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Mathai-Haubold Interval Entropy and Related Inequalities

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Abstract. The paper introduces a generalized interval entropy measure that provides a comprehensive framework for understanding and quantifying the uncertainty in systems with doubly truncated random variables. By characterizing well-known lifetime distributions (exponential, Pareto, and finite range distributions), deriving a lower bound for the entropy, and exploring stochastic comparisons, the paper demonstrates the usefulness of this measure in reliability modelling, survival analysis and information theory.

2020 Mathematics Subject Classifications: 62B10, 62R07, 60G35, 62N05

Key Words and Phrases: Entropy, residual entropy, interval entropy, generalized failure rate, characterization

1. Introduction

Differential entropy, also referred to as the Shannon information measure [1] is the traditional measure of uncertainty. Shannon outlined the features of information sources and communication channels in order to evaluate the outputs of different sources. In addition to providing a framework for tackling a wide range of statistical problems, statisticians have played a significant role in the development of information theory. The use of information measures for doubly truncated random variables have been investigated by Sunoj et al. [2] and this work is important for understanding the many components of a system's failure between two time points. Most existing studies on information and entropy measures, including Shannon entropy and its variants, have primarily focused on classical settings or extensions such as Rényi and Tsallis entropies.

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Interval entropy is still a relatively new and developing measure in information theory and reliability modeling, so the research gaps mainly lie in its generalizations, applications, and connections to other measures. The associated inequalities, bounds, and related properties have not been systematically developed or studied in depth. In particular, there is a lack of results connecting interval entropy to fundamental mathematical inequalities and their implications for statistical or applied problems.

Some Definitions:

i) Probability density function: A probability density function (PDF) is a function that describes the likelihood of a continuous random variable taking on a particular value.

For a continuous random variable X, the PDF is denoted by $f(x)$ and has the following properties:

a) $f(x) \geq 0$

b) $P(a \leq X \leq b) = \int_a^b f(x)dx$

c) $\int_{-\infty}^{\infty} f(x)dx = 1$

ii) Survival function: For a non-negative random variable X (representing lifetime or time-to-event), the survival function is defined as $\mathrm{R_X(x) = P(X > x)}$

iii) Hazard rate function: For a non-negative lifetime random variable X, the hazard rate function $h(x)$ is defined as

$$h(x) = \lim_{\Delta x \to 0} \frac{P(x < X < x + \Delta x / X \geq x)}{\Delta x}, x \geq 0.$$

iv) Mean residual function: It represents the expected remaining lifetime of an item, given that it has already survived up to time t and is given by

$$r(x) = \frac{1}{R(t)} \int_t^{\infty} R(x)dx$$

Also following relation holds:

$$h(x) = \frac{f(x)}{R(x)}$$

Let X be a non- negative random variable denoting the life time of a system, a component or living organism with probability density function f, distribution function F and reliability function R. It is assumed that the component is functioning at $\mathrm{t} = 0$ and it will fail to some $\mathrm{t} > 0$, so that $\mathrm{R(0) = 1}$.

The functional relationship between the hazard rate function and mean residual life function is given by

$$\mathrm{h(t) = \frac{r'(t) + 1}{r(t)}} \tag{1.1}$$

The average amount of uncertainty associated with the random variable X as given by Shannon entropy, is

$$H(X) = -\int_0^{\infty} f(x) \log f(x)dx \tag{1.2}$$

Mathai- Haubold [3] introduced the generalized information measure

$$\mathrm{K}_{\alpha}(\mathrm{X})=\frac{1}{(\alpha-1)}\int_{0}^{\infty}\left(\mathrm{f}^{(2-\alpha)}(\mathrm{x})-1\right)dx,\quad \alpha\neq 1, 0<\alpha<2 \tag{1.3}$$

Ebrahimi [10] argues that (1.1) is no longer effective in evaluating the uncertainty about the remaining life time of the unit if a unit is known to have survived up to an age t. A unit with high uncertainty is less reliable than one with low uncertainty. In order to account for this, he developed a measure of uncertainty for the residual life time distribution known as residual entropy. The residual entropy of a continuous random variable X is defined as

$$\mathrm{K}(\mathrm{X};\mathrm{t})=-\int_{\mathrm{t}}^{\infty}\frac{\mathrm{f}(\mathrm{x})}{\mathrm{R}(\mathrm{t})}\log\frac{\mathrm{f}(\mathrm{x})}{\mathrm{R}(\mathrm{t})}\mathrm{dx} \tag{1.4}$$

Corollary: For $\mathrm{t}=0$, (1.4) reduces to (1.1).

Ebrahimi [4] established that a specific dynamic measure can uniquely determine the underlying distribution function. This implies that by using this dynamic measure, one can fully understand the distribution of data or a random variable. Belzunce et al. [5], Asadi et al. [6], Dar et al. [7] and Anant et al.[8] extended Ebrahimi's work, but in the context of a generalized residual entropy. This type of entropy is a variation or extension of classical entropy measures and is useful for studying the uncertainty or information content of a system, particularly in cases where there may be a residual or remaining uncertainty after some event. Nair and Rajesh [9] and Asadi and Ebrahimi [10] further explored characterizations of distributions using dynamic entropies. In particular, they might have looked at how various dynamic entropy measures can be used to derive useful properties of distributions that are relevant in engineering, decision-making, or risk assessment. Di Crescenzo and M. Longobardi [11, 12] and Nanda, and P. Paul [13] introduces a new measure of uncertainty related to the past lifetime of a system or component, given that failure has already occurred before a certain time. N. Ebrahimi and F. Pellerey [14] introduce a partial ordering of survival functions that reflects the degree of uncertainty or randomness associated with the lifetime of systems. This ordering helps in characterizing and distinguishing different reliability behaviors, such as aging and variability in lifetimes. In many situations, we only have information between two points, so we should study the statistical measures under the condition of doubly truncated random variables. The doubly truncated measures are applicable to engineering systems when the observations are measured after it starts operating and before it fails. If the random variable X denotes the lifetime of a unit, then the random variable $(X/t_1<X<t_2)$; where $t_1,t_2\in D=\left\{(u,v)\in R^2:F(u)<F(v)\right\}$ is called a doubly truncated lifetime variable. Another extension of Shannon entropy is based on a doubly truncated random variable ($\mathrm{X}/\mathrm{t}_1<\mathrm{X}<\mathrm{t}_2$), which is defined as

$$\mathrm{K}\left(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2\right)=-\int_{\mathrm{t}_1}^{\mathrm{t}_2}\frac{\mathrm{f}(\mathrm{x})}{\mathrm{R}\left(\mathrm{t}_1\right)-\mathrm{R}\left(\mathrm{t}_2\right)}\log\frac{\mathrm{f}(\mathrm{x})}{\mathrm{R}\left(\mathrm{t}_1\right)-\mathrm{R}\left(\mathrm{t}_2\right)}\mathrm{dx} \tag{1.5}$$

Given that a system has survived up to time t_1 and has been found to be down at time t_2, then $K(X; t_1, t_2)$ measures the uncertainty about its lifetimes between t_1 and t_2. Sunoj et al. [19] have explored the use of information measures for double truncated random variables. Furthermore, Misagh and Yari [15, 16] explored the use of weighted information measures for doubly truncated random variables. For various results on doubly truncated random variable, we refer to Kayal and Moharana [17], Kundu [18], Jalayeri et.al [19], Kumar [20] and in [21, 22]. Joshi and Dar [23] develop MathaiHaubold Fuzzy Entropy extends the MathaiHaubold Entropy framework to the domain of fuzzy set theory, providing a generalized measure of uncertainty and fuzziness in imprecise or vague systems. This formulation integrates the pathway model of Mathai and Haubold with the principles of fuzzy entropy, allowing for a more flexible quantification of uncertainty when dealing with fuzzy membership functions. Oindrali Das, Siddhartha Chakraborty and Biswabrata Pradhan [24] proposed the MathaiHaubold and study its properties.

Since generalized entropy plays an important role, in the field of reliability theory and survival analysis, when a system has lifetime between two time points ($\mathrm{t}_1, \mathrm{t}_2$), the concept of doubly truncated data is also applied to medical research, specifically in the study of tumor progression in cancer patients who receive chemotherapy. In these studies, patients' times to progression (i.e., the time it takes for their condition to worsen) are truncated: they are observed between two points in time, possibly between when they start treatment and when they either progress or die. This doubly truncated data can be modelled effectively using generalized entropy.

2. Mathai-Haubold Interval Entropy

Based on the measure defined in (1.3), Dar and Bander [25], introduce the information measure that takes the current age of the system into consideration and generalizes (1.4) as

$$\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}) = \frac{1}{(\alpha-1)}\left[\frac{\int_{\mathrm{t}}^{\infty} \mathrm{f}^{2-\alpha}(\mathrm{x})\mathrm{dx}}{\mathrm{R}^{2-\alpha}(\mathrm{t})} - 1\right], \alpha \neq 1, 0<\alpha<2 \tag{2.1}$$

Using equation (2.1),the interval entropy of order α (now onwards MHIE) of the double truncated random variable ($\mathrm{X}/\mathrm{t}_1 < \mathrm{X} < \mathrm{t}_2$) is given by:

$$\mathrm{K}_\alpha\left(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2\right) = \frac{1}{(\alpha-1)}\left[\int_{\mathrm{t}_1}^{\mathrm{t}_2}\left(\frac{\mathrm{f}(\mathrm{x})}{\mathrm{F}\left(\mathrm{t}_2\right)-\mathrm{F}\left(\mathrm{t}_1\right)}\right)^{2-\alpha}\mathrm{dx} - 1\right], \alpha \neq 1, 0<\alpha<2 \tag{2.2}$$

When the system has age t_1, (2.2) provides the information spectrum of the remaining life of the system until age t_2.

Corollary: For $\mathrm{t}_1 = \mathrm{t}$ and $\mathrm{t}_2 = \infty$, (2.2) reduces to (2.1).

Definition 2.1: The general failure rate (GFR) of a doubly truncated random variable $(\mathrm{X}/\mathrm{t}_1 < \mathrm{X} < \mathrm{t}_2)$ is defined as $\mathrm{h}_{\mathrm{i}}\left(\mathrm{t}_1,\mathrm{t}_2\right) = \frac{\mathrm{f}(\mathrm{t}_{\mathrm{i}})}{\mathrm{F}(\mathrm{t}_2)-\mathrm{F}(\mathrm{t}_1)}, \mathrm{i}=1,2.$

Based on (2.2), we derive the interval entropy of some well-known lifetime distributions:
(i) Uniform distribution:

The probability density function and corresponding cdf of uniform distribution is given by
$\mathrm{f(x)}=\frac{1}{\mathrm{b-a}},\mathrm{a}<\mathrm{x}<\mathrm{b},\mathrm{F(x)}=1-\frac{\mathrm{b-x}}{\mathrm{b-a}}$ respectively.
The MHIE is given by
$(\alpha-1)\mathrm{K}_\alpha\left(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2\right)=\frac{1}{(\mathrm{t}_2-\mathrm{t}_1)^{2-\alpha}}\left(\int_{\mathrm{t}_1}^{\mathrm{t}_2}\mathrm{dx}\right)-1$
or

$$\mathrm{K}_\alpha\left(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2\right)=\frac{1}{(\alpha-1)}\left[\mathrm{t}_2\ \mathrm{h}_2^{2-\alpha}\left(\mathrm{t}_1,\mathrm{t}_2\right)-\mathrm{t}_1\ \mathrm{h}_1^{2-\alpha}\left(\mathrm{t}_1,\mathrm{t}_2\right)-1\right]$$

(ii) Exponential distribution:

The probability density function and corresponding cdf of exponential distribution is given by
$f(x)=\theta e^{-\theta x},x>0,\theta>0,F(x)=1-e^{-\theta x}$ respectively.

$$\begin{aligned}(\alpha-1)\mathrm{K}_\alpha\left(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2\right)&=\left(\frac{\theta}{\mathrm{e}^{-\theta\mathrm{t}_1}-\mathrm{e}^{-\theta\mathrm{t}_2}}\right)^{2-\alpha}\left[\int_{\mathrm{t}_1}^{\mathrm{t}_2}\mathrm{e}^{-(2-\alpha)\theta\mathrm{x}}\mathrm{dx}\right]-1\\&=\frac{1}{\theta(2-\alpha)}\left[\left(\frac{\theta\mathrm{e}^{-\theta\mathrm{t}_1}}{\mathrm{e}^{-\theta\mathrm{t}_1}-\mathrm{e}^{-\theta\mathrm{t}_2}}\right)^{2-\alpha}-\left(\frac{\theta\mathrm{e}^{-\theta\mathrm{t}_2}}{\mathrm{e}^{-\theta\mathrm{t}_1}-\mathrm{e}^{-\theta\mathrm{t}_2}}\right)^{2-\alpha}\right]-1\end{aligned}$$

or

$$\mathrm{K}_\alpha\left(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2\right)=\frac{1}{(\alpha-1)}\left[\frac{1}{\theta(2-\alpha)}\left(\mathrm{h}_1^{2-\alpha}\left(\mathrm{t}_1,\mathrm{t}_2\right)-\mathrm{h}_2^{2-\alpha}\left(\mathrm{t}_1,\mathrm{t}_2\right)\right)-1\right]$$

(iii) Finite range distribution:

The probability density function and corresponding cdf of finite range distribution is given by
$\mathrm{f(x)}=\mathrm{ab}(1-\mathrm{ax})^{\mathrm{b}-1},0<\mathrm{x}<\frac{1}{\mathrm{b}},\mathrm{a},\mathrm{b}>0,\ \mathrm{F(x)}=1-(1-\mathrm{ax})^{\mathrm{b}}$ respectively.

$$(\alpha-1)\mathrm{K}_\alpha\left(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2\right)=\left(\frac{\mathrm{ab}}{(1-\mathrm{at}_1)^{\mathrm{b}}-(1-\mathrm{at}_2)^{\mathrm{b}}}\right)^{2-\alpha}\left[\int_{\mathrm{t}_1}^{\mathrm{t}_2}(1-\mathrm{ax})^{(\mathrm{b}-1)(2-\alpha)}\mathrm{dx}\right]-1$$

$$\mathrm{K}_\alpha\left(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2\right)=\frac{1}{(\alpha-1))}\left[\frac{1}{\mathrm{a}((2-\alpha)(\mathrm{b}-1)+1}\left((1-\mathrm{at}_1)\,\mathrm{h}_1^{2-\alpha}\left(\mathrm{t}_1,\mathrm{t}_2\right)-(1-\mathrm{at}_2)\ \mathrm{h}_2^{2-\alpha}\left(\mathrm{t}_1,\mathrm{t}_2\right)\right)-1\right]$$

(iv) Power distribution: The density function and cdf of power distribution is given as

$$f(x) = \frac{b}{a}\left(\frac{x}{a}\right)^{b-1}, 0 < x < b, b > 0, F(x) = \left(\frac{x}{a}\right)^{b}$$

$$(\alpha-1)K_\alpha\left(X;t_1,t_2\right) = \left(\frac{b}{t_2^b - t_1^b}\right)^{2-\alpha}\left[\int_{t_1}^{t_2} x^{(b-1)(2-\alpha)}dx\right] - 1$$

$$= \frac{1}{((2-\alpha)(b-1)+1)}\left[t_2\left(\frac{bt_2^{b-1}}{t_2^b - t_1^b}\right)^{2-\alpha} - t_1\left(\frac{bt_1^{b-1}}{t_2^b - t_1^b}\right)^{2-\alpha}\right] - 1$$

or

$$\mathrm{K}_\alpha\left(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2\right) = \frac{1}{(\alpha-1)}\left[\frac{1}{((2-\alpha)(\mathrm{b}-1)+1)}\left(\mathrm{t}_2\ \mathrm{h}_2^{2-\alpha} - \mathrm{t}_1\ \mathrm{h}_1^{2-\alpha}\left(\mathrm{t}_1,\mathrm{t}_2\right)\right) - 1\right].$$

To gain further insights into the behavior of the Mathai-Haubold Interval Entropy (MHIE), we provide graphical representations of MHIE for several standard lifetime distributions.

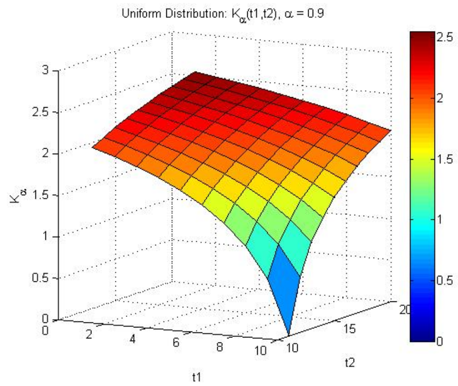

Figure 1: Uniform Distribution

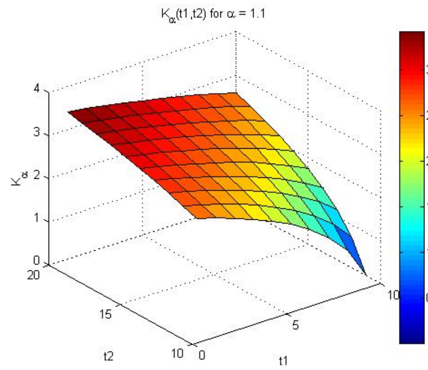

Figure 2: Uniform Distribution

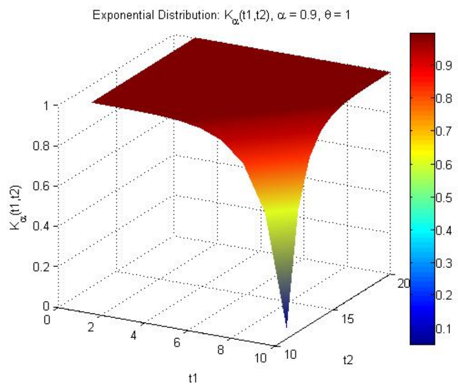

Figure 3: Exponential Distribution

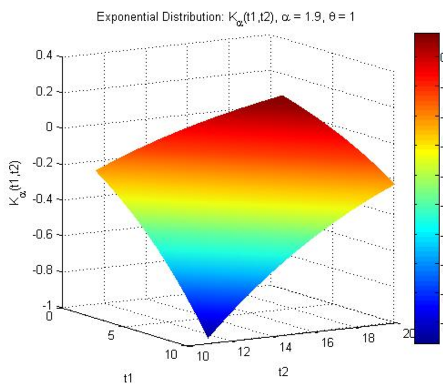

Figure 4: Exponential Distribution

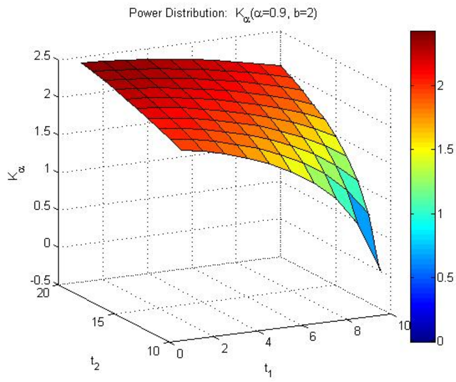

Figure 5: Power Distribution

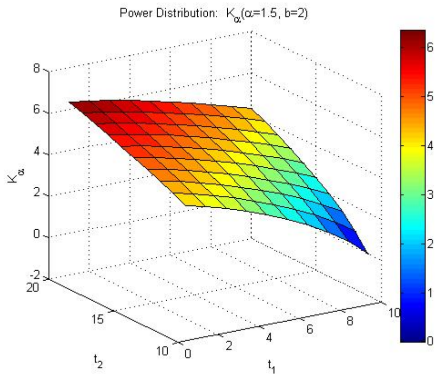

Figure 6: Power Distribution

3. Some characterization Results:

Theorem 3.1. *Let X be the non-negative random variable having continuous density function* $\mathrm{f(x)}$ *and distribution function* $\mathrm{F(x)}$. *Assume that* $\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2)$ *is increasing with respect to* t_1 *and* t_2, *then* $K_\alpha(X;t_1,t_2)$ *uniquely determine the distribution function* $\mathrm{F(x)}$.

Proof:
Differentiating (2.2) partially with respect to t_1 and t_2, we get

$$(\alpha-1)\frac{\partial}{\partial \mathrm{t}_1}\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2)=(2-\alpha)\int_{\mathrm{t}_1}^{\mathrm{t}_2}\frac{\mathrm{f}^{2-\alpha}(\mathrm{x})\mathrm{f}(\mathrm{t}_1)\,\mathrm{dx}}{(\mathrm{F}(\mathrm{t}_2)-\mathrm{F}(\mathrm{t}_1))^{3-\alpha}}-\left(\frac{\mathrm{f}(\mathrm{t}_1)}{\mathrm{F}(\mathrm{t}_2)-\mathrm{F}(\mathrm{t}_1)}\right)^{2-\alpha}$$

and

$$(\alpha-1)\frac{\partial}{\partial \mathrm{t}_2}\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2)=(2-\alpha)\int_{\mathrm{t}_1}^{\mathrm{t}_2}\frac{\mathrm{f}^{2-\alpha}(\mathrm{x})\mathrm{f}(\mathrm{t}_1)\,\mathrm{dx}}{(\mathrm{F}(\mathrm{t}_2)-\mathrm{F}(\mathrm{t}_1))^{3-\alpha}}-\left(\frac{\mathrm{f}(\mathrm{t}_2)}{\mathrm{F}(\mathrm{t}_2)-\mathrm{F}(\mathrm{t}_1)}\right)^{2-\alpha}$$

After simplification, we get

$$(\alpha-1)\frac{\partial}{\partial \mathrm{t}_1}\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2)=-\mathrm{h}_1^{2-\alpha}(\mathrm{t}_1,\mathrm{t}_2)+(2-\alpha)(\alpha-1)\mathrm{h}_1(\mathrm{t}_1,\mathrm{t}_2)\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2)$$
$$(\alpha-1)\frac{\partial}{\partial \mathrm{t}_2}\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2)=-\mathrm{h}_2^{2-\alpha}(\mathrm{t}_1,\mathrm{t}_2)+(2-\alpha)(\alpha-1)\mathrm{h}_2(\mathrm{t}_1,\mathrm{t}_2)\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2)$$

or

$$\mathrm{h}_1^{2-\alpha}(\mathrm{t}_1,\mathrm{t}_2)=(2-\alpha)(\alpha-1)\mathrm{h}_1(\mathrm{t}_1,\mathrm{t}_2)\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2)-(\alpha-1)\frac{\partial}{\partial \mathrm{t}_1}\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2)$$

$$\mathrm{h}_2^{2-\alpha}(\mathrm{t}_1,\mathrm{t}_2)=(2-\alpha)(\alpha-1)\mathrm{h}_2(\mathrm{t}_1,\mathrm{t}_2)\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2)+(\alpha-$$

(i) $\frac{\partial}{\partial \mathrm{t}_2} \mathrm{K}_\alpha(\mathrm{X}; \mathrm{t}_1, \mathrm{t}_2)$

Now for any fixed t_1 and $t_2, h_1(t_1, t_2)$ and $h_2(t_1, t_2)$ is a positive solution of the equation $\xi(x_{t_2}) = 0$ and $\psi(y_{t_1}) = 0$, where

$$\xi(\mathrm{x}_{\mathrm{t}_2}) = (\mathrm{x}_{\mathrm{t}_2})^{2-\alpha} - (2-\alpha)(\alpha-1)\mathrm{x}_{\mathrm{t}_2} \mathrm{K}_\alpha(\mathrm{X}; \mathrm{t}_1, \mathrm{t}_2) - (\alpha-1)\frac{\partial}{\partial \mathrm{t}_2} \mathrm{K}_\alpha(\mathrm{X}; \mathrm{t}_1, \mathrm{t}_2) \tag{3.1}$$

and

$$\psi(\mathrm{y}_{\mathrm{t}_1}) = (\mathrm{y}_{\mathrm{t}_1})^{2-\alpha} - (2-\alpha)(\alpha-1)\mathrm{y}_{\mathrm{t}_1} \mathrm{K}_\alpha(\mathrm{X}; \mathrm{t}_1, \mathrm{t}_2) + (\alpha-1)\frac{\partial}{\partial \mathrm{t}_1} \mathrm{K}_\alpha(\mathrm{X}; \mathrm{t}_1, \mathrm{t}_2) \tag{3.2}$$

Differentiating partially with respect to x_{t_2} and y_{t_1}, we obtain

$$\xi'(\mathrm{x}_{\mathrm{t}_2}) = (2-\alpha)(\mathrm{x}_{\mathrm{t}_2})^{1-\alpha} - (2-\alpha)(\alpha-1)\mathrm{K}_\alpha(\mathrm{X}; \mathrm{t}_1, \mathrm{t}_2) \tag{3.3}$$

$$\psi'(\mathrm{y}_{\mathrm{t}_1}) = (2-\alpha)(\mathrm{y}_{\mathrm{t}_1})^{1-\alpha} - (2-\alpha)(\alpha-1)\mathrm{K}_\alpha(\mathrm{X}; \mathrm{t}_1, \mathrm{t}_2) \tag{3.4}$$

For extreme values, $\xi'(\mathrm{x}_{\mathrm{t}_2}) = 0, \psi'(\mathrm{y}_{\mathrm{t}_1}) = 0$, which gives

$$\mathrm{x}_{\mathrm{t}_2} = [(\alpha-1)\mathrm{K}_\alpha(\mathrm{x}; \mathrm{t}_1, \mathrm{t}_2)]^{\frac{1}{1-\alpha}} = \mathrm{y}_{\mathrm{t}_1}.$$

Further more

$$\xi''(\mathrm{x}_{\mathrm{t}_2}) = (2-\alpha)(1-\alpha)(\mathrm{x}_{\mathrm{t}_2})^{-\alpha}$$

and

$$\psi''(\mathrm{y}_{\mathrm{t}_1}) = (2-\alpha)(1-\alpha)(\mathrm{y}_{\mathrm{t}_1})^{-\alpha}$$

Two cases arise:

Case I: $\alpha < 1$, then $\xi''(\mathrm{x}_{\mathrm{t}_2}) = \psi''(\mathrm{y}_{\mathrm{t}_1}) > 0$. Thus both $\xi(\mathrm{x}_{\mathrm{t}_2})$ and $\psi(\mathrm{y}_{\mathrm{t}_1})$ are minimized at $\mathrm{x}_{\mathrm{t}_2}$ and $\mathrm{y}_{\mathrm{t}_1}$ respectively. Also, $\xi(0) < 0$ and $\xi(\infty) = \infty$. Similarly, $\psi(0) > 0, \psi(\infty) = \infty$.

Hence both the equations $\xi(\mathrm{x}_{\mathrm{t}_2}) = 0$ and $\psi(\mathrm{y}_{\mathrm{t}_1}) = 0$ have unique positive solutions $\mathrm{h}_1(\mathrm{t}_1, \mathrm{t}_2)$ and $\mathrm{h}_2(\mathrm{t}_1, \mathrm{t}_2)$ respectively.

Case II: $\alpha > 1$, then $\xi''(\mathrm{x}_{\mathrm{t}_2}) = \psi''(\mathrm{y}_{\mathrm{t}_1}) < 0$. Thus both $\xi(\mathrm{x}_{\mathrm{t}_2})$ and $\psi(\mathrm{y}_{\mathrm{t}_1})$ are maximized at $\mathrm{x}_{\mathrm{t}_2}$ and $\mathrm{y}_{\mathrm{t}_1}$ respectively. Also, $\xi(0) < 0$ and $\xi(\infty) = \infty$. Similarly, $\psi(0) > 0, \psi(\infty) = \infty$.

Hence both the equations $\xi(\mathrm{x}_{\mathrm{t}_2}) = 0$ and $\psi(\mathrm{y}_{\mathrm{t}_1}) = 0$ have unique positive solutions $\mathrm{h}_1(\mathrm{t}_1, \mathrm{t}_2)$ and $\mathrm{h}_2(\mathrm{t}_1, \mathrm{t}_2)$ respectively.

Combining both the cases, we conclude $K_\alpha(X; t_1, t_2)$ determines the $h_i(t_1, t_2), i = 1, 2$ uniquely.

Theorem 3.2. *Let X be the non-negative random variable follows uniform distribution over (* a, b *),* a < b *if and only if*

$$K_\alpha(X;t_1,t_2)=\frac{1}{\alpha-1}\left[\frac{t_2-t_1}{(t_2-t_1)^{2-\alpha}}-1\right] \tag{3.5}$$

Proof: The probability density function and corresponding cdf of uniform distribution is given by

$$f(x)=\frac{1}{b-a}, a<x<b,\ F(x)=1-\frac{b-x}{b-a} \text{ respectively.}$$

Using (2.2), we get

$$K_\alpha(X;t_1,t_2)=\frac{1}{\alpha-1}\left[\frac{t_2-t_1}{(t_2-t_1)^{2-\alpha}}-1\right], \text{ which proves only if part of the theorem.}$$

To prove the if part, let (3.5) is valid.
Thus from (2.2) and (3.5), we get

$$\int_{t_1}^{t_2} f^{2-\alpha}(x)dx=\frac{(t_2-t_1)^{2-\alpha}}{\left((F(t_2)-F(t_1)\right)^{\alpha-2}}$$

Differentiating with respect t_1 and t_2, we obtain

$$h_1^{2-\alpha}(t_1,t_2)=(\alpha-1)(t_2-t_1)^{\alpha-2}+(2-\alpha)(t_2-t_1)^{\alpha-1}\ h_1(t_1,t_2)$$

and

$$h_2^{2-\alpha}(t_1,t_2)=(\alpha-1)(t_2-t_1)^{\alpha-2}+(2-\alpha)(t_2-t_1)^{\alpha-1}\ h_2(t_1,t_2)$$

Now for any fixed t_1 and t_2, $h_1(t_1,t_2)$ and $h_2(t_1,t_2)$ is a positive solution of the equation $\xi(x_{t_2})=0$ and $\psi(y_{t_1})=0$, where

$$\xi(x_{t_2})=(x_{t_2})^{2-\alpha}-\frac{(\alpha-1)}{(t_2-t_1)^{2-\alpha}}-\frac{(2-\alpha)}{(t_2-t_1)^{1-\alpha}}x_{t_2} \tag{3.6}$$

and

$$\psi(y_{t_1})=(y_{t_1})^{2-\alpha}-\frac{(\alpha-1)}{(t_2-t_1)^{2-\alpha}}-\frac{(2-\alpha)}{(t_2-t_1)^{1-\alpha}}y_{t_1} \tag{3.7}$$

Differentiating both side of (3.6) and (3.7) with respect to x_{t_2} and y_{t_1} respectively, we get

$$\xi'(x_{t_2})=(2-\alpha)(x_{t_2})^{1-\alpha}-\frac{(2-\alpha)}{(t_2-t_1)^{1-\alpha}}$$

$$\psi'(y_{t_1})=(2-\alpha)(y_{t_1})^{1-\alpha}-\frac{(2-\alpha)}{(t_2-t_1)^{1-\alpha}}$$

For extreme values, $\xi'\left(\mathrm{x}_{\mathrm{t}_2}\right)=0, \psi'\left(\mathrm{y}_{\mathrm{t}_1}\right)=0$, which gives

$$\mathrm{x}_{\mathrm{t}_2}=\frac{1}{\mathrm{t}_2-\mathrm{t}_1}=\mathrm{h}_1\left(\mathrm{t}_1, \mathrm{t}_2\right)$$

and

$$\mathrm{y}_{\mathrm{t}_1}=\frac{1}{\mathrm{t}_2-\mathrm{t}_1}=\mathrm{h}_2\left(\mathrm{t}_1, \mathrm{t}_2\right), \text{ which proves the theorem.}$$

Theorem 3.3. *Let X be the non-negative random variable having continuous density function $f(x)$ and distribution function $F(x)$, then a relation of the form*

$$(\alpha-1)\mathrm{K}_\alpha\left(\mathrm{X}; \mathrm{t}_1, \mathrm{t}_2\right)=\frac{1}{\mathrm{c}}\left[\left(1+\mathrm{kt}_2\right) \mathrm{h}_2^{2-\alpha}\left(\mathrm{t}_1, \mathrm{t}_2\right)-\left(1+\mathrm{kt}_1\right) \mathrm{h}_1^{2-\alpha}\left(\mathrm{t}_1, \mathrm{t}_2\right)\right]-1 \tag{3.8}$$

where c is a constant holds for all t_1, t_2. Then X has
(i) An Exponential distribution iff $\mathrm{c}=0$
(ii) A Pareto distribution iff $\mathrm{c}<0$
(iii) A finite range distribution iff $\mathrm{c}>0$.

Proof. (i) The probability density function and corresponding cdf of exponential distribution is given by
$f(x)=\theta e^{-\theta x}, x>0, \theta>0, F(x)=1-e^{-\theta x}$ respectively
Also,
$\mathrm{h}_{\mathrm{i}}\left(\mathrm{t}_1, \mathrm{t}_2\right)=\frac{\theta \mathrm{e}^{-\theta \mathrm{t}_{\mathrm{i}}}}{\mathrm{e}^{-\theta \mathrm{t}_1}-\mathrm{e}^{-\theta \mathrm{t}_2}}$
Using (2.2), we get

$$(\alpha-1)\mathrm{K}_\alpha\left(\mathrm{X}; \mathrm{t}_1, \mathrm{t}_2\right)=\frac{1}{\mathrm{c}}\left[\mathrm{h}_2^{2-\alpha}\left(\mathrm{t}_1, \mathrm{t}_2\right)-\mathrm{h}_1^{2-\alpha}\left(\mathrm{t}_1, \mathrm{t}_2\right)\right]-1 \tag{3.9}$$

where $c=\theta(2-\alpha)$.
Thus (3.9) is equivalent to (3.8) for $\mathrm{k}=0$.
(ii) The probability density function and corresponding cdf of uniform distribution is given by

$$\mathrm{f(x)}=\mathrm{rs}(1+\mathrm{rx})^{-(\mathrm{s}+1)}, \mathrm{r},\ \mathrm{s}>0,\ \mathrm{F}_{\mathrm{X}}(\mathrm{x})=1-(1+\mathrm{rx})^{\mathrm{s}}$$

Thus using (2.2), we have

$$(\alpha-1)\mathrm{K}_\alpha\left(\mathrm{X}; \mathrm{t}_1, \mathrm{t}_2\right)=\frac{1}{\mathrm{c}}\left[\left(1+\mathrm{rt}_2\right) \mathrm{h}_2^{2-\alpha}\left(\mathrm{t}_1, \mathrm{t}_2\right)-\left(1+\mathrm{rt}_1\right) \mathrm{h}_1^{2-\alpha}\left(\mathrm{t}_1, \mathrm{t}_2\right)\right]-1$$

where $\mathrm{c}=\frac{1}{\mathrm{r}[1-(\mathrm{s}+1)(2-\alpha)]}$ and $\mathrm{k}=\mathrm{r}>0$. Thus (3.8) holds
(iii) The probability density function and corresponding cdf of uniform distribution is given by

$f(x) = ab(1-ax)^{b-1}, 0 < x < \frac{1}{b}, a, b > 0,\ F(x) = 1-(1-ax)^{b}$ respectively.
Using (2.2), one will get

$$(\alpha-1)K_{\alpha}\left(X; t_1, t_2\right) = \frac{1}{c}\left[(1-at_2)\,h_1^{2-\alpha}\left(t_1, t_2\right) - (1-at_1)\,h_2^{2-\alpha}\left(t_1, t_2\right)\right] - 1$$

where $c = \frac{1}{-a[1+(b-1)(2-\alpha)]}$ and $k = -a < 0$. Thus (3.8) holds.
Conversely, let (3.8) holds. Using (2.2) in (3.8), we get
$c\int_{t_1}^{t_2} f^{2-\alpha}(x)dx = (1+kt_2)^{2-\alpha} f^{2-\alpha}(t_2) - (1+kt_1)^{2-\alpha} f^{2-\alpha}(t_1) - 1.$
Differentiating with respect to t_2, keeping t_1 fixed
$\frac{f'(t_2)}{f(t_2)} = \left(\frac{c-k}{2-\alpha}\right)\left(\frac{1}{1+kt_2}\right).$
Similarly differentiating with respect to t_1, keeping t_2 fixed
$\frac{f'(t_1)}{f(t_1)} = \left(\frac{c-k}{2-\alpha}\right)\left(\frac{1}{1+kt_1}\right).$

Generally,
$\frac{f'(t_i)}{f(t_i)} = \left(\frac{c-k}{2-\alpha}\right)\left(\frac{1}{1+kt_i}\right), i = 1, 2.$
This gives

$$\frac{d}{dt}\log f\left(t_i\right) = \left(\frac{c-k}{2-\alpha}\right)\left(\frac{1}{1+kt_i}\right), i = 1, 2 \tag{3.10}$$

(3.10) clearly shows that the underlying distribution is exponential if $k = 0$, Pareto distribution for $k > 0$ and finite rage distribution for $k < 0$. Hence the theorem is proved.

4. Some Inequalities and stochastic comparison:

Theorem 4.1. *Let X be an absolutely continuous random variable density function $f(x)$ and distribution function* $F(x)$*, then for $\alpha > 1(\alpha < 1)$, then* $K_{\alpha}\left(X; t_1, t_2\right)$ *is increasing (decreasing) in t_2 if $F\left(t_1\right) = 0$*

Proof: $K_{\alpha}\left(X; t_1, t_2\right) = \frac{1}{(\alpha-1)}\left[\int_{t_1}^{t_2}\left(\frac{f(x)}{F(t_2)-F(t_1)}\right)^{2-\alpha} dx - 1\right]$
For $F\left(t_1\right) = 0$, we have
$K_{\alpha}\left(X; t_1, t_2\right) = \frac{1}{(\alpha-1)}\left[\int_{t_1}^{t_2}\left(\frac{f(x)}{F(t_2)}\right)^{2-\alpha} dx - 1\right]$
Differentiating with respect t_2, we get
$K'_{\alpha}\left(X; t_1, t_2\right) = \frac{F(t_2)}{F^{3-\alpha}(t_2)}\left[f^{1-\alpha}\left(t_2\right)F\left(t_2\right) - (2-\alpha)\int_{t_1}^{t_2} f^{2-\alpha}(x)dx\right]$
Define
$K\left(t_2\right) = f^{1-\alpha}\left(t_2\right)F\left(t_2\right) - (2-\alpha)\int_{t_1}^{t_2} f^{2-\alpha}(x)dx$
Differentiating again with respect t_2, we have
$K'\left(t_2\right) = (\alpha-1)f^{-\alpha}\left(t_2\right)\left(f^2\left(t_2\right) - F\left(t_2\right)f'\left(t_2\right)\right.$.
If $\alpha > 1$, then $K'\left(t_2\right) \geq 0$, which in turns gives that $K_{\alpha}\left(X; t_1, t_2\right)$ is an increasing function of t_2.

If $\alpha < 1$, then $K'\left(t_2\right) \leq 0$, which in turns gives that $K_{\alpha}\left(X; t_1, t_2\right)$ is an decreasing function of t_2.

Theorem 4.2. *For an absolutely continuous random variable X , if $\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2)$ is increasing (decreasing) in t_1 for fixed t_2, then*
$\mathrm{h}_1(\mathrm{t}_1,\mathrm{t}_2) \leq (\geq) \left[((2-\alpha)(\alpha-1)\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2))\right]^{\frac{1}{1-\alpha}}$
for $\alpha > 1$
$\mathrm{h}_1(\mathrm{t}_1,\mathrm{t}_2) \geq (\leq) \left[((2-\alpha)(\alpha-1)\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2))\right]^{\frac{1}{1-\alpha}}$
for $\alpha < 1$.

Proof: We Know that
$(\alpha-1)\frac{\partial}{\partial \mathrm{t}_1}\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2) = -\mathrm{h}_1^{2-\alpha}(\mathrm{t}_1,\mathrm{t}_2) + (2-\alpha)(\alpha-1)\mathrm{h}_1(\mathrm{t}_1,\mathrm{t}_2)\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2)$
if $K_\alpha(X;t_1,t_2)$ is increasing (decreasing) in t_1 for fixed t_2, then
$-\mathrm{h}_1^{2-\alpha}(\mathrm{t}_1,\mathrm{t}_2) + (2-\alpha)(\alpha-1)\mathrm{h}_1(\mathrm{t}_1,\mathrm{t}_2)\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2) \geq (\leq) 0$
$\mathrm{h}_1^{2-\alpha}(\mathrm{t}_1,\mathrm{t}_2) \leq (\geq)(2-\alpha)(\alpha-1)\mathrm{h}_1(\mathrm{t}_1,\mathrm{t}_2)\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2)$
$\mathrm{h}_1^{1-\alpha}(\mathrm{t}_1,\mathrm{t}_2) \leq (\geq)(2-\alpha)(\alpha-1)\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2)$
For $\alpha > 1$
$\mathrm{h}_1(\mathrm{t}_1,\mathrm{t}_2) \leq (\geq) \left[((2-\alpha)(\alpha-1)\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2))\right]^{\frac{1}{1-\alpha}}$
For $\alpha < 1$
$\mathrm{h}_1(\mathrm{t}_1,\mathrm{t}_2) \geq (\leq) \left[((2-\alpha)(\alpha-1)\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2))\right]^{\frac{1}{1-\alpha}}$

Theorem 4.3. *For an absolutely continuous random variable X , if $\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2)$ is increasing (decreasing) in t_2 for fixed t_1, then*
$\mathrm{h}_2(\mathrm{t}_1,\mathrm{t}_2) \leq (\geq) \left[((2-\alpha)(\alpha-1)\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2))\right]^{\frac{1}{1-\alpha}}$
for $\alpha > 1$
$\mathrm{h}_2(\mathrm{t}_1,\mathrm{t}_2) \geq (\leq) \left[((2-\alpha)(\alpha-1)\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2))\right]^{\frac{1}{1-\alpha}}$
for $\alpha < 1$.

Proof: The proof follows under similar lines as in theorem 4.2.
Let X and Y be two random variables. Also, the distribution function and density function of X are indicated by $F(x)$ and $f(x)$ and those of Y are denoted by $G(x)$ and $g(x)$ respectively, then X is said to be less than or equal to Y in likelihood ratio ordering, if $F(x) \geq G(x), x \geq 0$. We write $X \leq^{st} Y$.

Theorem 4.4. *Let X and Y be an absolutely continuous random variable density function with cdf $\mathrm{F(x)}$ and $\mathrm{G(x)}$ respectively. If $X \leq^{st} Y$ for all $t_1, t_2 \geq 0$, then and distribution function $\mathrm{F(x)}$, then for $\alpha > 1 (0 < \alpha < 1)$, then $\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2) \geq (\leq) \mathrm{K}_\alpha(\mathrm{Y};\mathrm{t}_1,\mathrm{t}_2)$ for $\alpha > 1$ and for*

Proof: Since $X \leq^{st} Y$, therefore

$$\frac{1}{(\alpha-1)}\left[\int_{t_1}^{t_2}\left(\frac{f(x)}{F(t_2)-F(t_1)}\right)^{2-\alpha}dx - 1\right] \geq \frac{1}{(\alpha-1)}\left[\int_{t_1}^{t_2}\left(\frac{g(x)}{G(t_2)-G(t_1)}\right)^{2-\alpha}dx - 1\right]$$

Thus for $\alpha > 1$, we have

$$\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2) \geq \mathrm{K}_\alpha(\mathrm{Y};\mathrm{t}_1,\mathrm{t}_2)$$

and for $0 < \alpha < 1$, it is obvious

$$\mathrm{K}_\alpha(\mathrm{X};\mathrm{t}_1,\mathrm{t}_2) \leq \mathrm{K}_\alpha(\mathrm{Y};\mathrm{t}_1,\mathrm{t}_2)$$

5. Conclusion:

Generalized interval entropy plays a critical role in information theory, reliability modeling and survival analysis by quantifying the uncertainty associated with a system's lifetime, particularly when it has already survived between two time points. The measure is useful in characterizing different lifetime distributions and establishing relationships with reliability measures such as survival functions and hazard rates. Additionally, the connection between generalized interval entropy and stochastic ordering provides a powerful tool for comparing different lifetime distributions. Overall, the application of generalized interval entropy in the context of doubly truncated random variables and its generalization of previous results represents an important advancement in understanding and modeling the uncertainty inherent in lifetime distributions. These developments have important implications for reliability engineering, survival analysis, risk management, and other fields where the timing of events plays a crucial role.

Conflict of Interest: The authors declare that they have no Conflict of interest

References

[1] CE Shannon. A mathematical theoryof communication, bell system tech. *J*, 27:379–423, 1948.

[2] SM Sunoj, PG Sankaran, and SS Maya. Characterizations of life distributions using conditional expectations of doubly (interval) truncated random variables. *Communications in statistics-Theory and Methods*, 38(9):1441–1452, 2009.

[3] AM Mathai and Hans J Haubold. Pathway model, superstatistics, tsallis statistics, and a generalized measure of entropy. *Physica A: Statistical Mechanics and its Applications*, 375(1):110–122, 2007.

[4] Nader Ebrahimi. How to measure uncertainty in the residual life time distribution. *Sankhyā: The Indian Journal of Statistics, Series A*, pages 48–56, 1996.

[5] Felix Belzunce, Jorge Navarro, José M Ruiz, and Yolanda del Aguila. Some results on residual entropy function. *Metrika*, 59(2):147–161, 2004.

[6] Majid Asadi, Nader Ebrahimi, and Ehsan S Soofi. Dynamic generalized information measures. *Statistics & probability letters*, 71(1):85–98, 2005.

[7] Mirza Abdul Khalique Baig and Javid Gani Dar. Generalized residual entropy function and its applications. *European Journal of Pure and Applied Mathematics*, 1(4), 2008.

[8] Anant Hegde, Tian Lan, and Deniz Erdogmus. Order statistics based estimator for renyi's entropy. In *2005 IEEE Workshop on Machine Learning for Signal Processing*, pages 335–339. IEEE, 2005.

[9] G Rajesh and KR Muraleedharan Nair. *Characterization of probability distributions using the residual entropy function.* PhD thesis, Cochin University of Science and Technology, 2001.

[10] Majid Asadi and Nader Ebrahimi. Residual entropy and its characterizations in

terms of hazard function and mean residual life function. *Statistics & probability letters*, 49(3):263–269, 2000.

[11] Antonio Di Crescenzo and Maria Longobardi. Entropy-based measure of uncertainty in past lifetime distributions. *Journal of Applied probability*, 39(2):434–440, 2002.

[12] Antonio Di Crescenzo and Maria Longobardi. A measure of discrimination between past lifetime distributions. *Statistics & probability letters*, 67(2):173–182, 2004.

[13] Asok K Nanda and Prasanta Paul. Some properties of past entropy and their applications. *Metrika*, 64(1):47–61, 2006.

[14] Nader Ebrahimi and Franco Pellerey. New partial ordering of survival functions based on the notion of uncertainty. *Journal of applied probability*, 32(1):202–211, 1995.

[15] F Misagh and GH Yari. On weighted interval entropy. *Statistics & probability letters*, 81(2):188–194, 2011.

[16] Fakhroddin Misagh and Gholamhossein Yari. Interval entropy and informative distance. *Entropy*, 14(3):480–490, 2012.

[17] Suchandan Kayal and Rajesh Moharana. Some results on a doubly truncated generalized discrimination measure. *Applications of mathematics*, 61(5):585–605, 2016.

[18] Chanchal Kundu. On weighted measure of inaccuracy for doubly truncated random variables. *Communications in Statistics-Theory and Methods*, 46(7):3135–3147, 2017.

[19] Samira Jalayeri, GR Mohtashami Borzadaran, and M Khorashadizadeh. Some properties of tsallis entropy based on a doubly truncated (interval) random variable. *Reliability: Theory & Applications*, 19(1 (77)):448–464, 2024.

[20] Vikas Kumar and Nirdesh Singh. Characterization of lifetime distribution based on generalized interval entropy. *Statistics, Optimization & Information Computing*, 6(4):547–559, 2018.

[21] Rebecca A Betensky and Emily C Martin. Commentary: failure-rate functions for doubly-truncated random variables. *IEEE Transactions on Reliability*, 52(1):7–8, 2003.

[22] Vikas Kumar and Nirdesh Singh. Characterization of lifetime distribution based on generalized interval entropy. *Statistics, Optimization & Information Computing*, 6(4):547–559, 2018.

[23] Vaishali Manish Joshi and Javid Dar. Some results on mathai-haubold fuzzy entropy. *European Journal of Pure and Applied Mathematics*, 17(3):2349–2360, 2024.

[24] Oindrali Das, Siddhartha Chakraborty, and Biswabrata Pradhan. On mathai–haubold past entropy measure. *Journal of the Indian Society for Probability and Statistics*, 25(1):327–342, 2024.

[25] Javid Gani Dar and Bander Al-Zahrani. On some characterization results of life time distributions using mathai–haubold residual entropy. *IOSR Journal of Mathematics*, 3:56–60, 2013.