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Fermatean Double-Valued Neutrosophic Soft Topological Spaces

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Abstract. This paper introduces the concept of Fermatean Double Valued Neutrosophic Soft Set (FDVNSS), an advanced mathematical framework for modeling uncertainty, vagueness, and indeterminacy in complex systems. A FDVNSS extends traditional neutrosophic and soft set theories by incorporating two layers of truth and falsity degrees: absolute and relative. This multi-dimensional representation allows for more nuanced decision-making, particularly in domains characterized by conflicting or imprecise information, such as medical diagnostics. The formal definition of FDVNSS is established, including the conditions under which membership degrees satisfy the Fermatean constraint, i.e., the sum of the cubes of the degrees lies within a defined range. Key operations such as subset, null and absolute FDVNSS, complement, union, and intersection are rigorously defined and exemplified. Furthermore, the algebraic properties of FDVNSSs under these operations are explored, and foundational propositions-analogous to classical set theory- are proven to hold. Examples drawn from health care contexts, such as evaluating diseases, symptoms, or medical responses, are provided to illustrate the applicability of FDVNSS in real-world problems. Overall, this framework provides a more expressive and flexible tool for reasoning under high levels of uncertainty and offers significant potential for application in intelligent decision support systems. The notion of a Fermatean Double Valued Neutrosophic Soft Topological Space (FDVNSTS) is introduced. The concepts of semi-open (s-open), pre-open (p-open), and b-open sets are defined within the context of FDVNSTS. Among these generalized open sets, the pre-open set is selected for further exploration, and several fundamental topological notions are developed based on this definition. These include the closure, exterior, boundary, and interior in FDVNSTS.

2020 Mathematics Subject Classifications: 54A05

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1. Introduction

In the 1870s, set theory emerged through the groundbreaking work of Georg Cantor and Richard Dedekind [1], revealing its profound significance across a wide range of real-world applications. Classical set theory, based on the concept of crisp sets, defines membership in absolute terms an element either belongs to a set or it does not. However, this rigid framework was unable to capture the nuances of partial membership. To address this limitation, Lotfi Zadeh introduced fuzzy set theory in 1965, allowing elements to have varying degrees of membership [2]. Fuzzy sets were later generalized by Pawlak as rough sets in 1982 [3], and by Molodtsov as soft sets in 1999 [4]. These extensions have demonstrated significant utility in managing uncertainty across a wide range of real-world applications, including engineering, economics, social sciences, environmental sciences, and medical sciences [5, 8].

Fuzzy soft sets [7], intuitionistic fuzzy sets [9], intuitionistic fuzzy soft sets [10], hesitant fuzzy sets [11], hesitant fuzzy soft sets [12], picture fuzzy sets [13], picture fuzzy soft sets [14], hyper soft sets [16], neutrosophic soft sets [16], and neutrosophic hyper soft sets [17] are several notable variants developed through the generalization of truth (membership), falsity (non-membership), and hesitancy (indeterminacy). In this context, we highlight key scholarly contributions related to soft sets, neutrosophic sets, and Fermatean sets. In 2003, Maji introduced the foundational concepts of soft set theory, including basic elements and operators [18], and in 2009, Ali et al. proposed modified operators [19]. Subsequently, the theory evolved as Cagman and Enginoglu introduced the concept of the soft matrix [20]; Babitha and Sunil defined relations and functions on soft sets along with their properties [21, 22]; and Yang and Guo explored closure and kernel concepts within soft relations and soft mappings [23]. Further contributions to soft set theory have been made by various mathematicians [26, 27]. As the theory continued to evolve through the aforementioned developments, researchers began exploring its connections with algebraic structures. Aktas¸ and Cagman introduced the concept of soft groups [28], Acer defined soft rings [5], and Aslam and Qurashi investigated sub algebraic structures associated with soft groups [29]. Drawing inspiration from philosophical logic?particularly notions of relative and absolute truth and falsity?as well as real-world contexts such as game outcomes (win, loss, tie), voting results (in favor, against, abstention), numerical classifications (positive, negative, neutral), binary responses (yes, no, not applicable), and decision-making scenarios (acceptance, rejection, hesitation, pending decisions), Smarandache introduced neutrosophic sets. These tri-component sets, rooted in the concept of 'neutral wisdom,' account for membership, non-membership, and indeterminacy simultaneously [30, 31]. Neutrosophic set theory has continued to evolve through numerous significant contributions. Wang et al. proposed single-valued and interval-valued neutrosophic sets [32, 33], while Salama and Alblowi introduced neutrosophic topological spaces [34]. Georgiou developed the concept of soft topological spaces [35], and Bera and Mahapatra extended this to neutrosophic soft topological spaces [36]. Various mathematical constructs have been formulated in relation to neutrosophic sets, including measures and integrals [37], lattices [38], vector spaces [39], continuous functions [44], entropy [41], groups and subgroups [42], and algebraic structures such as soft rings and soft fields [43].

Mathematicians have also explored a wide range of applications for neutrosophic techniques, including image processing [44], medical diagnosis [45, 46], and multi-criteria decision-making [47, 49], often employing similarity measures, neutrosophic logic, and hyper soft graphs [50]. To address limitations in expressing degrees of membership and non-membership in intuitionistic and Pythagorean fuzzy sets, Senapati and Yager introduced the Fermatean fuzzy set [51]. This development opened new avenues for research: Broumi et al. applied complex Fermatean neutrosophic graphs to decision-making [52]; Bilgin et al. introduced Fermatean neutrosophic topological spaces [53]; and Salsabeela and John explored the use of the TOPSIS technique on Fermatean fuzzy soft sets [54]. Further in-depth studies on neutrosophic theory and its generalizations can be found in [[63]-[70]]. The foundational work on supra soft topological spaces by El-Sheikh and Abd El-Latif [71] introduced decompositions and notions of soft continuity,

setting a precedent for analyzing generalized topologies under uncertainty. This was further developed through the introduction of soft supra compactness [72], providing compactness criteria applicable to soft topological frameworks. Additionally, Abd El-Latif and Hosny [73] explored supra open soft sets and related separation axioms, deepening the theoretical basis for soft separation in such topologies. These contributions collectively offer a conceptual and structural foundation for extending topological properties to Fermatean Double Valued Neutrosophic Soft Sets (FDVNSS), where uncertainty is characterized by both degrees and interrelated membership values, enabling more robust modeling in complex decision and AI-driven environments.
Structure of the Paper: Section 2 presents the basic definitions required for understanding the subsequent sections. Section 3 introduces key concepts related to FDVNSS. This includes definitions and examples of the FDVNSS itself, its subset, null set, absolute set, complement, union, and intersection. Additionally, several propositions are discussed, along with the FDVNSS ?AND? and ?OR? operators, each accompanied by relevant examples to ensure a clear understanding of these concepts.

2. Preliminaries

This section is devoted to the basic definitions necessary for the subsequent sections.

Definition 1. *[2] Let X be a non-empty set. A fuzzy set A in X is characterized by its membership function $\mu_A : X \to [0,1]$ and $\mu_A(X)$ is interpreted as the degree of membership of element X in fuzzy set A ,for each $x \in X$. It is clear that A is completely determined by the set of tuples*

$$A = \{(x, \mu_A(x)) : x \in X\}.$$

Definition 2. *[10] The intuitionistic fuzzy sets defined on a non?empty set X as objects having the form $A = \{(x, F_A(x), G_A(x)) : x \in X\}$, where the functions $F_A(x) : X \to [0,1]$ and $G_A(x) : X \to [0,1]$, denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set A respectively, and the*

$$0 \leq F_A(x) + G_A(x) \leq 1, \ forall \ x \in X.$$

Clearly, when $G_A(x) = 1 - F_A(x)$, for every $x \in X$, the set A becomes a fuzzy set.

Definition 3. *[51]Fermatean fuzzy set F in the universe set X is an object with the type $F = \{(u, F(x), G(x)) : x \in X\}$ where $F : U \to [0,1]$ and $G : X \to [0,1]$ with the condition*

$$0 \leq (F(x))^3 + (G(x))^c \leq 1 \forall \ x \in X$$

Definition 4. *[32]A single valued neutrosophic set A on the universe of discourse X is defined as $A = (x : T_A(x), I_A(x), F_A(x)) : x \in X$ where $T : I : X \to [0,1]$ and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.*

Definition 5. *[4]Let X be the universal set and E be the attributes with respect to X. Let $P(X)$ be the power set of X and $A \subseteq E$. Pair (F, A) is called a soft set over U and its mapping is given as $F : A \to P(X)$. It is defined as, $(F, A) = \{F(e)P(X) : e \in E, \ F(e) = \phi \ if \ e \notin A\}$.*

Definition 6. *[54] Let E be any set of deferent parameters, and let X be the universe, $A \subseteq E$. A Fermatean fuzzy soft set (FFSS) and X is defined as the pair (F, A) where F is mapping given by $F : A \to FFS(X)$, Where FFS(X) is the set of all Fermatean fuzzy sets over X. Here for any parameters $e \in A$, $F(e)$ is the Fermatean fuzzy set over X, defined as:*

$$F(e) = \{(x, \alpha_{F(e)}(x), \beta_{F(e)} : x \in X\}.$$

Where, $\alpha_{F(e)}(x)$ *is the degree of membership of* x *with respect to the parameter* e *and* $\beta_{F(e)}$ *is the degree of non-membership of* x *with respect to the parameter* e*. These degrees satisfy the condition:*

$$0 \leq (\alpha_{F(e)}(x))^3 + (\beta_{F(e)})^3 \leq 1.$$

Thus the Fermatean fuzzy soft set (F, E) *can be expressed as:*

$$(F, E) = \{(e, \{x, (\alpha_{F(e)}(x), \beta_{F(e)})\}) : e \in A, x \in X\}.$$

Definition 7. *[59] Let* X *be an initial universe set and* E *be a set of parameters . Consider* $A \subseteq E$*. Let* $P(X)$ *denotes the set of all neutrosophic sets of* X*. The collection* (F, A) *is termed to be the soft neutrosophic set over* X*, where* F *is a mapping given by* $F : A \to P(X)$*.*

Definition 8. *[61] Consider* X *and* E *as a universe set and a set of parameters, respectively. Let* $P(X)$ *denotes the set Intuitionistic fuzzy set of* X*. Let* $A \subseteq E$*. A pair* (F, A) *is an intuitionistic fuzzy soft set over* X*. Where* F *is a mapping given by* $F : A \to P(X)$*.*

Definition 9. *[62] Let* $\acute{E}$ *be the set of parameters and* X *be the key set. Let* $P(X)$ *represent the power set of* X*. Then, a quadri-partitioned neutrosophic soft set* $(\tilde{F}, \acute{E})$ *over* X *is a mapping* $\tilde{F} : \acute{E} \to P(X)$*, where* $\tilde{F}$ *is the function of* $(\tilde{F}, \acute{E})$*. Symbolically,*

$$(\tilde{F}, \acute{E}) = \left\{ \left(\grave{e}, \langle s, AbT_{\tilde{F}(\grave{e})}(s), ReT_{\tilde{F}(\grave{e})}(s), ReF_{\tilde{F}(\grave{e})}(s), AbF_{\tilde{F}(\grave{e})}(s) : s \in X \rangle \right) : \grave{e} \in \acute{E} \right\}.$$

Where, $AbT_{\tilde{F}(\grave{e})}(s)$*,* $ReT_{\tilde{F}(\grave{e})}(s)$*,* $ReF_{\tilde{F}(\grave{e})}(s)$*, and* $AbF_{\tilde{F}(\grave{e})}(s) \in [0, 1]$ *respectively,* $AbT_{\tilde{F}(\grave{e})}(s)$*,* $ReT_{\tilde{F}(\grave{e})}(s)$*,* $ReF_{\tilde{F}(\grave{e})}(s)$*,* $AbF_{\tilde{F}(\grave{e})}(s)$ *are called the absolute true-membership, relative true-membership,unknown membership, confusion-membership, ignorance-membership, relative false-membership, and absolute false-membership function of* $\tilde{F}(\grave{e})$*. Since the supremum of each function is 1 and the infimum is 0 so the inequality;*

$$0 \leq AbT_{\tilde{F}(\grave{e})}(s) + ReT_{\tilde{F}(\grave{e})}(s) + ReF_{\tilde{F}(\grave{e})}(s) + AbF_{\tilde{F}(\grave{e})}(s) \leq 4.$$

Definition 10. *[62] Let* $(\tilde{F}, \acute{E})$ *be a QPNSS over the key set* X*. Then, the complement of* $(\tilde{F}, \acute{E})$ *is denoted by* $(\tilde{F}, \acute{E})^c$ *and is defined as follows:*

$$(\tilde{F}, \acute{E})^c = \left\{ \left(\grave{e}, \langle s, AbF_{\tilde{F}(\grave{e})}(s), ReF_{\tilde{F}(\grave{e})}(s), ReT_{\tilde{F}(\grave{e})}(s), AbT_{\tilde{F}(\grave{e})}(s) \rangle : s \in X \right) : \grave{e} \in \acute{E} \right\}.$$

So, $\left((\tilde{F}, \acute{E})^c\right)^c = (\tilde{F}, \acute{E})$*.*

Definition 11. *[62] Let* $(\tilde{F}, \acute{E})$ *and* $(\tilde{G}, \acute{E})$ *be two QPNSSs over the key set* X*. Then,* $(\tilde{F}, \acute{E}) \tilde{\subseteq} (\tilde{G}, \acute{E})$ *if*

$$\begin{aligned} AbT_{\tilde{F}(\grave{e})}(s) &\preceq AbT_{\tilde{G}(\grave{e})}(s), \\ ReT_{\tilde{F}(\grave{e})}(s) &\preceq ReT_{\tilde{G}(\grave{e})}(s), \\ ReF_{\tilde{F}(\grave{e})}(s) &\succeq ReF_{\tilde{G}(\grave{e})}(s), \\ AbF_{\tilde{F}(\grave{e})}(s) &\succeq AbF_{\tilde{G}(\grave{e})}(s), \end{aligned}$$

for all $\grave{e} \in \acute{E}$ *and* $s \in X$*. If* $(\tilde{F}, \acute{E}) \tilde{\subseteq} (\tilde{G}, \acute{E})$ *and* $(\tilde{F}, \acute{E}) \tilde{\supseteq} (\tilde{G}, \acute{E})$*, then* $(\tilde{F}, \acute{E}) = (\tilde{G}, \acute{E})$*.*

Definition 12. *[62] Let* $(\tilde{F}, \acute{E})$ *and* $(\tilde{G}, \acute{E})$ *be two QPNSSs over the key set* X *such that* $(\tilde{F}, \acute{E}) \neq (\tilde{G}, \acute{E})$*. Then their union is denoted by* $(\tilde{F}, \acute{E}) \Cup (\tilde{G}, \acute{E}) = (\tilde{H}, \acute{E})$ *and is defined as:*

$$(\tilde{H}, \acute{E}) = \left\{ \left(\grave{e}, \langle s, AbT_{\tilde{H}(\grave{e})}(s), ReT_{\tilde{H}(\grave{e})}(s), ReF_{\tilde{H}(\grave{e})}(s), AbF_{\tilde{H}(\grave{e})}(s) \rangle : s \in X \right) : \grave{e} \in \acute{E} \right\}.$$

where

$$AbT_{\tilde{H}(\grave{e})}(s) = \max\left\{AbT_{\tilde{F}(\grave{e})}(s), AbT_{\tilde{G}(\grave{e})}(s)\right\},$$
$$ReT_{\tilde{H}(\grave{e})}(s) = \max\left\{ReT_{\tilde{F}(\grave{e})}(s), ReT_{\tilde{G}(\grave{e})}(s)\right\},$$
$$ReF_{\tilde{H}(\grave{e})}(s) = \min\left\{ReF_{\tilde{F}(\grave{e})}(s), ReF_{\tilde{G}(\grave{e})}(s)\right\},$$
$$AbF_{\tilde{H}(\grave{e})}(s) = \min\left\{AbF_{\tilde{F}(\grave{e})}(s), AbF_{\tilde{G}(\grave{e})}(s)\right\}.$$

Definition 13. *[62] Let $(\tilde{F},\acute{E})$ and $(\tilde{G},\acute{E})$ be two QPNSSs over the key set X such that $(\tilde{F},\acute{E}) \neq (\tilde{G},\acute{E})$. Then their intersection is denoted by $(\tilde{F},\acute{E}) \widetilde{\Cap} (\tilde{G},\acute{E}) = (\tilde{H},\acute{E})$ and is defined as:*

$$(\tilde{H},\acute{E}) = \left\{\left(\grave{e}, \langle s, AbT_{\tilde{H}(\grave{e})}(s), ReT_{\tilde{H}(\grave{e})}(s), ReF_{\tilde{H}(\grave{e})}(s), AbF_{\tilde{H}(\grave{e})}(s)\rangle : s \in X\right) : \grave{e} \in \acute{E}\right\}.$$

where

$$AbT_{\tilde{H}(\grave{e})}(s) = \min\left\{AbT_{\tilde{F}(\grave{e})}(s), AbT_{\tilde{G}(\grave{e})}(s)\right\},$$
$$ReT_{\tilde{H}(\grave{e})}(s) = \min\left\{ReT_{\tilde{F}(\grave{e})}(s), ReT_{\tilde{G}(\grave{e})}(s)\right\},$$
$$ReF_{\tilde{H}(\grave{e})}(s) = \max\left\{ReF_{\tilde{F}(\grave{e})}(s), ReF_{\tilde{G}(\grave{e})}(s)\right\},$$
$$AbF_{\tilde{H}(\grave{e})}(s) = \max\left\{AbF_{\tilde{F}(\grave{e})}(s), AbF_{\tilde{G}(\grave{e})}(s)\right\}.$$

Definition 14. *[62] Let $(\tilde{F},\acute{E})$ and $(\tilde{G},\acute{E})$ be two QPNSSs over the key set X such that $(\tilde{F},\acute{E}) \neq (\tilde{G},\acute{E})$. Then, their difference is $(\tilde{H},\acute{E}) = (\tilde{F},\acute{E}) \setminus (\tilde{G},\acute{E})$ and is defined as:*

$$(\tilde{H},\acute{E}) = (\tilde{F},\acute{E}) \Cap (\tilde{G},\acute{E})^c,$$

such that

$$(\tilde{H},\acute{E}) = \left\{\left(\grave{e}, \langle s, AbT_{\tilde{H}(\grave{e})}(s), ReT_{\tilde{H}(\grave{e})}(s), ReF_{\tilde{H}(\grave{e})}(s), AbF_{\tilde{H}(\grave{e})}(s)\rangle\right) : s \in X, \grave{e} \in \acute{E}\right\}.$$

where

$$AbT_{\tilde{H}(\grave{e})}(s) = \min\left\{AbT_{\tilde{F}(\grave{e})}(s), AbT_{\tilde{G}(\grave{e})}(s)\right\},$$
$$ReT_{\tilde{H}(\grave{e})}(s) = \min\left\{ReT_{\tilde{F}(\grave{e})}(s), ReT_{\tilde{G}(\grave{e})}(s)\right\},$$
$$ReF_{\tilde{H}(\grave{e})}(s) = \max\left\{ReF_{\tilde{F}(\grave{e})}(s), ReF_{\tilde{G}(\grave{e})}(s)\right\},$$
$$AbF_{\tilde{H}(\grave{e})}(s) = \max\left\{AbF_{\tilde{F}(\grave{e})}(s), AbF_{\tilde{G}(\grave{e})}(s)\right\}.$$

Definition 15. *[62] Let $(\tilde{F},\acute{E})$ and $(\tilde{G},\acute{E})$ be two QPNSSs over the key set X. Then, the "AND" operation on them is denoted by $(\tilde{F},\acute{E}) \wedge (\tilde{G},\acute{E}) = (\tilde{H},\acute{E}\times\acute{E})$ and is defined as:*

$$(\tilde{H},\acute{E}\times\acute{E}) = \Bigg\{\Big((\grave{e}_1,\grave{e}_2), \langle s, AbT_{\tilde{H}}(\grave{e}_1,\grave{e}_2)(s), ReT_{\tilde{H}}(\grave{e}_1,\grave{e}_2)(s),$$
$$ReF_{\tilde{H}}(\grave{e}_1,\grave{e}_2)(s), AbF_{\tilde{H}}(\grave{e}_1,\grave{e}_2)(s) : s \in X\rangle\Big) : (\grave{e}_1,\grave{e}_2) \in \acute{E}\times\acute{E}\Bigg\}.$$

where

$$AbT_{\tilde{H}(\grave{e})}(s) = \min\left\{AbT_{\tilde{F}(\grave{e})}(s), AbT_{\tilde{G}(\grave{e})}(s)\right\},$$

$$ReT_{\tilde{H}(\grave{e})}(s) = \min\left\{ReT_{\tilde{F}(\grave{e})}(s), ReT_{\tilde{G}(\grave{e})}(s)\right\},$$
$$ReF_{\tilde{H}(\grave{e})}(s) = \max\left\{ReF_{\tilde{F}(\grave{e})}(s), ReF_{\tilde{G}(\grave{e})}(s)\right\},$$
$$AbF_{\tilde{H}(\grave{e})}(s) = \max\left\{AbF_{\tilde{F}(\grave{e})}(s), AbF_{\tilde{G}(\grave{e})}(s)\right\}.$$

Definition 16. *[62]Let $(\tilde{F}, \acute{E})$ and $(\tilde{G}, \acute{E})$ be two QPNSSs over the key set X. Then, the "OR" operation on them is denoted by $(\tilde{F}, \acute{E}) \vee (\tilde{G}, \acute{E}) = (\tilde{H}, \acute{E} \times \acute{E})$ and is defined as:*

$$(\tilde{H}, \acute{E} \times \acute{E}) = \Bigg\{ \Big((\grave{e}_1, \grave{e}_2), \langle s, AbT_{\tilde{H}(\grave{e}_1,\grave{e}_2)}(s), ReT_{\tilde{H}(\grave{e}_1,\grave{e}_2)}(s),$$
$$ReF_{\tilde{H}(\grave{e}_1,\grave{e}_2)}(s), AbF_{\tilde{H}(\grave{e}_1,\grave{e}_2)}(s) : s \in X \rangle \Big) : (\grave{e}_1, \grave{e}_2) \in \acute{E} \times \acute{E} \Bigg\}.$$

where

$$AbT_{\tilde{H}(\grave{e})}(s) = \max\left\{AbT_{\tilde{F}(\grave{e})}(s), AbT_{\tilde{G}(\grave{e})}(s)\right\},$$
$$ReT_{\tilde{H}(\grave{e})}(s) = \max\left\{ReT_{\tilde{F}(\grave{e})}(s), ReT_{\tilde{G}(\grave{e})}(s)\right\},$$
$$ReF_{\tilde{H}(\grave{e})}(s) = \min\left\{ReF_{\tilde{F}(\grave{e})}(s), ReF_{\tilde{G}(\grave{e})}(s)\right\},$$
$$AbF_{\tilde{H}(\grave{e})}(s) = \min\left\{AbF_{\tilde{F}(\grave{e})}(s), AbF_{\tilde{G}(\grave{e})}(s)\right\}.$$

3. Fundamental of Fermatean Double Valued Neutrosophic Soft Sets

This section is devoted to new of Fermatean double valued neutrosophic soft sets. The concept indeterminacy refers to the inherent uncertainty present in all aspects of life. Recognizing and incorporating this uncertainty makes research and scientific inquiry more realistic and sensitive by acknowledging the complex, often ambiguous nature of reality. In real-world scenarios, indeterminacy can sometimes exhibit a tendency toward truth?it holds more truth value than falsehood?yet it cannot be definitively classified as true. Conversely, it may lean more toward falsehood than truth, without being conclusively false. These meticulous cases require a more exact categorization to improve understanding and interpretation. To address this, indeterminacy can be classified into two distinct types:
1. Indeterminacy leaning toward truth (I_T) *or* $ReT(x)$**:** When the indeterminate state has a greater affinity with truth than falsehood but still falls short of being classified as true.
2. Indeterminacy leaning toward falsehood (I_F) *or* $ReF(x)$**:** When the indeterminate state is closer to falsehood than truth but cannot be conclusively labeled as false. This refined classification I_T *or* $ReT(x)$ and I_F *or* $ReF(x)$allows for a more accurate and detailed representation of uncertainty. It provides deeper insight into the spectrum of indeterminacy, enabling researchers and theorists to engage with complex phenomena more precisely and thoughtfully. In this section, the Fermatean double valued neutrosophic soft set, Fermatean double valued neutrosophic soft subset, FDVNSS null set and its example, FDVNSS absolute set and its example, FDVNSS complement and its example, FDVNSS union and its example, FDVNSS intersection and its example, several propositions, the FDVNSS ?AND? operator and its example, and the FDVNSS ?OR? operator and its examples are presented for a clearer understanding of these concepts.

Definition 17. *Let X be a non-empty set (universal). A Fermatean double valued neutrosophic soft set (DVNSS) is defined as*

$$\{x, T(x), ReT(x), ReF(x), F(x) : x \in X\},$$

Where $T(x), ReT(x), ReF(x), F(x) \in [0,1]$, $0 \leq (T(x))^3 + (ReT(x))^3 + (ReF(x))^3 + (F(x))^3 \leq 2$. *Then* $0 \leq (T(x))^3 + (ReF(x))^3 + (ReF(x))^3 + (F(x))^3 \leq 2$, *for all* $x \in X$. $T(x)$ *is the degree of true membership,* $ReT(x)$ *is the degree of relative truth membership,* $ReT(x$ *is the degree of relative false membership and* $F(x)$ *is the degree of false membership. Here* $T(x)$ *and* $F(x)$ *are dependent components and* $ReT(x)$ *and* $ReF(x)$ *is an independent component.*

Example 1. *Let* $X = \{x_1, x_2, x_3\}$ *to be set of three diseases. Consider a set of characteristics or symptoms* $E = \{e_1, e_2, e_3\}$, *where* $e_1 =$ *contagious,* $e_2 =$ *severity and* $e_3 =$ *treatment availability. We defined a function* $F : E \to FDVNSS(X)$ *as follows:*

$$(F,E) = \left\{ \begin{matrix} e_1 = \{\langle x_1, 0.2, 0.4, 0.2, 0.1\rangle, \langle x_2, 0.4, 0.1, 0.1, 0.6\rangle, \langle x_3, 0.3, 0.03, 0.07, 0.7\rangle\}, \\ e_2 = \{\langle x_1, 0.3, 0.2, 0.2, 0.5\rangle, \langle x_2, 0.2, 0.2, 0.1, 0.4\rangle, \langle x_3, 0.7, 0.4, 0.4, 0.2\rangle\}, \\ e_3 = \{\langle x_1, 0.6, 0.1, 0.1, 0.1\rangle, \langle x_2, 0.3, 0.3, 0.3, 0.9\rangle, \langle x_2, 0.1, 0.1, 0.1, 0.3\rangle\} \end{matrix} \right\}$$

$$F(e_1) = \{\langle x_1, 0.2, 0.4, 0.2, 0.1\rangle, \langle x_2, 0.4, 0.1, 0.1, 0.6\rangle, \langle x_3, 0.3, 0.03, 0.07, 0.7\rangle\};$$

$$F(e_2) = \{\langle x_1, 0.3, 0.2, 0.2, 0.5\rangle, \langle x_2, 0.2, 0.2, 0.1, 0.4\rangle, \langle x_3, 0.7, 0.4, 0.4, 0.2\rangle\};$$

$$F(e_3) = \{\langle x_1, 0.6, 0.1, 0.1, 0.1\rangle, \langle x_2, 0.3, 0.3, 0.3, 0.9\rangle, \langle x_3, 0.1, 0.1, 0.1, 0.3\rangle\}.$$

Then the FDVNSS over (X, E) *in matrix form can be expressed as:*

$$(F,E) = \begin{bmatrix} (\langle 0.2, 0.4, 0.2, 0.1\rangle), (\langle 0.4, 0.1, 0.1, 0.6\rangle), (\langle 0.3, 0.03, 0.07, 0.7\rangle) \\ (\langle 0.3, 0.2, 0.2, 0.5\rangle), (\langle 0.2, 0.2, 0.1, 0.4\rangle), (\langle 0.7, 0.4, 0.4, 0.2\rangle) \\ (\langle 0.6, 0.1, 0.1, 0.1\rangle), (\langle 0.3, 0.3, 0.3, 0.9\rangle), (\langle 0.1, 0.1, 0.1, 0.3\rangle) \end{bmatrix}.$$

Where i^{th} *row vector represents* $F(e_i)$, *the* i^{th} *column vector represent* x_i. *Its clearly that* $(0.2)^3 + (0.1)^3 \leq 1$ *and* $(0.2)^3 + (0.4)^3 + (0.2)^3 + (0.1)^3 \leq 2$ *which satisfy the FDVNSS condition.*

Definition 18. *Let* (F, E) *and* (G, E) *be two FDVNSS on* (X, E). *Then* (F, E)*is called to be a Fermatean double valued neutrosophic soft subset of* (G, E) *if* $F(e) \subseteq G(e)$, $\forall\ e \in E$, *here we say that* $(F, E) \subseteq (G, E))$.

Example 2. *Let* $X = \{x_1, x_2, x_3\}$ *to be set of three types of cancer diseases. Let* $E = \{e_1, e_2, e_3\}$ *be a set of medical parameters, where* $e_1 = early - stage\ diagnosis$, $e_2 = high\ mortality\ rate$ *and* $e_3 = responsive\ to\ treatment$. *Let* (F, E) *is FDVNSS over* (X, E) *defined as follows:*

$$F(e_1) = \{\langle x_1, 0.2, 0.2, 0.1, 0.2\rangle, \langle x_2, 0.4, 0.3, 0.3, 0.4\rangle, \langle x_3, 0.1, 0.1, 0.1, 0.9\rangle\};$$

$$F(e_2) = \{\langle x_1, 0.5, 0.1, 0.2, 0.8\rangle, \langle x_2, 0.7, 0.1, 0.2, 0.6\rangle, \langle x_3, 0.1, 0.1, 0.1, 0.6\rangle\};$$

$$F(e_3) = \{\langle x_1, 0.2, 0.2, 0.4, 0.3\rangle, \langle x_2, 0.3, 0.1, 0.4, 0.8\rangle, \langle x_3, 0.4, 0.1, 0.2, 0.7\rangle\}.$$

Let (G, E) *is FDVNSS over* (X, E) *defined as follows:*

$$G(e_1) = \{\langle x_1, 0.3, 0.2, 0.2, 0.1\rangle, \langle x_2, 0.6, 0.2, 0.5, 0.2\rangle, \langle x_3, 0.7, 0.1, 0.1, 0.8\rangle\};$$

$$G(e_2) = \{\langle x_1, 0.6, 0.2, 0.2, 0.7\rangle, \langle x_2, 0.9, 0.2, 0.2, 0.4\rangle, \langle x_3, 0.3, 0.4, 0.1, 0.3\rangle\};$$

$$G(e_3) = \{\langle x_1, 0.4, 0.2, 0.5, 0.1\rangle, \langle x_2, 0.6, 0.3, 0.3, 0.7\rangle, \langle x_3, 0.5, 0.2, 0.2, 0.5\rangle\}.$$

It follows that $(F, E) \subseteq (G, E)$.

Definition 19. *A FDVNSS is said to be a null FDVNSS, denoted by F_ϕ such that*

$$E_\phi = \{(e, \{x, 0, 0, 0, 1\}) : \ e \in E, \ c \in X\}.$$

Example 3. *Let $X = \{x_1, x_2, x_3\}$ to be set of three COVID-19 variants and $E = \{e_1, e_2, e_3\}$ be a set of medical characteristics, where $e_1 =$ high transmission rate, $e_2 =$ severe symptoms and $e_3 =$ resistant to treatment. We defined a function $F : E \to FDVNSS(X)$ which is a FDVNSS over (X, E) defined as follows:*

$$F(e_1) = \{\langle x_1, 0, 0, 0, 1\rangle, \langle x_2, 0, 0, 0, 1\rangle, \langle x_3, 0, 0, 0, 1\rangle\};$$

$$F(e_2) = \{\langle x_1, 0, 0, 0, 1\rangle, \langle x_2, 0, 0, 0, 1\rangle, \langle x_3, 0, 0, 0, 1\rangle\};$$

$$F(e_3) = \{\langle x_1, 0, 0, 0, 1\rangle, \langle x_2, 0, 0, 0, 1\rangle, \langle x_3, 0, 0, 0, 1\rangle\}.$$

$$(F, E) = \begin{bmatrix} (\langle 0,0,0,1\rangle), (\langle 0,0,0,1\rangle), (\langle 0,0,0,1\rangle) \\ (\langle 0,0,0,1\rangle), (\langle 0,0,0,1\rangle), (\langle 0,0,0,1\rangle) \\ (\langle 0,0,0,1\rangle), (\langle 0,0,0,1\rangle), (\langle 0,0,0,1\rangle) \end{bmatrix}.$$

This means (F, E) is null FDVNSS.

Definition 20. *A FDVNSS is said to be an absolute FDVNSS , denoted by X_E such that*

$$X_E = \{(e, \{x, 1, 0, 0, 0\}) : \ e \in E, \ x \in X\}.$$

Example 4. *Let $X = \{x_1, x_2, x_3\}$ to be set of three AIDS case. Let $E = \{e_1, e_2, e_3\}$ be a set of medical parameters, where $e_1 =$ low immunity level, $e_2 =$ presence of opportunistic infections and $e_3 =$ response to antiretroviral therapy. Let (F, E) be FDVNSS over (X, E) defined as follows:*

$$F(e_1) = \{\langle x_1, 1, 0, 0, 0\rangle, \langle x_2, 1, 0, 0, 0\rangle, \langle x_3, 1, 0, 0, 0\rangle\};$$

$$F(e_2) = \{\langle x_1, 1, 0, 0, 0\rangle, \langle x_2, 1, 0, 0, 0\rangle, \langle x_3, 1, 0, 0, 0\rangle\};$$

$$F(e_3) = \{\langle x_1, 1, 0, 0, 0\rangle, \langle x_2, 1, 0, 0, 0\rangle, \langle x_3, 1, 0, 0, 0\rangle\}.$$

$$(F, E) = \begin{bmatrix} (\langle 1,0,0,0\rangle), (\langle 1,0,0,0\rangle), (\langle 1,0,0,0\rangle) \\ (\langle 1,0,0,0\rangle), (\langle 1,0,0,0\rangle), (\langle 1,0,0,0\rangle) \\ (\langle 1,0,0,0\rangle), (\langle 1,0,0,0\rangle), (\langle 1,0,0,0\rangle) \end{bmatrix}.$$

This means (F, E) is absolute FDVNSS.

Definition 21. *Let (F, E) be a FDVNSS over (X, E). Then the complement of (F, E), denoted by $(F, E)^c$ is defined by $F^c(e) = D(e)$, $\forall\ e \in E$ where $D(e)$ denoted the Fermatean Double Valued Neutrosophic complement, where*

$$(\tilde{F}, E)^c = \{\langle x, F_{\tilde{F}(e)}(x), ReF_{\tilde{F}(e)}(x), ReT_{\tilde{F}(e)}(x), T_{\tilde{F}(e)}(x)\rangle : x \in X\}.$$

Obvious that $((\tilde{F}, E)^c)^c = (\tilde{F}, E)$.

Example 5. *Refer to the matrix form illustrated in Example 1.*

$$(F,E)=\begin{cases} e_1=\{\langle x_1,0.1,0.2,0.4,0.2\rangle,\langle x_2,0.6,0.1,0.1,0.4\rangle,\langle x_3,0.7,0.07,0.03,0.3\rangle\},\\ e_2=\{\langle x_1,0.5,0.2,0.2,0.3\rangle,\langle x_2,0.4,0.1,0.2,0.2\rangle,\langle x_3,0.2,0.4,0.4,0.7\rangle\},\\ e_3=\{\langle x_1,0.1,0.1,0.1,0.6\rangle,\langle x_2,0.9,0.3,0.3,0.3\rangle,\langle x_3,0.3,0.1,0.1,0.1\rangle\}\end{cases}\Bigg\}$$

By using the FDVNSS complement , we have $(F,E)^c=(G,E)$, where (G,E) is:

$$(G,E)=\begin{bmatrix} (\langle 0.2,0.4,0.2,0.1\rangle),(\langle 0.4,0.1,0.1,0.6\rangle),(\langle 0.3,0.03,0.07,0.7\rangle)\\ (\langle 0.3,0.2,0.2,0.5\rangle),(\langle 0.2,0.2,0.1,0.4\rangle),(\langle 0.7,0.4,0.4,0.2\rangle)\\ (\langle 0.6,0.1,0.1,0.1\rangle),(\langle 0.3,0.3,0.3,0.9\rangle),(\langle 0.1,0.1,0.1,0.3\rangle)\end{bmatrix}.$$

Definition 22. *The union of two FDVNSSs (F,E) and (G,E), denoted by $(F,E)\tilde{\cup}(G,E)$, is a $FDVNSS(H,E)$ such that $H:E\to FDVNSS(X)$ defined by $H(e)=(F(e)\tilde{\cup}G(e))$ is a Fermatean double valued neutrosophic soft union, where:*

$$(F,E)\tilde{\cup}(G,E)=\left\{x,(\max T_{\tilde{F}(e_1)}(x),\max ReT_{\tilde{F}(e_2)}(x),\min ReF_{\tilde{F}(e_1)}(x),\min F_{\tilde{F}(e_2)}(x))\right\}.$$

Example 6. *A universal set of objects: $X=\{x_1,x_2,x_3\}$ possibly individuals whose blood pressure is being analyzed. A set of parameters: $E=\{e_1,e_2,e_3\}$ likely representing different blood pressure characteristics, such as e_1 : Systolic pressure, e_2 : Diastolic pressure, e_3 : Pulse pressure or any relevant metric. Then the FDVNSS (F,E) is defined as follows:*

$$F(e_1)=\{\langle x_1,0.2,0.4,0.2,0.1\rangle),(\langle x_2,0.4,0.1,0.1,0.6\rangle,\langle x_3,0.3,0.03,0.07,0.7\rangle\};$$

$$F(e_2)=\{\langle x_1,0.3,0.2,0.2,0.5\rangle,\langle x_2,0.2,0.2,0.1,0.4\rangle,\langle x_3,0.7,0.4,0.4,0.2\rangle\};$$

$$F(e_3)=\{\langle x_1,0.6,0.1,0.1,0.1\rangle,\langle x_2,0.3,0.3,0.3,0.9\rangle,\langle x_3,0.1,0.1,0.1,0.3\rangle\}.$$

Let us define another FDVNSS (G,E) as follows:

$$G(e_1)=\{\langle x_1,0.3,0.1,0.1,0.1\rangle,\langle x_2,0.2,0.2,0.2,0.3\rangle,\langle x_3,0.3,0.1,0.1,0.2\rangle\};$$

$$G(e_2)=\{\langle x_1,0.2,0.2,0.1,0.3\rangle,\langle x_2,0.1,0.4,0.3,0.4\rangle,\langle x_3,0.1,0.1,0.1,0.3\rangle\};$$

$$G(e_3)=\{\langle x_1,0.4,0.2,0.1,0.4\rangle,\langle x_2,0.1,0.4,0.3,0.4\rangle,\langle x_3,0.1,0.5,0.4,0.3\rangle\}.$$

Then their FDVNSS union is given as follows:

$$(F,E)\tilde{\cup}(G,E)=\begin{cases} e_1=\{\langle x_1,0.3,0.4,0.1,0.1\rangle,\langle x_2,0.4,0.2,0.1,0.3\rangle,\langle x_3,0.3,0.1,0.07,0.2\rangle\},\\ e_2=\{\langle x_1,0.3,0.2,0.1,0.3\rangle,\langle x_2,0.2,0.4,0.1,0.4\rangle,\langle x_3,0.7,0.4,0.1,0.2\rangle\},\\ e_3=\{\langle x_1,0.6,0.2,0.1,0.1\rangle,\langle x_2,0.3,0.4,0.3,0.4\rangle,\langle x_3,0.1,0.5,0.1,0.3\rangle\}\end{cases}\Bigg\}$$

In matrix notation, we write it as follows:

$$(H,E)=\begin{bmatrix} (\langle 0.3,0.4,0.1,0.1\rangle),(\langle 0.4,0.2,0.1,0.3\rangle),(\langle 0.3,0.1,0.07,0.2\rangle)\\ (\langle 0.3,0.2,0.1,0.3\rangle),(\langle 0.2,0.4,0.1,0.4\rangle),(\langle 0.7,0.4,0.1,0.2\rangle)\\ (\langle 0.6,0.2,0.1,0.1\rangle),(\langle 0.3,0.4,0.3,0.4\rangle),(\langle 0.1,0.5,0.1,0.3\rangle)\end{bmatrix}.$$

Definition 23. *The intersection of two FDVNSSs (F,E) and (G,E), denoted by $(F,E)\tilde{\cap}(G,E)$, is a $FDVNSS(H,E)$ such that $H:E\to FDVNSS(X)$ defined by $H(e)=(F(e)\tilde{\cap}G(e))$ is a Fermatean double valued neutrosophic soft intersection, where:*

$$(F,E)\tilde{\cap}(G,E)=\left\{x,(\min T_{\tilde{F}(e_1)}(x),\min ReT_{\tilde{F}(e_2)}(x),\max ReF_{\tilde{F}(e_1)}(x),\max F_{\tilde{F}(e_2)}(x))\right\}.$$

Example 7. *Let us consider Example 1.*

$$F(e_1) = \{\langle x_1, 0.2, 0.4, 0.2, 0.1\rangle, \langle x_2, 0.4, 0.1, 0.1, 0.6\rangle, \langle x_3, 0.3, 0.03, 0.07, 0.7\rangle\};$$

$$F(e_2) = \{\langle x_1, 0.3, 0.2, 0.2, 0.5\rangle, \langle x_2, 0.2, 0.2, 0.1, 0.4\rangle, \langle x_3, 0.7, 0.4, 0.4, 0.2\rangle\};$$

$$F(e_3) = \{\langle x_1, 0.6, 0.1, 0.1, 0.1\rangle, \langle x_2, 0.3, 0.3, 0.3, 0.9\rangle, \langle x_3, 0.1, 0.1, 0.1, 0.3\rangle\}.$$

Let us define another FDVNSS (G, E) as follows:

$$G(e_1) = \{\langle x_1, 0.3, 0.1, 0.1, 0.1\rangle, \langle x_2, 0.2, 0.2, 0.2, 0.3\rangle, \langle x_3, 0.3, 0.1, 0.1, 0.2\rangle\};$$

$$G(e_2) = \{\langle x_1, 0.2, 0.2, 0.1, 0.3\rangle, \langle x_2, 0.1, 0.4, 0.3, 0.4\rangle, \langle x_3, 0.1, 0.1, 0.1, 0.3\rangle\};$$

$$G(e_3) = \{\langle x_1, 0.4, 0.2, 0.1, 0.4\rangle, \langle x_2, 0.1, 0.4, 0.3, 0.4\rangle, \langle x_3, 0.1, 0.5, 0.4, 0.3\rangle\}.$$

Then their FDVNSS intersection is given as follows:

$$(F, E)\tilde{\cap}(G, E) = \left\{ \begin{matrix} e_1 = \{\langle x_1, 0.2, 0.1, 0.2, 0.1\rangle, \langle x_2, 0.2, 0.1, 0.2, 0.6\rangle, \langle x_3, 0.3, 0.03, 0.1, 0.7\rangle\}, \\ e_2 = \{\langle x_1, 0.2, 0.2, 0.2, 0.5\rangle, \langle x_2, 0.1, 0.2, 0.3, 0.4\rangle, \langle x_3, 0.1, 0.1, 0.4, 0.3\rangle\}, \\ e_3 = \{\langle x_1, 0.4, 0.1, 0.1, 0.4\rangle, \langle x_2, 0.1, 0.3, 0.3, 0.9\rangle, \langle x_3, 0.1, 0.1, 0.4, 0.3\rangle\} \end{matrix} \right\}$$

In matrix notation, we write it as follows:

$$(H, E) = \begin{bmatrix} (\langle 0.2, 0.1, 0.2, 0.1\rangle), (\langle 0.2, 0.1, 0.2, 0.6\rangle), (\langle 0.3, 0.03, 0.1, 0.7\rangle) \\ (\langle 0.2, 0.2, 0.2, 0.5\rangle), (\langle 0.1, 0.2, 0.3, 0.4\rangle), (\langle 0.1, 0.1, 0.4, 0.3\rangle) \\ (\langle 0.4, 0.1, 0.1, 0.4\rangle), (\langle 0.1, 0.3, 0.3, 0.9\rangle), (\langle 0.1, 0.1, 0.4, 0.3\rangle) \end{bmatrix}.$$

Proposition 1. *Let(F, E), (G, E) and (H, E) are three FDVNSSs over (X, E) then the following holds: 1.*

$$(F, E)\tilde{\cup}(G, E) = (G, E)\tilde{\cup}(F, E)$$

2.

$$(F, E)\tilde{\cap}(G, E) = (G, E)\tilde{\cap}(F, E)$$

3.

$$(F, E)\tilde{\cup}((G, E)\tilde{\cup}(H, E)) = ((F, E)\tilde{\cup}(G, E))\tilde{\cup}(H, E)$$

4.

$$(F, E)\tilde{\cap}((G, E)\tilde{\cap}(H, E)) = ((F, E)\tilde{\cap}(G, E))\tilde{\cap}(H, E)$$

Proof. Obvious.

Proposition 2. *Let(F, E) and (G, E) are two FDVNSSs over (X, E) then the following holds: 1.*

$$((F, E)\tilde{\cup}(G, E))^c = ((F, E))^c\tilde{\cup}((G, E))^c$$

2.

$$((F, E)\tilde{\cap}(G, E))^c = ((F, E))^c\tilde{\cap}((G, E))^c$$

Proof. 1. $((F, E)\tilde{\cup}(G, E))^c = ((F, E))^c\tilde{\cup}((G, E))^c$.
2. $((F, E)\tilde{\cap}(G, E))^c = ((F, E))^c\tilde{\cap}((G, E))^c$.

Definition 24. *(F,A) and (G,B) are two FDVNSSs then "$(F,A)AND(G,B)$", defined by $(F,A)\wedge(G,B)$ is denoted by $(F,A)\wedge(G,B)=(H,A\times B)$. Where $H(\alpha,\beta)=(H(e_1,e_2)(x))$, for all $(e_1,e_2)\in A\times B$, such that*

$$H(e_1,e_2)=(F(e_1)\tilde{\cap}G(e_2)),\ \forall\ (e_1,e_2)\in A\times B,$$

where $\tilde{\cap}$ is Fermatean Double Valued neutrosophic soft intersection.

Example 8. *Assume the universe consists of three malaria patients, that is, $X=\{x_1,x_2,x_2\}$ and there are three parameters, $E=\{e_1,e_2,e_3\}$, which describe their health conditions based on specific malaria-related factors (e.g., fever level, parasite count, and response to medication). Suppose Dr. X wants to evaluate a patient based on only one of these parameters. Let there be two observations, (F,A) and (G,A) made by two experts, defined as follows:*

$$(F,E)\wedge(G,E)=\left\{x,(\min T_{\tilde{F}(e_1)}(x),\min ReT_{\tilde{F}(e_2)}(x),\max ReF_{\tilde{F}(e_1)}(x),\max F_{\tilde{F}(e_2)}(x))\right\}.$$

We have

$$F(e_1)=\{\langle x_1,0.2,0.4,0.2,0.1\rangle,\langle x_2,0.4,0.1,0.1,0.6\rangle,\langle x_3,0.3,0.03,0.07,0.7\rangle\};$$

$$F(e_2)=\{\langle x_1,0.3,0.2,0.2,0.5\rangle,\langle x_2,0.2,0.2,0.1,0.4\rangle,\langle x_3,0.7,0.4,0.4,0.2\rangle\};$$

$$F(e_3)=\{\langle x_1,0.6,0.1,0.1,0.1\rangle,\langle x_2,0.3,0.3,0.3,0.9\rangle,\langle x_3,0.1,0.1,0.1,0.3\rangle\}.$$

and

$$G(e_1)=\{\langle x_1,0.3,0.1,0.1,0.1\rangle,\langle x_2,0.2,0.2,0.2,0.3\rangle,\langle x_3,0.3,0.1,0.1,0.2\rangle\};$$

$$G(e_2)=\{\langle x_1,0.2,0.2,0.1,0.3\rangle,\langle x_2,0.1,0.4,0.3,0.4\rangle,\langle x_3,0.1,0.1,0.1,0.3\rangle\};$$

$$G(e_3)=\{\langle x_1,0.4,0.2,0.1,0.4\rangle,\langle x_2,0.1,0.4,0.3,0.4\rangle,\langle x_3,0.1,0.5,0.4,0.3\rangle\}.$$

So,

$$(F,E)\wedge(G,E)=\left\{\begin{matrix}(e_1,e_1)=\{\langle x_1,0.2,0.1,0.2,0.1\rangle,\langle x_2,0.2,0.1,0.2,0.6\rangle,\langle x_3,0.3,0.03,0.1,0.7\rangle\},\\(e_1,e_2)=\{\langle x_1,0.2,0.2,0.2,0.3\rangle,\langle x_2,0.1,0.03,0.1,0.6\rangle,\langle x_3,0.1,0.03,0.1,0.7\rangle\},\\(e_1,e_3)=\{\langle x_1,0.2,0.2,0.2,0.4\rangle,\langle x_2,0.1,0.1,0.3,0.6\rangle,\langle x_3,0.1,0.03,0.4,0.7\rangle\}\end{matrix}\right\}$$

$$=\left\{\begin{matrix}(e_2,e_1)=\{\langle x_1,0.3,0.1,0.2,0.5\rangle,\langle x_2,0.2,0.2,0.2,0.4\rangle,\langle x_3,0.3,0.1,0.2,0.4\rangle\},\\(e_2,e_2)=\{\langle x_1,0.2,0.2,0.2,0.5\rangle,\langle x_2,0.1,0.2,0.3,0.4\rangle,\langle x_3,0.1,0.1,0.4,0.3\rangle\},\\(e_2,e_3)=\{\langle x_1,0.3,0.2,0.2,0.5\rangle,\langle x_2,0.1,0.2,0.3,0.4\rangle,\langle x_3,0.2,0.2,0.2,0.5\rangle\}\end{matrix}\right\}$$

$$=\left\{\begin{matrix}(e_3,e_1)=\{\langle x_1,0.3,0.1,0.1,0.1\rangle,\langle x_2,0.2,0.2,0.3,0.9\rangle,\langle x_3,0.1,0.1,0.1,0.3\rangle\},\\(e_3,e_2)=\{\langle x_1,0.2,0.1,0.1,0.3\rangle,\langle x_2,0.1,0.03,0.3,0.9\rangle,\langle x_3,0.1,0.1,0.1,0.3\rangle\},\\(e_3,e_3)=\{\langle x_1,0.4,0.1,0.1,0.4\rangle,\langle x_2,0.1,0.3,0.3,0.9\rangle,\langle x_3,0.1,0.1,0.4,0.3\rangle\}\end{matrix}\right\}$$

In matrix notation, we write it as follows:

$$(H,E)=\begin{bmatrix}(\langle 0.2,0.1,0.2,0.1\rangle),(\langle 0.2,0.1,0.2,0.6\rangle),(\langle 0.3,0.03,0.1,0.7\rangle)\\(\langle 0.2,0.2,0.2,0.3\rangle),(\langle 0.1,0.03,0.1,0.6\rangle),(\langle 0.1,0.03,0.1,0.7\rangle)\\(\langle 0.2,0.2,0.2,0.4\rangle),(\langle 0.1,0.1,0.3,0.6\rangle),(\langle 0.1,0.03,0.4,0.7\rangle)\\(\langle 0.3,0.1,0.2,0.5\rangle),(\langle 0.2,0.2,0.2,0.4\rangle),(\langle 0.3,0.1,0.2,0.4\rangle)\\(\langle 0.2,0.2,0.2,0.5\rangle),(\langle 0.1,0.2,0.3,0.4\rangle),(\langle 0.1,0.1,0.4,0.3\rangle)\\(\langle 0.3,0.2,0.2,0.5\rangle),(\langle 0.1,0.2,0.3,0.4\rangle),(\langle 0.2,0.2,0.2,0.5\rangle)\\(\langle 0.3,0.1,0.1,0.1\rangle),(\langle 0.2,0.2,0.3,0.9\rangle),(\langle 0.1,0.1,0.1,0.3\rangle)\\(\langle 0.2,0.1,0.1,0.3\rangle),(\langle 0.1,0.03,0.3,0.9\rangle),(\langle 0.1,0.1,0.1,0.3\rangle)\\(\langle 0.4,0.1,0.1,0.4\rangle),(\langle 0.1,0.3,0.3,0.9\rangle),(\langle 0.1,0.1,0.4,0.3\rangle)\end{bmatrix}.$$

Definition 25. *(F, A) and (G, B) are two FDVNSSs then "$(F, A)OR(G, B)$", defined by $(F, A)\vee (G, B)$ is denoted by $(F, A) \vee (G, B) = (H, A \times B)$. Where $H(\alpha, \beta) = (H(e_1, e_2)(x))$, for all $(e_1, e_2) \in A \times B$, such that*

$$H(e_1, e_2) = (F(e_1)\tilde{\cup}G(e_2)),\ \forall\ (e_1, e_2) \in A \times B,$$

Example 9. *Let us consider Example 1.*

$$F(e_1) = \{\langle x_1, 0.2, 0.4, 0.2, 0.1\rangle, \langle x_2, 0.4, 0.1, 0.1, 0.6\rangle, \langle x_3, 0.3, 0.03, 0.07, 0.7\rangle\};$$

$$F(e_2) = \{\langle x_1, 0.3, 0.2, 0.2, 0.5\rangle, \langle x_2, 0.2, 0.2, 0.1, 0.4\rangle, \langle x_3, 0.7, 0.4, 0.4, 0.2\rangle\};$$

$$F(e_3) = \{\langle x_1, 0.6, 0.1, 0.1, 0.1\rangle, \langle x_2, 0.3, 0.3, 0.3, 0.9\rangle, \langle x_3, 0.1, 0.1, 0.1, 0.3\rangle\}.$$

and

$$G(e_1) = \{\langle x_1, 0.3, 0.1, 0.1, 0.1\rangle, \langle x_2, 0.2, 0.2, 0.2, 0.3\rangle, \langle x_3, 0.3, 0.1, 0.1, 0.2\rangle\};$$

$$G(e_2) = \{\langle x_1, 0.2, 0.2, 0.1, 0.3\rangle, \langle x_2, 0.1, 0.4, 0.3, 0.4\rangle, \langle x_3, 0.1, 0.1, 0.1, 0.3\rangle\};$$

$$G(e_3) = \{\langle x_1, 0.4, 0.2, 0.1, 0.4\rangle, \langle x_2, 0.1, 0.4, 0.3, 0.4\rangle, \langle x_3, 0.1, 0.5, 0.4, 0.3\rangle\}.$$

So,

$$(F, E)\vee(G, E) = \left\{\begin{matrix} (e_1, e_1) = \{\langle x_1, 0.3, 0.4, 0.1, 0.1\rangle, \langle x_2, 0.4, 0.2, 0.1, 0.3\rangle, \langle x_3, 0.3, 0.1, 0.07, 0.2\rangle\}, \\ (e_1, e_2) = \{\langle x_1, 0.2, 0.4, 0.1, 0.1\rangle, \langle x_2, 0.4, 0.1, 0.07, 0.2\rangle, \langle x_3, 0.3, 0.1, 0.07, 0.3\rangle\}, \\ (e_1, e_3) = \{\langle x_1, 0.4, 0.4, 0.1, 0.1\rangle, \langle x_2, 0.4, 0.4, 0.1, 0.4\rangle, \langle x_3, 0.3, 0.5, 0.07, 0.3\rangle\} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} (e_2, e_1) = \{\langle x_1, 0.3, 0.2, 0.1, 0.1\rangle, \langle x_2, 0.2, 0.2, 0.1, 0.3\rangle, \langle x_3, 0.7, 0.4, 0.1, 0.2\rangle\}, \\ (e_2, e_2) = \{\langle x_1, 0.3, 0.2, 0.1, 0.3\rangle, \langle x_2, 0.2, 0.2, 0.07, 0.2\rangle, \langle x_3, 0.3, 0.1, 0.07, 0.3\rangle\}, \\ (e_2, e_3) = \{\langle x_1, 0.4, 0.2, 0.1, 0.4\rangle, \langle x_2, 0.2, 0.4, 0.1, 0.4\rangle, \langle x_3, 0.7, 0.5, 0.4, 0.2\rangle\} \end{matrix}\right\}$$

$$= \left\{\begin{matrix} (e_3, e_1) = \{\langle x_1, 0.6, 0.1, 0.1, 0.1\rangle, \langle x_2, 0.3, 0.3, 0.2, 0.3\rangle, \langle x_3, 0.3, 0.1, 0.1, 0.2\rangle\}, \\ (e_3, e_2) = \{\langle x_1, 0.6, 0.2, 0.1, 0.1\rangle, \langle x_2, 0.3, 0.3, 0.07, 0.2\rangle, \langle x_3, 0.1, 0.1, 0.1, 0.3\rangle\}, \\ (e_3, e_3) = \{\langle x_1, 0.6, 0.2, 0.1, 0.1\rangle, \langle x_2, 0.3, 0.4, 0.3, 0.4\rangle, \langle x_3, 0.1, 0.5, 0.1, 0.3\rangle\} \end{matrix}\right\}$$

In matrix notation, we write it as follows:

$$(H, E) = \begin{bmatrix} (\langle 0.3, 0.4, 0.1, 0.1\rangle), (\langle 0.4, 0.2, 0.1, 0.3\rangle), (\langle 0.3, 0.03, 0.1, 0.7\rangle) \\ (\langle 0.2, 0.4, 0.1, 0.1\rangle), (\langle 0.14, 0.1, 0.07, 0.2\rangle), (\langle 0.3, 0.1, 0.07, 0.3\rangle) \\ (\langle 0.4, 0.4, 0.1, 0.1\rangle), (\langle 0.4, 0.4, 0.1, 0.4\rangle), (\langle 0.3, 0.5, 0.07, 0.3\rangle) \\ (\langle 0.3, 0.2, 0.1, 0.1\rangle), (\langle 0.2, 0.2, 0.1, 0.3\rangle), (\langle 0.7, 0.4, 0.1, 0.2\rangle) \\ (\langle 0.3, 0.2, 0.1, 0.3\rangle), (\langle 0.2, 0.2, 0.07, 0.2\rangle), (\langle 0.3, 0.1, 0.07, 0.3\rangle) \\ (\langle 0.4, 0.2, 0.1, 0.4\rangle), (\langle 0.2, 0.4, 0.1, 0.4\rangle), (\langle 0.7, 0.4, 0.4, 0.2\rangle) \\ (\langle 0.6, 0.1, 0.1, 0.1\rangle), (\langle 0.3, 0.3, 0.2, 0.3\rangle), (\langle 0.3, 0.1, 0.1, 0.2\rangle) \\ (\langle 0.6, 0.2, 0.1, 0.1\rangle), (\langle 0.3, 0.3, 0.07, 0.2\rangle), (\langle 0.1, 0.1, 0.1, 0.3\rangle) \\ (\langle 0.6, 0.2, 0.1, 0.1\rangle), (\langle 0.3, 0.4, 0.3, 0.4\rangle), (\langle 0.1, 0.5, 0.1, 0.3\rangle) \end{bmatrix}.$$

Example 10. *Assume the universe consists of three tuberculosis (TB) patients, that is,* $X = \{x_1, x_2, x_2\}$ *and there are three parameters,* $E = \{e_1, e_2, e_3\}$ *is a set of parameters representing medical indicators of TB severity, such as:* $e_1 : Chest\ X{-}ray\ findings$, $e_2 : Sputum\ smear\ results$, $e_3 : Clinical\ symptoms\ (e.g. fever\ weight\ loss\ cough)$. *Suppose Dr. X wants to evaluate a patient based on only one of these parameters. Let there be two observations,* (F, A) *and* (G, A) *made by two experts, defined as follows:*

$$F(e_1) = \{\langle x_1, 0.2, 0.4, 0.2, 0.1\rangle, \langle x_2, 0.4, 0.1, 0.1, 0.6\rangle, \langle x_3, 0.3, 0.03, 0.07, 0.7\rangle\};$$

$$F(e_2) = \{\langle x_1, 0.3, 0.2, 0.2, 0.5\rangle, \langle x_2, 0.2, 0.2, 0.1, 0.4\rangle, \langle x_3, 0.7, 0.4, 0.4, 0.2\rangle\};$$

$$F(e_3) = \{\langle x_1, 0.6, 0.1, 0.1, 0.1\rangle, \langle x_2, 0.3, 0.3, 0.3, 0.9\rangle, \langle x_3, 0.1, 0.1, 0.1, 0.3\rangle\}.$$

and

$$G(e_1) = \{\langle x_1, 0.3, 0.1, 0.1, 0.1\rangle, \langle x_2, 0.2, 0.2, 0.2, 0.3\rangle, \langle x_3, 0.3, 0.1, 0.1, 0.2\rangle\};$$

$$G(e_2) = \{\langle x_1, 0.2, 0.2, 0.1, 0.3\rangle, \langle x_2, 0.1, 0.4, 0.3, 0.4\rangle, \langle x_3, 0.1, 0.1, 0.1, 0.3\rangle\};$$

$$G(e_3) = \{\langle x_1, 0.4, 0.2, 0.1, 0.4\rangle, \langle x_2, 0.1, 0.4, 0.3, 0.4\rangle, \langle x_3, 0.1, 0.5, 0.4, 0.3\rangle\}.$$

So,

$$(F,E)\wedge(G,E) = \left\{\begin{array}{l} (e_1, e_1) = \{\langle x_1, 0.2, 0.1, 0.2, 0.1\rangle, \langle x_2, 0.2, 0.1, 0.2, 0.6\rangle, \langle x_3, 0.3, 0.03, 0.1, 0.7\rangle\}, \\ (e_1, e_2) = \{\langle x_1, 0.2, 0.2, 0.2, 0.3\rangle, \langle x_2, 0.1, 0.03, 0.1, 0.6\rangle, \langle x_3, 0.1, 0.03, 0.1, 0.7\rangle\}, \\ (e_1, e_3) = \{\langle x_1, 0.2, 0.2, 0.2, 0.4\rangle, \langle x_2, 0.1, 0.1, 0.3, 0.6\rangle, \langle x_3, 0.1, 0.03, 0.4, 0.7\rangle\} \end{array}\right\}$$

$$= \left\{\begin{array}{l} (e_2, e_1) = \{\langle x_1, 0.3, 0.1, 0.2, 0.5\rangle, \langle x_2, 0.2, 0.2, 0.2, 0.4\rangle, \langle x_3, 0.3, 0.1, 0.2, 0.4\rangle\}, \\ (e_2, e_2) = \{\langle x_1, 0.2, 0.2, 0.2, 0.5\rangle, \langle x_2, 0.1, 0.2, 0.3, 0.4\rangle, \langle x_3, 0.1, 0.1, 0.4, 0.3\rangle\}, \\ (e_2, e_3) = \{\langle x_1, 0.3, 0.2, 0.2, 0.5\rangle, \langle x_2, 0.1, 0.2, 0.3, 0.4\rangle, \langle x_3, 0.2, 0.2, 0.2, 0.5\rangle\} \end{array}\right\}$$

$$= \left\{\begin{array}{l} (e_3, e_1) = \{\langle x_1, 0.3, 0.1, 0.1, 0.1\rangle, \langle x_2, 0.2, 0.2, 0.3, 0.9\rangle, \langle x_3, 0.1, 0.1, 0.1, 0.3\rangle\}, \\ (e_3, e_2) = \{\langle x_1, 0.2, 0.1, 0.1, 0.3\rangle, \langle x_2, 0.1, 0.03, 0.3, 0.9\rangle, \langle x_3, 0.1, 0.1, 0.1, 0.3\rangle\}, \\ (e_3, e_3) = \{\langle x_1, 0.4, 0.1, 0.1, 0.4\rangle, \langle x_2, 0.1, 0.3, 0.3, 0.9\rangle, \langle x_3, 0.1, 0.1, 0.4, 0.3\rangle\} \end{array}\right\}$$

In matrix notation, we write it as follows:

$$(H,E) = \begin{bmatrix} (\langle 0.2, 0.1, 0.2, 0.1\rangle), (\langle 0.2, 0.1, 0.2, 0.6\rangle), (\langle 0.3, 0.03, 0.1, 0.7\rangle) \\ (\langle 0.2, 0.2, 0.2, 0.3\rangle), (\langle 0.1, 0.03, 0.1, 0.6\rangle), (\langle 0.1, 0.03, 0.1, 0.7\rangle) \\ (\langle 0.2, 0.2, 0.2, 0.4\rangle), (\langle 0.1, 0.1, 0.3, 0.6\rangle), (\langle 0.1, 0.03, 0.4, 0.7\rangle) \\ (\langle 0.3, 0.1, 0.2, 0.5\rangle), (\langle 0.2, 0.2, 0.2, 0.4\rangle), (\langle 0.3, 0.1, 0.2, 0.4\rangle) \\ (\langle 0.2, 0.2, 0.2, 0.5\rangle), (\langle 0.1, 0.2, 0.3, 0.4\rangle), (\langle 0.1, 0.1, 0.4, 0.3\rangle) \\ (\langle 0.3, 0.2, 0.2, 0.5\rangle), (\langle 0.1, 0.2, 0.3, 0.4\rangle), (\langle 0.2, 0.2, 0.2, 0.5\rangle) \\ (\langle 0.3, 0.1, 0.1, 0.1\rangle), (\langle 0.2, 0.2, 0.3, 0.9\rangle), (\langle 0.1, 0.1, 0.1, 0.3\rangle) \\ (\langle 0.2, 0.1, 0.1, 0.3\rangle), (\langle 0.1, 0.03, 0.3, 0.9\rangle), (\langle 0.1, 0.1, 0.1, 0.3\rangle) \\ (\langle 0.4, 0.1, 0.1, 0.4\rangle), (\langle 0.1, 0.3, 0.3, 0.9\rangle), (\langle 0.1, 0.1, 0.4, 0.3\rangle) \end{bmatrix}.$$

4. Fermatean Double Valued Neutrosophic Soft Set Topological Space

The notion of a Fermatean Double Valued Neutrosophic Soft Topological Space (FDVNSTS) is introduced in this section. The concepts of semi-open (s-open), pre-open (p-open), and b-open sets are defined within the context of FDVNSTS. Among these generalized open sets, the pre-open set is selected for further exploration, and several fundamental topological notions are developed based on this definition. These include the closure, exterior, boundary, and interior in FDVNSTS.

Definition 26. *Let FDVNSS $(\tilde{X}, \acute{E})$ be the family of all FDVNSSs, and let $\tau \subset FDVNSS(\tilde{X}, \acute{E})$, then τ is a Fermatean double-valued neutrosophic soft topology (FDVNST) on $\tilde{X}$ if:*

(i) $0_{(\tilde{X},\acute{E})}, 1_{(\tilde{X},\acute{E})} \in \tau$,

(ii) The union of any number of FDVNSSs in τ belongs to τ,

(iii) The intersection of a finite number of FDVNSSs in τ belongs to τ.

Then, $(\tilde{X}, \tau, \acute{E})$ is said to be a Fermatean double-valued neutrosophic soft topological space (FDVNSTS) over $\tilde{X}$.

Example 11. *A universal set of objects: $X = \{x_1, x_2, x_3\}$ possibly individuals whose blood pressure is being analyzed. A set of parameters: $E = \{e_1, e_2, e_3\}$ likely representing different blood pressure characteristics, such as e_1 : Systolic pressure, e_2 : Diastolic pressure, e_3 : Pulse pressure or any relevant metric. Then*

$$\tau^{FDVNSS} = \left\{0_{(X,\acute{E})}, 1_{(X,\acute{E})}, (F,E), (G,E), (H,E)\right\}$$

defines Fermatean double valued neutrosophic soft topology over X. Here, the Fermatean double valued neutrosophic soft sets (F,E), (G,E), (H,E) over X are defined as following: Then their FDVNSS is

$$(F,E) = \begin{Bmatrix} e_1 = \{\langle x_1, 0.6, 0.7, 0.3, 0.2\rangle, \langle x_2, 0.6, 0.7, 0.8, 0.8\rangle, \langle x_3, 0.6, 0.7, 0.8, 0.1\rangle\}, \\ e_2 = \{\langle x_1, 0.7, 0.8, 0.8, 0.8\rangle, \langle x_2, 0.7, 0.8, 0.8, 0.9\rangle, \langle x_3, 0.7, 0.8, 0.8, 0.8\rangle\} \end{Bmatrix}$$

$$(G,E) = \begin{Bmatrix} e_1 = \{\langle x_1, 0.5, 0.5, 0.8, 0.9\rangle, \langle x_2, 0.4, 0.4, 0.9, 0.9\rangle, \langle x_3, 0.2, 0.3, 0.9, 0.1\rangle\}, \\ e_2 = \{\langle x_1, 0.4, 0.5, 0.9, 0.8\rangle, \langle x_2, 0.6, 0.5, 0.9, 0.9\rangle, \langle x_3, 0.2, 0.7, 0.9, 0.8\rangle\} \end{Bmatrix}$$

$$(H,E) = \begin{Bmatrix} e_1 = \{\langle x_1, 0.2, 0.3, 0.9, 0.9\rangle, \langle x_2, 0.0, 0.0, 0.9, 0.9\rangle, \langle x_3, 0.0, 0.0, 0.9, 0.9\rangle\}, \\ e_2 = \{\langle x_1, 0.4, 0.5, 0.9, 0.8\rangle, \langle x_2, 0.6, 0.4, 0.9, 0.9\rangle, \langle x_3, 0.2, 0.7, 0.9, 0.8\rangle\} \end{Bmatrix}$$

Let us define FDVNSS (F,E) as follows:

$$F(e_1) = \{\langle x_1, 0.6, 0.7, 0.3, 0.2\rangle, \langle x_2, 0.6, 0.7, 0.8, 0.8\rangle, \langle x_3, 0.6, 0.7, 0.8, 0.1\rangle\};$$

$$F(e_2) = \{\langle x_1, 0.7, 0.8, 0.8, 0.8\rangle, \langle x_2, 0.7, 0.8, 0.8, 0.9\rangle, \langle x_3, 0.7, 0.8, 0.8, 0.8\rangle\};$$

Let us define FDVNSS (G,E) as follows:

$$G(e_1) = \{\langle x_1, 0.5, 0.5, 0.8, 0.9\rangle, \langle x_2, 0.4, 0.4, 0.9, 0.9\rangle, \langle x_3, 0.2, 0.3, 0.9, 0.1\rangle\};$$

$$G(e_2) = \{\langle x_1, 0.4, 0.5, 0.9, 0.8\rangle, \langle x_2, 0.6, 0.5, 0.9, 0.9\rangle, \langle x_3, 0.2, 0.7, 0.9, 0.8\rangle\};$$

Let us define another FDVNSS (H, E) as follows:

$$H(e_1) = \{\langle x_1, 0.2, 0.3, 0.9, 0.9\rangle, \langle x_2, 0.0, 0.0, 0.9, 0.9\rangle, \langle x_3, 0.0, 0.0, 0.9, 0.9\rangle\};$$

$$H(e_2) = \{\langle x_1, 0.4, 0.5, 0.9, 0.8\rangle, \langle x_2, 0.6, 0.4, 0.9, 0.9\rangle, \langle x_3, 0.2, 0.7, 0.9, 0.8\rangle\};$$

Then their FDVNSS union is given as follows:

$$(F,E)\tilde{\cup}(G,E) = (F,E) = \begin{Bmatrix} e_1 = \{\langle x_1, 0.6, 0.7, 0.3, 0.2\rangle, \langle x_2, 0.6, 0.7, 0.8, 0.8\rangle, \langle x_3, 0.6, 0.7, 0.8, 0.1\rangle\}, \\ e_2 = \{\langle x_1, 0.7, 0.8, 0.8, 0.8\rangle, \langle x_2, 0.7, 0.8, 0.8, 0.9\rangle, \langle x_3, 0.7, 0.8, 0.8, 0.8\rangle\} \end{Bmatrix}$$

$$(F,E)\tilde{\cup}(H,E) = (F,E) = \begin{Bmatrix} e_1 = \{\langle x_1, 0.6, 0.7, 0.3, 0.2\rangle, \langle x_2, 0.6, 0.7, 0.8, 0.8\rangle, \langle x_3, 0.6, 0.7, 0.8, 0.1\rangle\}, \\ e_2 = \{\langle x_1, 0.7, 0.8, 0.8, 0.8\rangle, \langle x_2, 0.7, 0.8, 0.8, 0.9\rangle, \langle x_3, 0.7, 0.8, 0.8, 0.8\rangle\} \end{Bmatrix}$$

$$(H,E)\tilde{\cup}(G,E) = (G,E) = \begin{Bmatrix} e_1 = \{\langle x_1, 0.5, 0.5, 0.8, 0.9\rangle, \langle x_2, 0.4, 0.4, 0.9, 0.9\rangle, \langle x_3, 0.2, 0.3, 0.9, 0.1\rangle\}, \\ e_2 = \{\langle x_1, 0.4, 0.5, 0.9, 0.8\rangle, \langle x_2, 0.6, 0.5, 0.9, 0.9\rangle, \langle x_3, 0.2, 0.7, 0.9, 0.8\rangle\} \end{Bmatrix}$$

Then their FDVNSS intersection is given as follows:

$$(F,E)\tilde{\cap}(G,E) = (G,E) = \begin{Bmatrix} e_1 = \{\langle x_1, 0.5, 0.5, 0.8, 0.9\rangle, \langle x_2, 0.4, 0.4, 0.9, 0.9\rangle, \langle x_3, 0.2, 0.3, 0.9, 0.1\rangle\}, \\ e_2 = \{\langle x_1, 0.4, 0.5, 0.9, 0.8\rangle, \langle x_2, 0.6, 0.5, 0.9, 0.9\rangle, \langle x_3, 0.2, 0.7, 0.9, 0.8\rangle\} \end{Bmatrix}$$

$$(F,E)\tilde{\cap}(H,E) = (H,E) = \begin{Bmatrix} e_1 = \{\langle x_1, 0.2, 0.3, 0.9, 0.9\rangle, \langle x_2, 0.0, 0.0, 0.9, 0.9\rangle, \langle x_3, 0.0, 0.0, 0.9, 0.9\rangle\}, \\ e_2 = \{\langle x_1, 0.4, 0.5, 0.9, 0.8\rangle, \langle x_2, 0.6, 0.4, 0.9, 0.9\rangle, \langle x_3, 0.2, 0.7, 0.9, 0.8\rangle\} \end{Bmatrix}$$

$$(G,E)\tilde{\cap}(H,E) = (H,E) = \begin{Bmatrix} e_1 = \{\langle x_1, 0.2, 0.3, 0.9, 0.9\rangle, \langle x_2, 0.0, 0.0, 0.9, 0.9\rangle, \langle x_3, 0.0, 0.0, 0.9, 0.9\rangle\}, \\ e_2 = \{\langle x_1, 0.4, 0.5, 0.9, 0.8\rangle, \langle x_2, 0.6, 0.4, 0.9, 0.9\rangle, \langle x_3, 0.2, 0.7, 0.9, 0.8\rangle\} \end{Bmatrix}$$

Theorem 1. *Let $(X, \tau_1, \acute{E})$ and $(X, \tau_2, \acute{E})$ be two FDVNSSTS. Then $\tau_1 \tilde{\cap} \tau_2$ is a FDVNSSTS on X.*

Proof. The first and third requirements are clear, and we move forward as follows for the second condition. Let $\{(\tilde{\mathcal{L}}_i, \acute{E}); i \in I\} \in \tau_1 \tilde{\cap} \tau_2$ then $(\tilde{\mathcal{L}}_i, \acute{E}) \in \tau_1$, $(\tilde{\mathcal{L}}_i, \acute{E}) \in \tau_1$ as τ_1 and τ_2 are FDVNSTSs on X, then $\tilde{\cup}_i(\tilde{\mathcal{L}}_i, \acute{E}) \in \tau_1$ and $\tilde{\cup}_i(\tilde{\mathcal{L}}_i, \acute{E}) \in \tau_2$. So $\tilde{\cup}_i(\tilde{\mathcal{L}}_i, \acute{E}) \in \tau_1 \tilde{\cap} \tau_2$.

Remark 1. *Let $(X, \tau_1, \acute{E})$ and $(X, \tau_2, \acute{E})$ be two FDVNSSTS. Then $\tau_1 \tilde{\cap} \tau_2$ is not necessarily to be FDVNSSTS on X.*

Definition 27. *Let $(X, \tau_1, \acute{E})$ be a FDVNSBTS over X, and $(\tilde{F}, \acute{E})$ be a FDVNSS then:*

(i) $(\tilde{F}, \acute{E})$ *is FDVNS semi-open if* $(\tilde{F}, \acute{E}) \subseteq NScl(NSint(\tilde{F}, \acute{E}))$.

(ii) $(\tilde{F}, \acute{E})$ *is FDVNS pre-open (p-open) if* $(\tilde{F}, \acute{E}) \subseteq NSint(NScl(\tilde{F}, \acute{E}))$.

(iii) $(\tilde{F}, \acute{E})$ *is FDVNS $*b$ open if*

$$(\tilde{F}, \acute{E}) \subseteq NScl(NSint(\tilde{F}, \acute{E})) \cup NSint(NScl(\tilde{F}, \acute{E})),$$

and FDVNS $*_b$ close if

$$(\tilde{F}, \acute{E}) \supseteq NScl(NSint(\tilde{F}, \acute{E})) \cap NSint(NScl(\tilde{F}, \acute{E})).$$

Definition 28. *Let $(X, \tau_1, \acute{E})$ be a FDVNSBTS over X, and $(\tilde{F}, \acute{E})$ be a FDVNS, then interior of $(\tilde{F}, \acute{E})$, denoted by $(\tilde{F}, \acute{E})^\circ$, is the union of all FDVNS p-open sets of $(\tilde{F}, \acute{E})$. Clearly, $(\tilde{F}, \acute{E})^\circ$ is the largest FDVNS p-open set contained in $(\tilde{F}, \acute{E})$.*

Definition 29. *Let $(X, \tau_1, \acute{E})$ be a FDVNSBTS, and $(\tilde{F}, \acute{E})$ be a FDVNS, the frontier of $(\tilde{F}, \acute{E})$ denoted by $Fr((\tilde{F}, \acute{E}))$, is a FDVNS point $s1^{\curlywedge}_{\langle\varphi_1,\varphi_2,\varphi_3,\varphi_4\rangle}$ is frontier of $(\tilde{F}, \acute{E})$ if every FDVNS p-open set comprising $s1^{\curlywedge}_{\langle\varphi_1,\varphi_2,\varphi_3,\varphi_4\rangle}$ comprises at least one point of $(\tilde{F}, \acute{E})$ and at least one FDVNS point of $(\tilde{F}, \acute{E})^c$.*

Definition 30. *If $(X, \tau_1, \acute{E})$ is a FDVNSBTS and $(\tilde{F}, \acute{E})$ is a FDVNS, then the exterior of $(\tilde{F}, \acute{E})$, denoted by $Ext((\tilde{F}, \acute{E}))$, is a FDVNS point $s1^{\curlywedge}_{\langle\varphi_1,\varphi_2,\varphi_3,\varphi_4\rangle}$ is called exterior of $(\tilde{F}, \acute{E})$ if $s1^{\curlywedge}_{\langle\varphi_1,\varphi_2,\varphi_3,\varphi_4\rangle}$ is in the interior of $(\tilde{F}, \acute{E})^c$, that is FDVNS p-open set $(\tilde{G}, \acute{E})$ such that*

$$s1^{\curlywedge}_{\langle\varphi_1,\varphi_2,\varphi_3,\varphi_4\rangle} \in (\tilde{G}, \acute{E}) \subseteq (\tilde{F}, \acute{E})^c.$$

Definition 31. *If $(X, \tau_1, \acute{E})$ and $(\langle\tilde{Y}\rangle, \mathfrak{F}_1, \acute{E})$ are FDVNSBTSs, and $(f, \phi) : (X, \tau_1, \acute{E}) \to (\langle\tilde{Y}\rangle, \mathfrak{F}_1, \acute{E})$ is a FDVNS mapping, then (f, ϕ) is said to be a FDVNS p-close mapping if the image $(f, \phi)(\tilde{F}, \acute{E})$ of each FDVNS p-closed set $(\tilde{F}, \acute{E})$ over $\tilde{X}$ is a FDVNS p-closed set in $\langle\tilde{Y}\rangle$.*

Theorem 2. *Let $(X, \tau_1, \acute{E})$ be a FDVNSBTSs over X and $(\tilde{F}, \acute{E})$ is FDVNS subset. Then, $(\tilde{F}, \acute{E})$ is a FDVNS p-open set if and only if $(\tilde{F}, \acute{E}) = (\tilde{F}, \acute{E})^\circ$.*

Proof. Let $(\tilde{F}, \acute{E})$ be a FDVNS p-open set. Then, the biggest FDVNS p-open set surrounded by $(\tilde{F}, \acute{E})$ is equal to $(\tilde{F}, \acute{E})$. Hence, $(\tilde{F}, \acute{E}) = (\tilde{F}, \acute{E})^\circ$.

Contrariwise, it is known that $(\tilde{F}, \acute{E})^\circ$ is a HPNS p-open set, and if $(\tilde{F}, \acute{E}) = (\tilde{F}, \acute{E})^\circ$, then $(\tilde{F}, \acute{E})$ is a FDVNS p-open set.

Theorem 3. *Let $(X, \tau_1, \acute{E})$ be a FDVNSBTS over X, and let $(\tilde{F}, \acute{E})$ and $(\tilde{G}, \partial)$ be FDVNS subsets then*

(i) $[(\tilde{F}, \acute{E})^\circ]^\circ = (\tilde{F}, \acute{E})^\circ$,

(ii) $(0_{(\langle\tilde{X}\rangle,\acute{E})})^\circ = 0_{(\langle\tilde{X}\rangle,\acute{E})}$ *and* $(1_{(\langle\tilde{X}\rangle,\acute{E})})^\circ = 1_{(\langle\tilde{X}\rangle,\acute{E})}$,

(iii) $(\tilde{F}, \acute{E}) \subseteq (\tilde{G}, \acute{E}) \Rightarrow (\tilde{F}, \acute{E})^\circ \subseteq (\tilde{G}, \acute{E})^\circ$,

(iv) $[(\tilde{F}, \acute{E}) \cap (\tilde{G}, \acute{E})]^\circ = (\tilde{F}, \acute{E})^\circ \cap (\tilde{G}, \acute{E})^\circ$,

(v) $(\tilde{F}, \acute{E})^\circ \cup (\tilde{G}, \acute{E})^\circ \subseteq [(\tilde{F}, \acute{E}) \cup (\tilde{G}, \acute{E})]^\circ$.

Proof.

(i) $(\tilde{F}, \acute{E})^\circ = (\tilde{G}, \acute{E})$ then $(\tilde{G}, \partial) \in \tilde{\tau}$ iff $(\tilde{G}, \acute{E}) = (\tilde{F}, \acute{E})^\circ$. So $[(\tilde{F}, \acute{E})^\circ]^\circ = (\tilde{F}, \acute{E})^\circ$

(ii) Since $0_{(\langle\tilde{X}\rangle,\acute{E})}$ and $1_{(\langle\tilde{X}\rangle,\acute{E})}$ are always FDVNS p-open sets, so

$$(0_{(\langle\tilde{X}\rangle,\acute{E})})^\circ = 0_{(\langle\tilde{X}\rangle,\acute{E})}, \quad \text{and} \quad (1_{(\langle\tilde{X}\rangle,\acute{E})})^\circ = 1_{(\langle\tilde{X}\rangle,\acute{E})}.$$

(iii) It is known that $(\tilde{F}, \acute{E})^\circ \subseteq (\tilde{F}, \acute{E}) \subseteq (\tilde{G}, \acute{E})$ and $(\tilde{G}, \acute{E})^\circ \subseteq (\tilde{G}, \acute{E})$. Since $(\tilde{G}, \acute{E})^\circ$ is the biggest FDVNS p-open set enclosed in $(\tilde{G}, \acute{E})$ and so, $(\tilde{F}, \acute{E})^\circ \subseteq (\tilde{G}, \acute{E})^\circ$.

(iv) Since $(\tilde{F}, \acute{E}) \cap (\tilde{G}, \acute{E}) \subseteq (\tilde{F}, \acute{E})$ and $(\tilde{F}, \acute{E}) \cap (\tilde{G}, \acute{E}) \subseteq (\tilde{G}, \acute{E})$, then

$$[(\tilde{F}, \acute{E}) \cap (\tilde{G}, \acute{E})]^\circ \subseteq (\tilde{F}, \acute{E})^\circ and [(\tilde{F}, \acute{E}) \cap (\tilde{G}, \acute{E})]^\circ \subseteq (\tilde{G}, \acute{E})^\circ.$$

so,

$$[(\tilde{F}, \acute{E}) \cap (\tilde{G}, \acute{E})]^\circ \subseteq (\tilde{F}, \acute{E})^\circ \cap (\tilde{G}, \acute{E})^\circ.$$

On other way, since $(\tilde{F}, \acute{E})^\circ \subseteq (\tilde{F}, \acute{E})$ and $(\tilde{G}, \acute{E})^\circ \subseteq (\tilde{G}, \acute{E})$, then

$$(\tilde{F}, \acute{E})^\circ \cap (\tilde{G}, \acute{E})^\circ \subseteq (\tilde{F}, \acute{E}) \cap (\tilde{G}, \acute{E}).$$

also $[(\tilde{F}, \acute{E}) \cap (\tilde{G}, \acute{E})]^\circ \subseteq (\tilde{F}, \acute{E}) \cap (\tilde{G}, \acute{E})$ is the biggest FDVNS p-open set.
$\Rightarrow (\tilde{F}, \acute{E})^\circ \cap (\tilde{G}, \acute{E})^\circ \subseteq [(\tilde{F}, \acute{E}) \cap (\tilde{G}, \acute{E})]^\circ$
Thus,

$$[(\tilde{F}, \acute{E}) \cap (\tilde{G}, \acute{E})]^\circ = (\tilde{F}, \acute{E})^\circ \cap (\tilde{G}, \acute{E})^\circ.$$

(v) Since $(\tilde{F}, \acute{E}) \subseteq (\tilde{F}, \acute{E}) \cup (\tilde{G}, \acute{E})$ and $(\tilde{G}, \acute{E}) \subseteq (\tilde{F}, \acute{E}) \cup (\tilde{G}, \acute{E})$, then

$$(\tilde{F}, \acute{E})^\circ \subseteq [(\tilde{F}, \acute{E}) \cup (\tilde{G}, \acute{E})]^\circ, \quad \text{and} \quad (\tilde{G}, \acute{E})^\circ \subseteq [(\tilde{G}, \acute{E}) \cup (\tilde{G}, \acute{E})]^\circ.$$

$\Rightarrow (\tilde{F}, \acute{E})^\circ \cup (\tilde{G}, \acute{E})^\circ \subseteq [(\tilde{F}, \acute{E}) \cup (\tilde{G}, \acute{E})]^\circ.$

Theorem 4. *Let $(X, \tau_1, \acute{E})$ be a FDVNSBTS over X, and if $(\tilde{F}, \acute{E})$ is a FDVNS subset, then $(\tilde{F}, \acute{E})$ is a FDVNS p-closed set if and only if $(\tilde{F}, \acute{E}) = \overline{(\tilde{F}, \acute{E})}$.*

Proof. Let $(\tilde{F}, \acute{E})$ is a FDVNS p-closer set, then:

$$(\tilde{F}, \acute{E})^d = (\tilde{F}, \acute{E})$$

this implies that

$$(\tilde{F}, \acute{E}) \cup (\tilde{F}, \acute{E})^d \cong (\tilde{F}, \acute{E})$$

$\Rightarrow$

$$\overline{(\tilde{F}, \acute{E})} \cong (\tilde{F}, \acute{E})$$

and conversely let $\overline{(\tilde{F}, \acute{E})} \cong (\tilde{F}, \acute{E})$, this implies that

$$(\tilde{F}, \acute{E}) \cup (\tilde{F}, \acute{E})^d \cong (\tilde{F}, \acute{E})$$

$\Rightarrow$

$$(\tilde{F}, \acute{E})^d = (\tilde{F}, \acute{E})$$

this implies, $(\tilde{F}, \acute{E})$ is a FDVNS p-closed set.

Theorem 5. *Let $(X, \tau_1, \acute{E})$ be a FDVNSBTS over X, and let $(\tilde{F}, \acute{E})$ and $(\tilde{G}, \acute{E})$ be HPNS subsets then*

(i) $\overline{(\tilde{F}, \acute{E})} = \overline{(\tilde{F}, \acute{E})}$,

(ii) $\overline{(0_{(\langle \tilde{X} \rangle, \acute{E})})} = 0_{(\tilde{X}, \acute{E})}$ *and* $\overline{(1_{(\langle \tilde{X} \rangle, \acute{E})})} = \overline{(1_{(\langle \tilde{X} \rangle, \acute{E})})}$,

(iii) $(\tilde{F}, \acute{E}) \subseteq \langle (\tilde{G}, \acute{E}) \rangle \Rightarrow \overline{(\tilde{F}, \acute{E})} \subseteq \overline{(\tilde{G}, \acute{E})}$,

(iv) $\overline{[(\tilde{F}, \acute{E}) \cup (\tilde{G}, \acute{E})]} = \overline{(\tilde{F}, \acute{E})} \cup \overline{(\tilde{G}, \acute{E})}$,

(v) $\overline{[(\tilde{F}, \acute{E}) \cap (\tilde{G}, \acute{E})]} \subseteq \overline{(\tilde{F}, \acute{E})} \cap \overline{(\tilde{G}, \acute{E})}$.

Proof.

(i) If $\overline{(\tilde{F},\acute{E})} = (\tilde{G},\acute{E})$ then $(\tilde{G},\acute{E})$ is a FDVNS p-closed set.Hence, if $(\tilde{G},\acute{E}) = \overline{(\tilde{G},\acute{E})}$. Therefore
$\overline{[\overline{(\tilde{F},\acute{E})}]} = \overline{(\tilde{F},\acute{E})}$

(ii) Since $0_{(\langle\tilde{X}\rangle,\acute{E})}$ and $1_{(\langle\tilde{X}\rangle,\acute{E})}$ are always FDVNS p-closure set so by the above result

$$\overline{(0_{(\langle\tilde{X}\rangle,\acute{E})})} = 0_{(\langle\tilde{X}\rangle,\acute{E})}, \quad \text{and} \quad \overline{(1_{(\langle\tilde{X}\rangle,\acute{E})})} = 1_{(\langle\tilde{X}\rangle,\acute{E})}.$$

(iii) Since $(\tilde{F},\acute{E}) \subseteq \overline{(\tilde{F},\acute{E})}$ and $(\tilde{G},\acute{E}) \subseteq \overline{(\tilde{G},\acute{E})}$, so $(\tilde{F},\acute{E}) \subseteq \underline{(\tilde{G},\acute{E})} \subseteq \overline{(\tilde{G},\acute{E})}$ .Since $\overline{(\tilde{G},\acute{E})}$ is the smallest FDVNS p-closure set covering in $(\tilde{F},\acute{E})$ then $\overline{(\tilde{F},\acute{E})} \subseteq \overline{(\tilde{G},\acute{E})}$.

(iv) Since $(\tilde{F},\acute{E}) \subseteq (\tilde{F},\acute{E}) \cup (\tilde{G},\acute{E})$ and $(\tilde{G},\acute{E}) \subseteq 0_{(\langle\tilde{X}\rangle,\acute{E})} \cup (\tilde{G},\acute{E})$ then

$$\overline{(\tilde{F},\acute{E})} \subseteq \overline{[(\tilde{F},\acute{E}) \cup (\tilde{G},\acute{E})]} and \overline{(\tilde{G},\acute{E})} \subseteq \overline{[(\tilde{F},\acute{E}) \cup (\tilde{G},\acute{E})]}.$$

so,

$$\overline{(\tilde{F},\acute{E})} \cup \overline{(\tilde{G},\acute{E})} \subseteq \overline{[(\tilde{F},\acute{E}) \cup (\tilde{G},\acute{E})]}.$$

Conversely, since $(\tilde{F},\acute{E}) \subseteq \overline{(\tilde{F},\acute{E})}$ and $(\tilde{G},\acute{E}) \subseteq \overline{(\tilde{G},\acute{E})}$, then

$$(\tilde{F},\acute{E}) \cup (\tilde{G},\acute{E}) \subseteq \overline{(\tilde{F},\acute{E})} \cup \overline{(\tilde{G},\acute{E})}.$$

Besides, $\overline{[(\tilde{F},\acute{E}) \cup (\tilde{G},\acute{E})]}$ is the smallest FDVNS p-closed set that enclosing $(\tilde{F},\acute{E}) \cup (\tilde{G},\acute{E})$ therefore, $\overline{[(\tilde{F},\acute{E}) \cup (\tilde{G},\acute{E})]} \subseteq \overline{(\tilde{F},\acute{E})} \cup \overline{(\tilde{G},\acute{E})}$.
Thus, $\overline{[(\tilde{F},\acute{E}) \cup (\tilde{G},\acute{E})]} = \overline{(\tilde{F},\acute{E})} \cup \overline{(\tilde{G},\acute{E})}$.

(v) Since $\langle(0_{(\langle\tilde{X}\rangle,\acute{E})})\rangle \cap X \subseteq \overline{(\tilde{F},\acute{E})} \cap \overline{X}$ and $\overline{[(\tilde{F},\acute{E}) \cap X]}$ is the smallest FDVNS p-closed set that enclosing $(\tilde{F},\acute{E}) \cap X$, then $\overline{[(\tilde{F},\acute{E}) \cap X]} \subseteq \overline{(\tilde{F},\acute{E})} \cap \overline{X}$.

Theorem 6. *Let $(X,\tau_1,\acute{E})$ be a FDVNSBTSs over X, and let $(\tilde{F},\acute{E})$ be a FDVNSS then,*

(i) $[(\tilde{F},\acute{E})]^c = [(\tilde{F},\acute{E})^c]^\circ$,

(ii) $[(\tilde{F},\acute{E})^\circ]^c = \overline{[(\tilde{F},\acute{E})^c]}$.

Proof.

(i)

$$\begin{aligned}
\overline{(\tilde{F},\acute{E})} &= \cap\{(\tilde{G},\acute{E}) \in (X,\tau_1,\acute{E})^c : (\tilde{G},\acute{E}) \supseteq (\tilde{F},\acute{E})\} \\
&\Rightarrow [\overline{(\tilde{F},\acute{E})}]^c = \Big[\cap\{(\tilde{G},\acute{E}) \in (X,\tau_1,\acute{E})^c : (\tilde{G},\acute{E}) \supseteq (\tilde{F},\acute{E})\}\Big]^c \\
&= \cup\{(\tilde{G},\acute{E})^c \in (X,\tau_1,\acute{E}) : (\tilde{G},\acute{E})^c \subseteq (\tilde{F},\acute{E})^c\} \\
&= [(\tilde{F},\acute{E})^c]^\circ.
\end{aligned}$$

(ii)

$$\begin{aligned}
(\tilde{F},\acute{E})^\circ &= \cup\{(\tilde{G},\acute{E}) \in (X,\tau_1,\acute{E}) : (\tilde{G},\acute{E}) \subseteq (\tilde{F},\acute{E})\} \\
&\Rightarrow [(\tilde{F},\acute{E})^\circ]^c = \Big[\cap\{(\tilde{G},\acute{E}) \in (X,\tau_1,\acute{E}) : (\tilde{G},\acute{E}) \subseteq (\tilde{F},\acute{E})\}\Big]^c \\
&= \cap\{(\tilde{G},\acute{E})^c \in (X,\tau_1,\acute{E})^c : (\tilde{G},\acute{E})^c \supseteq (\tilde{F},\acute{E})^c\} \\
&= \overline{[(\tilde{F},\acute{E})^c]}.
\end{aligned}$$

Theorem 7. *Let $(X, \tau_1, \acute{E})$ be a FDVNSBTS over X. If $(\tilde{F}, \acute{E})$ and $(\tilde{G}, \acute{E})$ are FDVNS subsets, then:*

(i) $\mathrm{Ext}((\tilde{F}, \acute{E}) \cup (\tilde{G}, \acute{E})) = \mathrm{Ext}((\tilde{F}, \acute{E})) \cup \mathrm{Ext}((\tilde{G}, \acute{E}))$.

(ii) $\mathrm{Ext}((\tilde{F}, \acute{E}) \cap (\tilde{G}, \acute{E})) \supseteq \mathrm{Ext}((\tilde{F}, \acute{E})) \cup \mathrm{Ext}((\tilde{G}, \acute{E}))$.

(iii) $\mathrm{Fr}((\tilde{F}, \acute{E}) \cup (\tilde{G}, \acute{E})) \subseteq \mathrm{Fr}(\tilde{F}, \acute{E}) \cup \mathrm{Fr}(\tilde{G}, \acute{E})$.

(iv) $\mathrm{Fr}((\tilde{F}, \acute{E}) \cap (\tilde{G}, \acute{E})) \subseteq \mathrm{Fr}(\tilde{F}, \acute{E}) \cup \mathrm{Fr}(\tilde{G}, \acute{E})$.

Proof.

(i) Since

$$\begin{aligned}
\mathrm{Ext}((\tilde{F}, \acute{E}) \cup (\tilde{G}, \acute{E})) &= (((\tilde{F}, \acute{E}) \cup (\Im, \partial))^c)^\circ \\
&= ((\tilde{F}, \acute{E})^c \cap (\tilde{G}, \acute{E})^c)^\circ \\
&= ((\tilde{F}, \acute{E})^c)^\circ \cap ((\tilde{G}, \acute{E})^c)^\circ \\
&= \mathrm{Ext}((\tilde{F}, \acute{E})) \cap \mathrm{Ext}((\tilde{G}, \acute{E})).
\end{aligned}$$

(ii)

$$\begin{aligned}
\mathrm{Ext}((\tilde{F}, \acute{E}) \cap (\tilde{G}, \acute{E})) &= (((\tilde{F}, \acute{E}) \cap (\tilde{G}, \acute{E}))^c)^\circ \\
&= (((\tilde{F}, \acute{E})^c \cup (\tilde{G}, \acute{E})^c))^\circ \\
&\supseteq ((\tilde{F}, \acute{E})^c)^\circ \cup ((\tilde{G}, \acute{E})^c)^\circ \\
&= \mathrm{Ext}((\tilde{F}, \acute{E})) \cup \mathrm{Ext}((\tilde{G}, \acute{E})).
\end{aligned}$$

(iii)

$$\begin{aligned}
&\mathrm{Fr}((\tilde{F}, \acute{E}) \cup (\tilde{G}, \acute{E})) = \overline{(\tilde{F}, \acute{E}) \cup (\tilde{G}, \acute{E})} \cap \overline{((\tilde{F}, \acute{E}) \cup (\tilde{G}, \acute{E}))^c} \\
&= (\overline{(\tilde{F}, \acute{E})} \cup \overline{(\tilde{G}, \acute{E})}) \cap \overline{(\tilde{F}, \acute{E})^c \cap (\tilde{G}, \acute{E})^c} \subseteq ((\tilde{F}, \acute{E})) \cup ((\tilde{G}, \acute{E})) \cap \overline{(\tilde{F}, \acute{E})^c} \cap \overline{(\tilde{G}, \acute{E})^c}) \\
&= \{(\overline{(\tilde{F}, \acute{E})}) \cup ((\tilde{G}, \acute{E})) \cap \overline{(\tilde{F}, \acute{E})^c} \cap \overline{(\tilde{G}, \acute{E})^c}\} \\
&= \{(\overline{(\tilde{F}, \acute{E})} \cap \overline{(\tilde{F}, \acute{E})^c}) \cup (\tilde{G}, \acute{E}) \cap \overline{(\tilde{F}, \acute{E})^c}\} \cup \{(\overline{(\tilde{G}, \acute{E})^c}) \cap (\overline{(\tilde{F}, \acute{E})^c} \cap \overline{(\tilde{G}, \acute{E})^c})\} \\
&= \{\mathrm{Fr}((\tilde{F}, \acute{E})) \cap \overline{((\tilde{G}, \acute{E})^c)}\} \cup \{\mathrm{Fr}((\tilde{G}, \acute{E})) \cap \overline{(\tilde{F}, \acute{E})^c}\} \\
&\subseteq \mathrm{Fr}((\tilde{F}, \acute{E})) \cup \mathrm{Fr}((\tilde{G}, \acute{E})).
\end{aligned}$$

(iv)

$$\begin{aligned}
&\mathrm{Fr}((\tilde{F}, \acute{E}) \cap (\tilde{G}, \acute{E})) = \overline{(\tilde{F}, \acute{E}) \cap (\tilde{G}, \acute{E})} \cap \overline{((\tilde{F}, \acute{E}) \cap (\tilde{G}, \acute{E}))^c} \\
&\subseteq (\tilde{F}, \acute{E}) \cap ((\tilde{F}, \acute{E})^c) \cup (\tilde{G}, \acute{E})^c) \\
&= \{(\overline{(\tilde{F}, \acute{E})}) \cup (\overline{(\tilde{G}, \acute{E})}) \cap \overline{(\tilde{F}, \acute{E})^c}\} \cup \{(\overline{(\tilde{F}, \acute{E})}) \cup (\overline{(\tilde{G}, \acute{E})}) \cap \overline{(\tilde{G}, \acute{E})^c}\} \\
&= \{\mathrm{Fr}((\tilde{F}, \acute{E})) \cap (\overline{(\tilde{G}, \acute{E})})\} \cup \{(\overline{(\tilde{F}, \acute{E})}) \cap \mathrm{Fr}((\tilde{G}, \acute{E}))\} \\
&\subseteq \mathrm{Fr}((\tilde{F}, \acute{E})) \cup \mathrm{Fr}((\tilde{G}, \acute{E})).
\end{aligned}$$

5. Conclusion and Future Work

In this paper, we introduced Fermatean Double-Valued Neutrosophic Soft Sets (FDVNSS), an advanced mathematical framework that significantly enhances the modeling of uncertainty, vagueness, and indeterminacy in complex environments. By incorporating both absolute and relative degrees of truth and falsity, FDVNSS provides a more nuanced and expressive structure compared to traditional neutrosophic and soft set approaches. The formal definition of FDVNSS was established under the Fermatean constraint, ensuring that the sum of the cubes of membership values lies within a bounded range. We also defined key operations?such as subset, null and absolute FDVNSS, complement, union, and intersection?and explored their algebraic properties, demonstrating alignment with classical set-theoretic behavior. Applications in medical decision-making were provided to illustrate the practical relevance of FDVNSS in handling imprecise, incomplete, or conflicting information, highlighting its potential in intelligent decision support systems. The notion of a Fermatean Double Valued Neutrosophic Soft Topological Space (FDVNSTS) is introduced. The concepts of semi-open (s-open), pre-open (p-open), and b-open sets are defined within the context of FDVNSTS. Among these generalized open sets, the pre-open set is selected for further exploration, and several fundamental topological notions are developed based on this definition. These include the closure, exterior, boundary, and interior in FDVNSTS.

Future work will focus on extending the FDVNSS framework to dynamic and real-time systems, integrating it with machine learning and data-driven inference mechanisms. Additional research will also explore optimization techniques, ranking algorithms, and hybrid models that combine FDVNSS with other uncertainty handling tools, such as fuzzy cognitive maps and rough sets. Moreover, domain-specific applications, particularly in healthcare, finance, and environmental risk assessment, will be pursued to further validate and refine the effectiveness of this framework in real-world scenarios. Future research could explore the integration of Double-Valued Neutrosophic Soft Topological Spaces (DV-NSTS) with machine learning techniques to enhance uncertainty handling in complex data analysis. The Epanechnikov-Pareto distribution [74] could be incorporated into DV-NSTS to develop a Neutrosophic Principal Component Analysis (N-PCA) that effectively manages indeterminacy during feature extraction, while concepts from lacunary statistical convergence [77] could provide robust criteria for analyzing neutrosophic eigenvalues in dimensionality reduction. The ranked set sampling approach [79] might be adapted for improved clustering of imbalanced data within DV-NSTS frameworks, potentially leading to advanced neutrosophic versions of k-means or fuzzy c-means algorithms that account for membership, indeterminacy, and non-membership as soft topological properties. Building on the stability analysis of Volterra integral equations [75], new neutrosophic regression models could be formulated where coefficients are represented as Double-Valued Neutrosophic Soft Sets (DV-NSS) and topological continuity ensures model stability, with potential applications in financial forecasting where market trends exhibit inherent vagueness. The reliability estimation methods using Pareto and Benktander distributions [78, 82] could be extended to create novel neutrosophic entropy measures for feature selection in machine learning pipelines, particularly valuable in IoT networks with varying data reliability. The modified Midzuno scheme [76] might be employed to optimize feature subset selection processes in high-dimensional neutrosophic datasets. Furthermore, the ratio estimator modifications [80] could inspire the development of specialized loss functions for Double-Valued Neutrosophic Neural Networks, while the fractional differential equations framework [81] might enable more sophisticated time-series analysis within DV-NSTS. Future work could also focus on implementing practical DV-NSS libraries in programming environments like Python or R to facilitate real-world applications in healthcare diagnostics, autonomous vehicle perception systems, and financial market prediction, where conventional methods struggle with incomplete or contradictory information. These directions would benefit from both theoretical advancements in defining topological convergence properties for DV-NSTS and empirical validation through case studies across various domains.

Future studies will investigate the integration of symbolic n-plithogenic intervals [83] with dimensionality reduction methods such as PCA and t-SNE, as well as clustering algorithms like K-means, to enhance pattern recognition and classification in complex fuzzy and neutrosophic environments. In the same way, the algebraic framework built on CCIFS in bisemirings [84] will be broadened with Fermatean double-valued neutrosophic sets. Merging this with PCA, t-SNE, and K-means could improve visualization and analysis in situations involving uncertain, high-dimensional data.

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