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Contractive Operators Controlled by Simulation Functions in Fuzzy Metric Spaces with Transitive $\mathcal{K}$-Closed Binary Relations

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Abstract. In this study, we establish a novel fuzzy functional contraction within fuzzy metric spaces equipped with a binary relation, relying on the weaker concept of $\mathcal{R}$-completeness rather than the classical completeness of the entire space or its subspaces. The usual continuity requirement on the mapping is relaxed and replaced by either $\mathcal{R}$-continuity or the $\mathfrak{P}$-self-closedness of the relation's restriction, employing a broad class of control functions $\mathfrak{S}$. The theoretical results are illustrated with examples and an application to solving an integral equation governed by a given binary relation, accompanied by several corollaries and derived consequences. This work extends the theory of relation-theoretic fuzzy fixed points and provides a rigorous basis for further study of coincidence and common fixed points, with potential applications to nonlinear operator equations in uncertain settings.

2020 Mathematics Subject Classifications: 47H10, 54H25

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1. Introduction and Preliminaries

Fixed point theory occupies a central role in nonlinear functional analysis, providing a rich toolkit for resolving diverse and intricate problems across many mathematical disciplines. Among its fundamental results is the Banach contraction principle, a cornerstone of metric fixed point theory, which has been extensively generalized and applied to numerous abstract metric frameworks. A recent advancement in this domain was introduced by Khojasteh et al. [1], who enriched fixed point theory by incorporating a novel class

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of control functions known as simulation functions, thereby broadening the theoretical horizon.

Another vibrant and rapidly growing facet of fixed point theory involves relation-theoretic results. This area traces its origins to the pioneering work of Turinici [2], who formulated the concept of order-theoretic fixed points. Building on this foundation, Ran and Reurings [3] developed an order-theoretic version of the Banach contraction principle in 2004, which they successfully applied to matrix equations, expanding the practical applicability of fixed point results. Later, Alam and Imdad [4] extended this framework by introducing a relation-theoretic version of the Banach contraction principle, unifying several classical order-theoretic theorems through arbitrary binary relations. This approach has since catalyzed a proliferation of fixed point theorems employing diverse notions of binary relations [5–10].

The introduction of fuzzy set theory by Zadeh in 1965 [11] marked a significant milestone, offering a robust mathematical apparatus to address uncertainty and vagueness beyond the scope of classical crisp sets. This innovation has become instrumental in modeling complex and imprecise phenomena. The extension of probabilistic metric spaces into the fuzzy context was pioneered by Kramosil and Michalek [12], who introduced fuzzy metric spaces a concept subsequently refined by George and Veeramani [13] to define a Hausdorff topology.

In the setting of fuzzy metric spaces, distance is conceptualized not as a fixed numerical quantity but as a degree of proximity influenced by a parameter $\Im > 0$. This parameter captures practical factors such as travel time or energy expenditure that influence the degree of proximity between two points. For instance, consider the journey by car from Marrakech (u) to Casablanca (v), the degree of proximity between these two cities can be modeled by parameters representing the duration of travel or the amount of fuel consumed along the route. These factors reflect real-world considerations affecting how "close" two locations are in a fuzzy metric sense, beyond mere physical distance. According to axiom ($\mathcal{M}2$), when the two points coincide ($u = v$), their degree of proximity attains the maximal value of 1, reflecting the intuitive concept of perfect proximity in practical scenarios where time, energy, or other resources play a critical role.

Recent years have witnessed growing interest in fixed point theory within fuzzy metric spaces. Early contributions include the introduction of fuzzy contractive mappings by Gregori and Sapena [14], which established foundational fixed point theorems. This line of research was further developed by Mihet [15] with ψ-contractive mappings, and by Wardowski [16] who introduced $\mathcal{H}$-contractive mappings. More recently, Melliani and Moussaoui [17] advanced the field by adapting the simulation function methodology to fuzzy metric spaces, proposing the notion of $\mathcal{FZ}$-contractions. An extensive corpus of literature on various contraction types in fuzzy metric spaces has since emerged, including important works such as [3, 5, 18–32].

To establish our main results, we begin by recalling essential concepts from the theory of binary relations, including definitions and key properties that will be used throughout the work.

Definition 1. *A binary relation $\mathcal{R}$ on a non-empty set $\mathcal{E}$ is defined as a subset of $\mathcal{E} \times \mathcal{E}$.*

For any $s,t \in \mathcal{E}$, if $(s,t) \in \mathcal{R}$, we say that s is related to t via $\mathcal{R}$, often denoted $s\mathcal{R}t$. Furthermore, if either $(s,t) \in \mathcal{R}$ or $(t,s) \in \mathcal{R}$, this is indicated by $[s,t] \in \mathcal{R}$.

Note that the Cartesian product $\mathcal{E} \times \mathcal{E}$ defines the universal relation on $\mathcal{E}$, while the empty set $\emptyset$ represents the empty relation.

A binary relation $\mathcal{R}$ on a nonempty set $\mathcal{E}$ is said to have the following properties: The relation $\mathcal{R}$ is *reflexive* if for every $s \in \mathcal{E}$, we have $s\mathcal{R}s$. It is *transitive* if whenever $s\mathcal{R}t$ and tRr, it follows that $s\mathcal{R}r$ for all $s,t,r \in \mathcal{E}$. The relation is *antisymmetric* if for all $s,t \in \mathcal{E}$, the conditions $s\mathcal{R}t$ and $t\mathcal{R}s$ imply $s=t$. Additionally, $\mathcal{R}$ is *complete* if for every $s,t \in \mathcal{E}$, the pair $[s,t]$ belongs to $\mathcal{R}$.

Finally, given a self-mapping $\mathcal{K} : \mathcal{E} \to \mathcal{E}$, the relation $\mathcal{R}$ is said to be $\mathcal{K}$-closed if whenever $(s,t) \in \mathcal{R}$, it follows that $(\mathcal{K}s, \mathcal{K}t) \in \mathcal{R}$ for all $s,t \in \mathcal{E}$.

Definition 2. *[4] Let $\mathcal{E}$ be a nonempty set and $\mathcal{R}$ a binary relation on $\mathcal{E}$. A sequence $\{s_q\} \subseteq \mathcal{E}$ is called $\mathcal{R}$-preserving if $(s_q, s_{q+1}) \in \mathcal{R}$ for every $q \in \mathbb{N}$.*

Definition 3. *[33] A continuous function $\curlywedge : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it is commutative, associative, and satisfies:*

(i) For every $\wp_1 \in [0,1]$, $\wp_1 \curlywedge 1 = \wp_1$,

(ii) For all $\wp_1, \wp_2, \wp_3, \wp_4 \in [0,1]$, if $\wp_1 \le \wp_3$ and $\wp_2 \le \wp_4$, then $\wp_1 \curlywedge \wp_2 \le \wp_3 \curlywedge \wp_4$.

Example 1. *Some standard continuous t-norms are:*

(1) *Minimum t-norm:* $\wp_1 \curlywedge_m \wp_2 = \min\{\wp_1, \wp_2\}$,

(2) *Łukasiewicz t-norm:* $\wp_1 \curlywedge_L \wp_2 = \max\{\wp_1 + \wp_2 - 1, 0\}$,

(3) *Product t-norm:* $\wp_1 \curlywedge_p \wp_2 = \wp_1 \cdot \wp_2$,

for all $\wp_1, \wp_2 \in [0,1]$.

Definition 4. *[13] Let $\mathcal{E}$ be a non-empty set, $\curlywedge$ a continuous t-norm, and $\mathfrak{P} : \mathcal{E}^2 \times (0,+\infty) \to [0,1]$ a fuzzy set. The triple $(\mathcal{E}, \mathfrak{P}, \curlywedge)$ is called a fuzzy metric space if for all $s,t,r \in \mathcal{E}$ and $\Im, \kappa > 0$, the following conditions hold:*

($\mathcal{M}1$) $\mathfrak{P}(s,t,\Im) > 0$,

($\mathcal{M}2$) $\mathfrak{P}(s,t,\Im) = 1$ *if and only if* $s = t$,

($\mathcal{M}3$) $\mathfrak{P}(s,t,\Im) = \mathfrak{P}(t,s,\Im)$,

($\mathcal{M}4$) $\mathfrak{P}(s,t,\Im) \curlywedge \mathfrak{P}(t,r,\kappa) \le \mathfrak{P}(s,r,\Im + \kappa)$,

($\mathcal{M}5$) *The function* $\mathfrak{P}(s,t,\cdot) : (0,+\infty) \to [0,1]$ *is continuous,*

Lemma 1. *[19] For all $s,t \in \mathcal{E}$, the function $\mathfrak{P}(s,t,\cdot)$ is nondecreasing on $(0,+\infty)$.*

Definition 5. *[13] Let $(\mathcal{E}, \mathfrak{P}, \curlywedge)$ be a fuzzy metric space.*

(i) A sequence $\{s_q\} \subseteq \mathcal{E}$ *is said to converge to* $s \in \mathcal{E}$ *if and only if*

$$\lim_{q \to +\infty} \mathfrak{P}(s_q, s, \Im) = 1 \quad \text{for all} \quad \Im > 0.$$

(ii) A sequence $\{s_q\} \subseteq \mathcal{E}$ *is called a Cauchy sequence if for each* $\alpha \in (0,1)$ *and* $\Im > 0$*, there exists* $q_0 \in \mathbb{N}$ *such that*

$$\mathfrak{P}(s_q, s_p, \Im) > 1 - \alpha \quad \text{for all} \quad q, p \geq q_0.$$

(iii) The fuzzy metric space $(\mathcal{E}, \mathfrak{P}, \curlywedge)$ *is said to be complete if every Cauchy sequence in* $\mathcal{E}$ *converges.*

The notion of fuzzy contractive mappings was initially formulated by Gregori and Sapena [14] as follows:

Definition 6. *[14] Let* $(\mathcal{E}, \mathfrak{P}, \curlywedge)$ *be a fuzzy metric space. A mapping* $\mathcal{K} : \mathcal{E} \to \mathcal{E}$ *is fuzzy contractive if there exists a constant* $\hbar \in (0,1)$ *such that for all* $s, t \in \mathcal{E}$ *and* $\Im > 0$*,*

$$\frac{1}{\mathfrak{P}(\mathcal{K}(s), \mathcal{K}(t), \Im)} - 1 \leq \hbar \left(\frac{1}{\mathfrak{P}(s, t, \Im)} - 1 \right).$$

Define Ψ as the family of continuous non-decreasing functions $\psi : (0,1] \to (0,1]$ satisfying $\psi(t) > t$ for all $t \in (0,1)$.

Definition 7. *[15] A mapping* $\mathcal{K} : \mathcal{E} \to \mathcal{E}$ *is fuzzy* ψ*-contractive if*

$$\mathfrak{P}(\mathcal{K}s, \mathcal{K}t, \Im) \geq \psi\big(\mathfrak{P}(s, t, \Im)\big) \quad \text{for all } s, t \in \mathcal{E}, \ \Im > 0.$$

Example 2. *[15] For each* $\hbar \in (0,1)$*, the function* $\psi_\hbar : (0,1] \to (0,1]$ *given by*

$$\psi_\hbar(s) = \frac{s}{s + \hbar(1 - s)}$$

belongs to Ψ*. Every* $\psi_\hbar$*-fuzzy contractive mapping is also a fuzzy contractive mapping as defined by Gregori and Sapena [14].*

Let $\mathcal{H}$ denote the set of strictly decreasing functions $\eta : (0,1] \to [0, +\infty)$ that map the interval $(0,1]$ onto the entire range $[0, +\infty)$.

Definition 8. *[16] A mapping* $\mathcal{K} : \mathcal{E} \to \mathcal{E}$ *is fuzzy* $\mathcal{H}$*-contractive with respect to* $\eta \in \mathcal{H}$ *if there exists a constant* $\hbar \in (0,1)$ *such that*

$$\eta\big(\mathfrak{P}(\mathcal{K}s, \mathcal{K}t, \Im)\big) \leq \hbar \cdot \eta\big(\mathfrak{P}(s, t, \Im)\big)$$

for all $s, t \in \mathcal{E}$ *and* $\Im > 0$*.*

Definition 9. *[17] (see also [23, 24]) A function* $\mathfrak{S} : (0,1] \times (0,1] \to \mathbb{R}$ *is termed an* $\mathcal{FZ}$*-simulation function if it satisfies:*

($\mathfrak{S}$1) $\mathfrak{S}(1,1)=0$,

($\mathfrak{S}$2) *For all* $s,t\in(0,1)$, *it holds that* $\mathfrak{S}(s,t)<\frac{1}{t}-\frac{1}{s}$,

($\mathfrak{S}$3) *For sequences* $\{s_q\}$ *and* $\{t_q\}$ *in* $(0,1]$ *with* $\lim_{q\to+\infty}s_q=\lim_{q\to+\infty}t_q<1$, *we have*

$$\lim_{q\to+\infty}\sup\mathfrak{S}(s_q,t_q)<0.$$

The collection of all such functions is denoted $\mathcal{FZ}$.

Definition 10. *[17] (see also [23, 24]) Let* $(\mathcal{E},\mathfrak{P},\curlywedge)$ *be a fuzzy metric space,* $\mathcal{K}:\mathcal{E}\to\mathcal{E}$ *a self-map, and* $\mathfrak{S}\in\mathcal{FZ}$. *The mapping* $\mathcal{K}$ *is an* $\mathcal{FZ}$*-contraction with respect to* $\mathfrak{S}$ *if for all* $s,t\in\mathcal{E}$ *and* $\Im>0$,

$$\mathfrak{S}\big(\mathfrak{P}(\mathcal{K}s,\mathcal{K}t,\Im),\mathfrak{P}(s,t,\Im)\big)\geq 0.$$

Example 3. *([24]) Consider the following functions* $\mathfrak{S}_i:(0,1]\times(0,1]\to\mathbb{R}$, $i=1,2,3$:

(i) $\mathfrak{S}_1(s,t)=\hbar\left(\frac{1}{t}-1\right)-\frac{1}{s}+1$, *for* $\hbar\in(0,1)$ *and all* $s,t\in(0,1]$,

(ii) $\mathfrak{S}_2(s,t)=\frac{1}{\psi(t)}-\frac{1}{s}$, *where* $\psi\in\Psi$, *for all* $s,t\in(0,1]$,

(iii) $\mathfrak{S}_3(s,t)=\frac{1}{\eta^{-1}(\hbar\cdot\eta(t))}-\frac{1}{s}$, *where* $\hbar\in(0,1)$, $\eta\in\mathcal{H}$, *for all* $s,t\in(0,1]$.

Each $\mathfrak{S}_i$ *is an* $\mathcal{FZ}$*-simulation function.*

Definition 11. *[34] A binary relation* $\mathcal{R}$ *on* $\mathcal{E}$ *is said to be* $\mathfrak{P}$*-self-closed if for any* $\mathcal{R}$*-preserving sequence* $\{s_q\}\subseteq\mathcal{E}$ *with*

$$\lim_{q\to+\infty}\mathfrak{P}(s_q,s,\Im)=1\quad\text{for all }\Im>0,$$

there exists a subsequence $\{s_{q_k}\}$ *of* $\{s_q\}$ *such that* $(s_{q_k},s)\in\mathcal{R}$.

Definition 12. *[10] Let* $(\mathcal{E},\mathfrak{P},\curlywedge)$ *be a fuzzy metric space and* $\mathcal{R}$ *a binary relation on* $\mathcal{E}$. *A sequence* $\{s_q\}\subseteq\mathcal{E}$ *is called an* $\mathcal{R}$*-Cauchy sequence if it is* $\mathcal{R}$*-preserving and for every* $\varepsilon\in(0,1)$ *and* $\Im>0$, *there exists* $q_0\in\mathbb{N}$ *such that*

$$\mathfrak{P}(s_{q+p},s_q,\Im)>1-\varepsilon\quad\text{for all }q\geq q_0,p\in\mathbb{N}.$$

Remark 1. *[10] For any binary relation* $\mathcal{R}$, *every standard Cauchy sequence is an* $\mathcal{R}$*-Cauchy sequence. The notions coincide when* $\mathcal{R}$ *is the universal relation on* $\mathcal{E}$.

Definition 13. *[10] A fuzzy metric space* $(\mathcal{E},\mathfrak{P},\curlywedge)$ *endowed with a binary relation* $\mathcal{R}$ *is said to be* $\mathcal{R}$*-complete if every* $\mathcal{R}$*-Cauchy sequence converges in* $\mathcal{E}$.

Remark 2. *[10] Every complete fuzzy metric space is* $\mathcal{R}$*-complete for any binary relation* $\mathcal{R}$.

Definition 14. *[35] Let* $\mathcal{E}$ *be a nonempty set and* $\mathcal{R}$ *a binary relation on* $\mathcal{E}$. *For* $s,t\in\mathcal{E}$, *a path of length* ℓ *from* s *to* t *in* $\mathcal{R}$ *is a finite sequence* $\{\Gamma_0,\Gamma_1,\ldots,\Gamma_\ell\}\subseteq\mathcal{E}$ *such that:*

($\mathcal{L}1$) $\Gamma_0 = s$ *and* $\Gamma_\ell = t$,

($\mathcal{L}2$) $(\Gamma_j, \Gamma_{j+1}) \in \mathcal{R}$ *for all* $j = 0, 1, \ldots, \ell - 1$.

Note that a path of length ℓ *contains* $\ell + 1$ *elements.*

Let $\mathcal{E}$ be a nonempty set, $\mathcal{U} \subseteq \mathcal{E}$, and $\mathcal{K} : \mathcal{E} \to \mathcal{E}$ be a self-mapping. We denote:

$$\mathcal{E}(\mathcal{K}, \mathcal{R}) := \{s \in \mathcal{E} : (s, \mathcal{K}s) \in \mathcal{R}\},$$

$$\mathcal{P}(s, t, \mathcal{R}) := \text{the set of all paths in } R \text{ connecting } s \text{ to } t,$$

and

$$\mathcal{R}|_{\mathcal{U}} := \mathcal{R} \cap (\mathcal{U} \times \mathcal{U}),$$

which is the restriction of $\mathcal{R}$ to $\mathcal{U}$, naturally induced by $\mathcal{R}$.

Definition 15. *Let* $(\mathcal{E}, \mathfrak{P}, \curlywedge)$ *be a fuzzy metric space, and* $\mathcal{R}$ *a binary relation on* $\mathcal{E}$. *A mapping* $\mathcal{K} : \mathcal{E} \to \mathcal{E}$ *is said to be* $\mathcal{R}$*-continuous at* $s \in \mathcal{E}$ *if for every* $\mathcal{R}$*-preserving sequence* $\{s_q\}$ *with*

$$\lim_{q \to +\infty} \mathfrak{P}(s_q, s, \Im) = 1 \quad \textit{for all } \Im > 0,$$

it follows that

$$\lim_{q \to +\infty} \mathfrak{P}(\mathcal{K}s_q, \mathcal{K}s, \Im) = 1 \quad \textit{for all } \Im > 0.$$

If $\mathcal{K}$ *is* $\mathcal{R}$*-continuous at every* $s \in \mathcal{E}$, *then* $\mathcal{K}$ *is* $\mathcal{R}$*-continuous.*

Remark 3. *Every continuous mapping is* $\mathcal{R}$*-continuous for any binary relation* $\mathcal{R}$. *In particular, if* $\mathcal{R}$ *is the universal relation on* $\mathcal{E}$, *then* $\mathcal{R}$*-continuity coincides with usual continuity.*

To support our main theorems, we use the following lemma.

Lemma 2. *[10] Let* $\mathcal{K} : \mathcal{E} \to \mathcal{E}$ *be a self-mapping, and let* $\mathcal{R}$ *be a binary relation on* $\mathcal{E}$ *that is transitive and* $\mathcal{K}$*-closed. Assume there exists* $s_0 \in \mathcal{E}$ *such that* $s_0 R \mathcal{K} s_0$. *Define the sequence* $\{s_q\}$ *recursively by*

$$s_q = \mathcal{K}s_{q-1}, \quad q \in \mathbb{N}.$$

Then, for all $p, q \in \mathbb{N}$ *with* $q < p$, *the relation*

$$s_p \mathcal{R} s_q$$

holds.

2. Main Results

In this section, we present the principal results. We begin by establishing a fundamental proposition that articulates a fuzzy functional contraction condition within fuzzy metric spaces endowed with a binary relation. This condition is framed through a control function $\mathfrak{S}$ belonging to the family $\mathcal{FZ}$, which governs the contractive behavior of the mapping $\mathcal{K}$ with respect to the fuzzy metric $\mathfrak{P}$. The proposition introduces complementary hypotheses that will be needed throughout the rest of the paper.

Proposition 1. *Let $(\mathcal{E}, \mathfrak{P}, \curlywedge)$ be a fuzzy metric space, $\mathcal{R}$ a binary relation on $\mathcal{E}$, and let $\mathcal{K} : \mathcal{E} \to \mathcal{E}$ be a mapping such that for every pair of elements $s, t \in \mathcal{E}$ with $(s, t) \in \mathcal{R}$ and every positive real number $\Im$, the following inequality holds with respect to the function $\mathfrak{S}$ in the family $\mathcal{FZ}$:*

$$\mathfrak{S}\big(\mathfrak{P}(\mathcal{K}s, \mathcal{K}t, \Im), \mathfrak{P}(s, t, \Im)\big) \geq 0. \tag{1}$$

Then the following two conditions complement each other:

($\mathcal{H}_1$) *For all $s, t \in \mathcal{E}$ such that $(s, t) \in \mathcal{R}$, $\mathfrak{S}\big(\mathfrak{P}(\mathcal{K}s, \mathcal{K}t, \Im), \mathfrak{P}(s, t, \Im)\big) \geq 0$,*

($\mathcal{H}_2$) *For all $s, t \in \mathcal{E}$ such that $[s, t] \in \mathcal{R}$, $\mathfrak{S}\big(\mathfrak{P}(\mathcal{K}s, \mathcal{K}t, \Im), \mathfrak{P}(s, t, \Im)\big) \geq 0$.*

Proof. The implication $(\mathcal{H}_2) \Rightarrow (\mathcal{H}_1)$ is trivial. Conversely, assuming that $(\mathcal{H}_1)$ holds, let $s, t \in \mathcal{E}$ such that $[s, t] \in \mathcal{R}$. In this case, $(\mathcal{H}_2)$ follows directly from $(\mathcal{H}_1)$. If, however, $(t, s) \in \mathcal{R}$, then by using the symmetry of the fuzzy metric $\mathfrak{P}$ and the assumption $(\mathcal{H}_1)$, we deduce the following:

$$\xi(\mathfrak{P}(\mathcal{K}s, \mathcal{K}t, \Im), \mathfrak{P}(t, s, \Im)) = \xi(\mathfrak{P}(\mathcal{K}t, \mathcal{K}s, \Im), \mathfrak{P}(s, t, \Im)) \geq 0.$$

Thus, we have shown that $(\mathcal{H}_1) \Rightarrow (\mathcal{H}_2)$.

Theorem 1. *Let $(\mathcal{E}, \mathfrak{P}, \curlywedge)$ be a fuzzy metric space equipped with a binary relation $\mathcal{R}$, and let $\mathcal{K} : \mathcal{E} \to \mathcal{E}$ be a self-mapping. Suppose the following conditions hold:*

(i) *There exists a subset $\mathcal{U} \subseteq \mathcal{E}$ such that $\mathcal{K}(\mathcal{E}) \subseteq \mathcal{U} \subseteq \mathcal{E}$, and $(\mathcal{U}, \mathfrak{P}, \curlywedge)$ is $\mathcal{R}$-complete.*

(ii) $\mathcal{E}(\mathcal{K}, \mathcal{R}) \neq \emptyset$.

(iii) *$\mathcal{R}$ is $\mathcal{K}$-closed and transitive.*

(iv) *For every pair of elements $s, t \in \mathcal{E}$ with $(s, t) \in \mathcal{R}$ and every positive real number $\Im$, the following inequality holds with respect to the function $\mathfrak{S}$ in the family $\mathcal{FZ}$:*

$$\mathfrak{S}\big(\mathfrak{P}(\mathcal{K}s, \mathcal{K}t, \Im), \mathfrak{P}(s, t, \Im)\big) \geq 0.$$

(v) *Either $\mathcal{R}|_{\mathcal{U}}$ is $\mathfrak{P}$-self-closed, or $\mathcal{K}$ is $\mathcal{R}$-continuous.*

Then, $\mathcal{K}$ has a fixed point.

Proof. The implication from $(\mathcal{H}_2)$ to $(\mathcal{H}_1)$ is straightforward. On the other hand, assuming that $(\mathcal{H}_1)$ holds, let $s, t \in \mathcal{E}$ with $[s,t] \in \mathcal{R}$. In this scenario, $(\mathcal{H}_2)$ follows immediately from $(\mathcal{H}_1)$. However, if $(t,s) \in \mathcal{R}$, applying the fuzzy metric symmetry $\mathfrak{P}$ and $(\mathcal{H}_1)$, we conclude:

$$\mathfrak{S}\big(\mathfrak{P}(\mathcal{K}t, \mathcal{K}s, \Im), \mathfrak{P}(t,s,\Im)\big) = \mathfrak{S}\big(\mathfrak{P}(\mathcal{K}s, \mathcal{K}t, \Im), \mathfrak{P}(s,t,\Im)\big) \geq 0,$$

where $\mathfrak{P}(s,t,\Im) = \mathfrak{P}(t,s,\Im)$. This shows that $(\mathcal{H}_1) \Rightarrow (\mathcal{H}_2)$.

Since $\mathcal{E}(\mathcal{K}, \mathcal{R}) \neq \emptyset$, let s_0 be an arbitrary element such that $s_0 \in \mathcal{E}(\mathcal{K}, \mathcal{R})$. Define the sequence $\{s_q\}$ by $s_{q+1} = \mathcal{K}s_q$ for all $q \in \mathbb{N}$. Since $\mathcal{R}$ is $\mathcal{K}$-closed and $(s_0, \mathcal{K}s_0) \in \mathcal{R}$, we have the following relations:

$$(s_0, \mathcal{K}s_0), (\mathcal{K}s_0, \mathcal{K}^2 s_0), (\mathcal{K}^2 s_0, \mathcal{K}^3 s_0), \ldots, (\mathcal{K}^q s_0, \mathcal{K}^{q+1} s_0) \in \mathcal{R}.$$

Thus, the sequence $\{s_q\}$ preserves the relation $\mathcal{R}$, as shown by:

$$(s_q, s_{q+1}) \in \mathcal{R}.$$

Given that $\mathcal{K}$ satisfies the contraction inequality (1) with respect to $\mathfrak{S} \in \mathcal{FZ}$, we have:

$$\mathfrak{S}\big(\mathfrak{P}(\mathcal{K}s_{q-1}, \mathcal{K}s_q, \Im), \mathfrak{P}(s_{q-1}, s_q, \Im)\big) \geq 0.$$

Thus, we deduce the following:

$$\begin{aligned} 0 &\leq \mathfrak{S}\big(\mathfrak{P}(\mathcal{K}s_{q-1}, \mathcal{K}s_q, \Im), \mathfrak{P}(s_{q-1}, s_q, \Im)\big) \\ &= \mathfrak{S}\big(\mathfrak{P}(s_q, s_{q+1}, \Im), \mathfrak{P}(s_{q-1}, s_q, \Im)\big) \\ &< \frac{1}{\mathfrak{P}(s_{q-1}, s_q, \Im)} - \frac{1}{\mathfrak{P}(s_q, s_{q+1}, \Im)}. \end{aligned}$$

This implies that $\mathfrak{P}(s_{q-1}, s_q, \Im) < \mathfrak{P}(s_q, s_{q+1}, \Im)$, which means that the sequence $\{\mathfrak{P}(s_{q-1}, s_q, \Im)\}$ is non-decreasing and bounded above by 1. Hence, there exists a limit $m(\Im) \leq 1$ such that:

$$\lim_{q \to +\infty} \mathfrak{P}(s_q, s_{q-1}, \Im) = m(\Im) \text{ for all } \Im > 0.$$

To prove that $m(\Im) = 1$, assume the contrary, that is, $m(\Im_0) < 1$ for some $\Im_0 > 0$. Applying the contraction condition and $(\mathfrak{S}1)$, we obtain

$$0 \leq \limsup_{q \to +\infty} \mathfrak{S}\big(\mathfrak{P}(\mathcal{K}s_{q-1}, \mathcal{K}s_q, \Im_0),\, \mathfrak{P}(s_{q-1}, s_q, \Im_0)\big) < 0.$$

We reach a contradiction, which implies that $m(\Im) = 1$. Therefore, we have:

$$\lim_{q \to +\infty} \mathfrak{P}(s_q, s_{q+1}, \Im) = 1 \text{ for all } \Im > 0. \tag{2}$$

Next, we show that $\{s_q\}$ is a Cauchy sequence. Assume the opposite, that is, that $\{s_q\}$ is not Cauchy. Then, there exist $\zeta \in (0,1)$, $\Im_0 > 0$ and two subsequences $\{s_{q_i}\}$ and $\{s_{p_i}\}$ such that for sufficiently large i, we have:

$$\mathfrak{P}(s_{p_i}, s_{q_i}, \Im_0) \leq 1 - \zeta.$$

Taking into consideration Lemma 1, we get

$$\mathfrak{P}\left(s_{p_i}, s_{q_i}, \frac{\Im_0}{2}\right) \leq 1 - \zeta. \tag{3}$$

By selecting p_i as the smallest index satisfying the above inequality, we obtain

$$\mathfrak{P}\left(s_{p_i}, s_{q_i-1}, \frac{\Im_0}{2}\right) > 1 - \zeta. \tag{4}$$

In view of Lemma 2 and condition (1), we have

$$\begin{aligned} 0 &\leq \mathfrak{S}\big(\mathfrak{P}(\mathcal{K}s_{p_i-1}, \mathcal{K}s_{q_i-1}, \Im_0), \mathfrak{P}(s_{p_i-1}, s_{q_i-1}, \Im_0)\big) \\ &= \mathfrak{S}\big(\mathfrak{P}(s_{p_i}, s_{q_i}, \Im_0), \mathfrak{P}(s_{p_i-1}, s_{q_i-1}, \Im_0)\big) \end{aligned}$$

the last equation together with property ($\mathfrak{S}2$), we obtain

$$\mathfrak{P}(s_{p_i-1}, s_{q_i-1}, \Im_0) < \mathfrak{P}(s_{p_i}, s_{q_i}, \Im_0). \tag{5}$$

By (3), (4), and ($\mathcal{M}4$), we have

$$\begin{aligned} 1 - \epsilon &\geq \mathfrak{P}(s_{p_i}, s_{q_i}, \Im_0) \\ &> \mathfrak{P}(s_{p_i-1}, s_{q_i-1}, \Im_0) \\ &\geq \mathfrak{P}\left(s_{p_i-1}, s_{p_i}, \frac{\Im_0}{2}\right) * \mathfrak{P}\left(s_{p_i}, s_{q_i-1}, \frac{\Im_0}{2}\right) \\ &> \mathfrak{P}\left(s_{q_i-1}, s_{q_i}, \frac{\Im_0}{2}\right) * (1 - \zeta). \end{aligned}$$

Taking limit as $i \to +\infty$ and using (2), we obtain

$$\lim_{i \to +\infty} \mathfrak{P}(s_{p_i}, s_{q_i}, \Im_0) = \lim_{i \to +\infty} \mathfrak{P}(s_{p_i-1}, s_{q_i-1}, \Im_0) = 1 - \zeta. \tag{6}$$

Based on the preceding analysis, we identify the two sequences

$$\{T_i\} = \{\mathfrak{P}(s_{p_i-1}, s_{q_i-1}, \Im_0)\}, \quad \{S_i\} = \{\mathfrak{P}(s_{p_i}, s_{q_i}, \Im_0)\}.$$

From the previous discussion, we consider the sequences

$$\{T_i\} = \{\mathfrak{P}(s_{p_i-1}, s_{q_i-1}, \Im_0)\}, \quad \{S_i\} = \{\mathfrak{P}(s_{p_i}, s_{q_i}, \Im_0)\},$$

both converging to the same limit $1 - \zeta < 1$. Moreover, noting that $\mathcal{K}$ fulfills the contraction condition (1) with respect to $\mathfrak{S} \in \mathcal{FZ}$, and property ($\mathfrak{S}3$), we get

$$0 \leq \lim_{i \to +\infty} \sup \mathfrak{S}\big(\mathfrak{P}(s_{p_i-1}, s_{q_i-1}, \Im_0), \mathfrak{P}(s_{p_i}, s_{q_i}, \Im_0)\big) < 0.$$

This leads to a contradiction, and therefore $\{s_q\}$ must be a Cauchy sequence. Since $\{s_q\} \subseteq \mathcal{K}(\mathcal{E}) \subseteq \mathcal{U}$, it follows that $\{s_q\}$ is an $\mathcal{R}$-preserving Cauchy sequence in $\mathcal{U}$. Given that $\mathcal{U}$ is $\mathcal{R}$-complete, the sequence converges to some point $\tilde{s} \in \mathcal{U}$. If $\mathcal{K}$ is $\mathcal{R}$-continuous, we obtain $\tilde{s} = \lim_{q\to+\infty} s_{q+1} = \lim_{q\to+\infty} \mathcal{K}s_q = \mathcal{K}\tilde{s}$. Thus, $\tilde{s}$ is a fixed point of $\mathcal{K}$.

Next, suppose that the binary relation $\mathcal{R}$ is $\mathfrak{P}$-self-closed. Since the sequence $\{s_q\}$ preserves $\mathcal{R}$ and converges to $\tilde{s}$, there exists a subsequence $\{s_{q_i}\}$ such that $[s_{q_i}, \tilde{s}] \in \mathcal{R}|_{\mathcal{U}}$ for all i, and by Proposition 1, $[s_{q_i}, \tilde{s}] \in \mathcal{R}$ and $s_{q_i} \to \tilde{s}$. Without losing generality, we assume that $s_{\mathrm{q_i}} \neq \tilde{s}$ for every $i \in \mathbb{N}$. Now making use contraction inequality (1), we obtain

$$0 \leq \mathfrak{S}\big(\mathfrak{P}(\mathcal{K}s_{q_i}, \mathcal{K}\tilde{s}, \Im), \mathfrak{P}(s_{q_i}, \tilde{s}, \Im)\big)$$

Finally, we will demonstrate that $\tilde{s}$ is a fixed point of $\mathcal{K}$. Suppose that $\mathcal{K}\tilde{s} \neq \tilde{s}$. Then, we have $\mathfrak{P}(\tilde{s}, \mathcal{K}\tilde{s}, \Im) < 1$. Now, by ($\mathfrak{S}2$) and ($\mathfrak{S}3$), we obtain

$$\begin{aligned} 0 &\leq \lim_{i\to+\infty} \sup \mathfrak{S}\big(\mathfrak{P}(s_{q_i+1}, \mathcal{K}\tilde{s}, \Im), \mathfrak{P}(s_{q_i}, \tilde{s}, \Im)\big) \\ &= \lim_{i\to+\infty} \sup \mathfrak{S}\big(\mathfrak{P}(\mathcal{K}s_{q_i}, \mathcal{K}\tilde{s}, \Im), \mathfrak{P}(s_{q_i}, \tilde{s}, \Im)\big) \\ &\leq \lim_{i\to+\infty} \sup \Big(\frac{1}{\mathfrak{P}(s_{q_i}, \tilde{s}, \Im)} - \frac{1}{\mathfrak{P}(\mathcal{K}s_{q_i}, \mathcal{K}\tilde{s}, \Im)}\Big) \\ &= 1 - \frac{1}{\mathfrak{P}(\tilde{s}, \mathcal{K}\tilde{s}, \Im)}. \end{aligned}$$

Consequentlly,

$$0 \leq \lim_{i\to+\infty} \sup \mathfrak{S}\big(\mathfrak{P}(s_{q_i+1}, \mathcal{K}\tilde{s}, \Im), \mathfrak{P}(s_{q_i}, \tilde{s}, \Im)\big) < 0.$$

This leads to a contradiction, which necessitates that $\mathfrak{P}(\tilde{s}, \mathcal{K}\tilde{s}, \Im) = 1$. As a result, we conclude that $\mathcal{K}\tilde{s} = \tilde{s}$, indicating that $\tilde{s}$ is a fixed point of the mapping $\mathcal{K}$.

Theorem 2. *Under the assumptions of Theorem 1, if for every pair of points $s, t \in \mathcal{E}$ the set $\mathcal{P}(s, t, \mathcal{R})$ is nonempty, then self-mapping $\mathcal{K}$ possesses a unique fixed point.*

Proof. We proceed by contradiction. Suppose that there exist two distinct fixed points, s and $\tilde{s}$, of the mapping $\mathcal{K}$. Since $\mathcal{P}(s, \tilde{s}, \mathcal{R})$ is nonempty, there exists a finite sequence $\{\Gamma_0, \Gamma_1, \ldots, \Gamma_q\}$ in $\mathcal{E}$ with length q, satisfying:

$$\Gamma_0 = s, \quad \Gamma_q = \tilde{s}, \quad \text{and} \quad (\Gamma_j, \Gamma_{j+1}) \in \mathcal{R} \quad \text{for } j = 0, 1, \ldots, q-1.$$

Due to the transitivity of $\mathcal{R}$, it follows that $(\Gamma_0, \Gamma_q) \in \mathcal{R}$. As $\mathcal{K}$ satisfies the contraction inequality (1) with respect to $\mathfrak{S} \in \mathcal{FZ}$, we have:

$$\begin{aligned} 0 &\leq \mathfrak{S}\big(\mathfrak{P}(\mathcal{K}\Gamma_0, \mathcal{K}\Gamma_q, \Im), \mathfrak{P}(\Gamma_0, \Gamma_q, \Im)\big) \\ &< \frac{1}{\mathfrak{P}(\Gamma_0, \Gamma_q, \Im)} - \frac{1}{\mathfrak{P}(\mathcal{K}\Gamma_0, \mathcal{K}\Gamma_q, \Im)} \\ &= \frac{1}{\mathfrak{P}(s, \tilde{s}, \Im)} - \frac{1}{\mathfrak{P}(s, \tilde{s}, \Im)}. \end{aligned}$$

Thus, $\mathfrak{P}(s, \tilde{s}, \Im) < \mathfrak{P}(s, \tilde{s}, \Im)$. This inequality leads to a contradiction. Therefore, the fixed point of $\mathcal{K}$ is unique.

Corollary 1. *Let $(\mathcal{E}, \mathfrak{P}, \curlywedge)$ be a fuzzy metric space endowed with a binary relation $\mathcal{R}$, and let $\mathcal{K} : \mathcal{E} \to \mathcal{E}$ be a self-mapping. Suppose the following conditions are satisfied:*

- **(i)** $(\mathcal{E}, \mathfrak{P}, \curlywedge)$ *is complete;*
- **(ii)** $\mathcal{E}(\mathcal{K}, \mathcal{R}) \neq \emptyset$*;*
- **(iii)** $\mathcal{R}$ *is $\mathcal{K}$-closed and transitive;*
- **(iv)** *For every pair $s, t \in \mathcal{E}$ with $(s, t) \in \mathcal{R}$ and each $\Im > 0$, the following inequality holds for some $\mathfrak{S} \in \mathcal{FZ}$:*
$$\mathfrak{S}\big(\mathfrak{P}(\mathcal{K}s, \mathcal{K}t, \Im), \mathfrak{P}(s, t, \Im)\big) \geq 0;$$
- **(v)** *Either $\mathcal{R}$ is $\mathfrak{P}$-self-closed or $\mathcal{K}$ is $\mathcal{R}$-continuous.*

Then, the mapping $\mathcal{K}$ has a fixed point.

Proof. The result follows directly by taking $\mathcal{U} = \mathcal{E}$ in Theorem 1.

Example 4. *Consider the metric space $(\mathcal{E}, \mathfrak{P})$, where $\mathcal{E} = \mathbb{R}$ is equipped with the fuzzy metric*
$$\mathfrak{P}(s, t, \Im) = \frac{\Im}{\Im + |s - t|},$$
for all $s, t \in \mathcal{E}$ and $\Im > 0$. This space is complete. Define the binary relation $\mathcal{R}$ on $\mathcal{E}$ by
$$\mathcal{R} = \{(s, t) \in \mathcal{E}^2 : (s - t)(s + t + 11) = 0\}.$$

Define the mapping $\mathcal{K} : \mathcal{E} \to \mathcal{E}$ by
$$\mathcal{K}(s) = s^2 + 11s + 16.$$

The mapping $\mathcal{K}$ is continuous. For any $s, t \in \mathcal{E}$ with $(s, t) \in \mathcal{R}$, it follows that
$$\mathcal{K}(s) = \mathcal{K}(t), \text{ hence } (\mathcal{K}(s), \mathcal{K}(t)) \in \mathcal{R},$$
which means $\mathcal{R}$ is $\mathcal{K}$-closed. Consider the point $-8 \in \mathcal{E}$. Then $\mathcal{K}(-8) = (-8)^2 + 11(-8) + 16 = -8$, therefore, $(-8, \mathcal{K}(-8)) \in \mathcal{R}$, confirming $\mathcal{E}(\mathcal{K}, \mathcal{R}) \neq \emptyset$.

Define the function $\mathfrak{S} : (0, 1] \times (0, 1] \to \mathbb{R}$ by
$$\mathfrak{S}(r, b) = \frac{7}{9}\left(\frac{1}{b} - 1\right) - \left(\frac{1}{r} - 1\right).$$

For any $r, b \in \mathcal{E}$ with $(r, b) \in \mathcal{R}$,
$$\begin{aligned}
\mathfrak{S}(\mathfrak{P}(\mathcal{K}s, \mathcal{K}t, \Im), \mathfrak{P}(s, t, \Im)) &= \frac{7}{9}\left(\frac{1}{\mathfrak{P}(s, t, \Im)} - 1\right) - \left(\frac{1}{\mathfrak{P}(\mathcal{K}s, \mathcal{K}t, \Im)} - 1\right). \\
&= \frac{7}{9}\frac{|\ s - t\ |}{\Im} - \frac{|\ \mathcal{K}s - \mathcal{K}t\ |}{\Im} \\
&= \frac{7}{9}\frac{|\ s - t\ |}{\Im} - \frac{|\ (s^2 + 11s + 16) - (t^2 + 11t + 16)\ |}{\Im} \\
&= \frac{7}{9}\frac{|\ s - t\ |}{\Im} \\
&\geq 0.
\end{aligned}$$

Hence,

$$\mathfrak{S}(\mathfrak{P}(\mathcal{K}s, \mathcal{K}t, \Im), \mathfrak{P}(s, t, \Im)) \geq 0,$$

confirming that $\mathcal{K}$ satisfies the contraction inequality (1) with respect to $\mathfrak{S}$.

By Theorem 1, $\mathcal{K}$ has at least one fixed point in $\mathcal{E}$.

Corollary 2. *Let $(\mathcal{E}, \mathfrak{P}, \curlywedge)$ be a fuzzy metric space endowed with a binary relation $\mathcal{R}$, and let $\mathcal{K} : \mathcal{E} \to \mathcal{E}$ be a self-mapping. Suppose the following conditions are satisfied:*

(i) *For all $s, t \in \mathcal{E}$ with $(s, t) \in \mathcal{R}$ and for each $\Im > 0$, the inequality*

$$\psi\big(\mathfrak{P}(s, t, \Im)\big) \leq \mathfrak{P}(\mathcal{K}s, \mathcal{K}t, \Im)$$

holds for $\psi \in \Psi$;

(ii) *There exists a subset $\mathcal{U} \subseteq \mathcal{E}$ such that $\mathcal{K}(\mathcal{E}) \subseteq \mathcal{U}$ and $(\mathcal{U}, \mathfrak{P}, \curlywedge)$ is $\mathcal{R}$-complete;*

(iii) *$\mathcal{E}(\mathcal{K}, \mathcal{R}) \neq \emptyset$;*

(iv) *The relation $\mathcal{R}$ is $\mathcal{K}$-closed and transitive;*

(v) *Either $\mathcal{R}|_{\mathcal{U}}$ is $\mathfrak{P}$-self-closed or $\mathcal{K}$ is $\mathcal{R}$-continuous.*

Then, the mapping $\mathcal{K}$ has a fixed point.

Proof. Define $\mathfrak{S} : (0, 1] \times (0, 1] \to \mathbb{R}$ by

$$\mathfrak{S}(r, b) = \frac{1}{\psi(b)} - \frac{1}{r}, \quad \text{for all } r, b \in (0, 1],$$

where $\eta \in \mathcal{H}$. The result then follows by applying Theorem 1.

Corollary 3. *Let $(\mathcal{E}, \mathfrak{P}, \curlywedge)$ be a fuzzy metric space endowed with a binary relation $\mathcal{R}$, and let $\mathcal{K} : \mathcal{E} \to \mathcal{E}$ be a self-mapping. Suppose the following conditions are satisfied:*

(i) *For all $s, t \in \mathcal{E}$ with $(s, t) \in \mathcal{R}$ and for each $\Im > 0$, the inequality*

$$\eta\big(\mathfrak{P}(\mathcal{K}s, \mathcal{K}t, \Im)\big) \leq \hbar \cdot \eta\big(\mathfrak{P}(s, t, \Im)\big)$$

holds for some $\hbar \in (0, 1)$ and a function $\eta \in \mathcal{H}$;

(ii) *There exists a subset $\mathcal{U} \subseteq \mathcal{E}$ such that $\mathcal{K}(\mathcal{E}) \subseteq \mathcal{U}$ and $(\mathcal{U}, \mathfrak{P}, \curlywedge)$ is $\mathcal{R}$-complete;*

(iii) *$\mathcal{E}(\mathcal{K}, \mathcal{R}) \neq \emptyset$;*

(iv) *The relation $\mathcal{R}$ is $\mathcal{K}$-closed and transitive;*

(v) *Either $\mathcal{R}|_{\mathcal{U}}$ is $\mathfrak{P}$-self-closed or $\mathcal{K}$ is $\mathcal{R}$-continuous.*

Then, the mapping $\mathcal{K}$ has a fixed point.

Proof. Define $\mathfrak{S} : (0,1] \times (0,1] \to \mathbb{R}$ by

$$\mathfrak{S}(r,b) = \frac{1}{\eta^{-1}(\hbar \cdot \eta(b))} - \frac{1}{r}, \quad \text{for all } r,b \in (0,1],$$

where $\eta \in \mathcal{H}$. The result then follows by applying Theorem 1.

Corollary 4. *If the $\mathcal{R}$-completeness of the subset $\mathcal{U}$ is replaced by the completeness of $\mathcal{E}$, and the $\mathcal{R}$-continuity of $\mathcal{K}$ is replaced by standard continuity, then the conclusion of Theorem 1 remains valid.*

Proof. The assertion follows immediately from Remark 2 and Remark 3.

Corollary 5. *Let $(\mathcal{E}, \mathfrak{P}, \curlywedge)$ be a fuzzy metric space equipped with a binary relation $\mathcal{R}$, and let $\mathcal{K} : \mathcal{E} \to \mathcal{E}$ be a self-mapping. Suppose the following conditions hold:*

(i) *For all $s,t \in \mathcal{E}$ with $(s,t) \in \mathcal{R}$ and for every $\Im > 0$,*

$$\frac{1}{\mathfrak{P}(\mathcal{K}s, \mathcal{K}t, \Im)} - 1 \leq \hbar \left(\frac{1}{\mathfrak{P}(s,t,\Im)} - 1 \right),$$

for some constant $\hbar \in (0,1)$;

(ii) *There exists a subset $\mathcal{U} \subseteq \mathcal{E}$ such that $\mathcal{K}(\mathcal{E}) \subseteq \mathcal{U}$ and $(\mathcal{U}, \mathfrak{P}, \curlywedge)$ is $\mathcal{R}$-complete;*

(iii) *$\mathcal{E}(\mathcal{K}, \mathcal{R}) \neq \emptyset$;*

(iv) *The relation $\mathcal{R}$ is both $\mathcal{K}$-closed and transitive;*

(v) *Either $\mathcal{R}|_{\mathcal{U}}$ is $\mathfrak{P}$-self-closed or $\mathcal{K}$ is $\mathcal{R}$-continuous.*

Then the mapping $\mathcal{K}$ admits a fixed point.

Proof. The result follows by defining $\mathfrak{S} : (0,1] \times (0,1] \to \mathbb{R}$ as

$$\mathfrak{S}(r,b) = \hbar \left(\frac{1}{b} - 1 \right) - \frac{1}{r} + 1 \quad \text{for all } r,b \in (0,1],$$

and applying Theorem 1.

Application

We solve an integral equation under a given binary relation by applying Theorem 1. Consider the integral equation

$$\delta(\ell) = r(\ell) + \beta \int_u^v h(e,\ell)\, g\big(e, \delta(e)\big)\, de, \quad \ell \in J := [u,v], \tag{7}$$

where δ is an arbitrary function defined on the interval $J = [u,v]$, $r : J \to \mathbb{R}$, $h : J \times J \to \mathbb{R}$, $g : J \times \mathbb{R} \to \mathbb{R}$, $\beta > 0$ is a positive parameter, and $\mathcal{E} := C(J, \mathbb{R})$ denotes the space of all continuous functions from J to $\mathbb{R}$.

Define Θ as the class of all mappings $\theta : [0,+\infty) \to [0,+\infty)$ satisfying:

(i) θ is non-decreasing;

(ii) For every $z \geq 0$, $\theta(z) \leq z$.

Consider the space $\mathcal{E} = C(J, \mathbb{R})$ of continuous functions on J, equipped with the supremum metric

$$d(\delta, z) = \sup_{\ell \in J} |\delta(\ell) - z(\ell)|.$$

Define the fuzzy metric function $\mathfrak{P} : \mathcal{E} \times \mathcal{E} \times (0, +\infty) \to [0, 1]$ as

$$\mathfrak{P}(\delta, z, \Im) = \frac{\Im}{\Im + d(\delta, z)}, \quad \Im > 0.$$

With the operation $\curlywedge$ on $[0, 1]$ defined by

$$\wp_1 \curlywedge \wp_2 = \wp_1 \times \wp_2, \quad \wp_1, \wp_2 \in [0, 1],$$

the triple $(\mathcal{E}, \mathfrak{P}, \curlywedge)$ forms a complete fuzzy metric space. We are now ready to present the main result of this section.

Theorem 3. *Assume the following assumptions for equation 7:*

$\mathfrak{S}1\rangle$ $\sup_{\ell \in J} \int_u^v |h(e, \ell)|\, de \leq \frac{2}{3\beta}$, *for all* $e, \ell \in J$,

$\mathfrak{S}1\rangle$ $|g(e, \delta_1(e)) - g(e, \delta_2(e))| \leq \theta(|\delta_1(e) - \delta_2(e)|)$, *for all* $\delta_1, \delta_2 \in \mathbb{R}$.

Then equation 7 admits at least one solution in $\mathcal{E}$.

Proof. Consider the operator $\mathcal{K} : \mathcal{E} \to \mathcal{E}$ defined by

$$\mathcal{K}\delta(\ell) = r(\ell) + \beta \int_a^b h(e, \ell)\, g(e, \delta(e))\, de, \quad \ell \in J := [a, b],$$

where $\mathcal{E} = C(J, \mathbb{R})$ and the binary relation

$$\mathcal{R} = \{(\delta, z) \in \mathcal{E} \times \mathcal{E} : \delta(\ell) \leq z(\ell) \text{ for all } \ell \in J\}$$

is defined. Equipped with the metric function $\mathfrak{P} : \mathcal{E} \times \mathcal{E} \times (0, +\infty) \to [0, 1]$, $\mathfrak{P}(\delta, z, \Im) = \frac{\Im}{\Im + d(\delta, z)}$, for all $\Im > 0$, where $d(\delta, z) = \sup_{\ell \in J} |\delta(\ell) - z(\ell)|$, the space $(\mathcal{E}, \mathfrak{P}, \curlywedge)$ is $\mathcal{R}$-complete. Next, Suppose a sequence $\{\delta_q\}$ in $\mathcal{E}$ is $\mathcal{R}$-preserving and converges to δ with respect to $\mathfrak{P}$, that is $\delta_q \xrightarrow{\mathfrak{P}} \delta$. Therefore, the sequence $\{\delta_q\}$ satisfies the monotonic chain

$$\delta_0(\ell) \leq \delta_1(\ell) \leq \delta_2(\ell) \leq \cdots \leq \delta_q(\ell) \leq \delta_{q+1}(\ell) \leq \cdots, \quad \ell \in J.$$

This convergence entails $\delta_n(\ell) \leq \delta(\ell)$ for all $q \in \mathbb{N}$, implying the existence of a subsequence $\{\delta_{n_i}\}$ such that $[\delta_{n_i}, \delta] \in \mathcal{R}$, $q \in \mathbb{N}$, and thus $\mathcal{R}$ is $\mathfrak{P}$-self-closed.

If $\delta, z \in \mathcal{E}$ satisfy $\delta(\ell) \leq z(\ell)$ for every ℓ and $h(\ell, e) \geq 0$, then $\mathcal{K}$ preserves the order

$$(\mathcal{K}\delta)(\ell) = r(\ell) + \int_u^v h(e,\ell)\, g(e,\delta(e))\, de \leq r(\ell) + \int_u^v h(e,\ell)\, g(e,z(e))\, de = (\mathcal{K}z)(\ell).$$

Therefore,

$$(\mathcal{K}\delta)(\ell) \leq (\mathcal{K}z)(\ell).$$

Hence, $(\mathcal{K}\delta, \mathcal{K}z) \in \mathcal{R}$, thereby establishing that $\mathcal{R}$ is $\mathcal{K}$-closed.

Next, consider the pair $(\delta, z) \in \mathcal{R}$, we have

$$\begin{aligned}
\frac{1}{\mathfrak{P}(\mathcal{K}s, \mathcal{K}t, \Im)} - 1 &= \frac{d(\mathcal{K}\delta, \mathcal{K}z)}{\Im} \\
&= \frac{\sup_{\ell \in J} |(\mathcal{K}\delta)(\ell) - (\mathcal{K}z)(\ell)|}{\Im} \\
&\leq \beta \sup_{\ell \in J} \int_u^v |h(e,\ell)|\, de \int_u^v |g(e,\delta(e)) - g(e,z(e))|\, de \\
&\leq \frac{2}{3\Im} \int_u^v |g(e,\delta(e)) - g(e,z(e))|\, de \\
&\leq \frac{2}{3\Im} \int_u^v \theta(|\delta(e) - z(e)|)\, de \\
&\leq \frac{2}{3\Im} \theta(d(\delta, z)) \\
&\leq \frac{2}{3} \frac{d(\delta, z)}{\Im} \\
&= \frac{2}{3} \Big(\frac{1}{\mathfrak{P}(\delta, z, \Im)} - 1 \Big).
\end{aligned}$$

Consequently, the operator $\mathcal{K}$ satisfies the contractive condition (iv) in Theorem 1 by employing the control function $\mathfrak{S}(\ell, e) \in \mathcal{FZ}$ defined as

$$\mathfrak{S}(\ell, e) = \frac{2}{3} \left(\frac{1}{\ell} - 1 \right) - \left(\frac{1}{e} - 1 \right).$$

Let $\gamma \in \mathcal{E}$ be such that $\gamma(\ell) \leq (\mathcal{K}\gamma)(\ell)$, which implies $(\gamma, \mathcal{K}\gamma) \in \mathcal{R}$, so the set $\mathcal{E}(\mathcal{K}, \mathcal{R})$ is nonempty. Now, Define $\varpi = \max\{\delta, z\}$. Then $\delta(\ell) \leq \varpi(\ell)$ and $z(\ell) \leq \varpi(\ell)$, which shows

$$(\delta, \varpi) \in \mathcal{R} \quad \text{and} \quad (z, \varpi) \in \mathcal{R}.$$

Hence, the triple $\{\delta, \varpi, z\}$ forms a path in $\mathcal{R}$ connecting δ and z. All assumptions of Theorem 2 are thus satisfied, concluding that $\mathcal{K}$ has a unique fixed point that solves the integral equation.

3. Conclusion

This study introduced a novel fuzzy contraction inequality essential for establishing the existence and uniqueness of fixed points in fuzzy metric spaces endowed with a binary relation. Notably, unlike many classical approaches, we did not require the completeness of the entire space or any of its subspaces. Instead, our results rely on the relatively weaker concept of $\mathcal{R}$-completeness within subspaces of the whole space. Moreover, we relaxed the traditional continuity assumption on the involved mapping, replacing it with the more generalized notions of $\mathcal{R}$-continuity or the $\mathfrak{P}$-self-closedness of the restriction of $\mathcal{R}$ to a subset $\mathcal{U}$.

Our framework, leveraging versatile control functions $\mathfrak{S}$, enriches the theory of relation-theoretic and fuzzy fixed points. These findings also pave the way for future research on coincidence and common fixed points within relation-theoretic fuzzy metric spaces, highlighting the broad potential and applicability of the approach.

Future work may extend this framework to broader contraction classes, including vector-valued and multi-dimensional fuzzy metric spaces. The results provide a solid basis for analyzing nonlinear operator equations and related problems involving uncertainty modeled by fuzzy metrics and relations.

In conclusion, this study unifies relational structures and fuzzy metric theory to establish robust fixed point results, opening pathways for further research on coincidence and common fixed points in abstract metric and relational frameworks.

Declarations:

Competing interests

The author reports no conflicts of interest

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Authors' contributions

All authors read and approved the final manuscript.

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