\mathcal{E}_{V^1\times V^2}, \mathcal{E}_{E^1\times E^2}, \vartheta_V, \vartheta_E)$, where*

$$\mathcal{E}_{E^1\times E^2} = \big\{((r_1,w_1),(r_2,w_2)) \mid (r_1 = w_1,\, (r_2,w_2) \in \mathcal{E}_{E^2}) \;\vee\; ((r_1,w_1) \in \mathcal{E}_{E^1},\, r_2 = w_2)\big\},$$

and the exponential memberships are

$$\vartheta_{V^1\times V^2}(r_1,w_1) = \min\{\vartheta_{V^1}(r_1),\, \vartheta_{V^2}(w_1)\},$$

$$\vartheta_{E^1\times E^2}\big((r_1,w_1),(r_2,w_2)\big) = \begin{cases} \min\{\vartheta_{V^1}(r_1),\, \vartheta_{E^2}(r_2,w_2)\}, & r_1 = w_1,\, (r_2,w_2) \in \mathcal{E}_{E^2}, \\ \min\{\vartheta_{E^1}(r_1,w_1),\, \vartheta_{V^2}(r_2)\}, & (r_1,w_1) \in \mathcal{E}_{E^1},\, r_2 = w_2. \end{cases}$$

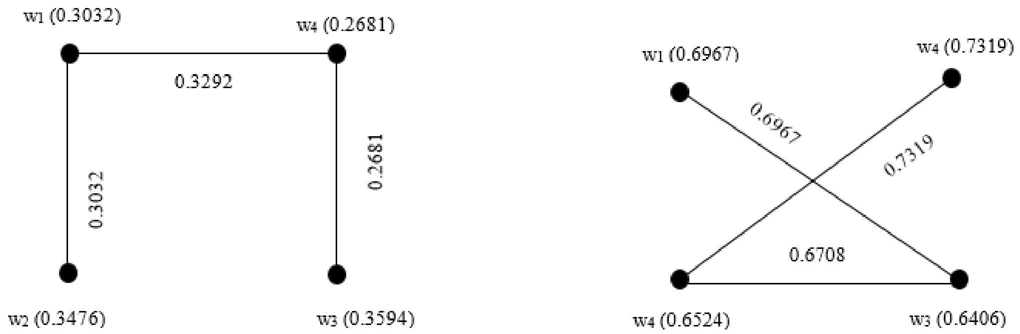

Figure 5: Exponential Fuzzy Graph

Definition 12 (Degree of Cartesian Product)**.** *Let* $\mathcal{G}^1 = (\mathcal{E}_{V^1}, \mathcal{E}_{E^1}, \vartheta_{V^1}, \vartheta_{E^1})$, $\mathcal{G}^2 = (\mathcal{E}_{V^2}, \mathcal{E}_{E^2}, \vartheta_{V^2}, \vartheta_{E^2})$ *be two exponential fuzzy graphs with* $\vartheta_{V^i}(w) = \alpha_{V^i}(w)\, e^{-\lambda\, \alpha_{V^i}(w)}$, $\quad \vartheta_{E^i}(r,w) = \alpha_{E^i}(r,w)\, e^{-\lambda\, \alpha_{E^i}(r,w)}$, $\quad i = 1, 2$. *Their cartesian product is* $\mathcal{G} = \mathcal{G}^1 \times \mathcal{G}^2 = (\mathcal{E}_{V^1 \times V^2}, \mathcal{E}_{E^1 \times E^2}, \vartheta_V, \vartheta_E)$. *Then the degree of vertex of cartesian product is*

$$deg_{\mathcal{G}^1 \times \mathcal{G}^2}(r_1, w_1), (r_2, w_2) = \sum_{r_1 = w_1,\, (r_2, w_2) \in \mathcal{E}_{E^2}} \min\{\vartheta_{V^1}(r_1),\ \vartheta_{E^2}(r_2, w_2)\} + \sum_{(r_1, w_1) \in \mathcal{E}_{E^1},\, r_2 = w_2} \min\{\vartheta_{E^1}(r_1, w_1),\ \vartheta_{V^2}(r_2)\}$$

Example 7. *Let* $V^1 = \{r_1, r_2, r_3\}, V^2 = \{w_1, w_2\}$, *with* $\lambda = 1$. *Base vertex-memberships and base edge-membership,*$\alpha_{V^1}(r_1) = 0.3, \alpha_{V^1}(r_2) = 0.7, \alpha_{V^1}(r_3) = 0.5, \alpha_{V^2}(w_1) = 0.4, \alpha_{V^2}(w_2) = 0.4, \alpha_{E^1}(r_1, r_3) = 0.3, \alpha_{E^1}(r_2, r_3) = 0.3, \alpha_{E^2}(w_1, w_2) = 0.3$. *Compute* $\vartheta_{V^1}(r_1) = 0.2222, \vartheta_{V^1}(r_2) = 0.3476, \vartheta_{V^1}(r_3) = 0.3032, \vartheta_{V^2}(w_1) = 0.2681, \vartheta_{V^2}(w_2) = 0.3292 \vartheta_{E^1}(r_1, r_3) = 0.1637, \vartheta_{E^1}(r_2, r_3) = 0.3032, \vartheta_{E^2}(w_1, w_2) = 0.2222$. *Then* $\mathcal{E}_V = \{(r_1, w_1), (r_1, w_2), (r_2, w_1), (r_2, w_2), (r_3, w_1), (r_3, w_2)\}$ *with*

$$\begin{aligned}
\vartheta_{V^1 \times V^2}(r_1, w_1) &= 0.2222, \quad \vartheta_{V^1 \times V^2}(r_1, w_2) = 0.2222 \\
\vartheta_{V^1 \times V^2}(r_2, w_1) &= 0.2681, \quad \vartheta_{V^1 \times V^2}(r_2, w_2) = 0.3292 \\
\vartheta_{V^1 \times V^2}(r_3, w_1) &= 0.2681, \quad \vartheta_{V^1 \times V^2}(r_3, w_2) = 0.3032
\end{aligned}$$

$$\begin{aligned}
\vartheta_{E^1 \times E^2}((r_1, w_1), (r_1, w_2)) &= 0.2222, \quad \vartheta_{E^1 \times E^2}((r_1, w_1), (r_2, w_1)) = 0.2222 \\
\vartheta_{E^1 \times E^2}((r_2, w_1), (r_2, w_2)) &= 0.2222, \quad \vartheta_{E^1 \times E^2}((r_2, w_2), (r_3, w_2)) = 0.3292, \\
\vartheta_{E^1 \times E^2}((r_3, w_1), (r_3, w_2)) &= 0.2222
\end{aligned}$$

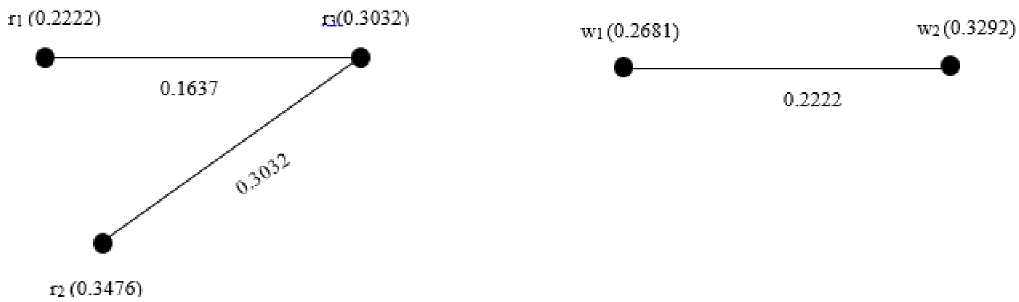

Figure 6: Exponential Fuzzy Graph

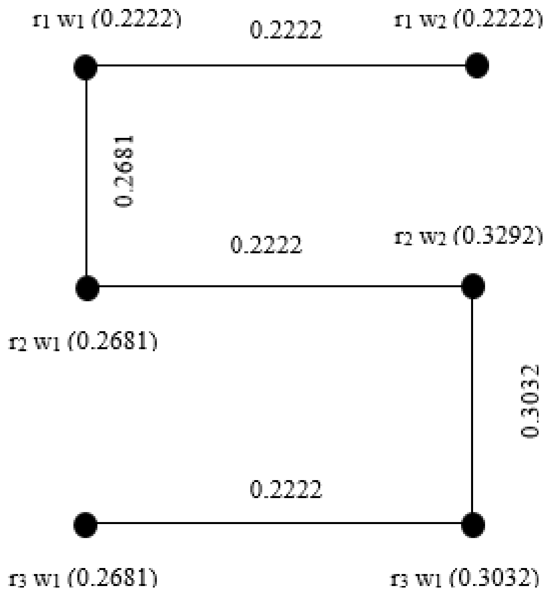

Figure 7: Exponential Fuzzy Graph

Proposition 1. *Let $\mathcal{G}$ be the cartesian product of EFGs $\mathcal{G}^1$ and $\mathcal{G}^2$. Let $(\vartheta_{V^i}, \vartheta_{E^i})$ be a exponential fuzzy subgraph of $\mathcal{G}^i, i = 1, 2$. Then the cartesian product $(\vartheta_{V^1 \times V^2}, \vartheta_{E^1 \times E^2})$ is a exponential fuzzy subgraph.*

Proof.

$$\begin{aligned}
\vartheta_{E^1 \times E^2}\big((r_1, w_1), (r_2, w_2)\big) &= min\{\vartheta_{V^1}(r_1),\ \vartheta_{E^2}(r_2, w_2)\}, r_1 = w_1,\ (r_2, w_2) \in \mathcal{E}_{E^2} \\
&= min\{\vartheta_{V^1}(r_1), min\{\vartheta_{V^2}(r_2), \vartheta_{V^2}(w_2)\}\} \\
&= min\{min\{\vartheta_{V^1}(r_1), \vartheta_{V^2}(r_2)\}, min\{\vartheta_{V^1}(r_1), \vartheta_{V^2}(w_2)\}\} \\
&= min\{\vartheta_{V^1 \times V^2}(r_1, r_2), \vartheta_{V^1 \times V^2}(r_1, r_2)\}
\end{aligned}$$

The exponential fuzzy graph $(\mathcal{E}_{V^1\times V^2}, \mathcal{E}_{E^1}\times\mathcal{E}_{E^2})$ of proposition is called the cartesian product of $(\vartheta_{V^1\times V^2}, \vartheta_{E^1\times E^2})$.

Theorem 1. *Assume that $\mathcal{G}$ is a cartesian product of EFGs $\mathcal{G}^1$ and $\mathcal{G}^2$. Let $(\vartheta_V, \vartheta_E)$ be a exponential fuzzy subgraph of $\mathcal{G}$. Then $(\vartheta_V, \vartheta_E)$ is a cartesian product of a exponential fuzzy subgraph of $\mathcal{G}^1$ and $\mathcal{G}^2$ and iff the following equations has many outputs for a_k, b_m, c_{mn}, d_{kl}, where $V^1 = \{w_{11}, w_{12}\cdots w_{1i}\}$ and $V^2 = \{w_{21}, w_{22}\cdots w_{2j}\}$:*

$$\begin{aligned}
min\{a_k, b_m\} =&\vartheta_V(w_{1k}, w_{2m}), k = 1,\cdots,i; m = 1,\cdots,j, && (1)\\
min\{a_k, c_{mn}\} =&\vartheta_E((w_{1k}, w_{2m}), (w_{1k}, w_{2n})),\\
&k = 1,\cdots,i; \text{ and } m, n \text{ are such that } w_{2m}w_{2n}\in\mathcal{E}_{E^2} && (2)\\
min\{b_m, d_{kl}\} =&\vartheta_E((w_{1k}, w_{2m}), (w_{1l}, w_{2m})),\\
&m = 1,\cdots,j; \text{ and } k, l \text{ are such that } w_{1k}w_{1l}\in\mathcal{E}_{E^1} && (3)
\end{aligned}$$

Proof. The given equations (1),(2) and (3) have a solution. Consider an arbitrary, but m, n fixed in equation (2) and k, l fixed in equation (3).
Let

$$\widehat{c}_{mn} = max\{\vartheta_E((w_{1k}, w_{2m}), (w_{1k}, w_{2n}))|k = 1,\cdots,i\}$$

and

$$\widehat{d}_{kl} = max\{\vartheta_E((w_{1k}, w_{2m}), (w_{1l}, w_{2m}))|m = 1,\cdots,j\}$$

.
Then the set of the subgraphs $M = \{(m, n)|m, n$ are such that $w_{2m}w_{2n}\in\mathcal{E}_{E^2}\}$ and $K = \{(k, l)|k, l$ are such that $w_{1k}w_{1l}\in\mathcal{E}_{E^1}\}$.
Now if

$$\{x_1,\cdots,x_n\}\cup\{c_{mn}|(m, n)\in J\}\cup\{y_1,\cdots,y_m\}\cup\{d_{kl}|(k, l)\in I\}$$

is any solution to (1),(2) and (3) then

$$\{x_1,\cdots,x_i\}\cup\{\widehat{c}_{mn}|(m, n)\in J\}\cup\{y_1,\cdots,y_j\}\cup\{\widehat{d}_{kl}|(k, l)\in I\}$$

is also a solution and in fact, $\widehat{c}_{mn}$ is the smallest possible c_{mn} and $\widehat{d}_{kl}$ is the smallest possible d_{kl}. Let us fix a solution of this kind and define the exponential fuzzy subsets $\vartheta_{V^1}, \vartheta_{V^2}, \vartheta_{E^1}$ and ϑ_{E^2} of $\mathcal{E}_{V^1}, \mathcal{E}_{V^2}, \mathcal{E}_{E^1}$ and $\mathcal{E}_{E^2}$ respectively, as follows:

$$\begin{aligned}
\vartheta_{V^1}(w_{1k}) =&x_k \text{ for } k = 1,\cdots,i\\
\vartheta_{V^2}(w_{2m}) =&y_m \text{ for } m = 1,\cdots,j\\
\vartheta_{E^2}(w_{2m}, w_{2n}) =& \widehat{c}_{mn} \text{ for } m, n \text{ are such that} w_{2m}w_{2n}\in\mathcal{E}_{E^2},\\
\vartheta_{E^1}(w_{1k}, w_{1l}) =& \widehat{d}_{kl} \text{ for } k, l \text{ are such that} w_{1k}w_{1l}\in\mathcal{E}_{E^1}
\end{aligned}$$

for any fixed m, n,

$$\vartheta_E((w_{1k}, w_{2m}), (w_{1k}, w_{2n}))\le min\{\vartheta_V(w_{1k}, w_{2m}), \vartheta_V(w_{1k}, w_{2n})\}$$

$$= min\{min\{\vartheta_{V^1}(w_{1k}), \vartheta_{V^2}(w_{2m}), min\{\vartheta_{V^1}(w_{1k}), \vartheta_{V^2}(w_{2n})\}\}$$
$$\leq min\{\vartheta_{V^2}(w_{2m}), \vartheta_{V^2}(w_{2n})\}, m, n = 1, \cdots, j; k = 1, \cdots, i$$

Thus

$$\begin{aligned}\widehat{c}_{mn} &= max\{\vartheta_E((w_{1k}, w_{2m}), (w_{1k}, w_{2n}))|k = 1, \cdots, i\} \\ &\leq min\{\vartheta_{V^2}(w_{2m}), \vartheta_{V^2}(w_{2n})\}.\end{aligned}$$

Hence $\vartheta_{E^2}(w_{2m}, w_{2n}) \leq min\{\vartheta_{V^2}(w_{2m}), \vartheta_{V^2}(w_{2n})\}$. Thus $(\vartheta_{w_2}, \vartheta_{E_2})$ is a exponential fuzzy subgraph of $\mathcal{G}^2$. Similarly, $(\vartheta_{w_1}, \vartheta_{E_1})$ is a exponential fuzzy subgraph of $\mathcal{G}^1$. Clearly, $\vartheta_V = \vartheta_{V^1 \times V^2}$ and $\vartheta_E = \vartheta_{E^1 \times E^2}$. Conversely, suppose that $(\vartheta_V, \vartheta_E)$ is a cartesian product of exponential fuzzy subgraph of $\mathcal{G}^1$ and $\mathcal{G}^2$. The solution of equations (1),(2) and (3) exists by definition of cartesian product.

Definition 13 (Composition of EFGs). *Let $\mathcal{G}^1 = (\mathcal{E}_V, \mathcal{E}_E, \vartheta_{V^1}, \vartheta_{E^1})$ and $\mathcal{G}^2 = (\mathcal{E}_V, \mathcal{E}_E, \vartheta_{V^2}, \vartheta_{E^2})$ be two exponential fuzzy graphs, where: $\vartheta_{V^i}(w) = \alpha_{V^i}(w) \cdot e^{-\lambda \alpha_{V^i}(w)}, ; \forall w \in \mathcal{E}_{V^i}, \vartheta_{E^i}(r, w) = \alpha_{E^i}(r, w) \cdot e^{-\lambda \alpha_{E^i}(r,w)}; \forall (r, w) \in \mathcal{E}_{V^i \times V^i}$, where i=1,2. Then, the composition of $\mathcal{G}^1$ and $\mathcal{G}^2$, denoted by $\mathcal{G}^1 \circ \mathcal{G}^2 = (\mathcal{E}_{V^1 \circ V^2}, \mathcal{E}_{E^1 \circ E^2})$, is defined as follows:*
The vertex set of the composition is,

$$\begin{aligned}\vartheta_{V^1 \circ V^2}(w_1, w_2) &= \min\{\vartheta_{V^1}(w_1), \vartheta_{V^2}(w_2)\} \\ &= \min\left\{\alpha_{V^1}(w_1)e^{-\lambda \alpha_{V^1}(w_1)}, \alpha_{V^2}(w_2)e^{-\lambda \alpha_{V^2}(w_2)}\right\}\end{aligned}$$

The edge set of the composition is,

$$\vartheta_{E^1 \circ E^2}((r_1, w_1), (r_2, w_2)) = \begin{cases} \min\{\vartheta_{V^1}(r_1), \vartheta_{E^2}(r_2, w_2)\}, & \text{if } r_1 = w_1,\ (r_2, w_2) \in \mathcal{E}_{E^2} \\ \min\{\vartheta_{E^1}(r_1, w_1), \vartheta_{V^2}(r_2)\}, & \text{if } r_2 = w_2,\ (r_1, w_1) \in \mathcal{E}_{E^1} \\ \min\{\vartheta_{E^1}(r_1, w_1), \vartheta_{E^2}(r_2, w_2)\}, & \text{if } (r_1, w_1) \in \mathcal{E}_{E^1},\ (r_2, w_2) \in \mathcal{E}_{E^2} \\ 0, & \text{otherwise} \end{cases}$$

Definition 14 (Degree of Composition). *Let $\mathcal{G}^1 = (\mathcal{E}_{V^1}, \mathcal{E}_{E^1}, \vartheta_{V^1}, \vartheta_{E^1})$, $\mathcal{G}^2 = (\mathcal{E}_{V^2}, \mathcal{E}_{E^2}, \vartheta_{V^2}, \vartheta_{E^2})$ be two exponential fuzzy graphs with $\vartheta_{V^i}(w) = \alpha_{V^i}(w)\, e^{-\lambda\, \alpha_{V^i}(w)}$, $\vartheta_{E^i}(r, w) = \alpha_{E^i}(r, w)\, e^{-\lambda\, \alpha_{E^i}(r,w)}$, $i = 1, 2$. Their composition is $\mathcal{G} = \mathcal{G}^1 \circ \mathcal{G}^2 = (\mathcal{E}_{V^1 \circ V^2}, \mathcal{E}_{E^1 \circ E^2}, \vartheta_V, \vartheta_E),$. Then the degree of vertex of composition is*

$$\begin{aligned}deg_{\mathcal{G}^1 \circ \mathcal{G}^2}(r_1, w_1), (r_2, w_2) = &\sum_{r_1 = w_1,\, (r_2, w_2) \in \mathcal{E}_{E^2}} \min\{\vartheta_{V^1}(r_1), \vartheta_{E^2}(r_2, w_2)\} \\ &+ \sum_{r_2 = w_2,\, (r_1, w_1) \in \mathcal{E}_{E^1}} \min\{\vartheta_{E^1}(r_1, w_1), \vartheta_{V^2}(r_2)\} \\ &+ \sum_{(r_1, w_1) \in \mathcal{E}_{E^1},\, (r_2, w_2) \in \mathcal{E}_{E^2}} \min\{\vartheta_{E^1}(r_1, w_1), \vartheta_{E^2}(r_2, w_2)\}\end{aligned}$$

Example 8. *Let* $\mathcal{E}_{V^1} = \{p, q\}, \mathcal{E}_{V^2} = \{s, t\}$, *with* $\lambda = 1$. *Base vertex-memberships:*

$$\alpha_{V^1}(p) = 0.6,\ \alpha_{V^1}(q) = 0.4; \quad \alpha_{V^2}(s) = 0.5,\ \alpha_{V^2}(t) = 0.7.$$

Compute

$$\vartheta_{V^1}(p) = 0.6e^{-0.6} \approx 0.3293,\ \vartheta_{V^1}(q) = 0.4e^{-0.4} \approx 0.2681,$$
$$\vartheta_{V^2}(s) = 0.5e^{-0.5} \approx 0.3032,\ \vartheta_{V^2}(t) = 0.7e^{-0.7} \approx 0.3476.$$

Then $\mathcal{E}_V = \{(p, s), (p, t), (q, s), (q, t)\}$ *with*

$$\vartheta_V(p, s) = \min(0.3293, 0.3032) = 0.3032, \quad \vartheta_V(p, t) = \min(0.3293, 0.3476) = 0.3293,$$

$$\vartheta_V(q, s) = \min(0.2681, 0.3032) = 0.2681, \quad \vartheta_V(q, t) = \min(0.2681, 0.3476) = 0.2681.$$

Edges: since each $\mathcal{E}_{E^i}$ *has* (p, q),

$$\mathcal{E}_E = \{((p, s), (q, t)),\ ((p, t), (q, s))\ ((p, s), (q, s))\ ((q, s), (q, t))\}.$$

If $\alpha_{E^1}(p, q) = 0.5,\ \alpha_{E^2}(s, t) = 0.4$, *then*

$$\vartheta_{E^1}(p, q) = 0.3032,\ \vartheta_{E^2}(s, t) = 0.2681,$$

and

$$\vartheta_E((p, s), (q, t)) = 0.2681, \quad \vartheta_E((p, t), (q, s)) = 0.2681,$$
$$\vartheta_E((p, s), (q, s)) = 0.3032, \quad \vartheta_E((q, s), (q, t)) = 0.2681.$$

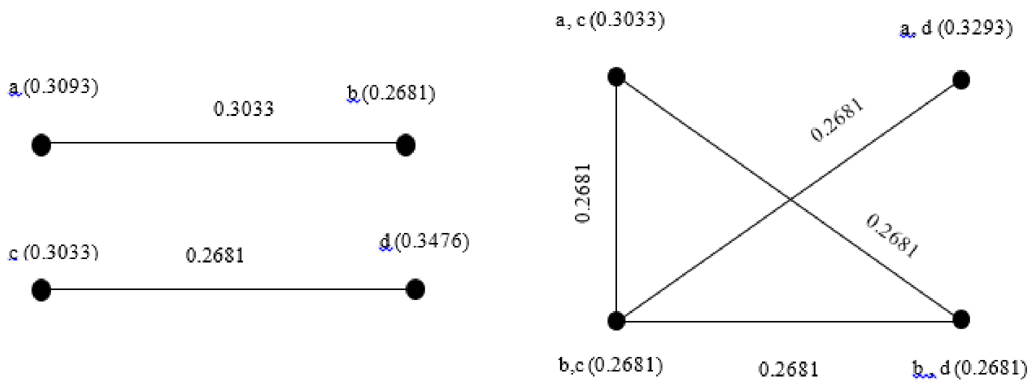

Figure 8: Exponential Fuzzy Graph

Theorem 2. *Suppose that* $\mathcal{G}$ *is the Composition of two exponential fuzzy graph* $\mathcal{G}^1$ *and* $\mathcal{G}^2$. *Let* $(\vartheta_V, \vartheta_E)$ *be a exponential fuzzy subgraph of* $\mathcal{G}$. *Then* $(\vartheta_V, \vartheta_E)$ *is the composition* *of exponential fuzzy subgraphs $\mathcal{G}^1$ and $\mathcal{G}^2$. consider the following equation (1),(2) and (3) of theorem 1 and*

$$min\{y_m, y_n, d_{kl}\} = \vartheta_E((w_{1k}, w_{2m}), (w_{1l}, w_{2n})), \quad where (w_{1k}, w_{2m}), (w_{1l}, w_{2n}) \in \mathcal{E}_{E^1 \circ E^2} \tag{4}$$

A necessary condition for $(\vartheta_V, \vartheta_E)$ to be a composition of exponential fuzzy subgraph of $\mathcal{G}^1$ and $\mathcal{G}^2$ is that a solution to equation (1) to (4) exists.
Suppose that a solution to equation (1) to (4) exists. If $\widehat{d}_{kl} \geq \vartheta_E((w_{1k}, w_{2m}), (w_{1l}, w_{2n})) \in \mathcal{E}_{E^1 \circ E^2}$, then $(\vartheta_V, \vartheta_E)$ is a composition of exponential fuzzy subgraphs of $\mathcal{G}^1$ and $\mathcal{G}^2$.

Proof. The necessary part of the theorem is clear. Suppose that a solution of equation (1) to (4) exists.
Let

$$\widehat{c}_{mn} = min\{\vartheta_E((w_{1k}, w_{2m}), (w_{1k}, w_{2n}))|k = 1, \cdots, i\}$$

and

$$\widehat{d}_{kl} = min\{\vartheta_E((w_{1k}, w_{2m}), (w_{1l}, w_{2m}))|m = 1, \cdots, j\}$$

.
Then there exists a solution to equations (1) to (4) as determined. Then the set of the subgraphs

$$M = \{(m, n)|m, n \text{ are such that } w_{2m}w_{2n} \in \mathcal{E}_{E^2}\}$$

and

$$K = \{(k, l)|k, l \text{ are such that } w_{1k}w_{1l} \in \mathcal{E}_{E^1}\}$$

.
Now if

$$\{x_1, \cdots, x_n\} \cap \{c_{mn}|(m, n) \in J\} \cap \{y_1, \cdots, y_m\} \cap \{d_{kl}|(k, l) \in I\}$$

is any solution to (4) then

$$\{x_1, \cdots, x_i\} \cap \{\widehat{c}_{mn}|(m, n) \in J\} \cap \{y_1, \cdots, y_j\} \cap \{\widehat{d}_{kl}|(k, l) \in I\}$$

because every $d_{kl} \geq \widehat{d}_{kl}$ and by the hypothesis concerning the $\widehat{d}_{kl}$. Thus if $(\vartheta_{V^i}, \vartheta_{E^i}), i = 1, 2$ are defined as in the proof of the Theorem 1 we have that $(\vartheta_{V^i}, \vartheta_{E^i})$ is a exponential fuzzy subgraph of $\mathcal{G}^i, i = 1, 2$ and $\vartheta_V = \vartheta_{V^1 \circ V^2}$ and $\vartheta_E = \vartheta_{E^1 \circ E^2}$.

Definition 15 (Union of EFG). *Let $\mathcal{G}^1 = (\mathcal{E}_V, \mathcal{E}_E, \vartheta_{V^1}, \vartheta_{E^1})$ and $\mathcal{G}^2 = (\mathcal{E}_V, \mathcal{E}_E, \vartheta_{V^2}, \vartheta_{E^2})$ be two EFGs with $\vartheta_{V^i}(w) = \alpha_{V^i}(w)e^{-\lambda\alpha_{V^i}(w)}$, $\vartheta_{E^i}(r, w) = \alpha_{E^i}(r, w)e^{-\lambda\alpha_{E^i}(r,w)}, i = 1, 2$. Then the union of EFG $(\mathcal{G}^1 \cup \mathcal{G}^2)$ is defined by*

$$\begin{aligned}\vartheta_{V^1 \cup V^2}(w) &= \max\{\vartheta_{V^1}(w), \vartheta_{V^2}(w)\} \\ \vartheta_{E^1 \cup E^2}(r, w) &= \max\{\vartheta_{E^1}(r, w), \vartheta_{E^2}(r, w)\}\end{aligned}$$

Definition 16 (Degree of Union). *Let* $\mathcal{G}^1 = (\mathcal{E}_V, \mathcal{E}_E, \vartheta_{V^1}, \vartheta_{E^1})$ *and* $\mathcal{G}^2 = (\mathcal{E}_V, \mathcal{E}_E, \vartheta_{V^2}, \vartheta_{E^2})$ *be two EFGs with* $\vartheta_{V^i}(w) = \alpha_{V^i}(w)e^{-\lambda\alpha_{V^i}(w)}$, $\vartheta_{E^i}(r, w) = \alpha_{E^i}(r, w)e^{-\lambda\alpha_{E^i}(r,w)}, i = 1, 2$. *Their union of EFG* $(\mathcal{G}^1 \cup \mathcal{G}^2)$. *Then the degree of vertex of union is*

$$deg_{\mathcal{G}^1\cup\mathcal{G}^2}(r) = \sum \max\{\vartheta_{E^1}(r, w), \vartheta_{E^2}(r, w)\}$$

Example 9. *Let* $V^1 = \{r_1, r_2, r_3\}, V^2 = \{r_1, r_4, r_3\}$, *with* $\lambda = 1$. *Base vertex-memberships and base edge-membership,*

$$\alpha_{V^1}(r_1) = 0.4, \alpha_{V^1}(r_2) = 0.8, \alpha_{V^1}(r_3) = 0.2, \alpha_{V^2}(r_1) = 0.3, \alpha_{V^2}(r_4) = 0.1, \alpha_{V^2}(r_3) = 0.5$$
$$\alpha_{E^1}(r_1, r_2) = 0.4, \alpha_{E^1}(r_2, r_3) = 0.2, \alpha_{E^2}(r_1, r_4) = 0.3, \alpha_{E^2}(r_4, r_3) = 0.1$$

Compute

$$\begin{aligned}
\vartheta_{V^1}(r_1) &= 0.2681, \vartheta_{V^1}(r_2) = 0.3594, \vartheta_{V^1}(r_3) = 0.1637\\
\vartheta_{V^2}(r_1) &= 0.2222, \vartheta_{V^2}(r_3) = 0.0904, \vartheta_{V^2}(r_2) = 0.3032\\
\vartheta_{E^1}(r_1, r_2) &= 0.2681, \vartheta_{E^1}(r_2, r_3) = 0.1637\\
\vartheta_{E^2}(r_1, r_4) &= 0.2222, \vartheta_{E^2}(r_4, r_3) = 0.0904
\end{aligned}$$

Then the union of EFG is

$$\begin{aligned}
\vartheta_{V^1\cup V^2}(r_1) &= 0.2222, \vartheta_{V^1\cup V^2}(r_2) = 0.2222\\
\vartheta_{V^1\cup V^2}(r_3) &= 0.2681, \vartheta_{V^1\cup V^2}(r_4) = 0.3292\\
\vartheta_{E^1\cup E^2}(r_1, r_2) &=0.2681, \vartheta_{E^1\cup E^2}(r_2, r_3) = 0.1637\\
\vartheta_{E^1\cup E^2}(r_1, r_4) &=0.0904, \vartheta_{E^1\cup E^2}(r_4, r_3) = 0.2222
\end{aligned}$$

Theorem 3. *If* $\mathcal{G}$ *is a union of two exponential fuzzy subgraphs* $\mathcal{G}^1$ *and* $\mathcal{G}^2$ *then every exponential fuzzy subgraphs* $(\vartheta_V, \vartheta_E)$ *is a union of a exponential fuzzy subgraphs of* $\mathcal{G}^1$ *and* $\mathcal{G}^2$.

Proof. Define the exponential fuzzy subsets $\vartheta_{V^1}, \vartheta_{V^2}, \vartheta_{E^1}$ and ϑ_{E^2} of $\mathcal{E}_{V^1}, \mathcal{E}_{V^2}, \mathcal{E}_{E^1}$ and $\mathcal{E}_{E^2}$ respectively, as follows:
$\vartheta_{V^i}(r)$ if $u \in \mathcal{E}_{V^i}$ and $\vartheta_{E^i}(rw)$ if $uv \in \mathcal{E}_{E^i}, i = 1, 2$. Then

$$\begin{aligned}
\vartheta_{E^i}(r_iw_i) &\leq max\{\vartheta_V(r_i), \vartheta_V(w_i)\}\\
&= max\{\vartheta_{V^i}(r_i), \vartheta_{V^i}(w_i)\} \ if \ u_iw_i \in \mathcal{E}_{E^i}, i = 1, 2.
\end{aligned}$$

Thus $\vartheta_{V^i}, \vartheta_{E^i}$ is a exponential fuzzy subgraph of $\mathcal{G}^i, i = 1, 2$. Clearly $\vartheta_V = \vartheta_{V^1\cup V^2}$ and $\vartheta_E = \vartheta_{E^1\cup E^2}$

Definition 17 (Join of EFGs). *Let* $\mathcal{G}^1 = (\mathcal{E}_{V^1}, \mathcal{E}_{E^1}, \vartheta_{V^1}, \vartheta_{E^1})$, $\mathcal{G}^2 = (\mathcal{E}_{V^2}, \mathcal{E}_{E^2}, \vartheta_{V^2}, \vartheta_{E^2})$ *be two exponential fuzzy graphs, where: for each vertex* $w \in \mathcal{E}_{V^i}$, *the membership function is given by* $\vartheta_{V^i}(w) = \alpha_{V^i}(w) \cdot e^{-\lambda\alpha_{V^i}(w)}, \quad \alpha_{V^i}(w) \in [0, 1], \quad \lambda > 0$. *For each edge* $(r, w) \in$

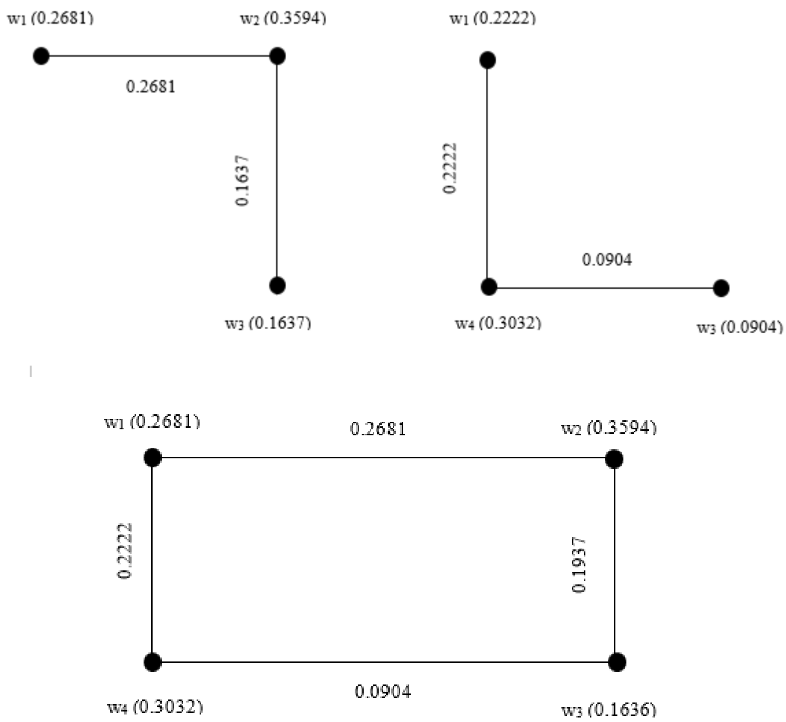

Figure 9: Exponential Fuzzy Graph

$\mathcal{E}_{V^i \times V^i}, \vartheta_{E^i}(r,w) = \alpha_{E^i}(r,w) \cdot e^{-\lambda \alpha_{E^i}(r,w)}$, *with* $\vartheta_{E^i}(r,w) \leq \min\{\vartheta_{V^i}(r), \vartheta_{V^i}(w)\}$, *for* $i = 1,2$. *The join of EFG* $\mathcal{G}^1 + \mathcal{G}^2 = \{(V^1 \cup V^2)(E^1 \cup E^2 \cup E')\}$ *where* E' *is the set of all edges joining the nodes* $\mathcal{E}_{V^1}, \mathcal{E}_{V^2}$. *Then,*

$$\begin{aligned}
\vartheta_{V^1+V^2}(w) &= \max\{\vartheta_{V^1}(w), \vartheta_{V^2}(w)\}; w \in \mathcal{E}_{V^1 \cup V^2} \\
\vartheta_{E^1+E^2}(r,w) &= \max\{\vartheta_{E^1}(r,w), \vartheta_{E^2}(r,w)\}; (r,w) \in \mathcal{E}_{E^1 \cup E^2} \\
\vartheta_{E^1+E^2}(r,w) &= \min\{\vartheta_{V^1 \cup V^2}(r), \vartheta_{V^1 \cup V^2}(w)\}; (r,w) \in \mathcal{E}_{E'}
\end{aligned}$$

Definition 18 (Degree of Join). *Let* $\mathcal{G}^1 = (\mathcal{E}_{V^1}, \mathcal{E}_{E^1}, \vartheta_{V^1}, \vartheta_{E^1})$, $\mathcal{G}^2 = (\mathcal{E}_{V^2}, \mathcal{E}_{E^2}, \vartheta_{V^2}, \vartheta_{E^2})$ *be two exponential fuzzy graphs with* $\vartheta_{V^i}(w) = \alpha_{V^i}(w)\, e^{-\lambda\, \alpha_{V^i}(w)}$, $\vartheta_{E^i}(r,w) = \alpha_{E^i}(r,w)\, e^{-\lambda\, \alpha_{E^i}(r,w)}$, $i = 1,2$. *Their Join is* $\mathcal{G} = \mathcal{G}^1 + \mathcal{G}^2 = (\mathcal{E}_{V^1+V^2}, \mathcal{E}_{E^1+E^2}, \vartheta_V, \vartheta_E)$,. *Then the Degree of vertex of Join is*

$$\begin{aligned}
deg_{\mathcal{G}^1+\mathcal{G}^2}(r_1,w_1),(r_2,w_2) = &\sum_{w \in \mathcal{E}_{V^1 \cup V^2}} \max\{\vartheta_{V^1}(w), \vartheta_{V^2}(w)\} + \sum_{(r,w) \in \mathcal{E}_{E^1 \cup E^2}} \max\{\vartheta_{E^1}(r,w), \vartheta_{E^2}(r,w)\} \\
&+ \sum_{(r,w) \in \mathcal{E}_{E'}} \min\{\vartheta_{V^1}(w), \vartheta_{V^2}(w)\}
\end{aligned}$$

Example 10. *Let* $\mathcal{E}_{V^1} = \{r_1, r_2, r_3, r_4\}, \mathcal{E}_{V^2} = \{r_1, r_2, r_3, r_4\}$, *with* $\lambda = 1$. *Base vertex-*

memberships and base edge-membership,

$$\alpha_{V^1}(r_1) = 0.3, \alpha_{V^1}(r_2) = 0.7, \alpha_{V^1}(r_3) = 0.9, \alpha_{V^1}(r_4) = 0.8$$
$$\alpha_{V^2}(r_1) = 0.9, \alpha_{V^2}(r_2) = 0.6, \alpha_{V^2}(r_3) = 0.4, \alpha_{V^2}(r_4) = 0.7$$
$$\alpha_{E^1}(r_1, r_2) = 0.3, \alpha_{E^1}(r_1, r_4) = 0.3, \alpha_{E^1}(r_3, r_4) = 0.8, \alpha_{E^1}(r_2, r_4) = 0.7$$
$$\alpha_{E^2}(r_1, r_2) = 0.6, \alpha_{E^2}(r_2, r_3) = 0.4, \alpha_{E^2}(r_3, r_4) = 0.4$$

Compute

$$\vartheta_{V^1}(r_1) = 0.2222, \vartheta_{V^1}(r_2) = 0.3476, \vartheta_{V^1}(r_3) = 0.3659, \vartheta_{V^1}(r_4) = 0.3594$$
$$\vartheta_{V^2}(r_1) = 0.3659, \vartheta_{V^2}(r_2) = 0.3292, \vartheta_{V^2}(r_3) = 0.2681, \vartheta_{V^2}(r_4) = 0.3476$$
$$\vartheta_{E^1}(r_1, r_2) = 0.2222, \vartheta_{E^1}(r_1, r_4) = 0.2222, \vartheta_{E^1}(r_3, r_4) = 0.3594, \vartheta_{E^1}(r_2, r_4) = 0.3476$$
$$\vartheta_{E^2}(r_1, r_2) = 0.3292, \vartheta_{E^2}(r_2, r_3) = 0.2681, \vartheta_{E^2}(r_3, r_4) = 0.2681$$

Then the union of EFG is

$$\vartheta_{V^1+V^2}(r_1) = 0.3659, \vartheta_{V^1+V^2}(r_2) = 0.3476$$
$$\vartheta_{V^1+V^2}(r_3) = 0.3659, \vartheta_{V^1+V^2}(r_4) = 0.3594$$
$$\vartheta_{E^1+E^2}(r_1, r_2) = 0.3292, \vartheta_{E^1+E^2}(r_2, r_3) = 0.2681$$
$$\vartheta_{E^1+E^2}(r_3, r_4) = 0.3594, \vartheta_{E^1+E^2}(r_4, r_1) = 0.2222$$
$$\vartheta_{E^1+E^2}(r_1, r_3) = 0.3659, \vartheta_{E^1+E^2}(r_2, r_4) = 0.3476$$

Theorem 4. *If $\mathcal{G}$ is the join of two exponential fuzzy subgraphs $\mathcal{G}^1$ and $\mathcal{G}^2$ then every strong exponential fuzzy subgraph $(\vartheta_V, \vartheta_E)$ of $\mathcal{G}$ is a join of a strong exponential fuzzy subgraph of $\mathcal{G}^1$ and $\mathcal{G}^2$.*

Proof. Let $\mathcal{G}$ is the join of two exponential fuzzy subgraphs $\mathcal{G}^1$ and $\mathcal{G}^2$. Define the exponential fuzzy subsets $\vartheta_{V^1}, \vartheta_{V^2}, \vartheta_{E^1}$ and ϑ_{E^2} of $\mathcal{E}_{V^1}, \mathcal{E}_{V^2}, \mathcal{E}_{E^1}$ and $\mathcal{E}_{E^2}$ respectively, as follows:

$$\vartheta_{V^i}(r) = \vartheta_V(r), r \in \mathcal{E}_{V^i}$$

and

$$\vartheta_{E^i}(rw) = \vartheta_E(rw), rw \in \mathcal{E}_{E^i}, i = 1, 2.$$

Then $(\vartheta_{V^i}, \vartheta_{E^i})$ is a exponential fuzzy subgraph of $\mathcal{G}^i, i = 1, 2$ and by the proof of theorem 3 we have $\vartheta_V = \vartheta_{V^1+V^2}$. If $uv \in \mathcal{E}_{E^1 \cup E^2}$, then $\vartheta_E(rw) = \vartheta_{E^1+E^2}(rw)$ as in the proof of theorem 3. suppose that $uv \in \mathcal{E}_{E'}$, where $u \in \mathcal{E}_{V^1}$ and $u \in \mathcal{E}_{V^2}$. Then

$$\begin{aligned}\vartheta_{E^1+E^2}(rw) &= min\{\vartheta_{(V^1)}(r), \vartheta_{(V^2)}(w)\}\\ &= min\{\vartheta_V(r), \vartheta_V(w)\} = \vartheta_E(rw),\end{aligned}$$

where the latter equality holds because $(\vartheta_V, \vartheta_E)$ is strong. Hence, every strong exponential fuzzy subgraph $(\vartheta_V, \vartheta_E)$ of $\mathcal{G}$ is a join of a strong exponential fuzzy subgraph of $\mathcal{G}^1 and \mathcal{G}^2$.

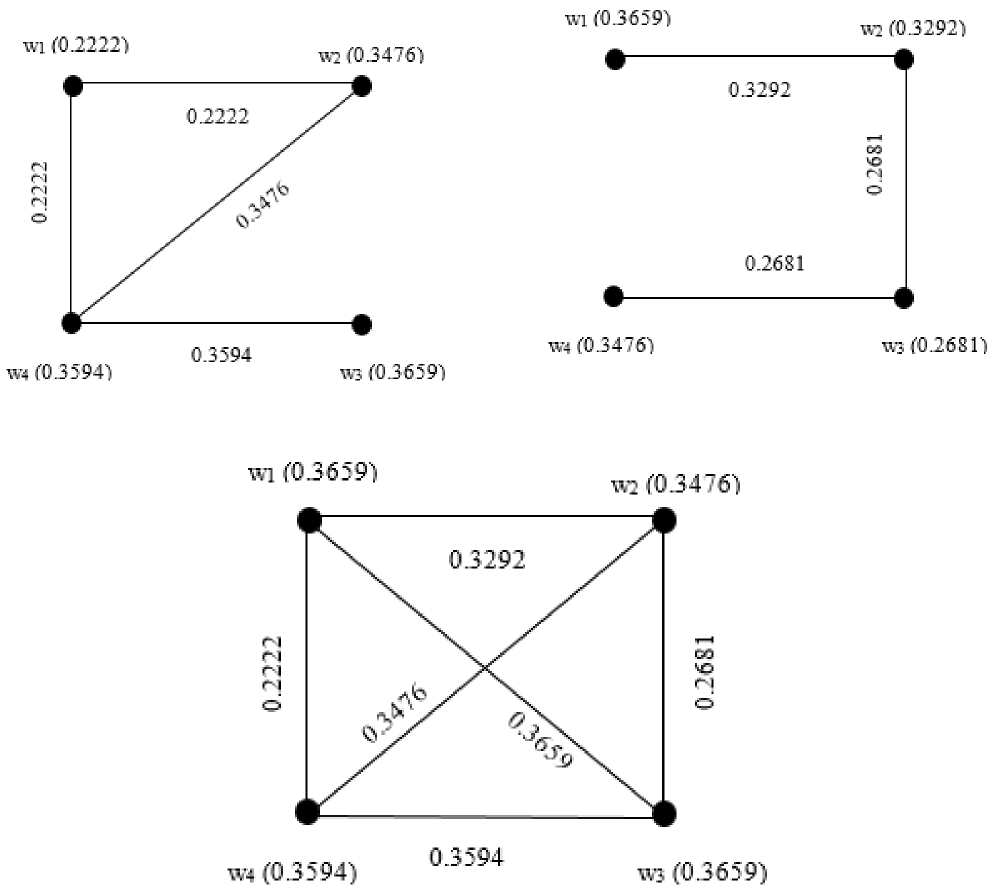

Figure 10: Exponential Fuzzy Graph

Theorem 5. *Let $\mathcal{G}^1 = (\mathcal{E}_{V^1}, \mathcal{E}_{E^1})$ and $\mathcal{G}^2 = (\mathcal{E}_{V^2}, \mathcal{E}_{E^2})$ be exponential fuzzy graphs. Suppose that $\mathcal{E}_{V^1 \cap V^2} = \phi$. Let $\vartheta_{V^1}, \vartheta_{V^2}, \vartheta_{E^1}$ and ϑ_{E^2} be exponential fuzzy subsets of $\mathcal{E}_{V^1}, \mathcal{E}_{V^2}, \mathcal{E}_{E^1}$ and $\mathcal{E}_{E^2}$ respectively. Then $(\vartheta_{V^1 \cup V^2}, \vartheta_{E^1 \cup E^2})$ is a exponential fuzzy subgraph of $\mathcal{G}^1 \cup \mathcal{G}^2$ if and only if $(\vartheta_{V^1}, \vartheta_{E^1})$ and $(\vartheta_{V^2}, \vartheta_{E^2})$ are exponential fuzzy subgraphs of $\mathcal{G}^1 and \mathcal{G}^2$ respectively.*

Proof. Suppose that $(\vartheta_{V^1 \cup V^2}, \vartheta_{E^1 \cup E^2})$ is a exponential fuzzy subgraph of $\mathcal{G}^1 \cup \mathcal{G}^2$. Let $uv \in \mathcal{E}_{E^2}$ and $r, w \in \mathcal{E}_{V^1 - V^2}$.
Hence

$$\begin{aligned}
\vartheta_{E^1} &= \vartheta_{E^1 \cup E^2}(rw) \\
&\leq min\{\vartheta_{V^1 \cup V^2}(r), \vartheta_{V^1 \cup V^2}(w)\} \\
&= min\{\vartheta_{V^1}(r), \vartheta_{V^1}(w)\}.
\end{aligned}$$

Thus $(\vartheta_{V^1}, \vartheta_{E^1})$ is a exponential fuzzy subgraph of $\mathcal{G}^1$. Similarly, $(\vartheta_{V^2}, \vartheta_{E^2})$ is a exponential fuzzy subgraph of $\mathcal{G}^2$.

Definition 19 (Semi-Strong Product). *Let $\mathcal{G}^1 = (\mathcal{E}_{V^1}, \mathcal{E}_{E^1}, \vartheta_{V^1}, \vartheta_{E^1})$ and $\mathcal{G}^2 = (\mathcal{E}_{V^2}, \mathcal{E}_{E^2}, \vartheta_{V^2}, \vartheta_{E^2})$ be two exponential fuzzy graphs, where: for each vertex $w \in \mathcal{E}_{V^i}$, the membership function is given by $\vartheta_{V^i}(w) = \alpha_{V^i}(w) \cdot e^{-\lambda\alpha_{V^i}(w)}$, $\alpha_{V^i}(w) \in [0,1]$, $\lambda > 0$. For each edge $(r,w) \in \mathcal{E}_{V^i \times V^i}, \vartheta_{E^i}(r,w) = \alpha_{E^i}(r,w) \cdot e^{-\lambda\alpha_{E^i}(r,w)}$, with $\vartheta_{E^i}(r,w) \leq \min\{\vartheta_{V^i}(r), \vartheta_{V^i}(w)\}$, for $i = 1,2$. The semi-strong product of $\mathcal{G}^1$ and $\mathcal{G}^2$, denoted by $\mathcal{G} = \mathcal{G}^1 \boxtimes_s \mathcal{G}^2$, is defined as follows: The vertex set is $\mathcal{E}_V = \mathcal{E}_{V^1 \boxtimes_s V^2}$. The vertex membership function is defined as*

$$\vartheta_V((r_1,r_2)) = \min\{\vartheta_{V^1}(r_1), \vartheta_{V^2}(r_2)\}.$$

The edge set consists of pairs $((r_1,r_2),(w_1,w_2)) \in \mathcal{E}_{V \times V}$ such that at least one of the following holds:

(i) $(r_1,w_1) \in \mathcal{E}_{E^1}$ *and* $r_2 = w_2$,

(ii) $r_1 = w_1$ *and* $(r_2,w_2) \in \mathcal{E}_{E^2}$,

(iii) $(r_1,w_1) \in \mathcal{E}_{E^1}$ *and* $(r_2,w_2) \in \mathcal{E}_{E^2}$.

The edge membership function is given by

$$\vartheta_E((r_1,w_1),(r_2,w_2)) = \max \begin{Bmatrix} \vartheta_{E^1}(r_1,w_1) \cdot \delta(r_2,w_2), \\ \vartheta_{E^2}(r_2,w_2) \cdot \delta(r_1,w_1), \\ \min\{\vartheta_{E^1}(r_1,w_1), \vartheta_{E^2}(r_2,w_2)\} \end{Bmatrix},$$

where

$$\delta(a,b) = \begin{cases} 1, & \text{if } a = b, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 20 (Degree of Semi-Strong Product). *Let $\mathcal{G}^1 = (\mathcal{E}_{V^1}, \mathcal{E}_{E^1}, \vartheta_{V^1}, \vartheta_{E^1})$, $\mathcal{G}^2 = (\mathcal{E}_{V^2}, \mathcal{E}_{E^2}, \vartheta_{V^2}, \vartheta_{E^2})$ be two exponential fuzzy graphs with $\vartheta_{V^i}(w) = \alpha_{V^i}(w)\, e^{-\lambda\, \alpha_{V^i}(w)}$, $\vartheta_{E^i}(r,w) = \alpha_{E^i}(r,w)\, e^{-\lambda\, \alpha_{E^i}(r,w)}$, $i = 1,2$. Their Semi-Strong product is $\mathcal{G} = \mathcal{G}^1 \boxtimes_s \mathcal{G}^2 = (\mathcal{E}_{V^1 \boxtimes_s V^2}, \mathcal{E}_{E^1 \boxtimes_s E^2}, \vartheta_V, \vartheta_E)$,. Then the degree of vertex of Semi-Strong product is*

$$\begin{aligned} deg_{\mathcal{G}^1 \boxtimes_s \mathcal{G}^2}(r_1,w_1),(r_2,w_2) = &\sum_{r_1 = w_1,\, (r_2,w_2) \in \mathcal{E}_{E^2}} \vartheta_{E^2}(r_2,w_2) \cdot \delta(r_1,w_1) + \sum_{r_2 = w_2,\, (r_1,w_1) \in \mathcal{E}_{E^1}} \vartheta_{E^1}(r_1,w_1) \cdot \delta(r_2,w_2) \\ &+ \sum_{(r_1,w_1) \in \mathcal{E}_{E^1},\, (r_2,w_2) \in \mathcal{E}_{E^2}} \min\{\vartheta_{E^1}(r_1,w_1), \vartheta_{E^2}(r_2,w_2)\} \end{aligned}$$

Theorem 6. *Let $\mathcal{G}^1 = (\mathcal{E}_{V^1}, \mathcal{E}_{E^1}, \vartheta_{V^1}, \vartheta_{E^1})$ and $\mathcal{G}^2 = (\mathcal{E}_{V^2}, \mathcal{E}_{E^2}, \vartheta_{V^2}, \vartheta_{E^2})$ be two exponential fuzzy graphs with $\vartheta_{V^i}(w) = \alpha_{V^i}(w)\, e^{-\lambda\, \alpha_{V^i}(w)}$, $\vartheta_{E^i}(r,w) = \alpha_{E^i}(r,w)\, e^{-\lambda\, \alpha_{E^i}(r,w)}$, $i = 1,2$. Define their cartesian product $\mathcal{G} = \mathcal{G}^1 \times \mathcal{G}^2$ by*

$$\mathcal{E}_V = \mathcal{E}_{V^1 \times V^2}, \quad \mathcal{E}_E = \big\{((r_1,w_1),(r_2,w_2)) \mid (r_1,w_1) \in \mathcal{E}_{E^1},\, (r_2,w_2) \in \mathcal{E}_{E^2}\big\},$$

and

$$\vartheta_V(r_1,w_1) = \min\{\vartheta_{V^1}(r_1),\, \vartheta_{V^2}(w_1)\}, \quad \vartheta_E\big((r_1,w_1),(r_2,w_2)\big) = \min\{\vartheta_{E^1}(r_1,w_1),\, \vartheta_{E^2}(r_2,w_2)\}.$$

Then $\mathcal{G}$ is an exponential fuzzy graph.

Proof. We must check two conditions:
(i) Range.
Since for each i, ϑ_{V^i} and ϑ_{E^i} take values in $[0,1]$, their minima also lie in $[0,1]$. Hence

$$\vartheta_V(r_1, w_1) \in [0,1], \quad \vartheta_E((r_1, w_1), (r_2, w_2)) \in [0,1].$$

(ii) Edge–vertex consistency.
We need $\vartheta_E\big((r_1, w_1), (r_2, w_2)\big) \ \le\ \min\{\vartheta_V(r_1, r_2),\, \vartheta_V(w_1, w_2)\}$.
Compute the left-hand side:

$$\vartheta_E\big((r_1, w_1), (r_2, w_2)\big) = \min\{\vartheta_{E^1}(r_1, w_1),\, \vartheta_{E^2}(r_2, w_2)\}.$$

Each factor satisfies the EFG condition in its own graph:

$$\vartheta_{E^1}(r_1, w_1) \le \min\{\vartheta_{V^1}(r_1),\, \vartheta_{V^1}(w_1)\}, \quad \vartheta_{E^2}(r_2, w_2) \le \min\{\vartheta_{V^2}(r_2),\, \vartheta_{V^2}(w_2)\}.$$

Therefore

$$\min\{\vartheta_{E^1}(r_1, w_1),\, \vartheta_{E^2}(r_2, w_2)\} \ \le\ \min\{\min\{\vartheta_{V^1}(r_1), \vartheta_{V^1}(w_1)\},\, \min\{\vartheta_{V^2}(r_2), \vartheta_{V^2}(w_2)\}\}.$$

By definition of ϑ_V on the product,

$$\min\{\vartheta_{V^1}(r_1), \vartheta_{V^2}(r_2)\} = \vartheta_V(r_1, r_2), \quad \min\{\vartheta_{V^1}(w_1), \vartheta_{V^2}(w_2)\} = \vartheta_V(w_1, w_2).$$

Thus

$$\vartheta_E\big((r_1, w_1), (r_2, w_2)\big) \ \le\ \min\{\vartheta_V(r_1, r_2),\, \vartheta_V(w_1, w_2)\},$$

as required. Since both conditions hold, $\mathcal{G}^1 \times \mathcal{G}^2$ is an exponential fuzzy graph.

4. Application

4.1. Algorithm of Exponential Fuzzy Graph

Step 1: Assign the base membership for vertices, $\alpha_V(w) \in [0,1]$.

Step 2: Compute the vertex membership value by using the EFG definition (5)

$$\vartheta_V(w) = \alpha_V(w) \cdot e^{-\lambda \alpha_V(w)}$$

.

Step 3: Compute the edge membership values by satisfying the condition

$$\vartheta_E(r, w) \le \min\{\vartheta_V(r), \vartheta_V(w)\}, \ \forall r, w \in V.$$

Step 4: Construct the Exponential fuzzy graph with the help of vertices and edges.

Step 5: From the given edge membership values, we have to find the degree of vertices by using the definition (6)

$$\deg_{\mathcal{G}}(w) = \sum_{\substack{u \in V \\ u \neq v}} \vartheta_E(v, u) = \sum_{\substack{u \in V \\ u \neq v}} \alpha_E(v, u) \cdot e^{-\lambda \alpha_E(v,u)}.$$

Step 6: Use the computed degrees $\deg_{\mathcal{G}}(w)$ to rank vertices. Higher degree indicates greater influence or sensitivity. This ranking supports decision-making processes, such as identifying critical pollution indicators or prioritizing interventions.

4.2. Pollution Impact Analysis for Decision-Making

Environmental pollution has turned into a serious worldwide problem because it greatly affects the health of both ecosystems and natural resources. Contaminated water from sewage, industry and agriculture is causing water quality to rapidly decrease, putting sensitive aquatic life and water safety at risk. Air pollution is increased by emissions from industry, exhaust fumes from vehicles and the burning of fossil fuels. This contributes to global warming and can lead to poor air quality and can cause lung diseases.

Inappropriate disposal of waste and using too many chemicals as fertilizers and pesticides harm the soil and can also make the food unsafe to consume. Due to pollutants, dams and growth near river banks, river ecosystems are becoming more at risk and freshwater species are being driven to extinction. As logging, cities and farms remove trees, the animals lose places to live and more carbon is released into the atmosphere. Several kinds of plant and animal species may go extinct because of the effects of pollution, environmental damage and climate change on biodiversity. Environmental indicators like air pollution, forest areas, types of contaminants, river health, water quality and level of biodiversity show a series of strong and uncertain associations. The exponential fuzzy graph model is strong in capturing uncertainties and the weakening influence of different environmental elements.

4.3. Problem Description

Let the set of environmental factors be Water Quality (W), Air Quality (A), Soil Contamination (S), River ecosystem (R), Forest Length (F) and Bio Diversity (B).
The vertex membership function is defined by

$$\vartheta_V(w) = \alpha_V(w) e^{\lambda_V(w)} \text{ and } \lambda = 1.$$

The membership value for each vertices is $\vartheta_V(w) = 0.7e^{-1(0.7)} = 0.3476, \vartheta_V(A) = 0.8e^{-1(0.8)} = 0.3594, \vartheta_V(S) = 0.4e^{-1(0.4)} = 0.2681, \vartheta_V(R) = 0.6e^{-1(0.6)} = 0.3292, \vartheta_V(F) = 0.5e^{-1(0.5)} = 0.3032$ and $\vartheta_V(B) = 0.3e^{-1(0.3)} = 0.2222$.
The Edge membership function is defined by

$$\vartheta_E(r, w) \leq min\left\{\vartheta_V(r), \vartheta_V(w)\right\}$$

The membership value for each edge is
$\vartheta_E(A,F) = 0.3476, \vartheta_E(A,S) = 0.2681, \vartheta_E(A,R) = 0.3292, \vartheta_E(A,W) = 0.3032, \vartheta_E(A,B) = 0.2222, \vartheta_E(F,S) = 0.2681, \vartheta_E(F,R) = 0.3292, \vartheta_E(F,W) = 0.3032, \vartheta_E(F,B) = 0.2222, \vartheta_E(S,R) = 0.2681, \vartheta_E(S,W) = 0.2681, \vartheta_E(S,B) = 0.2222, \vartheta_E(R,W) = 0.3032, \vartheta_E(R,B) = 0.2222, \vartheta_E(W,B) = 0.2222$, which is shown in the figure 11

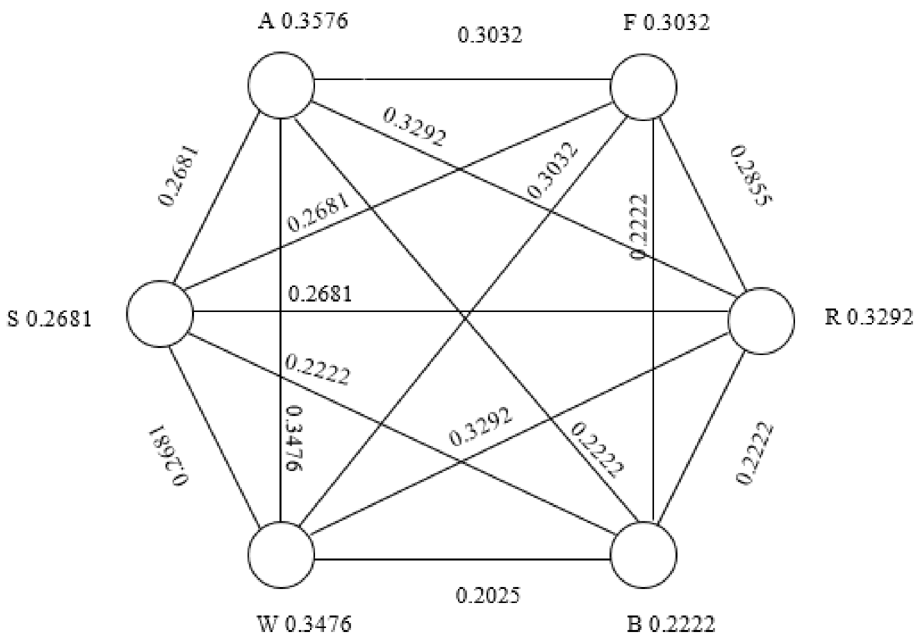

Figure 11: Fuzzy Graph

The exponential fuzzy graph in the application has an indicator at each vertex and edges show the fuzzy relationships among these indicators. Each edge membership function changes exponentially, which means that the influence linked to it decreases as we face greater uncertainty or lower levels of resistance or intensity factor λ (intensity or resistance factor).

Vertex	Membership $\lambda = 1$	Membership $\lambda = 2$	Membership $\lambda = 3$	Membership $\lambda = 4$
A(0.7)	0.3476	0.1726	0.0857	0.0425
F(0.8)	0.3594	0.1615	0.0725	0.0326
S(0.4)	0.2681	0.1797	0.1204	0.0807
R(0.6)	0.3292	0.1807	0.0991	0.0544
W(0.5)	0.3032	0.1839	0.1115	0.0676
B(0.3)	0.2222	0.1646	0.1219	0.0903

Table 2: Membership values of vertices from different λ values

This paragraph provides a thorough analysis of the vertex degrees for all values of $\lambda = 1$ *to* $\lambda = 4$.

Edge	Membership $\lambda = 1$	Membership $\lambda = 2$	Membership $\lambda = 3$	Membership $\lambda = 4$
AF	0.3476	0.1615	0.0725	0.0326
AS	0.2681	0.1726	0.0857	0.0425
AR	0.3292	0.1726	0.0857	0.0425
AW	0.3032	0.1726	0.0857	0.0425
AB	0.2222	0.1646	0.0857	0.0425
FS	0.2681	0.1615	0.0725	0.0326
FR	0.3292	0.1615	0.0725	0.0326
FW	0.3032	0.1615	0.0725	0.0326
FB	0.2222	0.1615	0.0725	0.0326
SR	0.2681	0.1797	0.0991	0.0544
SW	0.2681	0.1797	0.1115	0.0676
SB	0.2222	0.1646	0.1204	0.0807
RW	0.3032	0.1807	0.0991	0.0544
RB	0.2222	0.1646	0.0991	0.0544
WB	0.2222	0.1646	0.1115	0.0676

Table 3: Membership values of edges for different λ values

Description	Vertex	Degree $\lambda = 1$	Degree $\lambda = 2$	Degree $\lambda = 3$	Degree $\lambda = 4$
Air Quality	A (0.7)	1.4703	0.8436	0.4153	0.2026
Forest Length	F (0.8)	1.4703	0.8075	0.3625	0.1630
Soil Contamination	S (0.4)	1.2946	0.8581	0.4892	0.2778
River Ecosystem	R (0.6)	1.4519	0.8591	0.4555	0.2383
Water Quality	W (0.5)	1.3999	0.8591	0.4803	0.2647
Bio Diversity	B (0.3)	1.1110	0.8199	0.4892	0.2778

Table 4: Degree of vertices for various environmental parameters at different λ values

Vertex degree analysis across a range of λ values sheds light on how the relative significance or impact of various environmental elements varies under more demanding modelling circumstances. Air Quality (1.4703), Forest Length (1.4703) and River Ecosystem (1.4519) are the most dominating elements at $\lambda = 1$, where effect is widely dispersed with little decay, suggesting high connectedness and influence in the environmental system. The degree of each vertex decreases as λ rises, illustrating how their impact lessens under more conservative or decaying weight models.

River Ecosystem (0.8591) and Water Quality (0.8591) maintain the maximum degrees at $\lambda = 2$, indicating that these elements are structurally robust and preserve connectedness even in the face of greater limitations. Moderate persistence is also shown by soil contamination and biodiversity, suggesting more dispersed patterns of effect.

Vertex degrees further decrease for all factors when moving to $\lambda = 3$ and $\lambda = 4$, but the declines are more gradual for Soil Contamination (0.4892 to 0.2778), Forest Length (0.3625 to 0.1630) and Bio Diversity (0.4892 to 0.2778), suggesting greater robustness or long-term importance under stricter influence decay. On the other hand, even though it was the most important component at first, air quality drops dramatically (from 1.4703 to 0.2026), suggesting that conservative models greatly reduce its early influence.

Overall, the findings indicate that Forest Length, Soil Contamination and Bio Diversity acquire relative importance in longer-term or more stringent environmental evaluations, Air Quality, Water Quality and River ecosystem have the greatest influence in early-stage (low λ) models. While long-term resilience planning concentrates on forest ecosystems, soil health and biodiversity, initial efforts target high-impact components like air and water. This behaviour emphasises the dynamic nature of environmental interactions and can guide multi-stage policy design.

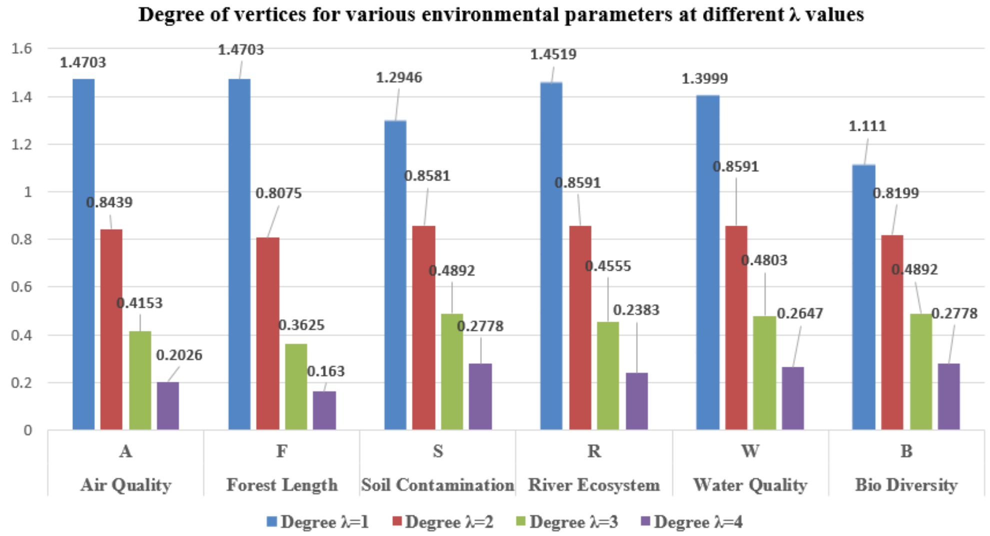

Figure 12: Fuzzy Graph

4.4. Comparative Analysis

Traditional fuzzy graphs (TFGs) provide a foundational approach to modeling uncertainty by assigning static membership values to vertices and edges within a graph structure. While this framework is effective in handling binary or constant uncertainty, it lacks the ability to capture dynamic changes or gradual decay in influence over time or through interaction. In contrast, the proposed Exponential Fuzzy Graph (EFG) model significantly advances this capability by incorporating an exponential decay function into the membership definitions. This allows the model to reflect real-world phenomena where relationships naturally weaken, such as environmental pollution, social influence spread, or biological interactions.

Unlike TFGs, which operate under fixed uncertainty levels, EFGs introduce a tunable decay parameter λ, enabling flexible control over the rate of influence attenuation. This results in a more adaptive and realistic representation of uncertain systems. Furthermore, EFGs maintain logical consistency by ensuring that edge memberships do not exceed the minimum of the associated vertex memberships, even after exponential transformation.

The EFG framework also redefines standard graph operations such as union, join and composition within the exponential context, allowing for more nuanced interpretations of connectivity and structure. All things considered, EFGs provide more thorough analytical insights and enhanced decision-making precision, particularly in fields where the strength of linkages varies over time. EFGs circumvent standard fuzzy graphs' static constraints by dynamically simulating the fading of influence, which better reflects the complex behavior seen in many real-world systems.

4.5. Sensitivity Analysis

The efficacy and conduct of the EFG model are profoundly affected by the selection of two critical parameters α (the fundamental membership value) and β (the decision threshold or tolerance level). An exponential decay function of the following kind is used in the EFG framework to simulate the membership values of vertices and edges:

$$\vartheta_V(w) = \alpha_V(w) \cdot e^{-\lambda \alpha_V(w)},$$
$$\vartheta_E(r,w) = \alpha_E(r,w) \cdot e^{-\lambda \alpha_E(r,w)} \ \forall \quad \alpha_V(w), \alpha_E(r,w) \in [0,1]$$

The strength or intensity of a vertex or edge's participation in the fuzzy graph structure is mostly determined by the parameter α. While a large α also results in a reduced membership because of the exponential decay, a small α produces a proportionately small membership number. The EFG model supports moderate values of α for maximal influence, as the function achieves its maximum at $\alpha = \frac{1}{\lambda}$. Therefore, properly adjusting α guarantees that important vertices and edges are preserved while weaker relationships are organically filtered out through decay.

Overall, the sensitivity analysis validates the dynamic adaptability of the EFG model and its effectiveness in capturing the nuanced impact of pollution indicators. It further highlights the importance of parameter tuning in practical implementations, ensuring the decision-making process remains both data sensitive and context-aware.

5. Conclusion

The Degree-Based EFG was presented in this work as a potent tool for simulating uncertainty and decay in real-world systems, particularly for the analysis of environmental pollutants. A dynamic improvement over conventional fuzzy graphs, EFGs incorporate exponential decay parameter $\lambda \geq 0$ into the membership functions of vertices and edges. We extended the concept to different graph operations with corresponding examples and theorems, and we looked at important aspects like degree, order, and size. The EFG's capacity to more accurately depict deteriorating interactions among environmental parameters was shown by its application to pollution effect assessments. The proposed model improves fuzzy graph theory's accuracy in ambiguous and time-sensitive situations, with great promise for use in sustainability, environmental monitoring, and decision-making.

Future Work Constructing exponential fuzzy networks with intuitionistic or neutrosophic membership functions to reflect indeterminacy, hesitancy and uncertainty in complicated systems. Integrating EFGs into frameworks for optimisation and multi-criteria decision-making (MCDM) to enhance the management of deteriorating and uncertain data in real-world situations such as risk assessment and environmental management.

References

[1] L. A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.

[2] A. Rosenfeld. Fuzzy graphs. In *Fuzzy Sets and Their Applications to Cognitive and Decision Processes*, pages 77–95. Academic Press, 1975.

[3] J. N. Mordeson and P. S. Nair. *Fuzzy Graph and Fuzzy Hypergraphs*. Physica-Verlag, Heidelberg, Germany, 2000.

[4] J. N. Mordeson and C. S. Peng. *Fuzzy Graph Theory with Applications to Human Trafficking*. Springer, 2000.

[5] L. N. McAllister. Fuzzy graphs and networks repairs. *International Journal of Computer Mathematics*, 80(11):1337–1342, 2003.

[6] R. Parvathi and G. Karunambigai. Bipolar fuzzy graphs. *International Journal of Computer Applications*, 8(4):1–5, 2009.

[7] S. Samanta and M. Pal. Fuzzy tolerance graphs. *Information Sciences*, 181(6):1101–1111, 2011.

[8] Muhammad Akram. Bipolar fuzzy graphs. *Information Sciences*, 181(24):5548–5564, 2011.

[9] Y. B. Jun, C. H. Kim, and S. Z. Song. Interval-valued intuitionistic fuzzy graphs. *Information Sciences*, 278:601–612, 2014.

[10] G. K. Thakur, B. Priya, and S. Pawan Kumar. A novel fuzzy graph theory-based approach for image representation and segmentation via graph coloring. *Journal of Applied Security Research*, 14(1):74–87, 2019.

[11] S. Mariappan, S. Ramalingam, S. Raman, and G. Bacak-Turan. Domination integrity and efficient fuzzy graphs. *Neural Computing and Applications*, 32:10263–10273, 2020.

[12] G. Muhiuddin, T. Mahapatra, M. Pal, O. Alshahrani, and A. Mahboob. Integrity on m-polar fuzzy graphs and its application. *Mathematics*, 11(6):1398, 2023.

[13] S. Ji, S. Pan, E. Cambria, P. Marttinen, and P. S. Yu. A survey on knowledge graphs: Representation, acquisition and applications. *IEEE Transactions on Neural Networks and Learning Systems*, 33(2):494–514, 2022.

[14] N. H. Tan, C. K. Long, T. M. Tuan, et al. A novel fuzzy knowledge graph structure for decision making of multimodal big data. *Applied Intelligence*, 55:490, 2025.

[15] D. Ramot, R. Milo, M. Friedman, and A. Kandel. Complex fuzzy sets. *IEEE Transactions on Fuzzy Systems*, 10(2):171–186, 2002.

[16] P. Thirunavukarasu, R. Suresh, and K. K. Viswanathan. Energy of a complex fuzzy graph. *International Journal of Mathematical Sciences and Engineering Applications*, 10(I):243–248, 2016.

[17] M. Shoaib, W. Mahmood, Q. Xin, and F. Tchier. Maximal product and symmetric difference of complex fuzzy graph with application. *Symmetry*, 14(6):1126, 2022.

[18] E. A. AbuHijleh. Complex hesitant fuzzy graph. *Fuzzy Information and Engineering*, 15(2):149–161, 2023.

[19] M. Kaviyarasu, Mohammed Alqahtani, and M. Rajeshwari. Exponential fuzzy sets and applications of ai-powered investment decision-making using the weighted mean method. *European Journal of Pure and Applied Mathematics*, 18(2):6050, 2025.

[20] W. Al-Omeri, Md. S. Noorani, and A. Al-Omari. New forms of contra-continuity in ideal topology spaces. *Missouri Journal of Mathematical Sciences*, 26(1):33–47, 2014.

[21] W. Al-Omeri, Md. S. M. Noorani, and A. Al-Omari. Weak open sets on simple extension ideal topological space. *Italian Journal of Pure and Applied Mathematics*, (33):333–344, 2014.

[22] Tareq M. Al-shami, Sandeep Kaur, Alkan Özkan, M. Hosny, and Abdelwaheb Mhemdi. Some characterizations of soft continuous mappings using soft graphs. *Journal of Intelligent & Fuzzy Systems*, 38(22):7823–7830, 2024.

[23] H. Z. Ibrahim, T. M. Al-Shami, M. Arar, and M. Hosny. Complex nth power root fuzzy sets: Theory and applications for multi-attribute decision making in uncertain environments. *PLoS One*, 20(5):e0319757, 2025.

[24] H. Z. Ibrahim, T. M. Al-shami, M. Arar, et al. ${k_m}^n$-rung picture fuzzy information in a modern approach to multi-attribute group decision-making. *Complex & Intelligent Systems*, 10:2605–2625, 2024.