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Tripolar Complex Fuzzy Lie Subalgebras of Lie Algebras

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Abstract. The tripolar complex fuzzy set ($\mathcal{TCFS}$) is an extension of the bipolar complex fuzzy set ($\mathcal{BCFS}$), which itself generalizes traditional fuzzy sets and bipolar fuzzy sets. In this paper, we further develop this framework by introducing the concept of tripolar complex fuzzy Lie brackets and investigating their algebraic properties. Additionally, we demonstrate that the scalar multiplication and addition of tripolar complex fuzzy Lie subalgebras yield another tripolar complex fuzzy Lie subalgebra. Moreover, we establish that the homomorphic image of a nilpotent (or solvable) tripolar complex fuzzy Lie ideal remains a nilpotent (or solvable) tripolar complex fuzzy Lie ideal. Finally, we establish that every nilpotent tripolar complex fuzzy Lie ideal is solvable.

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Key Words and Phrases: Lie algebra, Tripolar complex fuzzy set, Tripolar complex fuzzy Lie subalgebra, Tripolar complex fuzzy Lie ideal, Nilpotent, Solvable

1. Introduction

A Lie algebra is a fundamental mathematical framework in algebra that has important applications in many domains, including theoretical physics and mathematics. Lie algebras, named after Norwegian mathematician Sophus Lie [1], provide a robust framework for examining the algebraic properties of symmetries, vector fields, and transformations. Over time, academics have thoroughly investigated Lie algebras, revealing a variety of structural themes that apply to groups, rings, fields, and other algebraic systems. Lie algebras have applications in domains such as computer science [2] and coding theory [3].

Zadeh's introduction of fuzzy set theory ($\mathcal{FS}$) in 1965 [4] transformed the mathematical representation of uncertainty. Zhang introduced bipolar fuzzy sets ($\mathcal{BFS}$) in 1998 [5], and Lee [6] expanded on this by introducing the notion of bipolar-valued fuzzy sets with membership functions $\xi^P_{\tilde{\mathcal{B}}} : \tilde{\mathcal{L}} \to [0,1]$ and $\xi^N_{\tilde{\mathcal{B}}} : \tilde{\mathcal{L}} \to [-1,0]$.

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In the complex plane, Ramot et al. [7] developed the concept of complex fuzzy sets ($\mathcal{CFS}$), expanding the range $[0, 1]$ to the unit disk. Tamir et al. [8] expanded on this notion by mapping the range to $[0, 1] + i[0, 1]$. Mahmood et al. [9] proposed bipolar complex fuzzy sets ($\mathcal{BCFS}$), which combine $\mathcal{BFS}$ and $\mathcal{CFS}$ concepts. Rosenfeld [10] was the first to connect fuzzy set theory ($\mathcal{FS}$) to group theory. He invented the concept of fuzzy subgroups and investigated its fundamental features. Following this, other scholars expanded these ideas to various algebraic structures, including HX-subgroups, ordered semigroups, and BCK/BCI-algebras, in the setting of uncertainty (see [11–16]).

Yehia [17] introduced the concept of fuzzy Lie algebras by incorporating $\mathcal{FS}$ theory into the study of Lie algebraic ambiguity. Subsequent studies investigated Lie algebras in the context of $\mathcal{FS}$s (see [18, 19]). Atanassov [20] generalized fuzzy sets by introducing intuitionistic fuzzy sets ($\mathcal{IFS}$s), which incorporate degrees of membership, non-membership, and hesitation to model uncertainty more robustly. Akram and Shum [21] studied intuitionistic fuzzy Lie subalgebras and their properties. Akram [22] studied fuzzy Lie ideals with interval-valued membership functions, as well as solvable and nilpotent Lie $\mathcal{L}$-algebras utilizing bipolar fuzzy sets ($\mathcal{BFS}$) [23].

Shaqaqha [24] introduced the concept of complex fuzzy Lie subalgebras, examining their key properties. Kousar et al. [25] studied nilpotent and solvable Lie algebras in an image fuzzy environment. Al-Masarwah et al. [26] established the notion of crossing cubic Lie algebras, researching homomorphisms, isomorphism theorems, Cartesian products, and quotients within this framework. Jaleel et al. [27] proposed interval-valued bipolar complex fuzzy sets ($\mathcal{IVBCFS}$), whereas Qiyasi et al. [28] created the underlying theory of confidence-level-based bipolar complex fuzzy sets. Prommai et al. [29] presented tripolar fuzzy ideals in semigroups. Wattanasiripong et al. [30] introduced tripolar fuzzy pure ideals in ordered semigroups.

Muhiuddin et al. [14] developed tripolar picture fuzzy ideals of BCK-algebras. Balamurugan et al. [31] introduced complex fuzzy subalgebras and investigated their properties. Al-Masarwah et al. [32] investigated complex linear diophantine fuzzy ideals in BCK-algebras. The aim of a tripolar complex fuzzy set ($\mathcal{TCFS}$) is to extend the capabilities of traditional fuzzy sets, bipolar fuzzy sets, and complex fuzzy sets by incorporating three independent dimensions of information to model more complex and nuanced real-world scenarios. From the above literature we found the research gap and tripolar complex fuzzy set is introduced. The paper is organized as follows: Section 2 provides a concise set of basic definitions. Section 3 introduces the novel concept of tripolar complex fuzzy Lie bracket and examines its key properties. Section 4 develops the concept of tripolar complex fuzzy Lie subalgebras. Section 5 investigates nilpotent and solvable tripolar complex fuzzy Lie ideals. Section 6 concludes the study and suggests promising directions for future research. To ensure clarity, Table 1 summarizes all mathematical notation and terminology used throughout the paper.

Table 1: Acronyms list.

Acronyms	Representations
$\mathcal{L}$	Lie Algebra
$\mathcal{FS}(s)$	Fuzzy set(s)
$\mathcal{CFS}(s)$	Complex fuzzy set(s)
$\mathcal{IFS}(s)$	Complex Intuitionistic fuzzy set(s)
$\mathcal{BFS}(s)$	Bipolar Fuzzy set(s)
$\mathcal{BCFS}(s)$	Bipolar Complex Fuzzy set(s)
$\mathcal{TFS}(s)$	Tripolar Fuzzy set(s)
$\mathcal{TCFS}(s)$	Tripolar Complex Fuzzy set(s)
$\mathcal{TCFLS}(s)$	Tripolar Complex Fuzzy Lie Subalgebra(s)
$\mathcal{TCFLI}(s)$	Tripolar Complex Fuzzy Lie Ideal(s)
$\mathcal{NTCFLI}(s)$	Nilpotent Tripolar Complex Fuzzy Lie Ideal(s)
$\mathcal{STCFLI}(s)$	Solvable Tripolar Complex Fuzzy Lie Ideal(s)

2. Preliminaries

Definition 1. *[21] A Lie algebra $\tilde{\mathcal{L}}$ defined over a field $\mathcal{F}$ is a vector space together with a bilinear operation, called the Lie bracket, which adheres to certain key properties:*
(i) Bilinear: The Lie bracket $[,] : \tilde{\mathcal{L}} \times \tilde{\mathcal{L}} \to \tilde{\mathcal{L}}$ is a bilinear.
(ii) Skew-symmetry: For any $\tilde{\phi}, \tilde{\eta} \in \tilde{\mathcal{L}}$,

$$[\tilde{\phi}, \tilde{\eta}] = -[\tilde{\eta}, \tilde{\phi}].$$

(iii) Jacobi identity: For any $\tilde{\phi}, \tilde{\eta}, \tilde{\rho} \in \mathcal{L}$,

$$[\tilde{\phi}, [\tilde{\eta}, \tilde{\rho}]] + [\tilde{\eta}, [\tilde{\rho}, \tilde{\phi}]] + [\tilde{\rho}, [\tilde{\phi}, \tilde{\eta}]] = 0.$$

Definition 2. *[4] A $\mathcal{FS}$ $\tilde{\mathcal{F}} = \{(\tilde{\phi}, \xi_{\tilde{\mathcal{F}}}(\tilde{\phi})) \mid \tilde{\phi} \in \tilde{\mathcal{L}}\}$, where $\xi_{\tilde{\mathcal{F}}}(\tilde{\phi}) : \tilde{\mathcal{L}} \to [0, 1]$ is the $\mathcal{MD}$ of $\tilde{\phi}$ to $\mathcal{FS}$ $\tilde{\mathcal{F}}$.*

Definition 3. *[7] A $\mathcal{CFS}$ $\tilde{\mathcal{C}}$ in $\tilde{\mathcal{L}}$ is the structure*

$$\tilde{\mathcal{C}} = \{(\tilde{\phi}, \xi_{\tilde{\mathcal{C}}}(\tilde{\phi}) = \tilde{r}_{\tilde{\mathcal{C}}}(\tilde{\phi})e^{i2\pi\tilde{\omega}_{\tilde{\mathcal{C}}}(\tilde{\phi})}) \mid \tilde{\phi} \in \tilde{\mathcal{L}}\},$$

where $\tilde{r}_{\tilde{\mathcal{C}}}(\tilde{\phi}) \in [0, 1]$, $\tilde{\omega}_{\tilde{\mathcal{C}}}(\tilde{\phi}) \in [0, 2\pi]$.

Definition 4. *[20] An $\mathcal{IFS}$ $\tilde{\mathcal{I}}$ in $\tilde{\mathcal{L}}$ is the collection $\tilde{\mathcal{I}} = \{(\tilde{\phi}, \xi_{\tilde{\mathcal{I}}}(\tilde{\phi}), \eta_{\tilde{\mathcal{I}}}(\tilde{\phi})) \mid \tilde{\phi} \in \tilde{\mathcal{L}}\}$, where $\xi_{\tilde{\mathcal{I}}}(\tilde{\phi}) : \tilde{\mathcal{L}} \to [0, 1]$ and $\eta_{\tilde{\mathcal{I}}}(\tilde{\phi}) : \tilde{\mathcal{L}} \to [0, 1]$ denotes the $\mathcal{MD}$ and $\mathcal{NMD}$ degree of $\tilde{\phi}$, respectively with $0 \leq \xi_{\tilde{\mathcal{I}}}(\tilde{\phi}) + \eta_{\tilde{\mathcal{I}}}(\tilde{\phi}) \leq 1$ for all $\tilde{\phi} \in \tilde{\mathcal{L}}$.*

Definition 5. *[5] A $\mathcal{BFS}$ $\tilde{\mathcal{B}} = \{(\tilde{\phi}, \xi^{\mathcal{P}}_{\tilde{\mathcal{B}}}(\tilde{\phi}), \xi^{\mathcal{N}}_{\tilde{\mathcal{B}}}(\tilde{\phi})) \mid \tilde{\phi} \in \tilde{\mathcal{L}}\}$, where $\xi^{\mathcal{P}}_{\tilde{\mathcal{B}}}(\tilde{\phi}) : \tilde{\mathcal{L}} \to [0, 1]$ and $\xi^{\mathcal{N}}_{\tilde{\mathcal{B}}}(\tilde{\phi}) : \tilde{\mathcal{L}} \to$ [-1, 0] are $+\mathcal{MD}$ and $-\mathcal{MD}$ of $\tilde{\phi}$, respectively, with $-1 \leq \xi^{\mathcal{P}}_{\tilde{\mathcal{B}}}(\tilde{\phi}) + \xi^{\mathcal{N}}_{\tilde{\mathcal{B}}}(\tilde{\phi}) \leq 1$. The pair $(\mu^{\mathcal{P}}_{\tilde{\mathcal{B}}},\ \xi^{\mathcal{N}}_{\tilde{\mathcal{B}}})$ denotes the $\mathcal{BFN}$. The $+\mathcal{MD}$ is the degree of conformity of any property for an entity, while $-\mathcal{MD}$ shows the satisfaction degree of the inherent opposing characteristic.*

Definition 6. *[9] A $\mathcal{BCFS}$ $\tilde{\mathcal{B}}$ in $\tilde{\mathcal{L}}$ is the structure*

$$\tilde{\mathcal{B}} = \{(\tilde{\phi}, \xi^{\mathcal{P}}_{\tilde{\mathcal{B}}}(\tilde{\phi}) = \tilde{r}^{\mathcal{P}}_{\tilde{\mathcal{B}}}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\mathcal{B}}}(\tilde{\phi})}, \xi^{\mathcal{N}}_{\tilde{\mathcal{B}}}(\tilde{\phi}) = \tilde{r}^{\mathcal{N}}_{\tilde{\mathcal{B}}}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{N}}_{\tilde{\mathcal{B}}}(\tilde{\phi})}) \mid \tilde{\phi} \in \tilde{\mathcal{L}}\},$$

where $\xi^{\mathcal{P}}_{\tilde{\mathcal{B}}}(\tilde{\phi}) : \tilde{\mathcal{L}} \to [0, 1]$, $\xi^{\mathcal{N}}_{\tilde{\mathcal{B}}}(\tilde{\phi}) : \tilde{\mathcal{L}} \to [-1, 0]$.

Definition 7. *[29] A $\mathcal{TFS}$ $\tilde{\gimel} = \{\tilde{\phi}, \xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\phi}), \xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\phi}), \zeta_{\tilde{\gimel}}(\tilde{\phi})\}$ with $\tilde{\phi} \in \tilde{\mathcal{L}}$, where $\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\phi}) : \tilde{\mathcal{L}} \to [0, 1]$ is the $+\mathcal{MD}$, $\xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\phi}) : \tilde{\mathcal{L}} \to [-1, 0]$ is the $-\mathcal{MD}$ and $\zeta_{\tilde{\gimel}}(\tilde{\phi}) : \tilde{\mathcal{L}} \to [0, 1]$ is the $\mathcal{NM}$ degree with $-1 \leq \xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\phi}) + \eta_{\tilde{\gimel}}(\tilde{\phi}) \leq 1$ and $0 \leq \xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\phi}) + \zeta_{\tilde{\gimel}}(\tilde{\phi}) \leq 1$.*

Remark 1. *A $\mathcal{TFS}$ consists of the $\mathcal{MD}$(also called $+\mathcal{MD}$), $-\mathcal{MD}$, and $\mathcal{NMD}$ all together. The $+\mathcal{MD}(\mathcal{MD})$ denotes the somewhat satisfied property, $\mathcal{NMD}$ denotes the property that is not fulfilled, and $-\mathcal{MD}$ denotes some implicit-counter property (opposite to $\mathcal{MD}$).*

3. Tripolar Complex Fuzzy Bracket Product

Definition 8. *A $\mathcal{TCFS}$ $\tilde{\gimel}$ in $\tilde{\mathcal{L}}$ is the structure*

$$\tilde{\gimel} = \{(\tilde{\phi}, \xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\phi}) = \tilde{r}^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\phi})}, \xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\phi}) = \tilde{r}^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\phi})}, \zeta_{\tilde{\gimel}}(\tilde{\phi}) = \tilde{r}_{\tilde{\gimel}}(\tilde{\phi})e^{i2\pi\tilde{\omega}_{\tilde{\gimel}}(\tilde{\phi})}) \mid \tilde{\phi} \in \tilde{\mathcal{L}}\},$$

where $\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\phi}) : \tilde{\mathcal{L}} \to [0, 1], \xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\phi}) : \tilde{\mathcal{L}} \to [-1, 0], \zeta_{\tilde{\gimel}} : \tilde{\mathcal{L}} \to [0, 1]$.

Example 1. *Let $\tilde{\mathcal{L}} = \{\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3\}$ be a universe of discourse. A $\mathcal{TCFS}$ $\tilde{\gimel}$ in $\tilde{\mathcal{L}}$ is defined as:*

$$\tilde{\gimel} = \begin{Bmatrix} \left(\tilde{\phi}_1, 0.8e^{i2\pi(0.7)}, -0.5e^{i2\pi(0.3)}, 0.4e^{i2\pi(0.5)}\right), \\ \left(\tilde{\phi}_2, 0.6e^{i2\pi(0.6)}, -0.4e^{i2\pi(0.4)}, 0.3e^{i2\pi(0.4)}\right), \\ \left(\tilde{\phi}_3, 0.9e^{i2\pi(0.8)}, -0.7e^{i2\pi(0.2)}, 0.2e^{i2\pi(0.3)}\right) \end{Bmatrix}$$

Here:

- $\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\phi}_1) = 0.8e^{i2\pi(0.7)}$ *is the $+\mathcal{MD}$ of $\tilde{\phi}_1$ with amplitude* 0.8 *and phase* 0.7*,*
- $\xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\phi}_1) = -0.5e^{i2\pi(0.3)}$ *is the $-\mathcal{MD}$ of $\tilde{\phi}_1$ with amplitude* 0.5 *(but sign -1) and phase* 0.3*,*
- $\zeta_{\tilde{\gimel}}(\tilde{\phi}_1) = 0.4e^{i2\pi(0.5)}$ *is the $\mathcal{NMD}$ of $\tilde{\phi}_1$ with amplitude* 0.4 *and phase* 0.5*.*

Definition 9. *For any $\mathcal{TCFS}s$ $\tilde{\gimel} = (\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\phi}), \xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\phi}), \zeta_{\tilde{\gimel}}(\tilde{\phi}))$ and $\tilde{\daleth} = (\xi^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\phi}), \xi^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\phi}), \zeta_{\tilde{\daleth}}(\tilde{\phi}))$ of $\tilde{\mathcal{L}}$. Then $[\tilde{\gimel}, \tilde{\daleth}] = (\xi^{\mathcal{P}}_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\phi}), \xi^{\mathcal{N}}_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\phi}), \zeta_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\phi}))$, where*
(i) if $\tilde{\beta}_j \in \mathcal{F}, \tilde{\phi}_j, \tilde{\eta}_j \in \tilde{\mathcal{L}}$, then

$$\xi^{\mathcal{P}}_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\phi}) = \sup_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\beta_j[\tilde{\phi}_j,\tilde{\eta}_j]}\{\min_{j\in\mathcal{N}}\{\tilde{r}^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\phi}_j) \wedge \tilde{r}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}e^{i2\pi\min_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\phi}_j)\wedge\tilde{\omega}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}}$$

and if $\tilde{\phi} \neq \sum_{j\in\mathcal{N}} \beta_i[\tilde{\phi}_i, \tilde{\eta}_j]$*, then* $\xi^{\mathcal{P}}_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\phi}) = 0$*,*
(ii) if $\tilde{\beta}_j \in \mathcal{F}, \tilde{\phi}_j, \tilde{\eta}_j \in \tilde{\mathcal{L}}$*, then*

$$\xi^{\mathcal{N}}_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\phi}) = \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}} \beta_j[\tilde{\phi}_j,\tilde{\eta}_j]}\{\max_{j\in\mathcal{N}}\{\tilde{r}^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\phi}_j) \vee \tilde{r}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}e^{i2\pi \max_{j\in\mathcal{N}}\{\tilde{\omega}^{N}_{\tilde{\gimel}}(\tilde{\phi}_j)\vee\tilde{\omega}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}}$$

and if $\tilde{\phi} \neq \sum_{j\in\mathcal{N}} \beta_i[\tilde{\phi}_i, \tilde{\eta}_j]$*, then* $\xi^{\mathcal{N}}_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\phi}) = 0$*,*
(iii) if $\tilde{\beta}_j \in \mathcal{F}, \tilde{\phi}_j, \tilde{\eta}_j \in \tilde{\mathcal{L}}$*, then*

$$\zeta_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\phi}) = \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}} \beta_j[\tilde{\phi}_j,\tilde{\eta}_j]}\{\max_{j\in\mathcal{N}}\{\tilde{r}_{\tilde{\gimel}}(\tilde{\phi}_j) \vee \tilde{r}_{\tilde{\daleth}}(\tilde{\eta}_j)\}e^{i2\pi \max_{j\in\mathcal{N}}\{\tilde{\omega}_{\tilde{\gimel}}(\tilde{\phi}_j)\vee\tilde{\omega}_{\tilde{\daleth}}(\tilde{\eta}_j)\}}$$

and if $\tilde{\phi} \neq \sum_{j\in\mathcal{N}} \beta_i[\tilde{\phi}_i, \tilde{\eta}_j]$*, then* $\zeta_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\phi}) = 0$*.*

Remark 2. *If* $\tilde{\phi}, \tilde{\eta} \in \tilde{\mathcal{L}}$*, then*
(1) $\xi^{\mathcal{P}}_{[\tilde{\gimel},\tilde{\daleth}]}([\tilde{\phi},\tilde{\eta}]) = \tilde{r}^{\mathcal{P}}_{[\tilde{\gimel},\tilde{\daleth}]}([\tilde{\phi},\tilde{\eta}])e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{[\tilde{\gimel},\tilde{\daleth}]}([\tilde{\phi},\tilde{\eta}])}$, $\tilde{r}^{\mathcal{P}}_{[\tilde{\gimel},\tilde{\daleth}]}([\tilde{\phi},\tilde{\eta}]) \geq \tilde{r}^{\mathcal{P}}_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\phi}) \wedge \tilde{r}^{\mathcal{P}}_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\eta})$*, and* $\tilde{\omega}^{\mathcal{P}}_{[\tilde{\gimel},\tilde{\daleth}]}([\tilde{\phi},\tilde{\eta}]) \geq \tilde{\omega}^{\mathcal{P}}_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\phi}) \wedge \tilde{\omega}^{\mathcal{P}}_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\eta})$*,*
(2) $\xi^{\mathcal{N}}_{[\tilde{\gimel},\tilde{\daleth}]}([\tilde{\phi},\tilde{\eta}]) = \tilde{r}^{\mathcal{N}}_{[\tilde{\gimel},\tilde{\daleth}]}([\tilde{\phi},\tilde{\eta}])e^{i2\pi\tilde{\omega}^{\mathcal{N}}_{[\tilde{\gimel},\tilde{\daleth}]}([\tilde{\phi},\tilde{\eta}])}$, $\tilde{r}^{\mathcal{N}}_{[\tilde{\gimel},\tilde{\daleth}]}([\tilde{\phi},\tilde{\eta}]) \leq \tilde{r}^{\mathcal{N}}_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\phi}) \vee \tilde{r}^{\mathcal{N}}_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\eta})$*, and* $\tilde{\omega}^{\mathcal{N}}_{[\tilde{\gimel},\tilde{\daleth}]}([\tilde{\phi},\tilde{\eta}]) \leq \tilde{\omega}^{\mathcal{N}}_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\phi}) \vee \tilde{\omega}^{\mathcal{N}}_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\eta})$*,*
(3) $\zeta_{[\tilde{\gimel},\tilde{\daleth}]}([\tilde{\phi},\tilde{\eta}]) = \tilde{r}_{[\tilde{\gimel},\tilde{\daleth}]}([\tilde{\phi},\tilde{\eta}])e^{i2\pi\tilde{\omega}_{[\tilde{\gimel},\tilde{\daleth}]}([\tilde{\phi},\tilde{\eta}])}$, $\tilde{r}_{[\tilde{\gimel},\tilde{\daleth}]}([\tilde{\phi},\tilde{\eta}]) \leq \tilde{r}_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\phi}) \vee \tilde{r}_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\eta})$*, and* $\tilde{\omega}_{[\tilde{\gimel},\tilde{\daleth}]}([\tilde{\phi},\tilde{\eta}]) \leq \tilde{\omega}_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\phi}) \vee \tilde{\omega}^{N}_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\eta})$*.*

Theorem 1. *Let* $\tilde{\gimel} = (\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\phi}), \xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\phi}), \zeta_{\tilde{\gimel}}(\tilde{\phi}))$*,* $\tilde{\daleth} = (\xi^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\phi}), \xi^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\phi}), \zeta_{\tilde{\daleth}}(\tilde{\phi}))$*, and* $\tilde{\aleph} = (\xi^{\mathcal{P}}_{\tilde{\aleph}}(\tilde{\phi}), \xi^{\mathcal{N}}_{\tilde{\aleph}}(\tilde{\phi}), \zeta_{\tilde{\aleph}}(\tilde{\phi}))$ *be* $\mathcal{TCFS}$*s of* $\tilde{\mathcal{L}}$ *such that* $\tilde{\gimel} \subseteq \tilde{\aleph}$ *and* $\tilde{\aleph} \subseteq \tilde{\aleph}$*. Then* $\tilde{\gimel} + \tilde{\aleph} \subseteq \tilde{\aleph}$*.*

Proof. Let $\tilde{\phi} \in \tilde{\mathcal{L}}$. Then

$$\begin{aligned}
\xi^{\mathcal{P}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}) &= \tilde{r}^{\mathcal{P}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi})} \\
&\geq \sup_{\tilde{\phi}\ =\ \tilde{\varrho}+\tilde{\vartheta}}\{(\tilde{r}^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varrho}) \wedge \tilde{r}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\vartheta}))e^{i2\pi(\tilde{\omega}^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varrho})\wedge\tilde{\vartheta}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\vartheta}))}\} \\
&\geq \sup_{\tilde{\phi}\ =\ \tilde{\varrho}+\tilde{\vartheta}}\{(\tilde{r}^{\mathcal{P}}_{\tilde{\aleph}}(\tilde{\varrho}) \wedge \tilde{r}^{\mathcal{P}}_{\tilde{\aleph}}(\tilde{\vartheta}))e^{i2\pi(\tilde{\omega}^{\mathcal{P}}_{\tilde{\aleph}}(\tilde{\varrho})\wedge\tilde{\omega}^{\mathcal{P}}_{\tilde{\aleph}}(\tilde{\vartheta}))}\} \\
&\geq \sup_{\tilde{\phi}\ =\ \tilde{\varrho}+\tilde{\vartheta}}\{\tilde{r}^{\mathcal{P}}_{\tilde{\aleph}}(\tilde{\varrho}+\tilde{\vartheta})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\aleph}}(\tilde{\varrho}+\tilde{\vartheta})}\} \\
&= \tilde{r}^{\mathcal{P}}_{\tilde{\aleph}}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\aleph}}(\tilde{\phi})} \\
&= \xi^{\mathcal{P}}_{\tilde{\aleph}}(\tilde{\phi}),
\end{aligned}$$

$$\begin{aligned}
\xi^{\mathcal{N}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}) &= \tilde{r}^{\mathcal{N}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi})} \\
&\leq \inf_{\tilde{\phi}\ =\ \tilde{\varrho}+\tilde{\vartheta}}\{(\tilde{r}^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varrho}) \vee \tilde{r}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\vartheta}))e^{i2\pi(\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varrho})\vee\tilde{\omega}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\vartheta}))}\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \inf_{\tilde{\phi} \ = \ \tilde{\varrho}+\tilde{\vartheta}} \{(\tilde{r}^{\mathcal{N}}_{\tilde{\aleph}}(\tilde{\varrho}) \vee \tilde{r}^{\mathcal{N}}_{\tilde{\aleph}}(\tilde{\vartheta}))e^{i2\pi(\tilde{\omega}^{\mathcal{N}}_{\tilde{\aleph}}(\tilde{\varrho})\vee\tilde{\omega}^{\mathcal{N}}_{\tilde{\aleph}}(\tilde{\vartheta}))}\} \\
&\leq \inf_{\tilde{\phi} \ = \ \tilde{\varrho}+\tilde{\vartheta}} \{\tilde{r}^{\mathcal{N}}_{\tilde{\aleph}}(\tilde{\varrho}+\tilde{\vartheta})e^{i2\pi\tilde{\omega}^{\mathcal{N}}_{\tilde{\aleph}}(\tilde{\varrho}+\tilde{\vartheta})}\} \\
&= \tilde{r}^{\mathcal{N}}_{\tilde{\aleph}}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{N}}_{\tilde{\aleph}}(\tilde{\phi})} \\
&= \xi^{\mathcal{N}}_{\tilde{\aleph}}(\tilde{\phi}),
\end{aligned}
$$

and

$$
\begin{aligned}
\zeta_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}) &= \tilde{r}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi})e^{i2\pi\tilde{\omega}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi})} \\
&\leq \inf_{\tilde{\phi} \ = \ \tilde{\varrho}+\tilde{\vartheta}} \{(\tilde{r}_{\tilde{\gimel}}(\tilde{\varrho}) \vee \tilde{r}_{\tilde{\daleth}}(\tilde{\vartheta}))e^{i2\pi(\tilde{\omega}_{\tilde{\gimel}}(\tilde{\varrho})\vee\tilde{\omega}_{\tilde{\daleth}}(\tilde{\vartheta}))}\} \\
&\leq \inf_{\tilde{\phi} \ = \ \tilde{\varrho}+\tilde{\vartheta}} \{(\tilde{r}_{\tilde{\aleph}}(\tilde{\varrho}) \vee \tilde{r}_{\tilde{\aleph}}(\tilde{\vartheta}))e^{i2\pi(\tilde{\omega}_{\tilde{\aleph}}(\tilde{\varrho})\vee\tilde{\omega}_{\tilde{\aleph}}(\tilde{\vartheta}))}\} \\
&\leq \inf_{\tilde{\phi} \ = \ \tilde{\varrho}+\tilde{\vartheta}} \{\tilde{r}_{\tilde{\aleph}}(\tilde{\varrho}+\tilde{\vartheta})e^{i2\pi\tilde{\omega}_{\tilde{\aleph}}(\tilde{\varrho}+\tilde{\vartheta})}\} \\
&= \tilde{r}_{\tilde{\aleph}}(\tilde{\phi})e^{i2\pi\tilde{\omega}_{\tilde{\aleph}}(\tilde{\phi})} \\
&= \zeta_{\tilde{\aleph}}(\tilde{\phi}).
\end{aligned}
$$

Hence, $\tilde{\gimel}+\tilde{\daleth} \subseteq \tilde{\aleph}$.

Theorem 2. *Let* $\tilde{\gimel}_1 = (\xi^{\mathcal{P}}_{\tilde{\gimel}_1}(\tilde{\phi}), \xi^{\mathcal{N}}_{\tilde{\gimel}_1}(\tilde{\phi}), \zeta_{\tilde{\gimel}_1}(\tilde{\phi}))$, $\tilde{\gimel}_2 = (\xi^{\mathcal{P}}_{\tilde{\gimel}_2}(\tilde{\phi}), \xi^{\mathcal{N}}_{\tilde{\gimel}_2}(\tilde{\phi}), \zeta_{\tilde{\gimel}_2}(\tilde{\phi}))$ *and* $\tilde{\daleth}_1 = (\xi^{\mathcal{P}}_{\tilde{\daleth}_1}(\tilde{\phi}), \xi^{\mathcal{N}}_{\tilde{\daleth}_1}(\tilde{\phi}), \zeta_{\tilde{\daleth}_1}(\tilde{\phi}))$, $\tilde{\daleth}_2 = (\xi^{\mathcal{P}}_{\tilde{\daleth}_2}(\tilde{\phi}), \xi^{\mathcal{N}}_{\tilde{\daleth}_2}(\tilde{\phi}), \zeta_{\tilde{\daleth}_2}(\tilde{\phi}))$ *be* $\mathcal{TCFS}$*s of* $\tilde{\mathcal{L}}$ *such that* $\tilde{\gimel}_1 \subseteq \tilde{\gimel}_2$, $\tilde{\daleth}_1 \subseteq \tilde{\daleth}_2$. *Then* $[\tilde{\gimel}_1, \tilde{\daleth}] \subseteq [\tilde{\gimel}_2, \tilde{\daleth}]$ *and* $[\tilde{\gimel}, \tilde{\daleth}_1] \subseteq [\tilde{\gimel}, \tilde{\daleth}_2]$.

Proof. Let $\tilde{\phi} \in \tilde{\mathcal{L}}$. Then

$$
\begin{aligned}
\xi^{\mathcal{P}}_{[\tilde{\gimel}_1,\tilde{\daleth}_1]}(\tilde{\phi}) &= \sup_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\beta_j[\tilde{\phi}_j,\tilde{\eta}_j]} \{\min_{j\in\mathcal{N}}\{\tilde{r}^{\mathcal{P}}_{\tilde{\gimel}_1}(\tilde{\phi}_j) \wedge \tilde{r}^{\mathcal{P}}_{\tilde{\daleth}_1}(\tilde{\eta}_j)\}e^{i2\pi\max_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{P}}_{\tilde{\gimel}_1}(\tilde{\phi}_j)\wedge\tilde{\omega}^{\mathcal{P}}_{\tilde{\daleth}_1}(\tilde{\eta}_j)\}}\} \\
&\geq \sup_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\beta_j[\tilde{\phi}_j,\tilde{\eta}_j]} \{\min_{j\in\mathcal{N}}\{\tilde{r}^{\mathcal{P}}_{\tilde{\gimel}_2}(\tilde{\phi}_j) \wedge \tilde{r}^{\mathcal{P}}_{\tilde{\daleth}_2}(\tilde{\eta}_j)\}e^{i2\pi\min_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{P}}_{\tilde{\gimel}_2}(\tilde{\phi}_j)\wedge\tilde{\omega}^{\mathcal{P}}_{\tilde{\daleth}_2}(\tilde{\eta}_j)\}}\} \\
&= \xi^{\mathcal{P}}_{[\tilde{\gimel}_2,\tilde{\daleth}_2]}(\tilde{\phi}),
\end{aligned}
$$

$$
\begin{aligned}
\xi^{N}_{[\tilde{\gimel}_1,\tilde{\daleth}_1]}(\tilde{\phi}) &= \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\beta_j[\tilde{\phi}_j,\tilde{\eta}_j]} \{\max_{j\in\mathcal{N}}\{\tilde{r}^{\mathcal{N}}_{\tilde{\gimel}_1}(\tilde{\phi}_j) \vee \tilde{r}^{\mathcal{N}}_{\tilde{\daleth}_1}(\tilde{\eta}_j)\}e^{i2\pi\max_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}_1}(\tilde{\phi}_j)\vee\tilde{\omega}^{\mathcal{N}}_{\tilde{\daleth}_1}(\tilde{\eta}_j)\}}\} \\
&\leq \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\beta_j[\tilde{\phi}_j,\tilde{\eta}_j]} \{\max_{j\in\mathcal{N}}\{\tilde{r}^{\mathcal{N}}_{\tilde{\gimel}_2}(\tilde{\phi}_j) \vee \tilde{r}^{\mathcal{N}}_{\tilde{\daleth}_2}(\tilde{\eta}_j)\}e^{i2\pi\max_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}_2}(\tilde{\phi}_j)\vee\tilde{\omega}^{\mathcal{N}}_{\tilde{\daleth}_2}(\tilde{\eta}_j)\}}\} \\
&= \xi^{\mathcal{N}}_{[\tilde{\gimel}_2,\tilde{\daleth}_2]}(\tilde{\phi}),
\end{aligned}
$$

and

$$\begin{aligned}\zeta_{[\tilde{\gimel}_1,\tilde{\daleth}_1]}(\tilde{\phi}) &= \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\beta_j[\tilde{\phi}_j,\tilde{\eta}_j]}\{\max_{j\in\mathcal{N}}\{\tilde{r}_{\tilde{\gimel}_1}(\tilde{\phi}_j)\vee\tilde{r}_{\tilde{\daleth}_1}(\tilde{\eta}_j)\}e^{i2\pi\max_{j\in\mathcal{N}}\{\tilde{\omega}_{\tilde{\gimel}_1}(\tilde{\phi}_j)\vee\tilde{\omega}_{\tilde{\daleth}_1}(\tilde{\eta}_j)\}}\}\\
&\leq \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\beta_j[\tilde{\phi}_j,\tilde{\eta}_j]}\{\max_{j\in\mathcal{N}}\{\tilde{r}_{\tilde{\gimel}_2}(\tilde{\phi}_j)\vee\tilde{r}_{\tilde{\daleth}_2}(\tilde{\eta}_j)\}e^{i2\pi\max_{j\in\mathcal{N}}\{\tilde{\omega}_{\tilde{\gimel}_2}(\tilde{\phi}_j)\vee\tilde{\omega}_{\tilde{\daleth}_2}(\tilde{\eta}_j)\}}\}\\
&= \zeta_{[\tilde{\gimel}_2,\tilde{\daleth}_2]}(\tilde{\phi}).\end{aligned}$$

Theorem 3. *Let* $\tilde{\gimel}_1 = (\xi^{\mathcal{P}}_{\tilde{\gimel}_1}(\tilde{\phi}),\xi^{\mathcal{N}}_{\tilde{\gimel}_1}(\tilde{\phi}),\zeta_{\tilde{\gimel}_1}(\tilde{\phi}))$, $\tilde{\gimel}_2 = (\xi^{\mathcal{P}}_{\tilde{\gimel}_2}(\tilde{\phi}),\xi^{\mathcal{N}}_{\tilde{\gimel}_2}(\tilde{\phi}),\zeta_{\tilde{\gimel}_2}(\tilde{\phi}))$ *and* $\tilde{\daleth}_1 = (\xi^{\mathcal{P}}_{\tilde{\daleth}_1}(\tilde{\phi}),\xi^{\mathcal{N}}_{\tilde{\daleth}_1}(\tilde{\phi}),\zeta_{\tilde{\daleth}_1}(\tilde{\phi}))$, $\tilde{\daleth}_2 = (\xi^{\mathcal{P}}_{\tilde{\daleth}_2}(\tilde{\phi}),\xi^{\mathcal{N}}_{\tilde{\daleth}_2}(\tilde{\phi}),\zeta_{\tilde{\daleth}_2}(\tilde{\phi}))$ *and* $\tilde{\gimel} = (\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\phi}),\xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\phi}),\zeta_{\tilde{\gimel}}(\tilde{\phi}))$, $\tilde{\daleth} = (\xi^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\phi}),\xi^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\phi}),\zeta_{\tilde{\daleth}}(\tilde{\phi}))$ *be* $\mathcal{TCFS}$*s of* $\tilde{\mathcal{L}}$. *Then* $[\tilde{\gimel}_1+\tilde{\gimel}_2,\tilde{\daleth}] = [\tilde{\gimel}_1,\tilde{\daleth}]+[\tilde{\gimel}_2,\tilde{\daleth}]$ *and* $[\tilde{\gimel},\tilde{\daleth}_1+\tilde{\daleth}_2] = [\tilde{\gimel},\tilde{\daleth}_1]+[\tilde{\gimel},\tilde{\daleth}_2]$.

Proof. Let $\tilde{\phi}\in\tilde{\mathcal{L}}$ and $\overline{\sup} = \sup_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\beta_j[\tilde{\phi}_j,\tilde{\eta}_j]}$. Then

$$\begin{aligned}&\xi^{\mathcal{P}}_{[\tilde{\gimel}_1+\tilde{\gimel}_2,\tilde{\daleth}]}(\tilde{\phi})\\
&= \sup_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\beta_j[\tilde{\phi}_j,\tilde{\eta}_j]}\{\min_{j\in\mathcal{N}}\{\tilde{r}^{\mathcal{P}}_{\tilde{\gimel}_1+\tilde{\gimel}_2}(\tilde{\phi}_j)\wedge\tilde{r}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}e^{i2\pi\min_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{P}}_{\tilde{\gimel}_1+\tilde{\gimel}_2}(\tilde{\phi}_j)\wedge\tilde{\omega}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}}\}\\
&= \overline{\sup}\{\min_{j\in\mathcal{N}}\{\sup_{\tilde{\phi}_j=\tilde{\varrho}_j+\tilde{\vartheta}_j}\{\tilde{r}^{\mathcal{P}}_{\tilde{\gimel}_1}(\tilde{\varrho}_j)\wedge\tilde{r}^{\mathcal{P}}_{\tilde{\gimel}_2}(\tilde{\vartheta}_j)\}\wedge\tilde{r}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}e^{i2\pi\min_{j\in\mathcal{N}}\{\sup_{\tilde{\phi}_j=\tilde{\varrho}_j+\tilde{\vartheta}_j}\{\tilde{\omega}^{\mathcal{P}}_{\tilde{\gimel}_1}(\tilde{\varrho}_j)\wedge\tilde{\omega}^{\mathcal{P}}_{\tilde{\gimel}_2}(\tilde{\vartheta}_j)\}\wedge\tilde{\omega}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}}\}\\
&= \overline{\sup}\{\min_{j\in\mathcal{N}}\{\sup_{\tilde{\phi}_j=\tilde{\varrho}_j+\tilde{\vartheta}_j}\{\tilde{r}^{\mathcal{P}}_{\tilde{\gimel}_1}(\tilde{\varrho}_j)\wedge\tilde{r}^{\mathcal{P}}_{\tilde{\gimel}_2}(\tilde{\vartheta}_j)\wedge\tilde{r}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}\}e^{i2\pi\min_{j\in\mathcal{N}}\{\sup_{\tilde{\phi}_j=\tilde{\varrho}_j+\tilde{\vartheta}_j}\{\tilde{\omega}^{\mathcal{P}}_{\tilde{\gimel}_1}(\tilde{\varrho}_j)\wedge\tilde{\omega}^{\mathcal{P}}_{\tilde{\gimel}_2}(\tilde{\vartheta}_j)\wedge\tilde{\omega}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}\}}\}\\
&= \overline{\sup}\{\min_{j\in\mathcal{N}}\{\sup_{\tilde{\phi}_j=\tilde{\varrho}_j+\tilde{\vartheta}_j}\{\tilde{r}^{\mathcal{P}}_{\tilde{\gimel}_1}(\tilde{\varrho}_j)\wedge\tilde{r}^{\mathcal{P}}_{\tilde{\gimel}_2}(\tilde{\vartheta}_j)\wedge\tilde{r}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}\}\}e^{i2\pi\overline{\sup}\{\min_{j\in\mathcal{N}}\{\sup_{\tilde{\phi}_j=\tilde{\varrho}_j+\tilde{\vartheta}_j}\{\tilde{\omega}^{\mathcal{P}}_{\tilde{\gimel}_1}(\tilde{\varrho}_j)\wedge\tilde{\omega}^{\mathcal{P}}_{\tilde{\gimel}_2}(\tilde{\vartheta}_j)\wedge\tilde{\omega}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}\}\}}\\
&= \tilde{r}^{\mathcal{P}}_{[\tilde{\gimel}_1+\tilde{\gimel}_2,\tilde{\daleth}]}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{[\tilde{\gimel}_1+\tilde{\gimel}_2,\tilde{\daleth}]}(\tilde{\phi})}\\
&\geq \tilde{r}^{\mathcal{P}}_{[\tilde{\gimel}_1,\tilde{\daleth}]+[\tilde{\gimel}_2,\tilde{\daleth}]}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{[\tilde{\gimel}_1,\tilde{\daleth}]+[\tilde{\gimel}_2,\tilde{\daleth}]}(\tilde{\phi})}\\
&\geq \xi^{\mathcal{P}}_{[\tilde{\gimel}_1,\tilde{\daleth}]+[\tilde{\gimel}_2,\tilde{\daleth}]}(\tilde{\phi}),\end{aligned}$$

$$\begin{aligned}&\xi^{\mathcal{N}}_{[\tilde{\gimel}_1+\tilde{\gimel}_2,\tilde{\daleth}]}(\tilde{\phi})\\
&= \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\beta_j[\tilde{\phi}_j,\tilde{\eta}_j]}\{\max_{j\in\mathcal{N}}\{\tilde{r}^{\mathcal{N}}_{\tilde{\gimel}_1+\tilde{\gimel}_2}(\tilde{\phi}_j)\vee\tilde{r}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}e^{i2\pi\max_{j\in\mathcal{N}}\{\tilde{\omega}^{N}_{\tilde{\gimel}_1+\tilde{\gimel}_2}(\tilde{\phi}_j)\vee\tilde{\omega}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}}\}\\
&= \overline{\inf}\{\max_{j\in\mathcal{N}}\{\inf_{\tilde{\phi}_j=\tilde{\varrho}_j+\tilde{\vartheta}_j}\{\tilde{r}^{\mathcal{N}}_{\tilde{\gimel}_1}(\tilde{\varrho}_j)\vee\tilde{r}^{\mathcal{N}}_{\tilde{\gimel}_2}(\tilde{\vartheta}_j)\}\vee\tilde{r}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}e^{i2\pi\max_{j\in\mathcal{N}}\{\inf_{\tilde{\phi}_j=\tilde{\varrho}_j+\tilde{\vartheta}_j}\{\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}_1}(\tilde{\varrho}_j)\vee\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}_2}(\tilde{\vartheta}_j)\}\vee\tilde{\omega}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}}\}\\
&= \overline{\inf}\{\max_{j\in\mathcal{N}}\{\inf_{\tilde{\phi}_j=\tilde{\varrho}_j+\tilde{\vartheta}_j}\{\tilde{r}^{\mathcal{N}}_{\tilde{\gimel}_1}(\tilde{\varrho}_j)\vee\tilde{r}^{\mathcal{N}}_{\tilde{\gimel}_2}(\tilde{\vartheta}_j)\vee\tilde{r}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}\}e^{i2\pi\max_{j\in\mathcal{N}}\{\inf_{\tilde{\phi}_j=\tilde{\varrho}_j+\tilde{\vartheta}_j}\{\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}_1}(\tilde{\varrho}_j)\vee\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}_2}(\tilde{\vartheta}_j)\vee\tilde{\omega}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}\}}\}\\
&= \overline{\inf}\{\max_{j\in\mathcal{N}}\{\inf_{\tilde{\phi}_j=\tilde{\varrho}_j+\tilde{\vartheta}_j}\{\tilde{r}^{\mathcal{N}}_{\tilde{\gimel}_1}(\tilde{\varrho}_j)\vee\tilde{r}^{\mathcal{N}}_{\tilde{\gimel}_2}(\tilde{\vartheta}_j)\vee\tilde{r}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}\}\}e^{i2\pi\overline{\inf}\{\max_{j\in\mathcal{N}}\{\inf_{\tilde{\phi}_j=\tilde{\varrho}_j+\tilde{\vartheta}_j}\{\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}_1}(\tilde{\varrho}_j)\vee\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}_2}(\tilde{\vartheta}_j)\vee\tilde{\omega}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}\}\}}\\
&= \tilde{r}^{\mathcal{N}}_{[\tilde{\gimel}_1+\tilde{\gimel}_2,\tilde{\daleth}]}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{N}}_{[\tilde{\gimel}_1+\tilde{\gimel}_2,\tilde{\daleth}]}(\tilde{\phi})}\\
&\leq \tilde{r}^{\mathcal{N}}_{[\tilde{\gimel}_1,\tilde{\daleth}]+[\tilde{\gimel}_2,\tilde{\daleth}]}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{N}}_{[\tilde{\gimel}_1,\tilde{\daleth}]+[\tilde{\gimel}_2,\tilde{\daleth}]}(\tilde{\phi})}\\
&\leq \xi^{\mathcal{N}}_{[\tilde{\gimel}_1,\tilde{\daleth}]+[\tilde{\gimel}_2,\tilde{\daleth}]}(\tilde{\phi}),\end{aligned}$$

and

$$\zeta_{[\tilde{\gimel}_1+\tilde{\gimel}_2,\tilde{\daleth}]}(\tilde{\phi})$$

$$
\begin{aligned}
&= \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\beta_j[\tilde{\phi}_j,\tilde{\eta}_j]}\{\max_{j\in\mathcal{N}}\{\tilde{r}_{\tilde{\gimel}_1+\tilde{\gimel}_2}(\tilde{\phi}_j)\vee\tilde{r}_{\tilde{\daleth}}(\tilde{\eta}_j)\}e^{i2\pi\max_{j\in\mathcal{N}}\{\tilde{\omega}_{\tilde{\gimel}_1+\tilde{\gimel}_2}(\tilde{\phi}_j)\vee\tilde{\omega}_{\tilde{\daleth}}(\tilde{\eta}_j)\}}\}\\
&= \overline{\inf}\{\max_{j\in\mathcal{N}}\{\inf_{\tilde{\phi}_j=\tilde{\varrho}_j+\tilde{\vartheta}_j}\{\tilde{r}_{\tilde{\gimel}_1}(\tilde{\varrho}_j)\vee\tilde{r}_{\tilde{\gimel}_2}(\tilde{\vartheta}_j)\}\vee\tilde{r}_{\tilde{\daleth}}(\tilde{\eta}_j)\}e^{i2\pi\max_{j\in\mathcal{N}}\{\inf_{\tilde{\phi}_j=\tilde{\varrho}_j+\tilde{\vartheta}_j}\{\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}_1}(\tilde{\varrho}_j)\vee\tilde{\omega}_{\tilde{\gimel}_2}(\tilde{\vartheta}_j)\}\vee\tilde{\omega}_{\tilde{\daleth}}(\tilde{\eta}_j)\}}\}\\
&= \overline{\inf}\{\max_{j\in\mathcal{N}}\{\inf_{\tilde{\phi}_j=\tilde{\varrho}_j+\tilde{\vartheta}_j}\{\tilde{r}_{\tilde{\gimel}_1}(\tilde{\varrho}_j)\vee\tilde{r}_{\tilde{\gimel}_2}(\tilde{\vartheta}_j)\vee\tilde{r}_{\tilde{\daleth}}(\tilde{\eta}_j)\}\}e^{i2\pi\max_{j\in\mathcal{N}}\{\inf_{\tilde{\phi}_j=\tilde{\varrho}_j+\tilde{\vartheta}_j}\{\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}_1}(\tilde{\varrho}_j)\vee\tilde{\omega}_{\tilde{\gimel}_2}(\tilde{\vartheta}_j)\vee\tilde{\omega}_{\tilde{\daleth}}(\tilde{\eta}_j)\}\}}\}\\
&= \overline{\inf}\{\max_{j\in\mathcal{N}}\{\inf_{\tilde{\phi}_j=\tilde{\varrho}_j+\tilde{\vartheta}_j}\{\tilde{r}_{\tilde{\gimel}_1}(\tilde{\varrho}_j)\vee\tilde{r}_{\tilde{\gimel}_2}(\tilde{\vartheta}_j)\vee\tilde{r}_{\tilde{\daleth}}(\tilde{\eta}_j)\}\}\}e^{i2\pi\inf\{\max_{j\in\mathcal{N}}\{\inf_{\tilde{\phi}_j=\tilde{\varrho}_j+\tilde{\vartheta}_j}\{\tilde{\omega}_{\tilde{\gimel}_1}(\tilde{\varrho}_j)\vee\tilde{\omega}_{\tilde{\gimel}_2}(\tilde{\vartheta}_j)\vee\tilde{\omega}_{\tilde{\daleth}}(\tilde{\eta}_j)\}\}\}}\\
&= \tilde{r}_{[\tilde{\gimel}_1+\tilde{\gimel}_2,\tilde{\daleth}]}(\tilde{\phi})e^{i2\pi\tilde{\omega}_{[\tilde{\gimel}_1+\tilde{\gimel}_2,\tilde{\daleth}]}(\tilde{\phi})}\\
&\leq \tilde{r}_{[\tilde{\gimel}_1,\tilde{\daleth}]+[\tilde{\gimel}_2,\tilde{\daleth}]}(\tilde{\phi})e^{i2\pi\tilde{\omega}_{[\tilde{\gimel}_1,\tilde{\daleth}]+[\tilde{\gimel}_2,\tilde{\daleth}]}(\tilde{\phi})}\\
&\leq \zeta_{[\tilde{\gimel}_1,\tilde{\daleth}]+[\tilde{\gimel}_2,\tilde{\daleth}]}(\tilde{\phi}).
\end{aligned}
$$

This shows that $[\tilde{\gimel}_1+\tilde{\gimel}_2,\tilde{\daleth}]\subseteq[\tilde{\gimel}_1,\tilde{\daleth}]+[\tilde{\gimel}_2,\tilde{\daleth}]$. Let $\tilde{\phi}\in\tilde{\mathcal{L}}$. Then

$$
\begin{aligned}
\xi^{\mathcal{P}}_{\tilde{\gimel}_1+\tilde{\gimel}_2}(\tilde{\phi}) &= \tilde{r}^{\mathcal{P}}_{\tilde{\gimel}_1+\tilde{\gimel}_2}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\gimel}_1+\tilde{\gimel}_2}(\tilde{\phi})}\\
&= \sup_{\tilde{\phi}\,=\,\tilde{\varrho}+\tilde{\vartheta}}\{(\tilde{r}^{\mathcal{P}}_{\tilde{\gimel}_1}(\tilde{\varrho})\wedge\tilde{r}^{\mathcal{P}}_{\tilde{\gimel}_2}(\tilde{\vartheta}))e^{i2\pi(\tilde{\omega}^{\mathcal{P}}_{\tilde{\gimel}_1}(\tilde{\varrho})\wedge\tilde{\omega}^{\mathcal{P}}_{\tilde{\gimel}_2}(\tilde{\vartheta}))}\}\\
&\geq (\tilde{r}^{\mathcal{P}}_{\tilde{\gimel}_1}(\tilde{\phi})\wedge\tilde{r}^{\mathcal{P}}_{\tilde{\gimel}_2}(0))e^{i2\pi(\tilde{\omega}^{\mathcal{P}}_{\tilde{\gimel}_1}(\tilde{\phi})\wedge\tilde{\omega}^{\mathcal{P}}_{\tilde{\gimel}_2}(0))}\\
&= \tilde{r}^{\mathcal{P}}_{\tilde{\gimel}_1}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\gimel}_1}(\tilde{\phi})},
\end{aligned}
$$

$$
\begin{aligned}
\xi^{\mathcal{N}}_{\tilde{\gimel}_1+\tilde{\gimel}_2}(\tilde{\phi}) &= \tilde{r}^{\mathcal{N}}_{\tilde{\gimel}_1+\tilde{\gimel}_2}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}_1+\tilde{\gimel}_2}(\tilde{\phi})}\\
&= \inf_{\tilde{\phi}\,=\,\tilde{\varrho}+\tilde{\vartheta}}\{(\tilde{r}^{\mathcal{N}}_{\tilde{\gimel}_1}(\tilde{\varrho})\vee\tilde{r}^{\mathcal{N}}_{\tilde{\gimel}_2}(\tilde{\vartheta}))e^{i2\pi(\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}_1}(\tilde{\varrho})\vee\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}_2}(\tilde{\vartheta}))}\}\\
&\leq (\tilde{r}^{\mathcal{N}}_{\tilde{\gimel}_1}(\tilde{\phi})\vee\tilde{r}^{\mathcal{N}}_{\tilde{\gimel}_2}(0))e^{i2\pi(\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}_1}(\tilde{\phi})\vee\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}_2}(0))}\\
&= \tilde{r}^{\mathcal{N}}_{\tilde{\gimel}_1}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}_1}(\tilde{\phi})},
\end{aligned}
$$

and

$$
\begin{aligned}
\zeta_{\tilde{\gimel}_1+\tilde{\gimel}_2}(\tilde{\phi}) &= \tilde{r}_{\tilde{\gimel}_1+\tilde{\gimel}_2}(\tilde{\phi})e^{i2\pi\tilde{\omega}_{\tilde{\gimel}_1+\tilde{\gimel}_2}(\tilde{\phi})}\\
&= \inf_{\tilde{\phi}\,=\,\tilde{\varrho}+\tilde{\vartheta}}\{(\tilde{r}_{\tilde{\gimel}_1}(\tilde{\varrho})\vee\tilde{r}_{\tilde{\gimel}_2}(\tilde{\vartheta}))e^{i2\pi(\tilde{\omega}_{\tilde{\gimel}_1}(\tilde{\varrho})\vee\tilde{\omega}_{\tilde{\gimel}_2}(\tilde{\vartheta}))}\}\\
&\leq (\tilde{r}_{\tilde{\gimel}_1}(\tilde{\phi})\vee\tilde{r}_{\tilde{\gimel}_2}(0))e^{i2\pi(\tilde{\omega}_{\tilde{\gimel}_1}(\tilde{\phi})\vee\tilde{\omega}_{\tilde{\gimel}_2}(0))}\\
&= \tilde{r}_{\tilde{\gimel}_1}(\tilde{\phi})e^{i2\pi\tilde{\omega}_{\tilde{\gimel}_1}(\tilde{\phi})}.
\end{aligned}
$$

Therefore, $\tilde{\gimel}_1\subseteq\tilde{\gimel}_1+\tilde{\gimel}_2$. Similarly, $\tilde{\gimel}_2\subseteq\tilde{\gimel}_1+\tilde{\gimel}_2$. By Theorem 2, we have $[\tilde{\gimel}_1,\tilde{\daleth}]\subseteq[\tilde{\gimel}_1+\tilde{\gimel}_2,\tilde{\daleth}]$, $[\tilde{\gimel}_1,\tilde{\daleth}]\subseteq[\tilde{\gimel}_1+\tilde{\gimel}_2,\tilde{\daleth}]$. So, by Theorem 1, $[\tilde{\gimel}_1,\tilde{\daleth}]+[\tilde{\gimel}_2,\tilde{\daleth}]\subseteq[\tilde{\gimel}_1+\tilde{\gimel}_2,\tilde{\daleth}]$. Hence, $[\tilde{\gimel}_1+\tilde{\gimel}_2,\tilde{\daleth}]=[\tilde{\gimel}_1,\tilde{\daleth}]+[\tilde{\gimel}_2,\tilde{\daleth}]$.

Theorem 4. *Let* $\tilde{\gimel} = (\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\phi}), \xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\phi}), \zeta_{\tilde{\gimel}}(\tilde{\phi}))$, $\tilde{\daleth} = (\xi^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\phi}), \xi^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\phi}), \zeta_{\tilde{\daleth}}(\tilde{\phi}))$ *be* $\mathcal{TCFS}s$ *of* $\tilde{\mathcal{L}}$. *Then*

$$[\tilde{\beta}\tilde{\gimel}, \tilde{\daleth}] = \tilde{\beta}[\tilde{\gimel}, \tilde{\daleth}] \text{ and } [\tilde{\gimel}, \tilde{\beta}\tilde{\daleth}] = \tilde{\beta}[\tilde{\gimel}, \tilde{\daleth}],$$

for every $\tilde{\beta} \in \mathcal{F}$.

Proof. Let $\tilde{\phi} \in \tilde{\mathcal{L}}$. Then

$$\begin{aligned}
\xi^{\mathcal{P}}_{[\tilde{\beta}\tilde{\gimel},\tilde{\daleth}]}(\tilde{\phi}) &= \sup_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\tilde{\beta}_j[\tilde{\phi}_j,\tilde{\eta}_j]} \{\min_{j\in\mathcal{N}}\{\tilde{r}^{\mathcal{P}}_{\tilde{\beta}\tilde{\gimel}}(\tilde{\phi}_j) \wedge \tilde{r}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\} e^{i2\pi \min_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{P}}_{\tilde{\beta}\tilde{\gimel}}(\tilde{\phi}_j)\wedge\tilde{\omega}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}}\} \\
&= \sup_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\tilde{\beta}_j[\tilde{\phi}_j,\tilde{\eta}_j]} \{\min_{j\in\mathcal{N}}\{\tilde{r}^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\beta}^{-1}\tilde{\phi}_j) \wedge \tilde{r}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\} e^{i2\pi \min_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\beta}^{-1}\tilde{\phi}_j)\wedge\tilde{\omega}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}}\} \\
&= \sup_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\tilde{\beta}\tilde{\beta}_j[\tilde{\beta}^{-1}\tilde{\phi}_j,\tilde{\eta}_j]} \{\min_{j\in\mathcal{N}}\{\tilde{r}^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\beta}^{-1}\tilde{\phi}_j) \wedge \tilde{r}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\} e^{i2\pi \min_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\beta}^{-1}\tilde{\phi}_j)\wedge\tilde{\omega}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}}\} \\
&= \xi^{\mathcal{P}}_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\beta}^{-1}\tilde{\phi}) \\
&= \xi^{\mathcal{P}}_{\tilde{\beta}[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\phi}),
\end{aligned}$$

$$\begin{aligned}
\xi^{\mathcal{N}}_{[\tilde{\beta}\tilde{\gimel},\tilde{\daleth}]}(\tilde{\phi}) &= \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\tilde{\beta}_j[\tilde{\phi}_j,\tilde{\eta}_j]} \{\max_{j\in\mathcal{N}}\{\tilde{r}^{\mathcal{N}}_{\tilde{\beta}\tilde{\gimel}}(\tilde{\phi}_j) \vee \tilde{r}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\eta}_j)\} e^{i2\pi \max_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{N}}_{\tilde{\beta}\tilde{\gimel}}(\tilde{\phi}_j)\vee\tilde{\omega}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}}\} \\
&= \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\tilde{\beta}_j[\tilde{\phi}_j,\tilde{\eta}_j]} \{\max_{j\in\mathcal{N}}\{\tilde{r}^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\beta}^{-1}\tilde{\phi}_j) \vee \tilde{r}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\eta}_j)\} e^{i2\pi \max_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\beta}^{-1}\tilde{\phi}_j)\vee\tilde{\omega}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}}\} \\
&= \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\tilde{\beta}\tilde{\beta}_j[\tilde{\beta}^{-1}\tilde{\phi}_j,\tilde{\eta}_j]} \{\max_{j\in\mathcal{N}}\{\tilde{r}^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\beta}^{-1}\tilde{\phi}_j) \vee \tilde{r}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\eta}_j)\} e^{i2\pi \max_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\beta}^{-1}\tilde{\phi}_j)\vee\tilde{\omega}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}}\} \\
&= \xi^{\mathcal{N}}_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\beta}^{-1}\tilde{\phi}) \\
&= \xi^{\mathcal{N}}_{\tilde{\beta}[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\phi}),
\end{aligned}$$

and

$$\begin{aligned}
\zeta_{[\tilde{\beta}\tilde{\gimel},\tilde{\daleth}]}(\tilde{\phi}) &= \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\tilde{\beta}_j[\tilde{\phi}_j,\tilde{\eta}_j]} \{\max_{j\in\mathcal{N}}\{\tilde{r}_{\tilde{\beta}\tilde{\gimel}}(\tilde{\phi}_j) \vee \tilde{r}_{\tilde{\daleth}}(\tilde{\eta}_j)\} e^{i2\pi \max_{j\in\mathcal{N}}\{\tilde{\omega}_{\tilde{\beta}\tilde{\gimel}}(\tilde{\phi}_j)\vee\tilde{\omega}_{\tilde{\daleth}}(\tilde{\eta}_j)\}}\} \\
&= \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\tilde{\beta}_j[\tilde{\phi}_j,\tilde{\eta}_j]} \{\max_{j\in\mathcal{N}}\{\tilde{r}_{\tilde{\gimel}}(\tilde{\beta}^{-1}\tilde{\phi}_j) \vee \tilde{r}_{\tilde{\daleth}}(\tilde{\eta}_j)\} e^{i2\pi \max_{j\in\mathcal{N}}\{\tilde{\omega}_{\tilde{\gimel}}(\tilde{\beta}^{-1}\tilde{\phi}_j)\vee\tilde{\omega}_{\tilde{\daleth}}(\tilde{\eta}_j)\}}\} \\
&= \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\tilde{\beta}\tilde{\beta}_j[\tilde{\beta}^{-1}\tilde{\phi}_j,\tilde{\eta}_j]} \{\max_{j\in\mathcal{N}}\{\tilde{r}_{\tilde{\gimel}}(\tilde{\beta}^{-1}\tilde{\phi}_j) \vee \tilde{r}_{\tilde{\daleth}}(\tilde{\eta}_j)\} e^{i2\pi \max_{j\in\mathcal{N}}\{\tilde{\omega}_{\tilde{\gimel}}(\tilde{\beta}^{-1}\tilde{\phi}_j)\vee\tilde{\omega}_{\tilde{\daleth}}(\tilde{\eta}_j)\}}\} \\
&= \xi_{[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\beta}^{-1}\tilde{\phi}) \\
&= \xi_{\tilde{\beta}[\tilde{\gimel},\tilde{\daleth}]}(\tilde{\phi}).
\end{aligned}$$

If $\tilde{\beta} = 0, \tilde{\phi} \neq 0$, then

$$\xi^{\mathcal{P}}_{[\tilde{\beta}\tilde{\gimel},\tilde{\daleth}]}(\tilde{\phi}) = \sup_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\tilde{\beta}_j[\tilde{\phi}_j,\tilde{\eta}_j]} \{\min_{j\in\mathcal{N}}\{\tilde{r}^{\mathcal{P}}_{\tilde{\beta}\tilde{\gimel}}(\tilde{\phi}_j) \wedge \tilde{r}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\} e^{i2\pi \min_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{P}}_{\tilde{\beta}\tilde{\gimel}}(\tilde{\phi}_j)\wedge\tilde{\omega}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}}\},$$

$$\xi^{\mathcal{N}}_{[\tilde{\beta}\tilde{\Im},\tilde{\daleth}]}(\tilde{\phi}) = \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\tilde{\beta}_j[\tilde{\phi}_j,\tilde{\eta}_j]}\{\max_{j\in\mathcal{N}}\{\tilde{r}^{\mathcal{N}}_{\tilde{\beta}\tilde{\Im}}(\tilde{\phi}_j) \vee \tilde{r}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}e^{i2\pi \max_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{N}}_{\tilde{\beta}\tilde{\Im}}(\tilde{\phi}_j)\vee\tilde{\omega}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}}\},$$

and

$$\zeta_{[\tilde{\beta}\tilde{\Im},\tilde{\daleth}]}(\tilde{\phi}) = \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\tilde{\beta}_j[\tilde{\phi}_j,\tilde{\eta}_j]}\{\max_{j\in\mathcal{N}}\{\tilde{r}_{\tilde{\beta}\tilde{\Im}}(\tilde{\phi}_j) \vee \tilde{r}_{\tilde{\daleth}}(\tilde{\eta}_j)\}e^{i2\pi \max_{j\in\mathcal{N}}\{\tilde{\omega}_{\tilde{\beta}\tilde{\Im}}(\tilde{\phi}_j)\vee\tilde{\omega}_{\tilde{\daleth}}(\tilde{\eta}_j)\}}\},$$

there exists $\tilde{\phi} \neq 0$, which implies that $\tilde{r}^{\mathcal{P}}_{\tilde{\beta}\tilde{\Im}}(\tilde{\phi}_j) = 0, \tilde{\omega}^{\mathcal{P}}_{\tilde{\beta}\tilde{\Im}}(\tilde{\phi}_j) = 0, \tilde{r}^{\mathcal{N}}_{\tilde{\beta}\tilde{\Im}}(\tilde{\phi}_j) = 1, \tilde{\omega}^{\mathcal{N}}_{\tilde{\beta}\tilde{\Im}}(\tilde{\phi}_j) = 1$, and $\tilde{r}_{\tilde{\beta}\tilde{\Im}}(\tilde{\phi}_j) = 1, \tilde{\omega}_{\tilde{\beta}\tilde{\Im}}(\tilde{\phi}_j) = 1$. So, $\xi^{\mathcal{P}}_{[\tilde{\beta}\tilde{\Im},\tilde{\daleth}]}(\tilde{\phi}) = 0, \xi^{\mathcal{N}}_{[\tilde{\beta}\tilde{\Im},\tilde{\daleth}]}(\tilde{\phi}) = 1$ and $\zeta_{[\tilde{\beta}\tilde{\Im},\tilde{\daleth}]}(\tilde{\phi}) = 1$. If $\tilde{\beta} = 0$ and $\tilde{\phi} = 0$, then it is obvious. The second one can be obtained in the same way.

4. Tripolar Complex Fuzzy Lie Subalgebras

Definition 10. *A $\mathcal{TCFS}$ $\tilde{\Im} = \{(\tilde{\phi}, \xi^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi}) = \tilde{r}^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi})}, \xi^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\phi}) = \tilde{r}^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi})}, \zeta_{\tilde{\Im}}(\tilde{\phi}) = \tilde{r}_{\tilde{\Im}}(\tilde{\phi})e^{i2\pi\tilde{\omega}_{\tilde{\Im}}(\tilde{\phi})}) \mid \tilde{\phi} \in \tilde{\mathcal{L}}\}$ on $\tilde{\mathcal{L}}$ is termed as a $\mathcal{TCFLS}$ on $\tilde{\mathcal{L}}$ if $\tilde{\phi}, \tilde{\eta} \in \tilde{\mathcal{L}}, \tilde{\beta} \in \mathcal{F}$, the underneath holds:*

(i) $\xi^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi} + \tilde{\eta}) \geq \xi^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi}) \wedge \xi^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\eta}) \Rightarrow \tilde{r}^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi} + \tilde{\eta})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi}+\tilde{\eta})} \geq \tilde{r}^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi} + \tilde{\eta})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi})} \wedge \tilde{r}^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi} + \tilde{\eta})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\eta})}$,

(ii) $\xi^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\phi} + \tilde{\eta}) \leq \xi^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\phi}) \vee \xi^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\eta}) \Rightarrow \tilde{r}^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\phi} + \tilde{\eta})e^{i2\pi\tilde{\omega}^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\phi}+\tilde{\eta})} \leq \tilde{r}^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\phi} + \tilde{\eta})e^{i2\pi\tilde{\omega}^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\phi})} \vee \tilde{r}^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\phi} + \tilde{\eta})e^{i2\pi\tilde{\omega}^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\eta})}$,

(iii) $\zeta_{\tilde{\Im}}(\tilde{\phi} + \tilde{\eta}) \leq \zeta_{\tilde{\Im}}(\tilde{\phi}) \vee \zeta_{\tilde{\Im}}(\tilde{\eta}) \Rightarrow \tilde{r}_{\tilde{\Im}}(\tilde{\phi} + \tilde{\eta})e^{i2\pi\tilde{\omega}_{\tilde{\Im}}(\tilde{\phi}+\tilde{\eta})} \leq \tilde{r}_{\tilde{\Im}}(\tilde{\phi} + \tilde{\eta})e^{i2\pi\tilde{\omega}_{\tilde{\Im}}(\tilde{\phi})} \vee \tilde{r}_{\tilde{\Im}}(\tilde{\phi} + \tilde{\eta})e^{i2\pi\tilde{\omega}_{\tilde{\Im}}(\tilde{\eta})}$,

(iv) $\xi^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\beta}\tilde{\phi}) \geq \xi^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi}) \Rightarrow \tilde{r}^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\beta}\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\beta}\tilde{\phi}})} \geq \tilde{r}^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi}})}$,

(ϑ) $\xi^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\beta}\tilde{\phi}) \leq \xi^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\phi}) \Rightarrow \tilde{r}^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\beta}\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\beta}\tilde{\phi}})} \leq \tilde{r}^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\phi}})}$,

(vi) $\zeta_{\tilde{\Im}}(\tilde{\beta}\tilde{\phi}) \leq \zeta_{\tilde{\Im}}(\tilde{\phi}) \Rightarrow \tilde{r}_{\tilde{\Im}}(\tilde{\beta}\tilde{\phi})e^{i2\pi\tilde{\omega}_{\tilde{\Im}}(\tilde{\beta}\tilde{\phi}})} \leq \tilde{r}_{\tilde{\Im}}(\tilde{\phi})e^{i2\pi\tilde{\omega}_{\tilde{\Im}}(\tilde{\phi}})}$,

(vii) $\xi^{\mathcal{P}}_{\tilde{\Im}}([\tilde{\phi}, \tilde{\eta}]) \geq \xi^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi}) \wedge \xi^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\eta}) \Rightarrow \tilde{r}^{\mathcal{P}}_{\tilde{\Im}}([\tilde{\phi}, \tilde{\eta}])e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\Im}}([\tilde{\phi},\tilde{\eta}])} \geq \tilde{r}^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi})} \wedge \tilde{r}_{\tilde{\Im}}(\tilde{\eta})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\eta})}$,

(viii) $\xi^{\mathcal{N}}_{\tilde{\Im}}([\tilde{\phi}, \tilde{\eta}]) \leq \xi^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\phi}) \vee \xi^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\eta}) \Rightarrow \tilde{r}^{\mathcal{N}}_{\tilde{\Im}}([\tilde{\phi}, \tilde{\eta}])e^{i2\pi\tilde{\omega}^{\mathcal{N}}_{\tilde{\Im}}([\tilde{\phi},\tilde{\eta}])} \geq \tilde{r}^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\phi})} \vee \tilde{r}^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\eta})e^{i2\pi\tilde{\omega}^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\eta})}$,

(ix) $\zeta_{\tilde{\Im}}([\tilde{\phi}, \tilde{\eta}]) \leq \zeta_{\tilde{\Im}}(\tilde{\phi}) \vee \zeta_{\tilde{\Im}}(\tilde{\eta}) \Rightarrow \tilde{r}_{\tilde{\Im}}([\tilde{\phi}, \tilde{\eta}])e^{i2\pi\tilde{\omega}_{\tilde{\Im}}([\tilde{\phi},\tilde{\eta}])} \geq \tilde{r}_{\tilde{\Im}}(\tilde{\phi})e^{i2\pi\tilde{\omega}_{\tilde{\Im}}(\tilde{\phi})} \vee \tilde{r}_{\tilde{\Im}}(\tilde{\eta})e^{i2\pi\tilde{\omega}_{\tilde{\Im}}(\tilde{\eta})}$.

When the conditions (vii), (viii) and (ix) are modified to

(x) $\xi^{\mathcal{P}}_{\tilde{\Im}}([\tilde{\phi}, \tilde{\eta}]) \geq \xi^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi}) \vee \xi^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\eta}) \Rightarrow \tilde{r}^{\mathcal{P}}_{\tilde{\Im}}([\tilde{\phi}, \tilde{\eta}])e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\Im}}([\tilde{\phi},\tilde{\eta}])} \geq \tilde{r}^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi})} \vee \tilde{r}_{\tilde{\Im}}(\tilde{\eta})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\eta})}$,

(xi) $\xi^{\mathcal{N}}_{\tilde{\Im}}([\tilde{\phi}, \tilde{\eta}]) \leq \xi^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\phi}) \wedge \xi^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\eta}) \Rightarrow \tilde{r}^{\mathcal{N}}_{\tilde{\Im}}([\tilde{\phi}, \tilde{\eta}])e^{i2\pi\tilde{\omega}^{\mathcal{N}}_{\tilde{\Im}}([\tilde{\phi},\tilde{\eta}])} \geq \tilde{r}^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\phi})} \wedge \tilde{r}^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\eta})e^{i2\pi\tilde{\omega}^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\eta})}$,

(xii) $\zeta_{\tilde{\Im}}([\tilde{\phi}, \tilde{\eta}]) \leq \zeta_{\tilde{\Im}}(\tilde{\phi}) \wedge \zeta_{\tilde{\Im}}(\tilde{\eta}) \Rightarrow \tilde{r}_{\tilde{\Im}}([\tilde{\phi}, \tilde{\eta}])e^{i2\pi\tilde{\omega}_{\tilde{\Im}}([\tilde{\phi},\tilde{\eta}])} \geq \tilde{r}_{\tilde{\Im}}(\tilde{\phi})e^{i2\pi\tilde{\omega}_{\tilde{\Im}}(\tilde{\phi})} \wedge \tilde{r}_{\tilde{\Im}}(\tilde{\eta})e^{i2\pi\tilde{\omega}_{\tilde{\Im}}(\tilde{\eta})}$, *the $\mathcal{TCFS}$ $\tilde{\Im}$ is a tripolar complex fuzzy Lie ideal (shortly, $\mathcal{TCFLI}$) on $\tilde{\mathcal{L}}$.*

Definition 11. *Let $\tilde{\Im} = \{(\tilde{\phi}, \xi^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi}) = \tilde{r}^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi})}, \xi^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\phi}) = \tilde{r}^{\mathcal{N}}_{\tilde{\Im}}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\Im}}(\tilde{\phi})}, \zeta_{\tilde{\Im}}(\tilde{\phi}) = \tilde{r}_{\tilde{\Im}}(\tilde{\phi})e^{i2\pi\tilde{\omega}_{\tilde{\Im}}(\tilde{\phi})}) \mid \tilde{\phi} \in \tilde{\mathcal{L}}\}$ be a $\mathcal{TCFS}$ on $\tilde{\mathcal{L}}$. For $\tilde{\beta} \in F$ and $\tilde{\phi} \in \tilde{\mathcal{L}}$, define $\tilde{\beta}\tilde{\Im} = \{(\tilde{\phi}, \xi^{\mathcal{P}}_{\tilde{\beta}\tilde{\Im}}(\tilde{\phi}), \xi^{\mathcal{N}}_{\tilde{\beta}\mathcal{N}}(\tilde{\phi}), \zeta_{\tilde{\beta}\tilde{\Im}}(\tilde{\phi})) \mid \tilde{\phi} \in \tilde{\mathcal{L}}\}$, where*

$$\xi^{\mathcal{P}}_{\tilde{\beta}\tilde{\beth}}(\tilde{\phi}) = \begin{cases} \xi^{\mathcal{P}}_{\tilde{\beth}}(\tilde{\beta}^{-1}\tilde{\phi}) & \textit{if } \tilde{\beta} \neq 0 \\ 1 & \textit{if } \tilde{\beta} = 0, \ \tilde{\phi} = 0 \\ 0 & \textit{if } \tilde{\beta} = 0, \ \tilde{\phi} \neq 0, \end{cases}$$

$$\xi^{\mathcal{N}}_{\tilde{\beta}\tilde{\beth}}(\tilde{\phi}) = \begin{cases} \xi^{\mathcal{N}}_{\tilde{\beth}}(\tilde{\beta}^{-1}\tilde{\phi}) & \textit{if } \tilde{\beta} \neq 0 \\ 0 & \textit{if } \tilde{\beta} = 0, \ \tilde{\phi} = 0 \\ 1 & \textit{if } \tilde{\beta} = 0, \ \tilde{\phi} \neq 0, \end{cases}$$

and

$$\zeta_{\tilde{\beta}\tilde{\beth}}(\tilde{\phi}) = \begin{cases} \zeta_{\tilde{\beth}}(\tilde{\beta}^{-1}\tilde{\phi}) & \textit{if } \tilde{\beta} \neq 0 \\ 0 & \textit{if } \tilde{\beta} = 0, \ \tilde{\phi} = 0 \\ 1 & \textit{if } \tilde{\beta} = 0, \ \tilde{\phi} \neq 0. \end{cases}$$

Example 2. *Let $\tilde{\mathcal{L}} = \{\tilde{\phi}, \tilde{\eta}, \tilde{\rho}\}$ and define the $\mathcal{TCFS}$ $\tilde{\beth}$ on $\tilde{\mathcal{L}}$ as:*

$$\tilde{\beth} = \left\{ \begin{array}{l} (\tilde{\phi}, \ 0.5e^{i2\pi(0.2)}, \ 0.4e^{i2\pi(0.2)}, \ 0.6e^{i2\pi(0.3)}), \\ (\tilde{\eta}, \ 1e^{i2\pi(0)}, \ 0e^{i2\pi(0)}, \ 0e^{i2\pi(0)}), \\ (\tilde{\rho}, \ 0.8e^{i2\pi(0.1)}, \ 0.2e^{i2\pi(0.1)}, \ 0.7e^{i2\pi(0.25)}) \end{array} \right\}$$

Let $\tilde{\beta} = 1$. Then

$$\xi^{\mathcal{P}}_{\tilde{\beta}\tilde{\beth}}(\tilde{\phi}) = \xi^{\mathcal{P}}_{\tilde{\beth}}(\tilde{\beta}^{-1}\tilde{\phi}) = \xi^{\mathcal{P}}_{\tilde{\beth}}(\tilde{\phi})$$

and similarly for $\xi^{\mathcal{N}}$ and ζ. Therefore,

$$\tilde{\beta}\tilde{\beth} = \tilde{\beth} \quad (\textit{since } \tilde{\beta} = 1)$$

Now consider $\tilde{\beta} = 0$. Then for each $\tilde{\phi} \in \tilde{\mathcal{L}}$,

$$\xi^{\mathcal{P}}_{\tilde{\beta}\tilde{\beth}}(\tilde{\phi}) = \begin{cases} 1 & \textit{if } \tilde{\phi} = 0 \\ 0 & \textit{if } \tilde{\phi} \neq 0 \end{cases}, \quad \xi^{\mathcal{N}}_{\tilde{\beta}\tilde{\beth}}(\tilde{\phi}) = \begin{cases} 0 & \textit{if } \tilde{\phi} = 0 \\ 1 & \textit{if } \tilde{\phi} \neq 0 \end{cases},$$

$$\zeta_{\tilde{\beta}\tilde{\beth}}(\tilde{\phi}) = \begin{cases} 0 & \textit{if } \tilde{\phi} = 0 \\ 1 & \textit{if } \tilde{\phi} \neq 0 \end{cases}$$

So,

$$\tilde{\beta}\tilde{\beth} = \left\{ \begin{array}{l} (\tilde{\phi}, \ 0, \ 1, \ 1), \\ (\tilde{\eta}, \ 1, \ 0, \ 0), \\ (\tilde{\rho}, \ 0, \ 1, \ 1) \end{array} \right\}$$

Theorem 5. *Let $\tilde{\beth}$ be a $\mathcal{TCFLS}$ on $\tilde{\beth}$. Then $\tilde{\beta}\tilde{\beth}$ is also a $\mathcal{TCFLS}$ for any $\tilde{\beta} \in \mathcal{F}$.*

Proof. Let $\tilde{\beth} = \{(\tilde{\phi}, \xi^{\mathcal{P}}_{\tilde{\beth}}(\tilde{\phi}) = \tilde{r}^{\mathcal{P}}_{\tilde{\beth}}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\beth}}(\tilde{\phi})}, \xi^{\mathcal{N}}_{\tilde{\beth}}(\tilde{\phi}) = \tilde{r}^{\mathcal{N}}_{\tilde{\beth}}(\tilde{\phi})e^{i2\pi\tilde{\omega}^{\mathcal{P}}_{\tilde{\beth}}(\tilde{\phi})}, \zeta_{\tilde{\beth}}(\tilde{\phi}) = \tilde{r}_{\tilde{\beth}}(\tilde{\phi})e^{i2\pi\tilde{\omega}_{\tilde{\beth}}(\tilde{\phi})}) \mid \tilde{\phi} \in \tilde{\mathcal{L}}\}$ be a $\mathcal{TCFLS}$ on $\tilde{\mathcal{L}}$. Define $\tilde{\beta}\tilde{\beth} = \{(\tilde{\phi}, \xi^{\mathcal{P}}_{\tilde{\beta}\tilde{\beth}}(\tilde{\phi}), \xi^{\mathcal{N}}_{\tilde{\beta}\tilde{\mathcal{N}}}(\tilde{\phi}), \zeta_{\tilde{\beta}\tilde{\beth}}(\tilde{\phi})) \mid \tilde{\phi} \in \tilde{\mathcal{L}}\}$, where

$$\xi^{\mathcal{P}}_{\tilde{\beta}\tilde{\beth}}(\tilde{\phi}) = \xi^{\mathcal{P}}_{\tilde{\beth}}(\tilde{\beta}^{-1}\tilde{\phi}),$$

$$\xi^{\mathcal{N}}_{\tilde{\beta}\tilde{\beth}}(\tilde{\phi}) = \xi^{\mathcal{N}}_{\tilde{\beth}}(\tilde{\beta}^{-1}\tilde{\phi}),$$
$$\zeta_{\tilde{\beta}\tilde{\beth}}(\tilde{\phi}) = \zeta_{\tilde{\beth}}(\tilde{\beta}^{-1}\tilde{\phi}).$$

We need to verify that $\tilde{\beta}\tilde{\beth}$ is also a $\mathcal{TCFLS}$ for any $\tilde{\beta} \in \mathcal{F}$. Let $\tilde{\phi}, \tilde{\eta} \in \tilde{\mathcal{L}}$.

Case 1.

$$\begin{aligned}\xi^{\mathcal{P}}_{\tilde{\beta}\tilde{\beth}}(\tilde{\phi} + \tilde{\eta}) &= \xi^{\mathcal{P}}_{\tilde{\beth}}(\tilde{\beta}^{-1}(\tilde{\phi} + \tilde{\eta})) \\ &= \xi^{\mathcal{P}}_{\tilde{\beth}}(\tilde{\beta}^{-1}\tilde{\phi} + \tilde{\beta}^{-1}\tilde{\eta}) \\ &\geq \xi^{\mathcal{P}}_{\tilde{\beth}}(\tilde{\beta}^{-1}\tilde{\phi}) \wedge \xi^{\mathcal{P}}_{\tilde{\beth}}(\tilde{\beta}^{-1}\tilde{\eta}) \\ &= \xi^{\mathcal{P}}_{\tilde{\beta}\tilde{\beth}}(\tilde{\phi}) \wedge \xi^{\mathcal{P}}_{\tilde{\beta}\tilde{\beth}}(\tilde{\eta}).\end{aligned}$$

Case 2.

$$\begin{aligned}\xi^{\mathcal{N}}_{\tilde{\beta}\tilde{\beth}}(\tilde{\phi} + \tilde{\eta}) &= \xi^{\mathcal{N}}_{\tilde{\beth}}(\tilde{\beta}^{-1}(\tilde{\phi} + \tilde{\eta})) \\ &= \xi^{\mathcal{N}}_{\tilde{\beth}}(\tilde{\beta}^{-1}\tilde{\phi} + \tilde{\beta}^{-1}\tilde{\eta}) \\ &\leq \xi^{\mathcal{N}}_{\tilde{\beth}}(\tilde{\beta}^{-1}\tilde{\phi}) \vee \xi^{\mathcal{N}}_{\tilde{\beth}}(\tilde{\beta}^{-1}\tilde{\eta}) \\ &= \xi^{\mathcal{N}}_{\tilde{\beta}\tilde{\beth}}(\tilde{\phi}) \vee \xi^{\mathcal{N}}_{\tilde{\beta}\tilde{\beth}}(\tilde{\eta}).\end{aligned}$$

Case 3.

$$\begin{aligned}\zeta_{\tilde{\beta}\tilde{\beth}}(\tilde{\phi} + \tilde{\eta}) &= \zeta_{\tilde{\beth}}(\tilde{\beta}^{-1}(\tilde{\phi} + \tilde{\eta})) \\ &= \zeta_{\tilde{\beth}}(\tilde{\beta}^{-1}\tilde{\phi} + \tilde{\beta}^{-1}\tilde{\eta}) \\ &\leq \zeta_{\tilde{\beth}}(\tilde{\beta}^{-1}\tilde{\phi}) \vee \zeta_{\tilde{\beth}}(\tilde{\beta}^{-1}\tilde{\eta}) \\ &= \zeta_{\tilde{\beta}\tilde{\beth}}(\tilde{\phi}) \vee \zeta_{\tilde{\beta}\tilde{\beth}}(\tilde{\eta}).\end{aligned}$$

Case 4.

$$\begin{aligned}\xi^{\mathcal{P}}_{\tilde{\beta}\tilde{\beth}}(\tilde{\beta}\tilde{\phi}) &= \xi^{\mathcal{P}}_{\tilde{\beth}}(\tilde{\beta}^{-1}\tilde{\beta}\tilde{\phi}) \\ &= \xi^{\mathcal{P}}_{\tilde{\beth}}(\tilde{\beta}(\tilde{\beta}^{-1}\tilde{\phi})) \\ &\geq \xi^{\mathcal{P}}_{\tilde{\beth}}(\tilde{\beta}^{-1}\tilde{\phi}) \\ &= \xi^{\mathcal{P}}_{\tilde{\beta}\tilde{\beth}}(\tilde{\phi}).\end{aligned}$$

Case 5.

$$\begin{aligned}\xi^{\mathcal{N}}_{\tilde{\beta}\tilde{\beth}}(\tilde{\beta}\tilde{\phi}) &= \xi^{\mathcal{N}}_{\tilde{\beth}}(\tilde{\beta}^{-1}\tilde{\beta}\tilde{\phi}) \\ &= \xi^{\mathcal{N}}_{\tilde{\beth}}(\tilde{\beta}(\tilde{\beta}^{-1}\tilde{\phi})) \\ &\leq \xi^{\mathcal{N}}_{\tilde{\beth}}(\tilde{\beta}^{-1}\tilde{\phi}) \\ &= \xi^{\mathcal{N}}_{\tilde{\beta}\tilde{\beth}}(\tilde{\phi}).\end{aligned}$$

Case 6.

$$\begin{aligned}
\zeta_{\tilde{\beta}\tilde{\gimel}}(\tilde{\beta}\tilde{\phi}) &= \zeta_{\tilde{\gimel}}(\tilde{\beta}^{-1}\tilde{\beta}\tilde{\phi}) \\
&= \zeta_{\tilde{\gimel}}(\tilde{\beta}(\tilde{\beta}^{-1}\tilde{\phi})) \\
&\leq \zeta_{\tilde{\gimel}}(\tilde{\beta}^{-1}\tilde{\phi}) \\
&= \zeta_{\tilde{\beta}\tilde{\gimel}}(\tilde{\phi}).
\end{aligned}$$

Case 7.

$$\begin{aligned}
\xi^{\mathcal{P}}_{\tilde{\beta}\tilde{\gimel}}([\tilde{\phi},\tilde{\eta}]) &= \xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\beta}^{-1}[\tilde{\phi},\tilde{\eta}]) \\
&= \xi^{\mathcal{P}}_{\tilde{\gimel}}([\tilde{\beta}^{-1}\tilde{\phi},\tilde{\beta}^{-1}\tilde{\eta}]) \\
&\geq \xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\beta}^{-1}\tilde{\phi}) \wedge \xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\beta}^{-1}\tilde{\eta}) \\
&= \xi^{\mathcal{P}}_{\tilde{\beta}\tilde{\gimel}}(\tilde{\phi}) \wedge \xi^{\mathcal{P}}_{\tilde{\beta}\tilde{\gimel}}(\tilde{\eta}).
\end{aligned}$$

Case 8.

$$\begin{aligned}
\xi^{\mathcal{N}}_{\tilde{\beta}\tilde{\gimel}}([\tilde{\phi},\tilde{\eta}]) &= \xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\beta}^{-1}[\tilde{\phi},\tilde{\eta}]) \\
&= \xi^{\mathcal{N}}_{\tilde{\gimel}}([\tilde{\beta}^{-1}\tilde{\phi},\tilde{\beta}^{-1}\tilde{\eta}]) \\
&\leq \xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\beta}^{-1}\tilde{\phi}) \vee \xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\beta}^{-1}\tilde{\eta}) \\
&= \xi^{\mathcal{N}}_{\tilde{\beta}\tilde{\gimel}}(\tilde{\phi}) \vee \xi^{\mathcal{N}}_{\tilde{\beta}\tilde{\gimel}}(\tilde{\eta}).
\end{aligned}$$

Case 9.

$$\begin{aligned}
\zeta_{\tilde{\beta}\tilde{\gimel}}([\tilde{\phi},\tilde{\eta}]) &= \zeta_{\tilde{\gimel}}(\tilde{\beta}^{-1}[\tilde{\phi},\tilde{\eta}]) \\
&= \zeta_{\tilde{\gimel}}([\tilde{\beta}^{-1}\tilde{\phi},\tilde{\beta}^{-1}\tilde{\eta}]) \\
&\leq \zeta_{\tilde{\gimel}}(\tilde{\beta}^{-1}\tilde{\phi}) \vee \xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\beta}^{-1}\tilde{\eta}) \\
&= \zeta_{\tilde{\beta}\tilde{\gimel}}(\tilde{\phi}) \vee \zeta_{\tilde{\beta}\tilde{\gimel}}(\tilde{\eta}).
\end{aligned}$$

Hence, $\tilde{\beta}\tilde{\gimel}$ is indeed a $\mathcal{TCFLS}$ on $\tilde{\mathcal{L}}$.

Definition 12. *Let* $\tilde{\gimel} = \{(\tilde{\phi}, \xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\phi}), \xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\phi}), \zeta_{\tilde{\gimel}}(\tilde{\phi}))\}$ *and* $\tilde{\daleth} = \{(\tilde{\phi}, \xi^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\phi}), \xi^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\phi}), \zeta_{\tilde{\daleth}}(\tilde{\phi}))\}$ *be two* $\mathcal{TCFLS}$ *on* $\tilde{\mathcal{L}}$*. Then* $\tilde{\gimel} + \tilde{\daleth} = \{(\tilde{\phi}, \xi^{\mathcal{P}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}), \xi^{\mathcal{N}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}), \zeta_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}))\}$ *be* $\mathcal{TCFS}$ *on* $\tilde{\mathcal{L}}$ *given by*

$$\xi^{\mathcal{P}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}) = \begin{cases} \sup_{\tilde{\phi}=\tilde{\varepsilon}+\tilde{\varsigma}} \left\{\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varepsilon}) \wedge \xi^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\varsigma})\right\} & \textit{if } \tilde{\phi} = \tilde{\varepsilon} + \tilde{\varsigma} \\ 0 & \textit{otherwise}, \end{cases}$$

$$\xi^{\mathcal{N}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}) = \begin{cases} \inf_{\tilde{\phi}=\tilde{\varepsilon}+\tilde{\varsigma}} \left\{\xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varepsilon}) \vee \xi^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\varsigma})\right\} & \textit{if } \tilde{\phi} = \tilde{\varepsilon} + \tilde{\varsigma} \\ 1 & \textit{otherwise}, \end{cases}$$

and

$$\zeta_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}) = \begin{cases} \inf_{\tilde{\phi}=\tilde{\varepsilon}+\tilde{\varsigma}} \{\zeta_{\tilde{\gimel}}(\tilde{\varepsilon}) \vee \zeta_{\tilde{\daleth}}(\tilde{\varsigma})\} & \textit{if } \tilde{\phi} = \tilde{\varepsilon} + \tilde{\varsigma} \\ 1 & \textit{otherwise.} \end{cases}$$

Theorem 6. *Let* $\tilde{\gimel}$ *and* $\tilde{\daleth}$ *be a* $\mathcal{TCFLS}$ *of* $\tilde{\mathcal{L}}$. *Then* $\tilde{\gimel} + \tilde{\daleth}$ *is a* $\mathcal{TCFLS}$ *of* $\tilde{\mathcal{L}}$.

Proof. Let $\tilde{\gimel} = \{(\tilde{\phi}, \xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\phi}), \xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\phi}), \zeta_{\tilde{\gimel}}(\tilde{\phi}))\}$ and $\tilde{\daleth} = \{(\tilde{\phi}, \xi^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\phi}), \xi^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\phi}), \zeta_{\tilde{\daleth}}(\tilde{\phi}))\}$ be two $\mathcal{TCFLS}$ on $\tilde{\mathcal{L}}$. Define their sum as:

$$\tilde{\gimel} + \tilde{\daleth} = \{(\tilde{\phi}, \xi^{\mathcal{P}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}), \xi^{\mathcal{N}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}), \zeta_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}))\},$$

where for each $\tilde{\phi} \in \tilde{\mathcal{L}}$,

$$\begin{aligned} \xi^{\mathcal{P}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}) &= \sup_{\tilde{\phi}=\tilde{\varepsilon}+\tilde{\varsigma}} \left\{\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varepsilon}) \wedge \xi^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\varsigma})\right\} \\ \xi^{\mathcal{N}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}) &= \inf_{\tilde{\phi}=\tilde{\varepsilon}+\tilde{\varsigma}} \left\{\xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varepsilon}) \vee \xi^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\varsigma})\right\} \\ \zeta_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}) &= \inf_{\tilde{\phi}=\tilde{\varepsilon}+\tilde{\varsigma}} \left\{\zeta_{\tilde{\gimel}}(\tilde{\varepsilon}) \vee \zeta_{\tilde{\daleth}}(\tilde{\varsigma})\right\} \end{aligned}$$

To verify that $\tilde{\gimel} + \tilde{\daleth}$ is a $\mathcal{TCFLS}$ of $\tilde{\mathcal{L}}$. Let $\tilde{\phi}, \tilde{\eta} \in \tilde{\mathcal{L}}$.

Case 1.

$$\begin{aligned} \xi^{\mathcal{P}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}+\tilde{\eta}) &= \sup_{(\tilde{\phi}+\tilde{\eta})=\tilde{\varepsilon}+\tilde{\varsigma}} \left\{\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varepsilon}) \wedge \xi^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\varsigma})\right\} \\ &\geq \sup_{\substack{\tilde{\phi}=\tilde{\varepsilon}_1+\tilde{\varsigma}_1 \\ \tilde{\eta}=\tilde{\varepsilon}_2+\tilde{\varsigma}_2}} \left\{\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varepsilon}_1+\tilde{\varepsilon}_2) \wedge \xi^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\varsigma}_1+\tilde{\varsigma}_2)\right\} \\ &\geq \sup_{\substack{\tilde{\phi}=\tilde{\varepsilon}_1+\tilde{\varsigma}_1 \\ \tilde{\eta}=\tilde{\varepsilon}_2+\tilde{\varsigma}_2}} \left\{(\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varepsilon}_1) \wedge \xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varepsilon}_2)) \wedge (\xi^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\varsigma}_1) \wedge \xi^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\varsigma}_2))\right\} \\ &= \left(\sup_{\tilde{\phi}=\tilde{\varepsilon}_1+\tilde{\varsigma}_1} \{\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varepsilon}_1) \wedge \xi^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\varsigma}_1)\}\right) \wedge \left(\sup_{\tilde{\eta}=\tilde{\varepsilon}_2+\tilde{\varsigma}_2} \{\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varepsilon}_2) \wedge \xi^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\varsigma}_2)\}\right) \\ &= \xi^{\mathcal{P}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}) \wedge \xi^{\mathcal{P}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\eta}). \end{aligned}$$

Case 2.

$$\begin{aligned} \xi^{\mathcal{N}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}+\tilde{\eta}) &= \inf_{(\tilde{\phi}+\tilde{\eta})=\tilde{\varepsilon}+\tilde{\varsigma}} \left\{\xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varepsilon}) \vee \xi^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\varsigma})\right\} \\ &\leq \inf_{\substack{\tilde{\phi}=\tilde{\varepsilon}_1+\tilde{\varsigma}_1 \\ \tilde{\eta}=\tilde{\varepsilon}_2+\tilde{\varsigma}_2}} \left\{\xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varepsilon}_1+\tilde{\varepsilon}_2) \vee \xi^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\varsigma}_1+\tilde{\varsigma}_2)\right\} \\ &\leq \inf_{\substack{\tilde{\phi}=\tilde{\varepsilon}_1+\tilde{\varsigma}_1 \\ \tilde{\eta}=\tilde{\varepsilon}_2+\tilde{\varsigma}_2}} \left\{(\xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varepsilon}_1) \vee \xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varepsilon}_2)) \vee (\xi^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\varsigma}_1) \vee \xi^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\varsigma}_2))\right\} \\ &= \left(\inf_{\tilde{\phi}=\tilde{\varepsilon}_1+\tilde{\varsigma}_1} \{\xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varepsilon}_1) \vee \xi^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\varsigma}_1)\}\right) \vee \left(\inf_{\tilde{\eta}=\tilde{\varepsilon}_2+\tilde{\varsigma}_2} \{\xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varepsilon}_2) \vee \xi^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\varsigma}_2)\}\right) \end{aligned}$$

$$= \xi^{\mathcal{N}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}) \vee \xi^{\mathcal{N}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\eta}).$$

Case 3.

$$\begin{aligned}
\zeta_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}+\tilde{\eta}) &= \inf_{(\tilde{\phi}+\tilde{\eta})=\tilde{\varepsilon}+\tilde{\varsigma}} \left\{\zeta_{\tilde{\gimel}}(\tilde{\varepsilon}) \vee \zeta_{\tilde{\daleth}}(\tilde{\varsigma})\right\} \\
&\leq \inf_{\substack{\tilde{\phi}=\tilde{\varepsilon}_1+\tilde{\varsigma}_1 \\ \tilde{\eta}=\tilde{\varepsilon}_2+\tilde{\varsigma}_2}} \left\{\zeta_{\tilde{\gimel}}(\tilde{\varepsilon}_1+\tilde{\varepsilon}_2) \vee \zeta_{\tilde{\daleth}}(\tilde{\varsigma}_1+\tilde{\varsigma}_2)\right\} \\
&\leq \inf_{\substack{\tilde{\phi}=\tilde{\varepsilon}_1+\tilde{\varsigma}_1 \\ \tilde{\eta}=\tilde{\varepsilon}_2+\tilde{\varsigma}_2}} \left\{(\zeta_{\tilde{\gimel}}(\tilde{\varepsilon}_1) \vee \zeta_{\tilde{\gimel}}(\tilde{\varepsilon}_2)) \vee (\zeta_{\tilde{\daleth}}(\tilde{\varsigma}_1) \vee \zeta_{\tilde{\daleth}}(\tilde{\varsigma}_2))\right\} \\
&= \left(\inf_{\tilde{\phi}=\tilde{\varepsilon}_1+\tilde{\varsigma}_1} \{\zeta_{\tilde{\gimel}}(\tilde{\varepsilon}_1) \vee \zeta_{\tilde{\daleth}}(\tilde{\varsigma}_1)\}\right) \vee \left(\inf_{\tilde{\eta}=\tilde{\varepsilon}_2+\tilde{\varsigma}_2} \{\zeta_{\tilde{\gimel}}(\tilde{\varepsilon}_2) \vee \zeta_{\tilde{\daleth}}(\tilde{\varsigma}_2)\}\right) \\
&= \zeta_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}) \vee \zeta_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\eta}).
\end{aligned}$$

Case 4.

$$\begin{aligned}
\xi^{\mathcal{P}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\beta}\tilde{\phi}) &= \sup_{\tilde{\beta}\tilde{\phi}=\tilde{\varepsilon}+\tilde{\varsigma}} \left\{\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varepsilon}) \wedge \xi^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\varsigma})\right\} \\
&= \sup_{\tilde{\phi}=\tilde{\varepsilon}/\tilde{\beta}+\tilde{\varsigma}/\tilde{\beta}} \left\{\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\beta}(\tilde{\varepsilon}/\tilde{\beta})) \wedge \xi^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\beta}(\tilde{\varsigma}/\tilde{\beta}))\right\} \\
&\geq \sup_{\tilde{\phi}=\tilde{\varepsilon}'+\tilde{\varsigma}'} \left\{\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varepsilon}') \wedge \xi^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\varsigma}')\right\} \\
&= \xi^{\mathcal{P}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}).
\end{aligned}$$

Case 5.

$$\begin{aligned}
\xi^{\mathcal{N}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\beta}\tilde{\phi}) &= \inf_{\tilde{\beta}\tilde{\phi}=\tilde{\varepsilon}+\tilde{\varsigma}} \left\{\xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varepsilon}) \vee \xi^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\varsigma})\right\} \\
&= \inf_{\tilde{\phi}=\tilde{\varepsilon}/\tilde{\beta}+\tilde{\varsigma}/\tilde{\beta}} \left\{\xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\beta}(\tilde{\varepsilon}/\tilde{\beta})) \vee \xi^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\beta}(\tilde{\varsigma}/\tilde{\beta}))\right\} \\
&\leq \inf_{\tilde{\phi}=\tilde{\varepsilon}'+\tilde{\varsigma}'} \left\{\xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varepsilon}') \vee \xi^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\varsigma}')\right\} \\
&= \xi^{\mathcal{N}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi})
\end{aligned}$$

Case 6.

$$\begin{aligned}
\zeta_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\beta}\tilde{\phi}) &= \inf_{\tilde{\beta}\tilde{\phi}=\tilde{\varepsilon}+\tilde{\varsigma}} \left\{\zeta_{\tilde{\gimel}}(\tilde{\varepsilon}) \vee \zeta_{\tilde{\daleth}}(\tilde{\varsigma})\right\} \\
&= \inf_{\tilde{\phi}=\tilde{\varepsilon}/\tilde{\beta}+\tilde{\varsigma}/\tilde{\beta}} \left\{\zeta_{\tilde{\gimel}}(\tilde{\beta}(\tilde{\varepsilon}/\tilde{\beta})) \vee \zeta_{\tilde{\daleth}}(\tilde{\beta}(\tilde{\varsigma}/\tilde{\beta}))\right\} \\
&\leq \inf_{\tilde{\phi}=\tilde{\varepsilon}'+\tilde{\varsigma}'} \left\{\zeta_{\tilde{\gimel}}(\tilde{\varepsilon}') \vee \zeta_{\tilde{\daleth}}(\tilde{\varsigma}')\right\} \\
&= \zeta_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}).
\end{aligned}$$

Case 7.

$$\begin{aligned}
\xi^{\mathcal{P}}_{\tilde{\gimel}+\tilde{\daleth}}([\tilde{\phi},\tilde{\eta}]) &= \sup_{[\tilde{\phi},\tilde{\eta}]=\tilde{\varepsilon}+\tilde{\varsigma}} \left\{\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varepsilon}) \wedge \xi^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\varsigma})\right\} \\
&\geq \sup_{\substack{\tilde{\phi}=\tilde{\varepsilon}_1+\tilde{\varsigma}_1 \\ \tilde{\eta}=\tilde{\varepsilon}_2+\tilde{\varsigma}_2}} \left\{\xi^{\mathcal{P}}_{\tilde{\gimel}}([\tilde{\varepsilon}_1,\tilde{\varepsilon}_2]) \wedge \xi^{\mathcal{P}}_{\tilde{\daleth}}([\tilde{\varsigma}_1,\tilde{\varsigma}_2])\right\} \\
&\geq \sup_{\substack{\tilde{\phi}=\tilde{\varepsilon}_1+\tilde{\varsigma}_1 \\ \tilde{\eta}=\tilde{\varepsilon}_2+\tilde{\varsigma}_2}} \left\{(\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varepsilon}_1) \wedge \xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varepsilon}_2)) \wedge (\xi^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\varsigma}_1) \wedge \xi^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\varsigma}_2))\right\} \\
&= \xi^{\mathcal{P}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}) \wedge \xi^{\mathcal{P}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\eta}).
\end{aligned}$$

Case 8.

$$\begin{aligned}
\xi^{\mathcal{N}}_{\tilde{\gimel}+\tilde{\daleth}}([\tilde{\phi},\tilde{\eta}]) &= \inf_{[\tilde{\phi},\tilde{\eta}]=\tilde{\varepsilon}+\tilde{\varsigma}} \left\{\xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varepsilon}) \vee \xi^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\varsigma})\right\} \\
&\leq \inf_{\substack{\tilde{\phi}=\tilde{\varepsilon}_1+\tilde{\varsigma}_1 \\ \tilde{\eta}=\tilde{\varepsilon}_2+\tilde{\varsigma}_2}} \left\{\xi^{\mathcal{N}}_{\tilde{\gimel}}([\tilde{\varepsilon}_1,\tilde{\varepsilon}_2]) \vee \xi^{\mathcal{N}}_{\tilde{\daleth}}([\tilde{\varsigma}_1,\tilde{\varsigma}_2])\right\} \\
&\leq \inf_{\substack{\tilde{\phi}=\tilde{\varepsilon}_1+\tilde{\varsigma}_1 \\ \tilde{\eta}=\tilde{\varepsilon}_2+\tilde{\varsigma}_2}} \left\{(\xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varepsilon}_1) \vee \xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varepsilon}_2)) \vee (\xi^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\varsigma}_1) \vee \xi^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\varsigma}_2))\right\} \\
&= \xi^{\mathcal{N}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}) \vee \xi^{\mathcal{N}}_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\eta}).
\end{aligned}$$

Case 9.

$$\begin{aligned}
\zeta_{\tilde{\gimel}+\tilde{\daleth}}([\tilde{\phi},\tilde{\eta}]) &= \inf_{[\tilde{\phi},\tilde{\eta}]=\tilde{\varepsilon}+\tilde{\varsigma}} \left\{\zeta_{\tilde{\gimel}}(\tilde{\varepsilon}) \vee \zeta_{\tilde{\daleth}}(\tilde{\varsigma})\right\} \\
&\leq \inf_{\substack{\tilde{\phi}=\tilde{\varepsilon}_1+\tilde{\varsigma}_1 \\ \tilde{\eta}=\tilde{\varepsilon}_2+\tilde{\varsigma}_2}} \left\{\zeta_{\tilde{\gimel}}([\tilde{\varepsilon}_1,\tilde{\varepsilon}_2]) \vee \zeta_{\tilde{\daleth}}([\tilde{\varsigma}_1,\tilde{\varsigma}_2])\right\} \\
&\leq \inf_{\substack{\tilde{\phi}=\tilde{\varepsilon}_1+\tilde{\varsigma}_1 \\ \tilde{\eta}=\tilde{\varepsilon}_2+\tilde{\varsigma}_2}} \left\{(\zeta_{\tilde{\gimel}}(\tilde{\varepsilon}_1) \vee \zeta_{\tilde{\gimel}}(\tilde{\varepsilon}_2)) \vee (\zeta_{\tilde{\daleth}}(\tilde{\varsigma}_1) \vee \zeta_{\tilde{\daleth}}(\tilde{\varsigma}_2))\right\} \\
&= \zeta_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\phi}) \vee \zeta_{\tilde{\gimel}+\tilde{\daleth}}(\tilde{\eta}).
\end{aligned}$$

Hence, $\tilde{\gimel}+\tilde{\daleth}$ is indeed a $\tilde{\gimel}+\tilde{\daleth}$ is a $\mathcal{TCFLS}$ of $\tilde{\mathcal{L}}$.

5. Nilpotent and Solvable Tripolar Complex Fuzzy Lie Ideals

Definition 13. *A* $\mathcal{TCFLI}$ $\tilde{\gimel} = (\xi^{\mathcal{P}}_{\tilde{\gimel}}, \xi^{\mathcal{N}}_{\tilde{\gimel}}, \tilde{\zeta}_{\tilde{\gimel}})$ *is called nilpotent if there exists a positive integer* n *such that* $\tilde{\gimel}^n = 0$.

Theorem 7. *A homomorphic image of an* $\mathcal{NTCFLI}$ *is an* $\mathcal{NTCFLI}$.

Proof. Let $f : \tilde{\mathcal{L}}_1 \to \tilde{\mathcal{L}}_2$ be a homomorphism of $\tilde{\mathcal{L}}$, and let $\tilde{\gimel} = (\xi^{\mathcal{P}}_{\tilde{\gimel}}, \xi^{\mathcal{N}}_{\tilde{\gimel}}, \tilde{\zeta}_{\tilde{\gimel}})$ be an $\mathcal{NTCFLI}$ in $\tilde{\mathcal{L}}_1$. Assume $f(\tilde{\gimel}) = \tilde{\daleth}$. We proceed by induction to establish that $f(\tilde{\gimel}^{(n)}) \supseteq$

$\tilde{\daleth}^{(n)}$ for $n \in \tilde{\mathcal{N}}$. As a base case, we first demonstrate that $f([\tilde{\gimel},\tilde{\gimel}]) \supseteq [f(\tilde{\gimel}), f(\tilde{\gimel})] = [\tilde{\daleth},\tilde{\daleth}]$. Let $\tilde{\eta} \in \tilde{\mathcal{L}}_2$. Then

$$
\begin{aligned}
f\left(\langle\langle \xi^{\mathcal{P}}_{\tilde{\gimel}}, \xi^{\mathcal{P}}_{\tilde{\gimel}}\rangle\rangle\right)(\tilde{\eta}) &= \sup\left\{\langle\langle \xi^{\mathcal{P}}_{\tilde{\gimel}}, \xi^{\mathcal{P}}_{\tilde{\gimel}}\rangle\rangle(\tilde{\phi}) \mid f(\tilde{\phi}) = \tilde{\eta}\right\} \\
&= \sup\left\{\sup\left\{\min\left\{\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varepsilon}), \xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varsigma})\right\} \mid \tilde{\varepsilon}, \tilde{\varsigma} \in \tilde{\mathcal{L}}_1, [\tilde{\varepsilon}, \tilde{\varsigma}] = \tilde{\phi}, f(\tilde{\phi}) = \tilde{\eta}\right\}\right\} \\
&= \sup\left\{\min\left\{\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varepsilon}), \xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varsigma})\right\} \mid \tilde{\varepsilon}, \tilde{\varsigma} \in \tilde{\mathcal{L}}, [\tilde{\varepsilon}, \tilde{\varsigma}] = \tilde{\phi}, \quad f(\tilde{\phi}) = \tilde{\eta}\right\} \\
&= \sup\left\{\min\left\{\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varepsilon}), \xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varsigma})\right\} \mid \tilde{\varepsilon}, \tilde{\varsigma} \in \tilde{\mathcal{L}}_1, [f(\tilde{\varepsilon}), f(\tilde{\varsigma})] = \tilde{\eta}\right\} \\
&= \sup\left\{\min\left\{\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varepsilon}), \xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varsigma})\right\} \mid \tilde{\varepsilon}, \tilde{\varsigma} \in \tilde{\mathcal{L}}_1, f(\tilde{\varepsilon}) = \tilde{\varpi},\ f(\tilde{\varsigma}) = \tilde{\vartheta},\ [\tilde{\varpi}, \tilde{\vartheta}] = \tilde{\eta}\right\} \\
&\geq \sup\left\{\min\left\{\sup_{\tilde{\varepsilon} \in f^{-1}(\tilde{\varpi})} \xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varepsilon}), \sup_{\tilde{\varsigma} \in f^{-1}(\tilde{\vartheta})} \xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varsigma})\right\} \mid [\tilde{\varpi}, \tilde{\vartheta}] = \tilde{\eta}\right\} \\
&= \sup\left\{\min\left\{f(\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varpi})), f(\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\vartheta}))\right\} \mid [\tilde{\varpi}, \tilde{\vartheta}] = \tilde{\eta}\right\} \\
&= \langle\langle f(\xi^{\mathcal{P}}_{\tilde{\gimel}}), f(\xi^{\mathcal{P}}_{\tilde{\gimel}})\rangle\rangle(\tilde{\eta})
\end{aligned}
$$

and

$$
\begin{aligned}
f\left(\langle\langle \xi^{\mathcal{N}}_{\tilde{\gimel}}, \xi^{\mathcal{N}}_{\tilde{\gimel}}\rangle\rangle\right)(\tilde{\eta}) &= \inf\left\{\langle\langle \xi^{\mathcal{N}}_{\tilde{\gimel}}, \xi^{\mathcal{N}}_{\tilde{\gimel}}\rangle\rangle(\tilde{\phi}) \mid f(\tilde{\phi}) = \tilde{\eta}\right\} \\
&= \inf\left\{\inf\left\{\max\left\{\xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varepsilon}), \xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varsigma})\right\} \mid \tilde{\varepsilon}, \tilde{\varsigma} \in \tilde{\mathcal{L}}_1, [\tilde{\varepsilon}, \tilde{\varsigma}] = \tilde{\phi}, f(\tilde{\phi}) = \tilde{\eta}\right\}\right\} \\
&= \inf\left\{\max\left\{\xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varepsilon}), \xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varsigma})\right\} \mid \tilde{\varepsilon}, \tilde{\varsigma} \in \tilde{\mathcal{L}}_1, [\tilde{\varepsilon}, \tilde{\varsigma}] = \tilde{\phi}, f(\tilde{\phi}) = \tilde{\eta}\right\} \\
&= \inf\left\{\max\left\{\xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varepsilon}), \xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varsigma})\right\} \mid \tilde{\varepsilon}, \tilde{\varsigma} \in \tilde{\mathcal{L}}_1, [f(\tilde{\varepsilon}), f(\tilde{\varsigma})] = \tilde{\eta}\right\} \\
&= \inf\left\{\max\left\{\xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varepsilon}), \xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varsigma})\right\} \mid f(\tilde{\varepsilon}) = \tilde{\varpi}, f(\tilde{\varsigma}) = \tilde{\vartheta}, [\tilde{\varpi}, \tilde{\vartheta}] = \tilde{\eta}\right\} \\
&\leq \inf\left\{\max\left\{\inf_{\tilde{\varepsilon} \in f^{-1}(\tilde{\varpi})} \xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varepsilon}), \inf_{\tilde{\varsigma} \in f^{-1}(\tilde{\vartheta})} \xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varsigma})\right\} \mid [\tilde{\varpi}, \tilde{\vartheta}] = \tilde{\eta}\right\} \\
&= \inf\left\{\max\left\{f(\xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varpi})), f(\xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\vartheta}))\right\} \mid [\tilde{\varpi}, \tilde{\vartheta}] = \tilde{\eta}\right\} \\
&= \langle\langle f(\xi^{\mathcal{N}}_{\tilde{\gimel}}), f(\xi^{\mathcal{N}}_{\tilde{\gimel}})\rangle\rangle(\tilde{\eta}).
\end{aligned}
$$

Also,

$$
\begin{aligned}
f\left(\langle\langle \zeta_{\tilde{\gimel}}, \zeta_{\tilde{\gimel}}\rangle\rangle\right)(\tilde{\eta}) &= \inf\left\{\langle\langle \zeta_{\tilde{\gimel}}, \zeta_{\tilde{\gimel}}\rangle\rangle(\tilde{\phi}) \mid f(\tilde{\phi}) = \tilde{\eta}\right\} \\
&= \inf\left\{\inf\left\{\max\left\{\zeta_{\tilde{\gimel}}(\tilde{\varepsilon}), \zeta_{\tilde{\gimel}}(\tilde{\varsigma})\right\} \mid \tilde{\varepsilon}, \tilde{\varsigma} \in \tilde{\mathcal{L}}_1, [\tilde{\varepsilon}, \tilde{\varsigma}] = \tilde{\phi}, f(\tilde{\phi}) = \tilde{\eta}\right\}\right\} \\
&= \inf\left\{\max\left\{\zeta_{\tilde{\gimel}}(\tilde{\varepsilon}), \zeta_{\tilde{\gimel}}(\tilde{\varsigma})\right\} \mid \tilde{\varepsilon}, \tilde{\varsigma} \in \tilde{\mathcal{L}}_1, [\tilde{\varepsilon}, \tilde{\varsigma}] = \tilde{\phi}, f(\tilde{\phi}) = \tilde{\eta}\right\} \\
&= \inf\left\{\max\left\{\zeta_{\tilde{\gimel}}(\tilde{\varepsilon}), \zeta_{\tilde{\gimel}}(\tilde{\varsigma})\right\} \mid \tilde{\varepsilon}, \tilde{\varsigma} \in \tilde{\mathcal{L}}_1, [f(\tilde{\varepsilon}), f(\tilde{\varsigma})] = \tilde{\eta}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \inf\left\{\max\left\{\zeta_{\tilde{\gimel}}(\tilde{\varepsilon}), \zeta_{\tilde{\gimel}}(\tilde{\varsigma})\right\} \mid f(\tilde{\varepsilon}) = \tilde{\varpi}, f(\tilde{\varsigma}) = \tilde{\vartheta}, [\tilde{\varpi}, \tilde{\vartheta}] = \tilde{\eta}\right\} \\
&\leq \inf\left\{\max\left\{\inf_{\tilde{\varepsilon}\in f^{-1}(\tilde{\varpi})} \zeta_{\tilde{\gimel}}(\tilde{\varepsilon}), \inf_{\tilde{\varsigma}\in f^{-1}(\tilde{\vartheta})} \zeta_{\tilde{\gimel}}(\tilde{\varsigma})\right\} \mid [\tilde{\varpi}, \tilde{\vartheta}] = \tilde{\eta}\right\} \\
&= \inf\left\{\max\left\{f(\zeta_{\tilde{\gimel}}(\tilde{\varpi})), f(\zeta_{\tilde{\gimel}}(\tilde{\vartheta}))\right\} \mid [\tilde{\varpi}, \tilde{\vartheta}] = \tilde{\eta}\right\} \\
&= \langle\langle f(\zeta_{\tilde{\gimel}}), f(\zeta_{\tilde{\gimel}})\rangle\rangle(\tilde{\eta}).
\end{aligned}
$$

Thus,

$$
f([\tilde{\gimel}, \tilde{\gimel}]) \supseteq f(\langle\langle \tilde{\gimel}, \tilde{\gimel}\rangle\rangle) \supseteq \langle\langle f(\tilde{\gimel}), f(\tilde{\gimel})\rangle\rangle = [f(\tilde{\gimel}), f(\tilde{\gimel})].
$$

For $n > 1$, we get

$$
f(\tilde{\gimel}^n) = f(\tilde{\gimel} \cdot \tilde{\gimel}^{n-1}) \supseteq [f(\tilde{\gimel}), f(\tilde{\gimel}^{n-1})] \supseteq [\tilde{\daleth}, \tilde{\daleth}^{n-1}] = \tilde{\daleth}^n.
$$

Let m be a positive integer such that $\tilde{\gimel}^m = 0$. Then, for $0 \neq \tilde{\eta} \in \tilde{\mathcal{L}}_2$,

$$
\begin{aligned}
\xi^{\mathcal{P}}_{\tilde{\daleth}(m)}(\tilde{\eta}) &\leq f(\xi^{\mathcal{P}}_{\tilde{\gimel}^n})(\tilde{\eta}) = f(0)\tilde{\eta} = \sup\left\{0(\tilde{\varepsilon}) \mid f(\tilde{\varepsilon}) = \tilde{\eta}\right\} = 0, \\
\xi^{\mathcal{N}}_{\tilde{\daleth}^m}(\eta) &\leq f(\xi^{\mathcal{N}}_{\tilde{\gimel}^n})(\tilde{\eta}) = f(0)\tilde{\eta} = \inf\left\{1(\tilde{\varepsilon}) \mid f(\tilde{\varepsilon}) = \tilde{\eta}\right\} = 0.
\end{aligned}
$$

Also,

$$
\tilde{\zeta}_{\tilde{\daleth}^m}(\tilde{\eta}) \geq f(\tilde{\zeta}_{\tilde{\gimel}^n})(\tilde{\eta}) = \inf\left\{1(\tilde{\varepsilon}) \mid f(\tilde{\varepsilon}) = \tilde{\eta}\right\} = 0.
$$

Thus, $\tilde{\daleth}^m = 0$.

Definition 14. *Let $\tilde{\gimel} = (\xi^{\mathcal{P}}_{\tilde{\gimel}}, \xi^{\mathcal{N}}_{\tilde{\gimel}}, \tilde{\zeta}_{\tilde{\gimel}})$ and $\tilde{\daleth} = (\xi^{\mathcal{P}}_{\tilde{\daleth}}, \xi^{\mathcal{N}}_{\tilde{\daleth}}, \tilde{\zeta}_{\tilde{\daleth}})$ be two $\mathcal{TCFLI}$ of $\tilde{\mathcal{L}}$. The sum $\tilde{\gimel} \oplus \tilde{\daleth}$ is called a direct sum if $\tilde{\gimel} \cap \tilde{\daleth} = 0$.*

Theorem 8. *The direct sum of two $\mathcal{NTCFLI}$s is also an $\mathcal{NTCFLI}$.*

Proof. Suppose that $\tilde{\gimel} = (\xi^{\mathcal{P}}_{\tilde{\gimel}}, \xi^{\mathcal{N}}_{\tilde{\gimel}}, \tilde{\zeta}_{\tilde{\gimel}})$ and $\tilde{\daleth} = (\xi^{\mathcal{P}}_{\tilde{\daleth}}, \xi^{\mathcal{N}}_{\tilde{\daleth}}, \tilde{\zeta}_{\tilde{\daleth}})$ are two $\mathcal{TCFLI}$s such that $\tilde{\gimel} \cap \tilde{\daleth} = 0$. We prove that $\tilde{\gimel} \cap \tilde{\daleth} = 0$. Let $\tilde{\phi}(\neq 0) \in \tilde{\mathcal{L}}$. Then

$$
\begin{aligned}
\langle\langle \xi^{\mathcal{P}}_{\tilde{\gimel}}, \xi^{\mathcal{P}}_{\tilde{\daleth}}\rangle\rangle(\tilde{\phi}) &= \sup\left\{\min\left\{\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\varepsilon}), \xi^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\vartheta})\right\} \mid [\tilde{\varepsilon}, \tilde{\varsigma}] = \tilde{\phi}\right\} \\
&\leq \min\left\{\xi^{\mathcal{P}}_{\tilde{\gimel}}(\tilde{\phi}), \xi^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\phi})\right\} \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
\langle\langle \xi^{\mathcal{N}}_{\tilde{\gimel}}, \xi^{\mathcal{N}}_{\tilde{\daleth}}\rangle\rangle(\phi) &= \inf\left\{\max\left\{\xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\varepsilon}), \xi^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\vartheta})\right\} \mid [\tilde{\varepsilon}, \tilde{\varsigma}] = \tilde{\phi}\right\} \\
&\geq \max\left\{\xi^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\phi}), \xi^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\phi})\right\}
\end{aligned}
$$

$$= 0.$$

Also,

$$\begin{aligned}
\langle\langle \zeta_{\tilde{\gimel}}, \zeta_{\tilde{\daleth}}\rangle\rangle(\phi) &= \inf\left\{\max\left\{\zeta_{\tilde{\gimel}}(\tilde{\varepsilon}), \zeta_{\tilde{\daleth}}(\tilde{\vartheta})\right\} \mid [\tilde{\varepsilon}, \tilde{\varsigma}] = \tilde{\phi}\right\}\\
&\geq \max\left\{\zeta_{\tilde{\gimel}}(\tilde{\phi}), \zeta_{\tilde{\daleth}}(\tilde{\phi})\right\}\\
&= 0.
\end{aligned}$$

Therefore, $\tilde{\gimel} \cap \tilde{\daleth} = 0 \Rightarrow [\tilde{\gimel}^m, \tilde{\daleth}^n] = 0$, for all positive integers m, n. Also, we claim that $(\tilde{\gimel} \oplus \tilde{\daleth})^n \subseteq \tilde{\gimel}^n \oplus \tilde{\daleth}^n$ for $n \in \tilde{\mathcal{N}}$. We proceed by induction on n. For $n = 1$,

$$(\tilde{\gimel} \oplus \tilde{\daleth})^1 = [\tilde{\gimel} \oplus \tilde{\daleth}, \tilde{\gimel} \oplus \tilde{\daleth}] \subseteq [\tilde{\gimel}, \tilde{\gimel}] \oplus [\tilde{\gimel}, \tilde{\daleth}] \oplus [\tilde{\daleth}, \tilde{\gimel}] \oplus [\tilde{\daleth}, \tilde{\daleth}] = \tilde{\gimel}^1 + \tilde{\daleth}^1.$$

Now, for $n > 1$,

$$\begin{aligned}
(\tilde{\gimel} \oplus \tilde{\daleth})^n &= [\tilde{\gimel} \oplus \tilde{\daleth}, (\tilde{\gimel} \oplus \tilde{\daleth})^{n-1}]\\
&\subseteq [\tilde{\gimel} \oplus \tilde{\daleth}, \tilde{\gimel}^{n-1} \oplus \tilde{\daleth}^{n-1}]\\
&\subseteq [\tilde{\gimel}, \tilde{\gimel}^{n-1}] \oplus [\tilde{\gimel}, \tilde{\daleth}^{n-1}] \oplus [\tilde{\daleth}, \tilde{\gimel}^{n-1}] \oplus [\tilde{\daleth}, \tilde{\daleth}^{n-1}]\\
&= \tilde{\gimel}^n \oplus \tilde{\daleth}^n.
\end{aligned}$$

Since, there are two positive integers $r\&t$ such that $\tilde{\gimel}^r = \tilde{\daleth}^t = 0$, we have

$$(\tilde{\gimel} \oplus \tilde{\daleth})^{r+t} \subseteq \tilde{\gimel}^{r+t} \oplus \tilde{\daleth}^{r+t} = 0.$$

Definition 15. *A $\mathcal{TCFLI}$ $\tilde{\gimel} = (\xi^{\mathcal{P}}_{\tilde{\gimel}}, \xi^{\mathcal{N}}_{\tilde{\gimel}}, \tilde{\zeta}_{\tilde{\gimel}})$ is called solvable if there exists a positive integer n such that $\tilde{\daleth}^{(n)} = 0$.*

Theorem 9. *An $\mathcal{NTCFLI}$ is solvable.*

Proof. To establish the desired result, it suffices to show that $\tilde{\gimel}^{(n)} \subseteq \tilde{\gimel}^n$, for all positive integer n. We will prove this statement by induction on n, making use of Theorem 2,

$$\begin{aligned}
&\tilde{\gimel}^{(1)} = [\tilde{\gimel}, \tilde{\gimel}] = \tilde{\gimel}^{(1)}\\
&\tilde{\gimel}^{(2)} = [\tilde{\gimel}^{(1)}, \tilde{\gimel}^{(1)}] \subseteq [\tilde{\gimel}, \tilde{\gimel}^{(1)}] = \tilde{\gimel}^2\\
&\tilde{\gimel}^{(3)} = [\tilde{\gimel}^{(2)}, \tilde{\gimel}^{(2)}] \subseteq [\tilde{\gimel}, \tilde{\gimel}^{(2)}] = \tilde{\gimel}^3\\
&\quad\vdots\\
&\tilde{\gimel}^{(n)} = [\tilde{\gimel}^{(n-1)}, \tilde{\gimel}^{(n-1)}] \subseteq [\tilde{\gimel}, \tilde{\gimel}^{(n-1)}] \subseteq [\tilde{\gimel}, \tilde{\gimel}^{(n-1)}] = \tilde{\gimel}^n.
\end{aligned}$$

Definition 16. *Let $\tilde{\gimel} = (\xi^{\mathcal{P}}_{\tilde{\gimel}}, \xi^{\mathcal{N}}_{\tilde{\gimel}}, \zeta_{\tilde{\gimel}})$ be a $\mathcal{TCFLI}$ of $\tilde{\mathcal{L}}$. Construct a sequence of $\mathcal{TCFLI}$s of $\tilde{\mathcal{L}}$ by*

$$\tilde{\daleth}^{(0)} = \tilde{\daleth}, \tilde{\daleth}^{(1)} = [\tilde{\daleth}^{(0)}, \tilde{\daleth}^{(0)}], \tilde{\daleth}^{(2)} = [\tilde{\daleth}^{(1)}, \tilde{\daleth}^{(1)}], ..., \tilde{\daleth}^{(n)} = [\tilde{\daleth}^{(n-1)}, \tilde{\daleth}^{(n-1)}],$$

then $\tilde{\daleth}^{(n)}$ *is called the* n^{th} *derived TCFLI of* $\tilde{\mathcal{L}}$. *In which,* $\tilde{\daleth}^{(i+1)} = (\xi^{\mathcal{P}}_{\tilde{\daleth}^{(i+1)}}, \xi^{\mathcal{N}}_{\tilde{\daleth}^{(i+1)}}, \zeta_{\tilde{\daleth}^{(i+1)}})$, *where*

(i) if $\tilde{\beta}_j \in \mathcal{F}, \tilde{\phi}_j, \tilde{\eta}_j \in \tilde{\mathcal{L}}$, *then*

$\xi^{\mathcal{P}}_{\tilde{\daleth}^{(i+1)}}(\tilde{\phi}) = \sup_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\tilde{\beta}_j[\tilde{\phi}_j,\tilde{\eta}_j]}\{\min_{j\in\mathcal{N}}\{\tilde{r}^{\mathcal{P}}_{\tilde{\daleth}^{(i)}}(\tilde{\phi}_j) \wedge \tilde{r}^{\mathcal{P}}_{\tilde{\daleth}^{(i)}}(\tilde{\eta}_j)\}e^{i2\pi \min_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{P}}_{\tilde{\daleth}^{(i)}}(\tilde{\phi}_j)\wedge\tilde{\omega}^{\mathcal{P}}_{\tilde{\daleth}^{(i)}}(\tilde{\eta}_j)\}}\}$

and if $\tilde{\phi} \neq \sum_{j\in\mathcal{N}}\tilde{\beta}_i[\tilde{\phi}_i, \tilde{\eta}_j]$, *then* $\xi^{\mathcal{P}}_{\tilde{\daleth}^{(i+1)}}(\tilde{\phi}) = 0$,

(ii) if $\tilde{\beta}_j \in \mathcal{F}, \tilde{\phi}_j, \tilde{\eta}_j \in \tilde{\mathcal{L}}$, *then*

$\xi^{\mathcal{N}}_{\tilde{\daleth}^{(i+1)}}(\tilde{\phi}) = \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\tilde{\beta}_j[\tilde{\phi}_j,\tilde{\eta}_j]}\{\max_{j\in\mathcal{N}}\{\tilde{r}^{\mathcal{N}}_{\tilde{\daleth}^{(i)}}(\tilde{\phi}_j) \vee \tilde{r}^{\mathcal{N}}_{\tilde{\daleth}^{(i)}}(\tilde{\eta}_j)\}e^{i2\pi \max_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{N}}_{\tilde{\daleth}^{(i)}}(\tilde{\phi}_j)\vee\tilde{\omega}^{\mathcal{N}}_{\tilde{\daleth}^{(i)}}(\tilde{\eta}_j)\}}\}$

and if $\tilde{\phi} \neq \sum_{j\in\mathcal{N}}\tilde{\beta}_i[\tilde{\phi}_i, \tilde{\eta}_j]$, *then* $\xi^{\mathcal{N}}_{\tilde{\daleth}^{(i+1)}}(\tilde{\phi}) = 0$,

(iii) if $\tilde{\beta}_j \in \mathcal{F}, \tilde{\phi}_j, \tilde{\eta}_j \in \tilde{\mathcal{L}}$, *then*

$\zeta_{\tilde{\daleth}^{(i+1)}}(\tilde{\phi}) = \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\tilde{\beta}_j[\tilde{\phi}_j,\tilde{\eta}_j]}\{\max_{j\in\mathcal{N}}\{\tilde{r}_{\tilde{\daleth}^{(i)}}(\tilde{\phi}_j) \vee \tilde{r}_{\tilde{\daleth}^{(i)}}(\tilde{\eta}_j)\}e^{i2\pi \max_{j\in\mathcal{N}}\{\tilde{\omega}_{\tilde{\daleth}^{(i)}}(\tilde{\phi}_j)\vee\tilde{\omega}_{\tilde{\daleth}^{(i)}}(\tilde{\eta}_j)\}}\}$

and if $\tilde{\phi} \neq \sum_{j\in\mathcal{N}}\tilde{\beta}_i[\tilde{\phi}_i, \tilde{\eta}_j]$, *then* $\zeta_{\tilde{\daleth}^{(i+1)}}(\tilde{\phi}) = 0$.

Remark 3. *From the Definition 13, we can get* $\xi^{\mathcal{P}}_{\tilde{\daleth}^{(0)}} \supseteq \xi^{\mathcal{P}}_{\tilde{\daleth}^{(1)}} \supseteq \xi^{\mathcal{P}}_{\tilde{\daleth}^{(2)}} \supseteq ... \supseteq \xi^{\mathcal{P}}_{\tilde{\daleth}^{(n)}} \supseteq ...$, $\xi^{\mathcal{N}}_{\tilde{\daleth}^{(0)}} \subseteq \xi^{\mathcal{N}}_{\tilde{\daleth}^{(1)}} \subseteq \xi^{\mathcal{N}}_{\tilde{\daleth}^{(2)}} \subseteq ... \subseteq \xi^{\mathcal{N}}_{\tilde{\daleth}^{(n)}} \subseteq ...$, *and* $\zeta_{\tilde{\daleth}^{(0)}} \subseteq \zeta_{\tilde{\daleth}^{(1)}} \subseteq \zeta_{\tilde{\daleth}^{(2)}} \subseteq ... \subseteq \zeta_{\tilde{\daleth}^{(n)}} \subseteq ....$

Definition 17. *A* $\tilde{\daleth} = (\xi^{\mathcal{P}}_{\tilde{\daleth}}, \xi^{\mathcal{N}}_{\tilde{\daleth}}, \zeta_{\tilde{\daleth}})$ *be a* $\mathcal{TCFLI}$ *of* $\tilde{\mathcal{L}}$. *Then* $\tilde{\daleth} = (\xi^{\mathcal{P}}_{\tilde{\daleth}}, \xi^{\mathcal{N}}_{\tilde{\daleth}}, \zeta_{\tilde{\daleth}})$ *is* $\mathcal{STCFLI}$ *if and if there exists a positive integer n such that* $\xi^{\mathcal{P}}_{\tilde{\daleth}^{(m)}}(n) = 1_0$, $\xi^{\mathcal{N}}_{\tilde{\daleth}^m}(n) = (-1)_0$ *and* $\zeta_{\tilde{\daleth}^m}(n) = 0_0$, *for all positive integer* $m \geq n$.

Theorem 10. *A homomorphic images of* $\mathcal{STCFLI}$*s are* $\mathcal{STCFLI}$*s.*

Proof. Consider $f : \tilde{\mathcal{L}} \to \tilde{\mathcal{L}}'$ be a homomorphism of $\tilde{\mathcal{L}}$, and assume that $\tilde{\daleth} = (\xi^{\mathcal{P}}_{\tilde{\daleth}}, \xi^{\mathcal{N}}_{\tilde{\daleth}}, \zeta_{\tilde{\daleth}})$ is a $\mathcal{TCFLI}$ $\tilde{\mathcal{L}}$. Let $f(\tilde{\daleth}) = \tilde{\urcorner}$, i.e., $\xi^{\mathcal{P}}_{\tilde{\urcorner}} = \xi^{\mathcal{P}}_{f(\tilde{\daleth})}$, $\xi^{\mathcal{N}}_{\tilde{\urcorner}} = \xi^{\mathcal{N}}_{f(\tilde{\daleth})}$. We aim to prove, by induction on n, that $\xi^{\mathcal{P}}_{f(\tilde{\daleth}^{(n)})} = \xi^{\mathcal{P}}_{\tilde{\urcorner}^{(n)}}$ and $\xi^{\mathcal{N}}_{f(\tilde{\daleth}^{(n)})} = \xi^{\mathcal{N}}_{\tilde{\urcorner}^{(n)}}$, for all positive integer n. As the base case, consider $n = 1$. Let $\tilde{\eta} \in \tilde{\mathcal{L}}'$. Then

$$\begin{aligned}
&\xi^{\mathcal{P}}_{f(\tilde{\daleth}^{(1)})}(\tilde{\eta}) \\
&= \xi^{\mathcal{P}}_{f([\tilde{T},\tilde{T}])}(\tilde{\eta}) \\
&= \sup_{\tilde{\eta}=f(\tilde{\phi})}\left\{\xi^{\mathcal{P}}_{[\tilde{T},\tilde{T}]}(\tilde{\eta})\right\} \\
&= \sup_{\tilde{\eta}=f(\tilde{\phi})}\left\{\sup_{\tilde{\phi}=\sum\limits_{j\in\mathcal{N}}\tilde{\beta}_j[\tilde{\phi}_j,\tilde{\eta}_j]}\left\{\min_{j\in\mathcal{N}}\left\{\tilde{r}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\phi}_j) \wedge \tilde{r}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\right\} e^{i2\pi \min_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\phi}_j)\wedge\tilde{\omega}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}}\right\}\right\} \\
&= \sup_{\tilde{\phi}=\sum\limits_{j\in\mathcal{N}}\tilde{\beta}_j[\tilde{\phi}_j,\tilde{\eta}_j]}\left\{\min_{j\in\mathcal{N}}\left\{\tilde{r}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\phi}_j) \wedge \tilde{r}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\right\} e^{i2\pi \min_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\phi}_j)\wedge\tilde{\omega}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}}\right\} \\
&= \sup_{\tilde{\phi}=\sum\limits_{j\in\mathcal{N}}\tilde{\beta}_j[\tilde{\varepsilon}_j,\tilde{\varsigma}_j]}\left\{\min_{j\in\mathcal{N}}\left\{\tilde{r}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\phi}_j) \wedge \tilde{r}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\right\} e^{i2\pi \min_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\phi}_j)\wedge\tilde{\omega}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\eta}_j)\}} \mid f(\tilde{\phi}_j) = \tilde{\varepsilon}_j, f(\tilde{\eta}_j) = \tilde{\varsigma}_j\right\}
\end{aligned}$$

$$= \sup_{\sum_{j\in\mathcal{N}} \tilde{\beta}_j[\tilde{\varepsilon}_j,\tilde{\varsigma}_j]=\tilde{\phi}} \left\{ \min_{j\in\mathcal{N}} \left\{ \tilde{r}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\varepsilon}_j) \wedge \tilde{r}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\varsigma}_j) \right\} e^{i2\pi \min_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\varepsilon}_j)\wedge\tilde{\omega}^{\mathcal{P}}_{\tilde{\daleth}}(\tilde{\varsigma}_j)\}} \right\}$$

$$= \xi^{\mathcal{P}}_{[\tilde{\daleth},\tilde{\daleth}]}(\tilde{\eta})$$

$$= \xi^{\mathcal{P}}_{\tilde{\daleth}^{(1)}}(\tilde{\eta})$$

and

$$\xi^{\mathcal{N}}_{f(\tilde{\gimel}^{(1)})}(\tilde{\eta})$$

$$= \xi^{\mathcal{N}}_{f([\tilde{T},\tilde{T}])}(\tilde{\eta})$$

$$= \inf_{\tilde{\eta}=f(\tilde{\phi})} \left\{ \xi^{\mathcal{N}}_{[\tilde{T},\tilde{T}]}(\tilde{\eta}) \right\}$$

$$= \inf_{\tilde{\eta}=f(\tilde{\phi})} \left\{ \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}} \tilde{\beta}_j[\tilde{\phi}_j,\tilde{\eta}_j]} \left\{ \max_{j\in\mathcal{N}} \left\{ \tilde{r}^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\phi}_j) \wedge \tilde{r}^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\eta}_j) \right\} e^{i2\pi \max_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\phi}_j)\wedge\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\eta}_j)\}} \right\} \right\}$$

$$= \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}} \tilde{\beta}_j[\tilde{\phi}_j,\tilde{\eta}_j]} \left\{ \max_{j\in\mathcal{N}} \left\{ \tilde{r}^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\phi}_j) \wedge \tilde{r}^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\eta}_j) \right\} e^{i2\pi \max_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\phi}_j)\wedge\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\eta}_j)\}} \right\}$$

$$= \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}} \tilde{\beta}_j[\tilde{\varepsilon}_j,\tilde{\varsigma}_j]} \left\{ \max_{j\in\mathcal{N}} \left\{ \tilde{r}^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\phi}_j) \wedge \tilde{r}^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\eta}_j) \right\} e^{i2\pi \max_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\phi}_j)\wedge\tilde{\omega}^{\mathcal{N}}_{\tilde{\gimel}}(\tilde{\eta}_j)\}} \mid f(\tilde{\phi}_j)=\tilde{\varepsilon}_j, f(\tilde{\eta}_j)=\tilde{\varsigma}_j \right\}$$

$$= \inf_{\sum_{j\in\mathcal{N}} \tilde{\beta}_j[\tilde{\varepsilon}_j,\tilde{\varsigma}_j]=\tilde{\phi}} \left\{ \max_{j\in\mathcal{N}} \left\{ \tilde{r}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\varepsilon}_j) \wedge \tilde{r}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\varsigma}_j) \right\} e^{i2\pi \max_{j\in\mathcal{N}}\{\tilde{\omega}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\varepsilon}_j)\wedge\tilde{\omega}^{\mathcal{N}}_{\tilde{\daleth}}(\tilde{\varsigma}_j)\}} \right\}$$

$$= \xi^{\mathcal{N}}_{[\tilde{\daleth},\tilde{\daleth}]}(\tilde{\eta})$$

$$= \xi^{\mathcal{N}}_{\tilde{\daleth}^{(1)}}(\tilde{\eta}).$$

Also,

$$\zeta_{f(\tilde{\gimel}^{(1)})}(\tilde{\eta})$$

$$= \zeta_{f([\tilde{T},\tilde{T}])}(\tilde{\eta})$$

$$= \inf_{\tilde{\eta}=f(\tilde{\phi})} \left\{ \zeta^{\mathcal{N}}_{[\tilde{T},\tilde{T}]}(\tilde{\eta}) \right\}$$

$$= \inf_{\tilde{\eta}=f(\tilde{\phi})} \left\{ \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}} \tilde{\beta}_j[\tilde{\phi}_j,\tilde{\eta}_j]} \left\{ \max_{j\in\mathcal{N}} \left\{ \tilde{r}_{\tilde{\gimel}}(\tilde{\phi}_j) \wedge \tilde{r}_{\tilde{\gimel}}(\tilde{\eta}_j) \right\} e^{i2\pi \max_{j\in\mathcal{N}}\{\tilde{\omega}_{\tilde{\gimel}}(\tilde{\phi}_j)\wedge\tilde{\omega}_{\tilde{\gimel}}(\tilde{\eta}_j)\}} \right\} \right\}$$

$$= \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}} \tilde{\beta}_j[\tilde{\phi}_j,\tilde{\eta}_j]} \left\{ \max_{j\in\mathcal{N}} \left\{ \tilde{r}_{\tilde{\gimel}}(\tilde{\phi}_j) \wedge \tilde{r}_{\tilde{\gimel}}(\tilde{\eta}_j) \right\} e^{i2\pi \max_{j\in\mathcal{N}}\{\tilde{\omega}_{\tilde{\gimel}}(\tilde{\phi}_j)\wedge\tilde{\omega}_{\tilde{\gimel}}(\tilde{\eta}_j)\}} \right\}$$

$$= \inf_{\tilde{\phi}=\sum_{j\in\mathcal{N}}\tilde{\beta}_j[\tilde{\varepsilon}_j,\tilde{\varsigma}_j]} \left\{ \max_{j\in\mathcal{N}} \left\{ \tilde{r}_{\tilde{\gimel}}(\tilde{\phi}_j) \wedge \tilde{r}_{\tilde{\gimel}}(\tilde{\eta}_j) \right\} e^{i2\pi \max_{j\in\mathcal{N}}\{\tilde{\omega}_{\tilde{\gimel}}(\tilde{\phi}_j)\wedge\tilde{\omega}_{\tilde{\gimel}}(\tilde{\eta}_j)\}} \mid f(\tilde{\phi}_j) = \tilde{\varepsilon}_j, f(\tilde{\eta}_j) = \tilde{\varsigma}_j \right\}$$

$$= \inf_{\sum_{j\in\mathcal{N}}\tilde{\beta}_j[\tilde{\varepsilon}_j,\tilde{\varsigma}_j]=\tilde{\phi}} \left\{ \max_{j\in\mathcal{N}} \left\{ \tilde{r}_{\tilde{\daleth}}(\tilde{\varepsilon}_j) \wedge \tilde{r}_{\tilde{\daleth}}(\tilde{\varsigma}_j) \right\} e^{i2\pi \max_{j\in\mathcal{N}}\{\tilde{\omega}_{\tilde{\daleth}}(\tilde{\varepsilon}_j)\wedge\tilde{\omega}_{\tilde{\daleth}}(\tilde{\varsigma}_j)\}} \right\}$$

$$= \zeta_{[\tilde{\daleth},\tilde{\daleth}]}(\tilde{\eta})$$

$$= \zeta_{\tilde{\daleth}^{(1)}}(\tilde{\eta}).$$

The statement holds for the base case $n = 1$. Now, assume that it is true for some positive integer $n - 1$; that is, we assume the induction hypothesis holds for $n - 1$. Then

$$\xi^{\mathcal{P}}_{f(\tilde{\gimel}^{(n)})} = \xi^{\mathcal{P}}_{f([\tilde{\gimel}^{(n-1)},\tilde{\gimel}^{(n-1)}])} = \xi^{\mathcal{P}}_{[f(\tilde{\gimel}^{(n-1)}),f(\tilde{\gimel}^{(n-1)})]} = \xi^{\mathcal{P}}_{[\tilde{\daleth}^{(n-1)},\tilde{\daleth}^{(n-1)}]} = \xi^{\mathcal{P}}_{\tilde{\daleth}^{(n)}},$$
$$\xi^{\mathcal{N}}_{f(\tilde{\gimel}^{(n)})} = \xi^{\mathcal{N}}_{f([\tilde{\gimel}^{(n-1)},\tilde{\gimel}^{(n-1)}])} = \xi^{\mathcal{N}}_{[f(\tilde{\gimel}^{(n-1)}),f(\tilde{\gimel}^{(n-1)})]} = \xi^{\mathcal{N}}_{[\tilde{\daleth}^{(n-1)},\tilde{\daleth}^{(n-1)}]} = \xi^{\mathcal{N}}_{\tilde{\daleth}^{(n)}}$$

and

$$\zeta_{f(\tilde{\gimel}^{(n)})} = \zeta_{f([\tilde{\gimel}^{(n-1)},\tilde{\gimel}^{(n-1)}])} = \zeta_{[f(\tilde{\gimel}^{(n-1)}),f(\tilde{\gimel}^{(n-1)})]} = \zeta_{[\tilde{\daleth}^{(n-1)},\tilde{\daleth}^{(n-1)}]} = \zeta_{\tilde{\daleth}^{(n)}}.$$

Let $\xi^{\mathcal{P}}_{\tilde{\gimel}^{(m)}} = 1_0$, $\xi^{\mathcal{N}}_{\tilde{\gimel}^{(m)}} = (-1)_0$, and $\zeta_{\tilde{\gimel}^{(m)}} = 0_0$. Then

$$\xi^{\mathcal{P}}_{\tilde{\daleth}^{(m)}}(\tilde{\eta}) = \xi^{\mathcal{P}}_{f(\tilde{\gimel}^{m})}(\tilde{\eta}) = \sup_{\tilde{\eta}=f(\tilde{\phi})} \left\{ 1_0(\tilde{\phi}) \right\} = 0,$$
$$\xi^{\mathcal{N}}_{\tilde{\daleth}^{(m)}}(\tilde{\eta}) = \xi^{\mathcal{N}}_{f(\tilde{\gimel}^{m})}(\tilde{\eta}) = \inf_{\tilde{\eta}=f(\tilde{\phi})} \left\{ (-1)_0(\tilde{\phi}) \right\} = 0,$$

and

$$\zeta_{\tilde{\daleth}^{(m)}}(\tilde{\eta}) = \zeta_{f(\tilde{\gimel}^{m})}(\tilde{\eta}) = \inf_{\tilde{\eta}=f(\tilde{\phi})} \left\{ 0_0(\tilde{\phi}) \right\} = 0,$$

for every $0 \neq \tilde{\eta} \in \tilde{\mathcal{L}}'$. So $\xi^{\mathcal{P}}_{\tilde{\daleth}^{(m)}} = 1_0$, $\xi^{\mathcal{N}}_{\tilde{\daleth}^{(m)}} = (-1)_0$ and $\zeta_{\tilde{\daleth}^{(m)}} = 0_0$.

Theorem 11. *Let* $\tilde{\gimel} = (\xi^{\mathcal{P}}_{\tilde{\gimel}}, \xi^{\mathcal{N}}_{\tilde{\gimel}}, \tilde{\zeta})$ *be a* $\mathcal{TCFLI}$ *of* $\tilde{\mathcal{L}}$ *and suppose that the quotient* $\frac{\tilde{\gimel}}{\tilde{\mathcal{J}}}$ *forms a* $\mathcal{STCFLI}$ *in the quotient algebra* $\frac{\tilde{\mathcal{L}}}{\tilde{\mathcal{J}}}$. *Assume that* $\tilde{\daleth} = (\xi^{\mathcal{P}}_{\tilde{\daleth}}, \xi^{\mathcal{N}}_{\tilde{\daleth}}, \tilde{\zeta}_{\tilde{\daleth}})$ *is a* $\mathcal{STCFLI}$ *of* $\tilde{\mathcal{L}}$, *and also serves as a* $\mathcal{TCFLI}$ *of* $\tilde{\gimel} = (\xi^{\mathcal{P}}_{\tilde{\gimel}}, \xi^{\mathcal{N}}_{\tilde{\gimel}}, \tilde{\zeta})$. *If, in addition,* $\tilde{\daleth}(\tilde{\mathcal{J}}) = \tilde{\gimel}(\tilde{\mathcal{J}})$, *then* $\tilde{\gimel} = (\xi^{\mathcal{P}}_{\tilde{\gimel}}, \xi^{\mathcal{N}}_{\tilde{\gimel}}, \tilde{\zeta}_{\tilde{\gimel}})$ *is solvable.*

Proof. Let $f : \tilde{\mathcal{L}} \to \frac{\tilde{\mathcal{L}}}{\tilde{\mathcal{J}}}$ be the canonical map. According to the argument presented in the proof of Theorem 7,

$$\xi^{\mathcal{P}}_{\phi(\tilde{\gimel}^{(n)})} = \xi^{\mathcal{P}}_{(\frac{\tilde{\gimel}}{\tilde{\mathcal{J}}})^{(n)}}, \; \xi^{\mathcal{N}}_{\phi(\tilde{\gimel}^{(n)})} = \xi^{\mathcal{P}}_{(\frac{\tilde{\gimel}}{\tilde{\mathcal{J}}})^{(n)}} \text{ and } \zeta_{\phi(\tilde{\gimel}^{(n)})} = \zeta_{(\frac{\tilde{\gimel}}{\tilde{\mathcal{J}}})^{(n)}}.$$

Since $\frac{\tilde{\gimel}}{\tilde{\mathcal{J}}}$ is solvable, there exists a positive integer n such that $\xi^{\mathcal{P}}_{(\frac{\tilde{\gimel}}{\tilde{\mathcal{J}}})^{(n)}} = 1_0, \xi^{\mathcal{N}}_{(\frac{\tilde{\gimel}}{\tilde{\mathcal{J}}})^{(n)}} = (-1)_0$ and $\zeta_{(\frac{\tilde{\gimel}}{\tilde{\mathcal{J}}})^{(n)}} = 0_0$.

For $0 \neq \tilde{\eta} \in \frac{\tilde{\mathcal{L}}}{\tilde{\mathcal{J}}}$, we have

$$\sup_{m\in f^{-1}(\tilde{\eta})} \left\{\xi^{\mathcal{P}}_{\tilde{\gimel}^{(n)}}(m)\right\} = \xi^{\mathcal{P}}_{f(\tilde{\gimel}^{(n)})}(\tilde{\eta}) = \xi^{\mathcal{P}}_{(\frac{\tilde{\gimel}}{\tilde{\mathcal{J}}})^{(n)}}(\tilde{\eta}) = 0,$$
$$\inf_{m\in f^{-1}(\tilde{\eta})} \left\{\xi^{\mathcal{N}}_{\tilde{\gimel}^{(n)}}(m)\right\} = \xi^{\mathcal{N}}_{f(\tilde{\gimel}^{(n)})}(\tilde{\eta}) = \xi^{\mathcal{N}}_{(\frac{\tilde{\gimel}}{\tilde{\mathcal{J}}})^{(n)}}(\tilde{\eta}) = 0,$$

and

$$\inf_{m\in f^{-1}(\tilde{\eta})} \left\{\zeta_{\tilde{\gimel}^{(n)}}(m)\right\} = \zeta_{f(\tilde{\gimel}^{(n)})}(\tilde{\eta}) = \zeta_{(\frac{\tilde{\gimel}}{\tilde{\mathcal{J}}})^{(n)}}(\tilde{\eta}) = 0.$$

Note that $m \neq 0$ and $m \in \tilde{\mathcal{L}}$. Then $\xi^{\mathcal{P}}_{\tilde{\gimel}^{(n)}}(m) = 0, \xi^{\mathcal{N}}_{\tilde{\gimel}^{(n)}}(m) = 0$ and $\zeta_{\tilde{\gimel}^{(n)}}(m) = 0$.
For $\tilde{\eta} = 0$, we have

$$\sup_{m\in f^{-1}(0)} \left\{\xi^{\mathcal{P}}_{(\tilde{\gimel}^{(n)})}(m)\right\} = \xi^{\mathcal{P}}_{f(\tilde{\gimel}^{(n)})}(0) = 1,$$
$$\inf_{m\in f^{-1}(0)} \left\{\xi^{\mathcal{N}}_{(\tilde{\gimel}^{(n)})}(m)\right\} = \xi^{\mathcal{N}}_{f(\tilde{\gimel}^{(n)})}(0) = -1$$

and

$$\inf_{m\in f^{-1}(0)} \left\{\zeta_{(\tilde{\gimel}^{(n)})}(m)\right\} = \zeta_{f(\tilde{\gimel}^{(n)})}(0) = 0.$$

Since $f^{-1}(0) = \tilde{\mathcal{J}}$ and $\tilde{\daleth}(\tilde{\mathcal{J}}) = \tilde{\gimel}(\tilde{\mathcal{J}})$, we have

$$\xi^{\mathcal{P}}_{\tilde{\daleth}^{(n)}}(\tilde{\mathcal{J}}) = \xi^{\mathcal{P}}_{\tilde{\gimel}^{(n)}}(\tilde{\mathcal{J}}),\ \xi^{\mathcal{N}}_{\tilde{\daleth}^{(n)}}(\tilde{\mathcal{J}}) = \xi^{\mathcal{N}}_{\tilde{\gimel}^{(n)}}(\tilde{\mathcal{J}}) \text{ and } \zeta_{\tilde{\daleth}^{(n)}}(\tilde{\mathcal{J}}) = \zeta_{\tilde{\gimel}^{(n)}}(\tilde{\mathcal{J}}).$$

For any $\tilde{\phi} \in \tilde{\mathcal{J}}$, $\tilde{\daleth}$ is solvable, then there exists a positive integer n such that $\xi^{\mathcal{P}}_{\tilde{s}^{(n)}} = 1_0$ and $\xi^{\mathcal{N}}_{\tilde{s}^{(n)}} = (-1)_0$,

$$\xi^{\mathcal{P}}_{\tilde{\daleth}^{(n)}} = 1_0,\ \xi^{\mathcal{N}}_{\tilde{\gimel}^{(n)}} = (-1)_0 \text{ and } \zeta_{\tilde{\gimel}^{(n)}} = 0_0.$$

Hence, $\xi^{\mathcal{P}}_{\tilde{\gimel}^{(n)}} = 1_0$, $\xi^{\mathcal{N}}_{\tilde{\gimel}^{(n)}} = (-1)_0$ and $\zeta_{\tilde{\gimel}^{(n)}} = 0_0$, which implies that $\tilde{\gimel} = (\xi^{\mathcal{P}}_{\tilde{\gimel}}, \xi^{\mathcal{N}}_{\tilde{\gimel}}, \tilde{\zeta})$ is solvable.

6. Conclusion

In this paper, we extended the framework of tripolar complex fuzzy sets ($\mathcal{TCFS}$ by introducing the concept of tripolar complex fuzzy Lie brackets and investigating their fundamental algebraic properties. We demonstrated that the scalar multiplication and addition of tripolar complex fuzzy Lie subalgebras yield a tripolar complex fuzzy Lie subalgebra, reinforcing their algebraic closure properties. Also, we established that the homomorphic image of a nilpotent (or solvable) tripolar complex fuzzy Lie ideal remains nilpotent (or solvable), highlighting the structural preservation under homomorphisms. Finally, we proved that every nilpotent tripolar complex fuzzy Lie ideal is solvable, extending a fundamental result from classical Lie theory to the tripolar complex fuzzy setting. These results contribute to the growing body of research on fuzzy algebraic structures, particularly in the context of multipolar and complex-valued fuzzy systems. In future work, we will extend the framework to n-polar complex Pythagorean fuzzy Lie algebras to explore higher-dimensional algebraic properties.

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