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Asymptotic Solutions to a Singularly Perturbed Partial Integro-Differential Equation with a Rapidly Oscillating Right-Hand Side

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Abstract. The paper considers the Cauchy problem for a singularly perturbed integro-differential partial differential equation with a rapidly oscillating right-hand side. When considering such problems, it turned out that the existing technique for regularizing singularly perturbed equations is not effective and requires significant rethinking. The development of a new technique for constructing regularized asymptotic solutions for integro-differential equations with partial derivatives and exponential inhomogeneity constitutes the main content of this work. The problem was regularized and the normal and unique solvability of general iterative problems was proved. The asymptotic convergence of formal solutions is proved and a solution to the first iterative problem is constructed.

2020 Mathematics Subject Classifications: 45K05

Key Words and Phrases: Singular perturbation, partial integro-differential equation, rapidly oscillating right-hand side, solvability of iterative problems, regularization of an integral

1. Introduction

In the paper, we consider the Cauchy problem for the singularly perturbed integro-differential equation with partial derivatives:

$$\begin{aligned} &\varepsilon \tfrac{\partial y(x,t,\varepsilon)}{\partial x} = a(x)y(x,t,\varepsilon) + \int\limits_{x_0}^{x} K(x,t,s)y(s,t,\varepsilon)ds + h_1(x,t) + \\ &+ h_2(x,t)e^{\frac{i\beta(x)}{\varepsilon}}, \quad y(x_0,t,\varepsilon) = y^0(t) \quad ((x,t) \in [x_0,X] \times [0,T]) \end{aligned} \tag{1.1}$$

where $\beta'(x) > 0$, $a(x)$ is a scalar functions, $y^0(t)$ constant, $\varepsilon > 0$ is a small parameter. The purpose of the work is to generalize the algorithm of the regularization method [1, 2]

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on problems of type (1.1) and analysis of singularities in the solution $y(t, \varepsilon)$, introduced by the integral operator $\int_0^t K(x, t, s)y(s, t, \varepsilon)ds$ and the rapidly oscillating inhomogeneity $h_2(x, t)e^{\frac{i\beta(x)}{\varepsilon}}$. For the sake of simplicity, a scalar version of this problem is studied.

Lomov's regularization method [1, 2] was developed to construct regularized asymptotic solutions of ordinary differential equations in the case of stability of the spectrum of the limit operator. Problems devoted to the construction of regularized asymptotic solutions of Cauchy problems in the presence of weak turning points of the limit operator are considered in the works of [3–5], initialization in the work of [6]. The works of [7] considered the problems of constructing a regularized asymptotic solution to a nonlinear differential equation in a Banach space and the analytical aspects of the theory of Tikhonov systems [8]. Singularly perturbed ordinary differential equations with rapidly oscillating coefficients from the perspective of the regularization method were carried out in the work of [9]. The justification of the regularization method for linear and nonlinear integro-differential equations with a zero operator of the differential part was studied in the works of [10, 11].

Singularly perturbed integro-differential equations with rapidly oscillating coefficients and rapidly changing kernels in the case of a multiple spectrum were considered in the studies of [12–14], with rapidly oscillating coefficients and with rapidly oscillating inhomogeneities in the works of [15–21]. The Fredholm integro-differential equation with a rapidly decreasing kernel and an exponentially oscillating inhomogeneity was studied in the work of [22]. The integro-differential Cauchy problem with exponential inhomogeneity and with a spectral value that vanishes at an isolated point on a segment of an independent variable is considered in the work of [23]. The problem belongs to the class of singularly perturbed equations with an unstable spectrum and has not been considered previously in the presence of an integral operator. It is especially difficult to study it in the vicinity of zero spectral value of the inhomogeneity. In this case, it is not possible to apply the well-known procedure of the Lomov regularization method, so the researchers chose a method for constructing the asymptotics of the solution to the original problem, based on the use of the regularized asymptotics of the fundamental solution of the corresponding homogeneous equation, the construction of which from the standpoint of the regularization method has not been considered until now.

It should be noted that singularly perturbed differential and integro-differential equations with fractional derivatives in the absence and presence of rapidly oscillating components were considered in works [24–28]. In these works, the ideas of the regularization method were generalized for equations with fractional derivatives, regularized asymptotic solutions of problems were constructed, and the influence of rapidly oscillating coefficients on the leading term of the asymptotics was studied.

For the first time, singularly perturbed partial integro-differential equations from the standpoint of the Lomov regularization method [1] were studied in the works of [29, 30]. First, a system of integro-differential partial differential equations with slowly varying kernels is considered. It turned out that the regularization procedure significantly depends on the type of integral operator. It turns out that the most difficult case is when the upper limit of the integral is not a differentiation variable. The case is studied when the upper limit of the integral operator coincides with the differentiation variable. Next, we consider a system of integro-differential partial differential equations with rapidly changing kernels. The study revealed that the type of upper limit of the integral operator in such equations leads to two fundamentally different situations. The most difficult situation arises when the upper bound of the integration operator does not coincide with the differentiation variable. As studies have shown, in this case the integral operator can have characteristic values, and to construct the asymptotics, more stringent conditions on the initial data of the problem will be required. A singularly perturbed partial integro-differential equation with rapidly oscillating coefficients in the absence of resonance was studied in the works of [31–33].

Thus, in this paper, S. A. Lomov's regularization method [1] is generalized to integro-differential partial differential equations with exponentially oscillating right-hand side. The influence of oscillating components on the structure of the asymptotics of the solution to the original problem will be revealed.

Denote by $\lambda_1(x) = -a(x)$, $\beta'(x)$ is a frequency of rapidly oscillating inhomogeneity. In the function $\lambda_2(x) = \beta'(x)$ will be called the spectrum of a rapidly oscillating inhomogeneity. We assume that the conditions are fulfilled:

(i) $a(x), \beta(x) \in C^\infty([x_0, X], \mathbb{R})$, $h_j(x,t) \in C^\infty\left([x_0, X] \times [0,T], \mathbf{C^2}\right), j = 1, 2,$
$K(x,t,s) \in C^\infty\left(\{x_0 \le x \le s \le X,\ 0 \le t \le T\}, \mathbf{C^2}\right);$

(ii) $\lambda_1(x) \neq \lambda_2(x),\ \ \lambda_j(x) \neq 0\ (\forall x \in [x_0, X]),\ j = \overline{1,2};$

(iii) $\lambda_1(x) < 0\ \ (\forall x \in [x_0, X]).$

We will develop an algorithm for constructing a regularized [1] asymptotic solution of problem (1.1).

2. Regularization of the problem

Based on the spectrum $\lambda_1(x) = -a(x)$, the frequency of the rapidly oscillating inhomogeneity $\lambda_2(x) = i\beta'(x)$, we introduce regularizing functions of the form:

$$\tau_j = \frac{1}{\varepsilon}\int_{x_0}^{x} \lambda_j(\theta)d\theta \equiv \frac{\psi_j(x)}{\varepsilon}, \quad j = \overline{1,2}.$$

We introduce the notation $\tau = (\tau_1, \tau_2)$, $\psi_j(x,\varepsilon) = (\psi_1(x,\varepsilon), \psi_2(x,\varepsilon))$ and instead of the desired solution $y(x,t,\varepsilon)$ to problem (1.1), we will study some extended function $u(x,t,\tau,\varepsilon)$ such that its restriction identically

$$u(x,t,\tau,\varepsilon)\big|_{\tau=\frac{\psi(x)}{\varepsilon}} \equiv y(x,t,\varepsilon)$$

coincides with the desired solution to problem (1.1). We find the total derivative for the function $u\left(x,t,\frac{\psi(x)}{\varepsilon},\varepsilon\right)$, and instead of problem (1.1), consider the problem

$$\varepsilon\frac{\partial u}{\partial x}+\sum_{j=1}^{2}\lambda_j(x)\frac{\partial u}{\partial \tau_j}-\lambda_1(x)u-\int_{x_0}^{x}K(x,t,s)u(s,t,\frac{\psi(s)}{\varepsilon},\varepsilon)ds=h_1(x,t)+$$
$$+h_2(x,t)e^{\tau_2}\sigma,\;\; u(x_0,t,0,\varepsilon)=y^0(t),\;\; \left((x,t)\in[x_0,X]\times[0,T]\right),\;\; \sigma=e^{\frac{i\beta(x_0)}{\varepsilon}}. \tag{2.1}$$

However, it cannot be considered fully regularized, since it does not regularize the integral

$$Ju\equiv J\left(u(x,t,\tau,\sigma,\varepsilon)|_{x=s,\tau=\psi(s)/\varepsilon}\right)=\int_{x_0}^{x}K\left(x,t,s\right)u\left(s,t,\frac{\psi(s)}{\varepsilon},\sigma,\varepsilon\right)ds \tag{2.2}$$

has not been regularized in it. To regularize it, we introduce a class M_ε that is asymptotically invariant with respect to the operator (see [1], p. 62).

Definition 1. *We will say that a function $u(x,t,\tau,\sigma)$ belongs to the class U, if it can be represented as a sum*

$$U=\left\{u(x,t,\tau,\sigma):\;\; u=u_0(x,t,\sigma)+\sum_{j=1}^{2}u_j(x,t,\sigma)e^{\tau_j},\right.$$
$$\left. u_j(x,t,\sigma)\in C^{\infty}\left([0,X]\times[0,T],\mathbb{C}^2\right),j=\overline{0,2}\right\}. \tag{2.3}$$

As a class M_ε we take restrictions of the class M_ε for $\tau=\frac{\psi(x)}{\varepsilon}$. We prove that $U|_{\tau=\frac{\psi(x)}{\varepsilon}}$ is invariant with respect to the integral operator J.

Theorem 1. *Let conditions (i) - and (iii) be satisfied. Then the class $M_\varepsilon=U|_{\tau=\frac{\psi(x)}{\varepsilon}}$ is asymptotically invariant with respect to the integral operator J.*

Proof. Substituting (2.2) into Ju, we have:

$$Ju(x,t,\tau)=\int_{x_0}^{x}K(x,t,s)u_0(s,t)ds+\sum_{j=1}^{2}\int_{x_0}^{x}K(x,t,s)u_j(s,t)e^{\frac{1}{\varepsilon}\int_{x_0}^{s}\lambda_j(\theta)d\theta}ds.$$

It is necessary to show that integrals containing exponentials are expanded into an asymptotic series in powers ε (for $\varepsilon\to+0$). Applying the operation of integration by parts; we will have

$$J_j(x,t,\varepsilon)=\varepsilon\int_{x_0}^{x}\frac{K(x,t,s)u_j(s,t)}{\lambda_j(s)}d\left(e^{\frac{1}{\varepsilon}\int_{x_0}^{s}\lambda_j(\theta)d\theta}\right)=$$
$$=\varepsilon\left[\frac{K(x,t,s)u_j(s,t)}{\lambda_j(s)}\,e^{\frac{1}{\varepsilon}\int_{x_0}^{s}\lambda_j(\theta)d\theta}\Bigg|_{s=x_0}^{s=x}-\right.$$

$$
-\int\limits_{x_0}^{x}\left(\frac{\partial}{\partial s}\frac{K(x,t,s)u_j(s,t)}{\lambda_j(s)}\right)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}\lambda_j(\theta)d\theta}ds\Bigg] =
$$

$$
=\varepsilon\left[\frac{K(x,t,x)u_j(x,t)}{\lambda_j(x)}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_j(\theta)d\theta}-\frac{K(x,t,x_0)u_j(x_0,t)}{\lambda_j(x_0)}\right]-
$$

$$
-\varepsilon\int\limits_{x_0}^{x}\left(\frac{\partial}{\partial s}\frac{K(x,t,s)u_j(s,t)}{\lambda_j(s)}\right)e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}\lambda_j(\theta)d\theta}ds.
$$

Thus, after a single integration by parts, the terms outside the integral are distinguished, which for $\tau=\psi$ have the form of summands (2.2), and the integral term is again an integral of the type $J_j(t,\varepsilon)$. Multiple integration by parts leads to a formal series

$$
J_j(x,t,\varepsilon)=\sum_{\nu=0}^{\infty}\varepsilon^{\nu+1}\Bigg[\left(I_j^{\nu}(K(x,t,s)u_j(s,t))\right)_{s=x}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_j(\theta)d\theta}-
$$

$$
-\left(I_j^{\nu}(K(x,t,s)u_j(s,t))\right)_{s=x_0}\Bigg] \tag{2.4}
$$

where

$$
I_j^0=\frac{1}{\lambda_j(s)},\ I_1^{\nu}=\frac{1}{\lambda_j(s)}\frac{\partial}{\partial s}I_j^{\nu-1},(\nu\geq 1),j=1,2.
$$

The series (2.4) converge asymptotically as $\varepsilon\to+0$ (uniformly in $(x,t)\in[0,X]\times[0,T]$). The partial sum of this series

$$
S_N(x,t,\varepsilon)=\sum_{\nu=0}^{N-1}(-1)^{\nu}\varepsilon^{\nu+1}\Bigg[\left(I_j^{\nu}(K(x,t,s)u_j(s,t))\right)_{s=x}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_j(\theta)d\theta}-
$$

$$
-\left(I_j^{\nu}(K(x,t,s)u_j(s,t))\right)_{s=x_0}\Bigg]
$$

satisfies the equality

$$
J_j(x,t,\varepsilon)-S_N(x,t,\varepsilon)=(-\varepsilon)^N\int\limits_{x_0}^{x}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}\lambda_j(\theta)d\theta}\frac{\partial}{\partial s}\left(I_j^{N-1}(K(x,t,s)u_j(s,t))\right)ds.
$$

Let us integrate by parts the integral standing here, selecting one more power ε on the right-hand side:

$$
J_j(x,t,\varepsilon)-S_N(x,t,\varepsilon)=\varepsilon^{N+1}\Big[(-1)^N\Big\{\left(I_j^{N}(K(x,t,s)u_j(s,t))\right)_{s=x}e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_j(\theta)d\theta}-
$$

$$-\left(I_j^N(K(x,t,s)u_j(s,t))\right)_{s=x_0}\Big\} +$$

$$+(-1)^{N+1}\int\limits_{x_0}^{x} e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{s}\lambda_j(\theta)d\theta} I_j^{N+1}(K(x,t,s)u_j(s,t))ds\Bigg].$$

Due to the infinite differentiability of the function $K(x,t,s)$ on $(x,t,\tau)\in[x_0,X]\times[0,T]\times$ $\times(0,\varepsilon_0]$ and $\lambda_j(x)$ on $[x_0,X]$, and also due to the uniform boundedness of the function, the equality in the square brackets of the last one is uniformly bounded for $(x,t,\tau)\in$ $\in[x_0,X]\times[0,T]\times\{\tau : Re\,\tau_j, j=1,2\}\})$. Consequently,

$$\|J_j(x,t,\varepsilon)-S_N(x,t,\varepsilon)\|_{C[x_0,X]\times[0,T]}\leq C_N\varepsilon^{N+1}$$

where $C_N>0$ is a constant independent of ε for $\varepsilon\in(0,\varepsilon_0]$ ($\varepsilon_0>0$ is sufficiently small). This means that the series (2.4)) converges to the integral asymptotically for $\varepsilon\to+0$. The Theorem 1 is proved.

Thus, the image $Ju(t,\varepsilon)$ is expanded into an asymptotic series

$$Ju(t,\tau)=\int\limits_{x_0}^{x}K(x,t,s)u_0(s,t)ds+$$

$$+\sum_{j=1}^{2}\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[\left(I_j^{\nu}\left(K(x,t,s)u_j(s)\right)\right)_{s=x}e^{\tau_j}-\left(I_j^{\nu}\left(K(t,s)u_j(s)\right)\right)_{s=x_0}\right]$$

where $\tau=\frac{\psi(x)}{\varepsilon}$. This proves that the class M_ε is asymptotically invariant with respect to the integral operator J.

Let us now construct an extension of the operator (2.2).

If now the series

$$u(x,t,\tau,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k u_k(x,t,\tau),\quad u_k(x,t,\tau)\in U \tag{2.5}$$

on the narrowing $\tau=\frac{\psi(x)}{\varepsilon}$ is an asymptotic series for $\varepsilon\to+0$ (uniformly in $(x,t)\in$ $\in[x_0,X]\times[0,T]$), then the image of $Ju(x,t,\tau,\varepsilon)$, obviously, will be the same. To construct the series $Ju(t,\tau,\varepsilon)$, we introduce the so-called order operators. Let $u_k(x,t,\tau)\in U$ be an arbitrary element (2.3). Then we have the expansion (2.5), which can be written as

$$Ju(x,t,\tau)=R_0u(x,t,\tau)+\sum_{\nu=0}^{\infty}\varepsilon^{k+1}R_{\nu+1}u_\nu(x,t,\tau) \tag{2.6}$$

where $\tau=\frac{\psi(x)}{\varepsilon}$, and the order operators R_ν have the form:

$$R_0u(x,t,\tau)=\int\limits_{0}^{x}K(x,t,s)u_0\left(s,t\right)ds, \tag{2.7$_0$}$$

$$R_1 u(x,t,\tau) = \sum_{j=1}^{2}\Big[\big(I_j^0\left(K(x,t,s)u_j(s,t)\right)\big)_{s=x} e^{\tau_j} - \big(I_j^0\left(K(x,t,s)u_j(s,t)\right)\big)_{s=x_0}\Big], \quad (2.7_1)$$

$$R_{\nu+1} u(x,t,\tau) = \sum_{j=1}^{2}\left[\big(I_j^\nu\left(K(x,t,s)u_j(s,t)\right)\big)_{s=x} e^{\tau_j} - \big(I_j^\nu\left(K(x,t,s)u_j(s,t)\right)\big)_{s=x_0}\right], \nu \geq 1. \quad (2.7_{\nu+1})$$

The $R_\nu : U \to U$ operators are called order operators because they extract the sum of the terms of the order ν with respect to the parameter ε in the expression $Ju(t,\tau)$. Applying the operator J to the series (2.5), and then using formulas (2.6),$(2.7_0), \ldots, (2.7_{\nu+1})$, and collecting the coefficients at the same powers of ε, we arrive at the following expression for $Ju(x,t,\tau,\varepsilon)$:

$$Ju(x,t,\tau,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k J u_k(x,t,\tau) = \sum_{r=0}^{\infty} \varepsilon^r \sum_{s=0}^{r} R_{r-s} u_s(x,t,\tau)|_{\tau=\psi(x)/\varepsilon}.$$

This equality is the basis for defining the extension of the operator J.

Definition 2. *The formal extension of the integral operator (2.2) is the operator $\tilde{J}$, which acts on each function of the form (2.6) according to the law*

$$\tilde{J}u(x,t,\tau,\varepsilon) \equiv \tilde{J}\left(\sum_{k=0}^{\infty} \varepsilon^k u_k(x,t,\tau)\right) \stackrel{def}{=} \sum_{r=0}^{\infty} \varepsilon^r \sum_{s=0}^{r} R_{r-s} u_s(x,t,\tau). \quad (2.8)$$

The operator $\tilde{J}$ is defined at least in the class of series (2.5) (with coefficients $u_k(x,t,\tau) \in$ $\in U$) converging asymptotically at $\varepsilon \to +0$ (uniformly in $(x,t,\tau) \in [x_0, X] \times [0,T] \times$ $\times \{\tau : \mathrm{Re}\tau_j \leq 0, j = 1,2\}$.

Now we can write the system, completely regularized with respect to the original problem (1.1):

$$L_\varepsilon u(x,t,\tau,\varepsilon) \equiv \varepsilon \frac{\partial u}{\partial x} + \sum_{j=1}^{2} \lambda_j(x) \frac{\partial u}{\partial \tau_i} - \lambda_1(x)\tilde{u} - \tilde{J}u = h_1(x,t) +$$
$$+ h_2(x,t) e^{\tau_2}\sigma, \quad \tilde{u}(x_0,t,0,\varepsilon) = y^0(t), \quad ((x,t) \in [x_0, X] \times [0,T]) \quad (2.9)$$

where the operator $\tilde{J}$ has the form (2.8).

3. Solvability of iterative problems

We will define the solution of system (2.9) in the form of series (2.5). Substituting (2.5) into (2.9) and equating $(2.7_0), \ldots, (2.7_{\nu+1})$ the coefficients ε at the same powers (taking into account the formulas $(2.7_0), \ldots, (2.7_{\nu+1})$), we obtain the following systems of equations:

$$Lu_0(x,t,\tau) \equiv \sum_{j=1}^{2} \lambda_j(x)\tfrac{\partial u_0}{\partial \tau_j} - \lambda_1(x)u_0 - R_0u_0 = \tag{3.1_0}$$
$$= h_1(x,t) + h_2(x,t)e^{\tau_2}\sigma,\ \ u_0(x_0,t,0) = y^0(t);$$

$$Lu_1(x,t,\tau) = -\frac{\partial u_0}{\partial x} + R_1u_0, \quad u_1(x_0,t,0) = 0; \tag{3.1_1}$$

$$Lu_2(x,t,\tau) = -\frac{\partial u_1}{\partial x} + R_1u_1 + R_2u_0,\ \ u_2(x_0,t,0) = 0; \tag{3.1_2}$$

....

$$Lu_k(x,t,\tau) = -\tfrac{\partial u_{k-1}}{\partial x} + R_ku_0 + \ldots + R_1u_{k-1}, \tag{3.1_k}$$
$$u_k(x_0,t,0) = 0, \quad k \geq 1.$$

To calculate solutions to iterative problems of the (3.1_0),..., (3.1_k) we need to study the solvability of the general iterative system

$$Lu(x,t,\tau) \equiv \sum_{j=1}^{2} \lambda_j(x)\frac{\partial u}{\partial \tau_j} - \lambda_1(x)u - R_0u = H(x,t,\tau) \quad u(x_0,t,0) = y_*(t) \tag{3.2}$$

where $H(x,t,\tau) = H_0(x,t) + \sum\limits_{j=1}^{2} H_j(x,t)e^{\tau_j} \in U$ is the known vector function of space U, y_* is the known constant vector of the complex space $\mathbb{C}$, and the operator R_0 has the form (see (2.7_0)):

$$R_0u(x,t,\tau) \equiv R_0\left[u_0(x,t) + \sum_{j=1}^{2} u_j(x,t)e^{\tau_j}\right] \triangleq \int\limits_{x_0}^{x} K(x,t,s)u_0(s,t)ds.$$

We introduce scalar (for each $x \in [x_0,X],\ \ t \in [0,T]$) product in space U:

$$\langle u, w\rangle \equiv \left\langle u_0(x,t) + \sum_{j=1}^{2} u_j(x,t)e^{\tau_j}, w_0(x,t) + \sum_{j=1}^{2} w_j(x,t)e^{\tau_j}\right\rangle \triangleq \sum_{j=0}^{2}(u_j(x,t), w_j(x,t))$$

where we denote by $(*,*)$ the usual scalar product in the complex space $\mathbb{C}$. Let us prove the following statement.

Theorem 2. *Let conditions (i)-(ii) be fulfilled and the right-hand side* $H(x,t,\tau) = H_0(x,t) + + \sum\limits_{j=1}^{2} H_j(x,t)e^{\tau_j}$ *of equation (3.2) belongs to the space* U. *Then the equation(3.2) is solvable in* U, *if and only if*

$$H_1(x,t,\tau) \equiv 0 \ \ \forall (x,t) \in [x_0,X] \times [0,T]. \tag{3.3}$$

Proof. We will determine the solution of equation (3.2) as an element (2.3) of the space U. Substituting (2.3) into equation (3.2), we will have

$$\sum_{j=1}^{2}\left[\lambda_j(x)-\lambda_1(x)\right]u_j(x,t)e^{\tau_j}-$$

$$-\lambda_1(x)u_0(x,t)-\int\limits_{x_0}^{x}K(x,t,s)u_0(s,t)ds=H_0(x,t)+\sum_{j=1}^{2}H_j(x,t)e^{\tau_j}.$$

Equating here the free terms and coefficients separately for identical exponents, we obtain the following equations:

$$-\lambda_1(x)u_0(x,t)-\int\limits_{x_0}^{x}K(x,t,s)u_0(s,t)ds=H_0(x,t), \qquad (3.4_0)$$

$$\left[\lambda_j(x)-\lambda_1(x)\right]u_j(x,t)=H_j(x,t),\, j=\overline{1,2}. \qquad (3.4_j)$$

The equation (11_0) can be written as

$$u_0(x,t)=-\int\limits_{x_0}^{x}\lambda_1^{-1}(x)K(x,t,s)u_0(s,t)ds-\lambda_1^{-1}(x)(x)H_0(x,t). \qquad (3.5)$$

Due to the smoothness of the kernel $-\lambda_1^{-1}(x)K(x,t,s)$ and heterogeneity $-\lambda_1^{-1}(x)H_0(x,t)$, this Volterra integral equation has a unique solution $u_0(x,t)\in C^\infty[x_0,X]\times[0,T]$. The equations (3.4_2) have unique solutions

$$u_2(x,t)=\left[\lambda_2(x)-\lambda_1(x)\right]^{-1}H_2(x,t)\in C^\infty[x_0,X]\times[0,T].$$

Equation (3.4_1) are solvable in space $C^\infty[x_0,X]\times[0,T]$ if and only if there are identities $H_1(x,t,\tau)\equiv 0\ \forall(x,t)\in[x_0,X]\times[0,T]$. Thus, condition (3.3) is necessary and sufficient for the solvability of equation (3.2) in the space U. The Theorem 2 is proved.

Remark 1. *If identity (3.3) holds, then under conditions (i)-(ii), equation (3.2) has the following solution in the space* U :

$$u(x,t,\tau)=u_0(x,t)+\alpha_1(x,t)e^{\tau_1}+\left[\lambda_2(x)-\lambda_1(x)\right]^{-1}H_2(x,t)e^{\tau_2} \qquad (3.6)$$

where $\alpha_1(x,t)\in C^\infty[x_0,X]\times[0,T]$ *are arbitrary function,* $u_0(x,t)$ *is the solution of an integral equation* (3.5).

4. Unique solvability of the general iterative problem in the space U. Remainder theorem

As seen from (3.6), the solution of the equation (3.2) is determined ambiguously. However, if its solution satisfies to the additional conditions

$$\begin{aligned} &u(x_0,t,0)=y_*(t),\\ &\left\langle -\tfrac{\partial u}{\partial x}+R_1u+Q\left(x_0,t,\tau\right),e^{\tau_1}\right\rangle \equiv 0,\ \ \forall (x,t)\in[x_0,X]\times[0,T] \end{aligned} \tag{4.1}$$

where $Q(x,t,\tau)=H_0(x,t)+\sum\limits_{j=1}^{2}Q_j(x,t)e^{\tau_j}$ is a known function of the space U, y_* is a constant number of the complex space $\mathbb{C}$, then equation (3.2) will be uniquely solvable in the space U. More precisely, the following result holds.

Theorem 3. *Let conditions (i)–(ii) be satisfied, the right-hand side $H\left(x,t,\tau\right)$ of the equation (3.2) belongs to the space U and satisfies the orthogonality condition (3.3). Then equation (3.2) under additional conditions (4.1) is uniquely solvable in U.*

Proof. Under condition (3.3), equation (3.2) has a solution (3.6) in the space U, where the function $\alpha_1\left(x,t\right)\in C^\infty[x_0,X]\times[0,T]$, are still arbitrary. Subordinating (3.6) to the first condition (4.1), i.e. $u\left(x_0,t,0\right)=y_*(t)$, we obtain the equation

$$-a^{-1}(x_0,t)H_0(x_0,t)+\alpha_1(x_0,t)+\left[\lambda_2(x_0)-\lambda_1(x_0)\right]^{-1}H_2(x_0,t)=y^*(t)$$

and we find the values

$$\alpha_1(x_0,t)=y^*(t)+a^{-1}(x_0,t)H_0(x_0,t)-\left[\lambda_2(x_0)-\lambda_1(x_0)\right]^{-1}H_2(x_0,t). \tag{4.2}$$

Let us now subordinate solution (3.6) to the second condition (4.1). The right-hand side of this equation has the form

$$-\frac{\partial u_0}{\partial x}+R_1u_0+Q\left(x,t,\tau\right)=-\frac{\partial}{\partial x}\left(u_0(x,t)\right)-\frac{\partial}{\partial x}\left(\alpha_1(x,t)\right)e^{\tau_1}+\left(\frac{H_2(x,t)}{\lambda_2(x)-\lambda_1(x)}\right)^{\bullet}e^{\tau_2}+$$
$$+\sum_{j=1}^{2}\left[\frac{K(x,t,x)u_j(x,t)}{\lambda_j(x)}e^{\tau_j}\ -\frac{K(x,t,x_0)u_j(x_0,t)}{\lambda_j(x_0)}\right]+Q\left(x,t,\tau\right).$$

we obtain equations

$$\dot{\alpha}_1(x,t)-\frac{K(x,t,x)}{\lambda_1(x)}\alpha_1(x,t)-Q_1(x,t)=0.$$

Adding the initial conditions (4.2) to them, we can uniquely find the function $\alpha_1(x,t)$:

$$\alpha_1(x,t)=\alpha_1(x_0,t)e^{-\int\limits_{x_0}^{x}\frac{K(s,t,s)}{\lambda_1(s)}dx}+\int\limits_{x_0}^{x}e^{-\int\limits_{x_0}^{s}\frac{K(s,t,s)}{\lambda_1(s)}ds}Q_1(s,t)ds$$

and hence, we define the solution (3.6) of the equation (3.2) in the space in a unique way. The Theorem 3 is proved.

Applying Theorems 2 and 3 to iterative problems (3.1_k), we find uniquely their solutions in the space U and construct series (2.5). Let $u_{\varepsilon N}(x,t) = \sum\limits_{k=0}^{N} \varepsilon^k u_k\left(x, t, \frac{\varphi(x)}{\varepsilon}\right)$ is the restriction of the $N-$th partial sum of series (2.5) for $\tau = \frac{\varphi(x)}{\varepsilon}$. Same as in [1, 34], it is easy to prove the following statement.

Lemma 1. *Let conditions (i) - (iii) be satisfied. Then the partial sum* $u_{\varepsilon N}(x,t)$ *satisfies problem (1.1) up to* $O(\varepsilon^{N+1})$ $(\varepsilon \to +0)$, *i.e.*

$$\begin{gathered}\varepsilon\frac{du_{\varepsilon N}(x,t)}{dt} \equiv a(x)u_{\varepsilon N}(x,t) + \int\limits_{x_0}^{x} K(x,t,s)u_{\varepsilon N}(s,t)ds + h_2(x,t)e^{\frac{i\beta(x)}{\varepsilon}} + \\ +h_1(x,t) + \varepsilon^{N+1}R_N(x,t,\varepsilon), \quad u_{\varepsilon N}(x_0,t) = y^0(t), \quad \forall (x,t) \in [x_0, X] \times [0,T]\end{gathered} \tag{4.3}$$

where $||R_N(x,t,\varepsilon)||_{C[x_0,X]\times[0,T]} \le \bar{R}_N$ *for all* $(x,t) \in [x_0, X]\times[0,T]$ *and for all* $\varepsilon \in (0, \varepsilon_N]$.

Consider now the following problem:

$$\begin{gathered}\varepsilon\frac{\partial u(x,t,\varepsilon)}{\partial x} = a(x)u(x,t,\varepsilon) + \int\limits_{x_0}^{x} K(x,t,s)u(s,t,\varepsilon)ds + \\ +\Phi(x,t,\varepsilon), \quad u(x_0,t,\varepsilon) = 0, \quad (x,t) \in [x_0,X]\times[0,T].\end{gathered} \tag{4.4}$$

Let us show that this problem is solvable in the space $C^1[x_0,X]\times[0,T]$ (i.e. it has a solution for any right-hand side $\Phi(t,\varepsilon) \in C\left([x_0,X]\times[0,T], \mathbb{C}^2\right)$ and that in this case there is an estimate

$$||u(x,t,\varepsilon)||_{C[x_0,X]\times[0,T]} \le \frac{\nu_0}{\varepsilon}||\Phi(x,t,\varepsilon)||_{C[x_0,X]\times[0,T]}. \tag{4.5}$$

Theorem 4. *Let conditions (i) - (iii) be satisfied. Then, for sufficiently small* $\varepsilon \in (0, \varepsilon_0]$ *problem (4.4) for any right-hand side* $\Phi(x,t,\varepsilon) \in C[x_0,X]\times[0,T]$ *has a unique solution* $u(x,t,\varepsilon)$ *in the space* $C^1[x_0,X]\times[0,T]$ *and estimate (4.5) holds, where* ν_0 *is a constant independent of* $\varepsilon > 0$.

Proof. Introduce an additional unknown function

$$z(x,t,\varepsilon) = \int\limits_{x_0}^{x} K(x,t,s)z(s,t,\varepsilon)ds.$$

Differentiating it with respect to x, we will have

$$\varepsilon\frac{\partial z(x,t,\varepsilon)}{\partial x} = \varepsilon K(x,t,x)z(x,t,\varepsilon) + \varepsilon\int\limits_{x_0}^{x} \frac{\partial K(x,t,s)}{\partial x}z(s,t,\varepsilon)ds.$$

From this and (4.3) it follows that the vector function $w = \{z, u\}$ satisfies the following system:

$$\varepsilon\frac{\partial z}{\partial x} = \begin{pmatrix} a(x) & 1 \\ 0 & 0 \end{pmatrix} w + \varepsilon \begin{pmatrix} 0 \\ K(x,t,x)z + \int\limits_{x_0}^{x} \frac{\partial K(x,t,s)}{\partial x} z(x,s,\varepsilon)ds \end{pmatrix} + \\ + \begin{pmatrix} \Phi(x,t,\varepsilon) \\ 0 \end{pmatrix}, w(x_0,t,\varepsilon) = 0. \tag{4.6}$$

Denote by $Y(x,t,s,\varepsilon)$ the normal fundamental matrix of the homogeneous system

$$\varepsilon\frac{\partial w}{\partial x} = \begin{pmatrix} a(x) & 1 \\ 0 & 0 \end{pmatrix} w$$

i.e., the matrix satisfying the equation

$$\varepsilon\frac{\partial Y(x,\eta,\varepsilon)}{\partial x} = \begin{pmatrix} a(x) & 1 \\ 0 & 0 \end{pmatrix} Y(x,\eta,\varepsilon), \;\; Y(x,x,\varepsilon) = I, \quad x_0 \le \eta \le x \le X.$$

Since the matrix $\begin{pmatrix} a(x) & 1 \\ 0 & 0 \end{pmatrix}$ is a matrix of simple structure and its spectrum $\{\lambda_1(x), 0\}$ lies in the half-plane $\operatorname{Re}\lambda_1(x) \le 0$, then the Cauchy matrix $Y(x,\eta,\varepsilon)$ is uniformly bounded, i.e.,

$$\|Y(x,\eta,\varepsilon)\| \le c_0 \;\; \forall (x,\eta,\varepsilon) : x_0 \le \eta \le x \le X, \varepsilon > 0$$

where the constant $c_0 > 0$ does not depend on $\varepsilon > 0$ (see, for example, [1], pp. 119-120). We now write down an integral system equivalent to system (4.6):

$$w(x,t,\varepsilon) = \int\limits_{x_0}^{x} Y(x,\eta,\varepsilon) \begin{pmatrix} 0 \\ K(\eta,t,\eta)z(\eta,t,\varepsilon) + \int\limits_{x_0}^{\eta} \frac{\partial K(\eta,t,,s)}{\partial \eta} z(s,t,\varepsilon)ds \end{pmatrix} d\eta + \\ + \frac{1}{\varepsilon}\int\limits_{x_0}^{x} Y(x,\eta,\varepsilon) \begin{pmatrix} \Phi(\eta,t,\varepsilon) \\ 0 \end{pmatrix} d\eta. \tag{4.6$_0$}$$

Since for each $\varepsilon > 0$ there exists the solution $w(x,t,\varepsilon)$ of the system (4.6) in the space $C^1[x_0,X] \times [0,T]$ then substituting it into (4.6_0), we obtain the identity. Let's move on to the norms:

$$||w(x,t,\varepsilon)|| \le \int\limits_{x_0}^{x} ||Y(x,\zeta,\varepsilon)|| \cdot ||K(\zeta,\zeta)|| \cdot ||z(\zeta,\varepsilon)||d\zeta + \int\limits_{x_0}^{x} ||Y(x,\zeta,\varepsilon)|| \times$$

$$\times \int\limits_{x_0}^{\zeta} ||\frac{\partial K(\zeta,s)}{\partial x}|| \cdot ||z(s,\varepsilon)||dsd\zeta + \frac{1}{\varepsilon}\int\limits_{x_0}^{x} ||Y(x,\zeta,\varepsilon)|| \cdot ||\Phi(\zeta,\varepsilon)||d\zeta \le$$

$$\leq c_0 k_0 \int\limits_{x_0}^{x} ||w(\zeta,\varepsilon)||d\zeta + c_0 k_1 \int\limits_{x_0}^{x}\int\limits_{x_0}^{\zeta} ||w(s,\varepsilon)||dsd\zeta +$$

$$+\frac{X_0}{\varepsilon} c_0 ||\Phi(x,\varepsilon)||_{C[x_0,X]\times[0,T]} \leq c_0 k_0 \int\limits_{x_0}^{x} ||w(s,\varepsilon)||ds +$$

$$+c_0 k_1 \int\limits_{x_0}^{x}\int\limits_{x_0}^{x} ||w(s,\varepsilon)||dsd\zeta + \frac{c_0 X_0}{\varepsilon}||\Phi(x,\varepsilon)||_{C[x_0,X]\times[0,T]} \leq$$

$$\leq c_0 k_0 \int\limits_{x_0}^{x} ||w(s,\varepsilon)||ds + c_0 k_1 \int\limits_{x_0}^{X}\int\limits_{x_0}^{x} ||w(s,\varepsilon)||dsd\zeta +$$

$$+\frac{c_0 X_0}{\varepsilon}||\Phi(x,\varepsilon)||_{C[x_0,X]\times[0,T]} \leq (c_0 k_0 + c_0 k_1 X_0) \int\limits_{x_0}^{x} ||w(s,\varepsilon)||ds +$$

$$+\frac{c_0 X_0}{\varepsilon}||\Phi(x,\varepsilon)||_{C[x_0,X]\times[0,T]}$$

where $X_0 = X - x_0$, $||K(x,t,s)||_{C[x_0,X]\times[0,T]} = k_0$, $||\frac{\partial K(x,t,s)}{\partial x}||_{C[x_0,X]\times[0,T]} = k_1$. We got the inequality

$$||w(x,t,\varepsilon)|| \leq \frac{c_0 X_0}{\varepsilon}||\Phi(x,t,\varepsilon)||_{C[x_0,X]\times[0,T]} + (c_0 k_0 + c_0 k_1 X_0) \int\limits_{x_0}^{x} ||w(s,\varepsilon)||ds.$$

Applying the Gronwall-Bellman lemma to this inequality [35], we have

$$\|w(x,t,\varepsilon)\| \leq \frac{c_0 X_0}{\varepsilon}\|\Phi(x,t,\varepsilon)\|_{C[x_0,X]\times[0,T]} e^{(c_0 k_0 + c_0 k_1 X_0)\int\limits_{x_0}^{x} ds} =$$

$$= \frac{c_0 X_0}{\varepsilon}\|\Phi(x,t,\varepsilon)\|_{C[[x_0,X]\times[0,T]} e^{(c_0 k_0 + c_0 k_1 X_0)(x-x_0)} \leq$$

$$\leq \frac{\nu_0}{\varepsilon}\|\Phi(x,t,\varepsilon)\|_{C[x_0,X]\times[0,T]} \Rightarrow$$

$$\Rightarrow \|z(x,t,\varepsilon)\|_{C[x_0,X]\times[0,T]} \leq \frac{\nu_0}{\varepsilon}\|\Phi(x,t,\varepsilon)\|_{C[x_0,X]\times[0,T]}$$

where $\nu_0 = c_0 X_0 \cdot \max\limits_{t\in[x_0,X]\times[0,T]} e^{(c_0 k_0 + c_0 k_1 X_0)(x-x_0)}$. The Theorem 4 is proved.

Theorem 5. *Let conditions (i)–(iii) be satisfied. Then for any $\varepsilon \in (0, \varepsilon_0]$, where $\varepsilon_0 > 0$ is small enough, the problem (1.1) has a unique solution $y(x, t, \varepsilon) \in C^1 [x_0, X] \times [0, T]$; in this case, the estimate*

$$\|y(x, t, \varepsilon) - u_{\varepsilon N}(x, t)\|_{C[x_0,X]\times[0,T]} \leq C_N \varepsilon^{N+1} (N = 0, 1, 2 \ldots)$$

holds true, where the constant $C_N > 0$ does not depend on $\varepsilon \in (0, \varepsilon_0]$.

Proof. By the Lemma 1, the partial sum $u_{\varepsilon N}(x, t)$ satisfies the problem (4.3), so the remainder $r_N(x, t, \varepsilon) \equiv y(x, t, \varepsilon) - u_{\varepsilon N}(x, t)$ satisfies the following problem:

$$\varepsilon \frac{\partial r_N(x, t, \varepsilon)}{\partial x} = a(x) r_N(x, t, \varepsilon) + \int\limits_{x_0}^{x} K(x, t, s) r_N(s, t, \varepsilon) ds +$$

$$+\varepsilon^{N+1} R_N(x, t, \varepsilon),\ r_N(x_0, t, \varepsilon) = 0$$

where $\Phi(x, t, \varepsilon) = -\varepsilon^{N+1} \int\limits_{x_0}^{x} R_N(s, t, \varepsilon) ds$. By Theorem 4, we have the estimate

$$\|r_N(x, t, \varepsilon)\|_{C[x_0,X]\times[0,T]} \leq \varepsilon^N \bar{R}_N$$

for all $N = 0, 1, 2, \ldots$ and all $\varepsilon \in (0, \varepsilon_N]$, which means that the partial sum $u_{\varepsilon,N+1}(x, t) = = u_{\varepsilon N}(x, t) + \varepsilon^{N+1} u_{N+1}(x, t, \frac{\psi(t)}{\varepsilon})$ satisfies the inequality

$$||y(x, t, \varepsilon) - u_{\varepsilon,N+1}(x, t)||_{C[x_0,X]\times[0,T]} \equiv$$

$$\equiv ||(y(x, t, \varepsilon) - y(x, t)) - \varepsilon^{N+1} u_{N+1}(x, t, \frac{\psi(x)}{\varepsilon})||_{C[x_0,X]\times[0,T]} \leq \bar{C}_{N+1} \varepsilon^{N+1}.$$

Using the inequality $||a - b|| \geq |||a|| - ||b|||$, valid for any numbers a and b, we will have

$$||y(x, t, \varepsilon) - u_{\varepsilon N}(x, t)||_{C[x_0,X]\times[0,T]} \leq \left(\bar{C}_N + \left\| u_{N+1} \left(x, t, \frac{\psi(x)}{\varepsilon} \right) \right\|_{C[x_0,X]\times[0,T]} \right) \varepsilon^{N+1}$$

whence we derive the estimate

$$||y(x, t, \varepsilon) - u_{\varepsilon N}(x, t)||_{C[x_0,X]\times[0,T]} \leq C_N \varepsilon^{N+1}$$

where the constant $C_N > 0$ does not depend on $\varepsilon \in (0, \varepsilon_N]$.

5. Construction of the solution of the first iteration problem in the space *U*

Using Theorem 2, we will try to find a solution to the first iterative problem (3.1_k). Since the right-hand side $h_1(t) + h_2(x, t)e^{\tau_2}$ of the equation (3.1_0), satisfies the condition (3.3), this equation has (according to (3.6)) a solution in the space U in the form

$$u_0 (t, \tau) = u_0^{(0)} (x, t) + \alpha_1^{(0)} (x, t) e^{\tau_1} + [\lambda_2(x) - \lambda_1(x)]^{-1} h_2(x, t) e^{\tau_2} \tag{5.1}$$

where $\alpha_1^{(0)}(x,t) \in C^{\infty}[x_0, X] \times [t_0, T]$ are arbitrary function, $u_0^{(0)}(x,t)$ is the solution of the integral equation $-a(x)\, u_0(x,t) - \int\limits_{x_0}^{x} K(x,t,s)\, u_0(s,t)\, ds = h_1(x,t)$.

Subordinating (5.1) to the initial condition $u_0(x_0, t, 0) = y^0(t)$, we obtain values

$$u_0^{(0)}(x_0,t) + \alpha_1^{(0)}(x_0,t) + [\lambda_2(x_0) - \lambda_1(x_0)]^{-1}\, h_2(x_0,t) = y^0(t) \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \alpha_1^{(0)}(x_0,t) = y^0(t) + \lambda_1^{-1}(x_0)\, h_1(x_0,t) - [\lambda_2(x_0) - \lambda_1(x_0)]^{-1}\, h_2(x_0,t). \tag{5.2}$$

For a complete calculation of the function $\alpha_1^{(0)}(x,t)$, we pass to the next iterative problem (3.1_1). Substituting the solution (5.1) of the equation (3.1_0) into it, we arrive at the following equation:

$$\begin{aligned} Lu_1(t,\tau) = &-\frac{\partial}{\partial x}\left(u_0^{(0)}(x,t)\right) - \frac{\partial}{\partial x}\left(\alpha_1^{(0)}(x,t)\right) e^{\tau_1} - \\ &-\frac{\partial}{\partial x}\left([\lambda_2(x) - \lambda_1(x)]^{-1}\, h_2(x,t)\right) e^{\tau_2} + \\ &+\sum_{j=1}^{2}\left[\frac{\left(K(x,t,x)\,\alpha_j^{(0)}(x,t)\right)}{\lambda_j(x)} e^{\tau_i} - \frac{\left(K(x,t,x_0)\,\alpha_j^{(0)}(x_0,t)\right)}{\lambda_j(x_0)}\right]. \end{aligned} \tag{5.3}$$

Subordinating the right-hand side of this equation to the solvability conditions (3.3), we obtain the equation of ordinary differential equations

$$-\frac{\partial}{\partial x}\left(\alpha_1^{(0)}(x,t)\right) + \frac{K(x,t,x)}{\lambda_1(x)}\alpha_1^{(0)}(x,t) = 0.$$

Adding the initial condition (5.2) to them, we find

$$\alpha_1^{(0)}(x,t) = \alpha_1^{(0)}(x_0,t)\, e^{\int\limits_{x_0}^{x} \frac{K(\theta,t,\theta)}{\lambda_1(\theta)} d\theta}$$

and therefore, the solution (5.1) of the problem will be found uniquely in the space U. In this case, the leading term of the asymptotics has the following form:

$$\begin{aligned} y_{\varepsilon 0}(x,t) = &\, u_0^{(0)}(x,t) + \alpha_1^{(0)}(x_0,t)\, e^{\int\limits_{x_0}^{x} \frac{K(\theta,t,\theta)}{\lambda_1(\theta)} d\theta + \frac{1}{\varepsilon}\int\limits_{x_0}^{x} \lambda_1(\theta) d\theta} + \\ &+ [\lambda_2(x) - \lambda_1(x)]^{-1}\, h_2(x,t) e^{\frac{1}{\varepsilon}\int\limits_{x_0}^{x} \lambda_2(\theta) d\theta} \end{aligned} \tag{5.3$_0$}$$

where $u_0^{(0)}(x,t)$ is the solution of the integral equation

$$-a(t)\, u_0(x,t) - \int\limits_{x_0}^{x} K(x,t,s)\, u_0(s,t)\, ds = h_1(x,t).$$

Example 1. *Let us consider the following problem:*

$$\varepsilon\frac{\partial y(x,t,\varepsilon)}{\partial x}=-y+\int\limits_{x_0}^{x}xtsy\left(s,t,\varepsilon\right)ds+2xt+xte^{\frac{i\left(x+x^3\right)}{\varepsilon}},\quad y\left(x_0,t,\varepsilon\right)=y^0(t),\tag{5.4}$$

where

$$a(x)=-1,\ K\left(x,t,s\right)=xts,\ h_1(x,t)=2xt,\ h_2(x,t)=xt.$$

Let's try to construct its main term of the asymptotic solution. In this equation we has the following spectrum: $\{\lambda_1\left(t\right),\lambda_2\left(t\right)\}=\left\{-1,i\left(1+3x^2\right)\right\}.$

Regularizing problem (5.4) using the functions

$$\begin{aligned}&\tau_1=\tfrac{1}{\varepsilon}\int\limits_{x_0}^{x}\lambda_1\left(\theta\right)d\theta,\quad\Leftrightarrow\quad\tau_1=-\tfrac{1}{\varepsilon}(x-x_0),\\&\tau_2=\tfrac{i}{\varepsilon}\int\limits_{x_0}^{x}\left(1+3\theta^2\right)d\theta=\tfrac{i}{\varepsilon}\left[\left(x-x_0+x^3-{x_0}^3\right)\right]\end{aligned}$$

we get the following extended problem:

$$\begin{aligned}&L_\varepsilon u\left(x,t,\tau,\varepsilon\right)\equiv\varepsilon\tfrac{\partial u}{\partial x}+\lambda_1\tfrac{\partial u}{\partial\tau_1}+\lambda_2\tfrac{\partial u}{\partial\tau_2}-\lambda_1(s)\tilde{y}-\tilde{J}u=\\&=h_1(x,t)+h_2(x,t)e^{\tau_2}\sigma,\ u\left(x_0,t,\tau,\varepsilon\right)|_{\tau=(0,0)}=y^0(t)\end{aligned}\tag{5.5}$$

where $\tau=\left(\tau_1,\tau_2\right),\ \sigma=exp\frac{-i(x_0+{x_0}^3)}{\varepsilon}$, $\tilde{J}$ *is extension of the integral operator* J *on series of the form*

$$u\left(x,t,\tau,\varepsilon\right)=\sum_{k=0}^{\infty}\varepsilon^k u_k\left(x,t,\tau\right)\tag{5.6}$$

with coefficients $u_k\left(x,t,\tau\right)$ *from the space* U *of vector functions*

$$u\left(x,t,\tau\right)=u_0\left(x,t\right)+\sum_{j=1}^{2}u_j\left(x,t\right)e^{\tau_j},\ u_k\left(x,t\right)\in C^\infty\left(\left[x_0,X\right]\times\left[0,T\right],\mathbb{C}\right),k=\overline{0,2}.$$

This extension has the form

$$\tilde{J}u(x,t,\tau,\varepsilon)=\sum_{r=0}^{\infty}\varepsilon^r\sum_{s=0}^{r}R_{r-s}u_s(x,t,\tau)$$

where the operators $R_\nu:U\to U$ *are calculated by the formulas :*

$$\begin{aligned}&R_0u(x,t,\tau)=\int\limits_{x_0}^{x}xtsu_0(s,t)ds,\\&R_1u(x,t,\tau)=\sum_{j=1}^{2}\left[\tfrac{x^2t\cdot u_j(x,t)}{\lambda_j(x)}e^{\tau_j}-\tfrac{xtx_0\cdot u_j(x_0,t)}{\lambda_j(x_0)}\right],\\&\ldots\ldots.\end{aligned}$$

(operators R_ν at $\nu \geq 2$ we do not write out, because we do not need them when constructing the leading term of the asymptotics). Defining the solution of the problem (5.5) in the form of series (5.6), we obtain the following iterative problems:

$$\begin{gathered} Lu_0 \equiv \lambda_1 \tfrac{\partial u_0}{\partial \tau_1} + \lambda_2 \tfrac{\partial u_0}{\partial \tau_2} - \lambda_1 u_0 - Ru_0 = \\ = h_1(x,t) + h_2(x,t)\, e^{\tau_2}\sigma,\ u_0(x_0,t,\tau)\,|_{\tau=0} = y^0(t); \end{gathered} \tag{5.7}$$

$$Lu_1 = -\frac{\partial u_0}{\partial x} + R_1 y_0, \quad u_1(x_0,t,0) = 0; \tag{5.8}$$

$$\ldots$$

The solution of the first iterative problem (5.7) will be as follows:

$$u_0(x,t,\tau) = u_0(x,t) + \alpha_1(x,t)\, e^{\tau_1} + (\lambda_2(x) - \lambda_1)^{-1} h_2(x,t) e^{\tau_2} \tag{5.9}$$

where $\alpha_1(x,t) \in C^\infty[x_0,X] \times [t_0,T]$ are arbitrary function, $u_0(x,t)$ is the solution of the integral equation

$$-u_0(x,t) - x\int\limits_{x_0}^{x} tsu_0(s,t)\, ds = 2xt. \tag{5.10}$$

Subordinate (5.8) to the initial condition $u_0(x,t,\tau)\,|_{x=x_0,\tau=0} = y^0(t)$. Taking into account the form of function (5.10) and $h_2(x,t)$, we obtain the equation $\alpha_1(x_0,t) = y^0(t)$. For a complete calculation of the functions $\alpha_1(x,t)$, we pass to the next iterative problem (5.8). Taking into account that under the conditions of solvability (3.2) of problem (5.8) only exponentials e^{τ_1} and e^{τ_2} are involved we keep in its right-hand side only terms depending on these exponentials:

$$-\frac{\partial(\alpha_1(x,t))}{\partial x} e^{\tau_1} - x^2 t \cdot \alpha_1(x,t)\, e^{\tau_1} = 0.$$

We will have

$$\dot{\alpha}_1(x,t) + x^2 t \cdot \alpha_1(x,t) = 0$$

or

$$\dot{\alpha}_1(x,t) = -x^2 t \cdot \alpha_1(x,t)\,.$$

Adding to these equations the initial conditions $\alpha_1(x_0,t) = y^0(t)$, found earlier, we uniquely find the functions

$$\alpha_1(x,t) = y^0(t) e^{-\frac{x^3}{3}t}$$

and hence, we will uniquely construct solution (5.9) of the first iterative problem (5.7). Making a narrowing in it at $\tau_1 = -\frac{x-x_0}{\varepsilon}$, $\tau_2 = \frac{i}{\varepsilon}\left[(x - x_0 + x^3 - x_0{}^3)\right]$, we obtain the leading term of the asymptotic solution of the problem (23):

$$y_{\varepsilon 0}(x,t) = y_0(x,t) + u^0(t) e^{-\frac{x^3}{3}t} e^{-\frac{x-x_0}{\varepsilon}} + \left[i(1+3x^2) - 1\right] xte^{\frac{i}{\varepsilon}\left[(x-x_0+x^3-x_0{}^3)\right]} \tag{5.11}$$

where $u_0(x,t)$ is the solution of the integral equation (5.10). It is seen from (5.11) that, at the exact solution $y(x,t,\varepsilon)$ of the problem (5.4) does not tend to the solution $y_0(x,t)$ of the integral equation (5.10) at $\varepsilon \to +0$, but performs quick oscillations near it.

6. Conclusion

From the expression (5.3_0) for $y_{\varepsilon 0}(t)$, it can be seen that the construction of the leading term of the asymptotics of the solution to problem (1.1) is significantly influenced by both the rapidly oscillating inhomogeneity and the kernel of the integral operator.

Note that the application of other asymptotic methods (for example, the method of boundary functions [36–38]) to problems of type (1) with rapidly oscillating inhomogeneities is problematic, since many of them rely heavily on the fact that all points of the spectrum (including the spectral value of the inhomogeneity) lie in the open half-plane $\mathrm{Re}\ \lambda < 0$.

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