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Innovative Solitary Wave Solutions for the (3+1)-Dimensional Boussinesq Kadomtsev-Petviashvili-Type Equation Derived via the Improved Modified Extended Tanh-Function Method

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Abstract. The goal of this research is to create and analyze a novel (3+1) dimensional model that incorporates two different equations: a three-dimensional Kadomtsev-Petviashvili equation and a three-dimensional Boussinesq-KP-type equation. One of the unexpected outcomes of the idea of mixing integrable equations is a resonance of solitons. This paper presents a wide range of possible analytical solutions for the pKP–BKP equation in (3+1)-dimensions, including dark, bright, singular solitons, and other exact solutions like singular periodic, Jacobi elliptic function, rational, and exponential type. The (3+1)-dimensional B-KP-type model is subjected to the improved modified extended tanh-function approach in order to obtain novel traveling wave solutions. The employed equation plays a crucial role in describing and interpreting a broad range of nonlinear phenomena seen in fluid mechanics and other nonlinear engineering and physics issues due to the strong correlation and wide range of applications of the Boussinesq-type and KP equations. The approach can help to find other kinds of solutions to the chosen equation that have not been found and published in the literature before. These solutions can aid in the comprehension of wave propagation in water wave dynamics. To further facilitate learning, they are replicated through the use of contour graphics, 2D, and 3D symbolic calculations. Moreover, linear stability analysis is discussed for the obtained solutions.

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1. Introduction

A variety of nonlinear evolution equations, such as Boussinesq, Kadomtsev–Petviashvili, and nonlinear Schrödinger-type models, are examined in some latest research, with an emphasis on building precise soliton and wave solutions by analytical techniques [1–5]. Modeling nonlinear wave propagation, optical solitons, and interaction dynamics in a variety of physical media, including fluids, birefringent fibers, magneto-optic waveguides, and twin-core couplers, is made easier by the results [6–10]. Many recent researchers have committed their efforts to examining the integrability of various nonlinear evolution equations using a variety of methods and techniques because they recognize the importance and effectiveness of studying integrability prior to analyzing distinct nonlinear evolution equations and understanding their unique behavior. For example, in [11], the Painlevé property for partial differential equations (PDEs) was explained in detail, and its significance in determining the integrability was illustrated. The Painlevé property for the PDEs was explained in detail, and its significance in determining the integrability was illustrated. The Painlevé test was utilized by the authors to verify the integrability of numerous universal evolution equations, including Burgers, Sine-Gordon, Boussinesq (B-type), Korteweg-de Vries (KdV)-type, and the Kadomtsev-Petviashvili (KP) equations. In many scientific domains, these equations may be used to mimic a wide range of nonlinear processes. Several studies in the literature have been effective in providing an explanation for a wide range of nonlinear and enigmatic phenomena that appear in various physical systems due to numerous different evolution equations that have been studied for integrability. For instance, in [12–14], the properties of either modulated or unmodulated solitary waves (SWs) and cnoidal waves (CWs) that exist in fluid mechanics, seas, and oceans, as well as in narrow channels, have been effectively investigated using the KdV-type equation and other equations (such as the nonlinear Schrödinger equation (NLSE) and Boussinesq-type equations). These formulas have also been used to investigate nonlinear optical communications, solitons, and shocks in plasma physics. It is established that in the system under study, a balance between nonlinearity and dispersion can produce solitons. The SWs become shock waves when the dispersion and nonlinearity of the system are unbalanced. Assume that the system has a dissipation force, dispersion, and nonlinearity, and that the dissipation force predominates over the dispersion. The solitons will then change into shock waves in such scenarios.

Recent studies have applied a host of analysis techniques to obtain exact solutions of higher-dimensional and nonlinear fractional evolution equations that are significant in wave dynamics and nonlinear optics. For instance, Almatrafi [15, 16] applied the improved modified extended tanh-function method and other analytical techniques to obtain solitary wave solutions of fractional models with emphasis on lower-dimensional equations without taking into account the additional complexity arising from higher spatial dimensions. Murad et al. [17] examined the Sasa–Satsuma equation of higher-order time-fractional in optical fibers, and Younas et al. [18] considered interaction phenomena in the (2+1)-dimensional KdV–Sawada–Kotera–Ramani equation. Younas et al. [19, 20] also discussed soliton dynamics in optical systems and magneto-electro-elastic media. However, these studies were concentrated on specific physical models with a predetermined dimensionality or special nonlinear structures. On the other hand, the present work is concerned with the dynamics of a novel (3+1)-dimensional nonlinear wave equation with conformable fractional derivatives, in which a more generalized model frame is presented that can portray complex nonlinear features in higher-dimensional fluid and optical systems. This extension fills the existing gap in the literature by providing a more comprehensive class of exact solutions, including bright, dark, singular, and periodic solitons, and thus developing additional insight into the spatiotemporal dynamics of nonlinear waves in realistic multidimensional settings.

The Boussinesq equation (BE), developed by Boussinesq in 1871, is one of the most important evolution equations used in plasma physics and fluid mechanics. The propagation of small-amplitude shallow water waves traveling at a constant speed in a continuous depth water channel is described by this equation and certain extensions from it [21]. Numerous studies have tackled this equation, scrutinizing and resolving it using diverse methodologies. For example, Wazwaz [22] used a family of tanh methods to evaluate it and get only a periodic solution as well as a single soliton solution. To obtain multiple soliton solutions for this model, the author also used Hirota's direct approach in conjunction with its simplified form. In [23], the author obtained solutions in the form of single solitons and periodic waves by analyzing the behavior of this equation using the modified Adomian decomposition technique. Furthermore, a variety of nonlinear phenomena seen in a wide range of engineering and physical systems have been accurately modeled by the KdV equations.

The KP equation (KPE) belongs to the KdV family of evolutionary equations, particularly when two-dimensional propagation and perturbation are taken into account as in [24]. In this study, the two-dimensional KdV equation has an unlimited number of conserved quantities and is fully integrable, which relates to the simulation of many nonlinear structures in a wide range of real-world circumstances, including a harmonic lattice, a two-layer liquid with gradually varying depth, and plasma physics.

Inspired by several applications of both BE and KPE, the goal of our present work is to combine the two equations to produce a new model that may satisfactorily describe a variety of occurrences for which neither equation could account. As in [25], the work introduced a brand-new, three-dimensional Boussinesq-KP-type (B-KP-type) equation with four linear terms that were balanced with a higher order linear term of fourth order that represented the dispersion effect. This model was analyzed using the tanh-method family; however, periodic wave and single soliton solutions were only obtained.

In this study, the improved modified extended (IME) tanh-function method is suggested as one simple method to analyze the novel constructed evolution equations in [25], which is named as (3+1)-dimensional B-KP-type equation. Several traveling wave and soliton solutions are derived and discussed. A wide range of complex and nonlinear phenomena that develop and spread in different nonlinear engineering and physical systems, particularly the nonlinear phenomena found in fluid dynamics, were clearly explained through some 2D, 3D, and contour illustrations that show propagation properties and behaviors of some obtained traveling wave solutions. In addition, to discuss the stability of the obtained solutions, we present linear stability analysis in some cases, like neutrally stable and instability.

The (3+1)-dimensional B-KP-type equation is given as follows:

$$\mathbb{C}\phi_{tt} + \phi_{xxxx} + \alpha\phi_{xx} + \beta\phi_{xy} + \gamma\phi_{xz} + \vartheta\phi_{xt} + \varrho(\phi^2)_{xx} + \mu\phi_{yy} = 0, \tag{1}$$

where $\phi \equiv \phi(x, y, z, t)$ represents a real-valued function that must be sufficiently frequently differentiable, and subscripts denote partial derivatives. It refers to the height of a fluid's free surface; x, y, z refer to the variables of the space dimensions, while t denotes the time variation. $\mathbb{C}, \alpha, \beta, \gamma, \vartheta, \varrho$ and μ are arbitrary real values parameters that will be calculated later during this study. ϕ appears with its partial derivatives to describe the fluid wave profiles with significant amplitudes that are maintained for a brief duration of time, which signifies localization in both the space and the time domains. The (3+1)-dimensional pKP-BKP equation, which depicts the propagation of nonlinear waves in three spatial dimensions and one temporal dimension, is particularly significant for studying complex wave dynamics in systems like fluids, plasmas, and optical media. It extends the standard KP equation by including wave interactions that occur both longitudinally and transversely, and it shows how perturbations in one direction can propagate over many spatial dimensions. Understanding this equation is necessary for understanding phenomena such as shallow water waves, ion-acoustic waves in plasmas, and even optical pulses in nonlinear media, where the stability and growth of the wave crucially depend on the balance between nonlinearity and dispersion.

The main objective of this approach is to maximize the use of Eq. (1) by including its parameters to generate more types of analytical solutions when utilized in an appropriate, straightforward, easy manner, and with the assistance of symbolic computations. Numerous studies have employed the IME tanh-function approach to examine a broad spectrum of solitons and other precise wave solutions [13]. This study uses the IME tanh-function approach for Eq. (1) to compute various solitons and other exact wave solutions. This particular combination is novel research that has never been done before, and it uses the IME tanh-function approach to provide accurate analytical solutions to nonlinear partial differential equations (NPDEs) that are insightful, efficient, and versatile. Numerous solutions are obtained using this approach, such as the Jacobi epsilon function, exponential, singular periodic, rational, dark soliton, and bright soliton solutions. Furthermore, the recovered solutions confirm the effectiveness and strength of the used approach.

The structure of this article is as follows. An outline of the suggested model and its theoretical underpinnings is given in Section 1. The main components of the IME tanh-function algorithm are presented in Section 2. In Section 3, the symbolic computations are completed and the results are summarized using the Wolfram Mathematica program. Section 4 presents multiple dynamic wave patterns of different soliton solutions graphically using 3-D, contour, and 2-D simulations and their discussion. Section 5 presents a novel comparison with the most common methods in the literature. An implementation of the linear stability analysis on the governing model is discussed in Section 6. The study conclusions are reported in Section 7.

2. Improved modified extended tanh-function method

The general procedures of the IME tanh-function method [26, 27] will be shown in this section, in addition to its motivation and advantages in solving NPDE.

2.1. Quick view of the method

As we start looking at the NPDE that's shown below [28, 29]:

$$\mathcal{J}\left(\phi, \phi_x, \phi_y, \phi_z, \phi_t, \phi_{xx}, \phi_{xy}, \phi_{xz}, \ldots\right) = 0, \tag{2}$$

where $\mathcal{J}$ is a polynomial that is composed of $\phi(x, y, z, t)$ and some of ϕ partial derivatives with respect to both time (t) and the space dimensions of the used dynamic system (x, y, z).
Procedure-(I): Considering the following wave transformation

$$\phi(x, y, z, t) = \mathcal{R}(\eta); \qquad \eta = x + ky + \aleph z - \Omega t, \tag{3}$$

in which k, $\aleph$ and Ω are real valued constants. $\mathcal{R}$ acts as the function of the obtained solution.
Considering Eq. (3) and Eq. (2) together, with performing rearranging, the following nonlinear ordinary differential equation (NLODE) form is derived:

$$\mathcal{Z}(\mathcal{R},\ \mathcal{R}',\ \mathcal{R}'',\ \mathcal{R}''', \ldots) = 0. \tag{4}$$

Procedure-(II): In order to achieve the solution of Eq. (4) according to the applied method, the solution is proposed in the following truncated series:

$$\mathcal{R}(\eta) = \sum_{i=0}^{\mathbb{N}} \mathbf{d}_i \mathcal{W}^i(\eta) + \sum_{i=1}^{\mathbb{N}} \mathbf{f}_i \mathcal{W}^{-i}(\eta), \tag{5}$$

where $\mathbf{d}_0$, $\mathbf{d}_1$ ..., $\mathbf{d}_{\mathbb{N}}$ and $\mathbf{f}_1$, ...,$\mathbf{f}_{\mathbb{N}}$ are real constants to be calculated, under condition that $\mathbf{d}_{\mathbb{N}}$ and $\mathbf{f}_{\mathbb{N}}$ should not be zero, simultaneously.
Procedure-(III): By applying the homogeneous balance principle between the nonlinearity and the dispersion of Eq. (4) to determine the value of the balancing constant $\mathbb{N}$. In addition to considering the following constraint for $\mathcal{W}(\eta)$:

$$\mathcal{W}'(\eta) = \epsilon\sqrt{\tau_0 + \tau_1\mathcal{W}(\eta) + \tau_2\mathcal{W}^2(\eta) + \tau_3\mathcal{W}^3(\eta) + \tau_4\mathcal{W}^4(\eta)}, \tag{6}$$

where $\epsilon = \pm 1$ and τ_m $(0 \leq m \leq 4)$ are real-valued constants. Several types of fundamental solutions are obtained from Eq. (6) using the many potential values of $\tau_0, \tau_1, \tau_2, \tau_3$ and τ_4 which are shown in Appendix A.
Procedure-(IV): An equation in powers of $\mathcal{W}(\eta)$ is obtained by inserting Eqs. (5) and (6) into Eq. (4). The system of algebraic non-linear equations that results from applying algebraic polynomial operations on the coefficients of $\mathcal{W}(\eta)$ and equating them to zero may be solved with a variety of software applications, including the Wolfram Mathematica. This will finally lead to the production of several exact solutions for Eq. (2).
The flowchart in figure (1) shows a brief description of the used scheme.

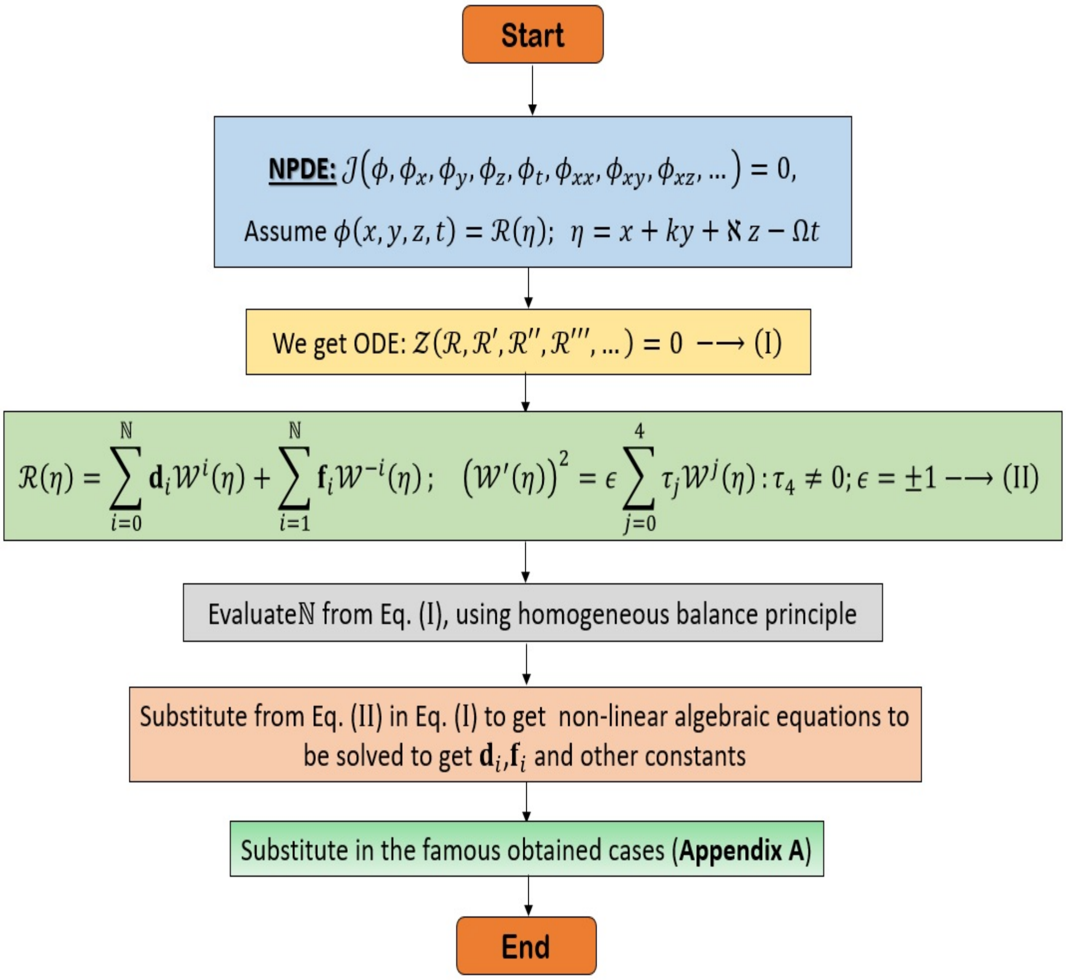


Figure 1: Flowchart of the IME tanh-function method.

2.2. Motivation and advantages of the method

The IME tanh-function method is highly advantageous for obtaining analytical solutions and gaining a deeper understanding of soliton dynamics when compared to the other most current approaches. However, it might be challenging to regulate because of its complexity and sensitivity to the initial conditions. Depending on the specifics of the model being used, such as the type of NPDEs, determining the boundary constraints, and the desired ratio of computational practicality to analytical proficiency, each technique is appropriate. Alternative techniques are more complex, but they also allow for greater flexibility and improvement.

3. Extraction of solitons and some other solutions

By applying Eq. (3) for Eq. (1), then Eq. (1) appears as a non-linear ordinary differential equation (NODE), having the following form:

$$2\varrho\left(\mathcal{R}'\right)^2+\left(\alpha+k(\beta+k\mu)+2\varrho\mathcal{R}-\vartheta\Omega+\mathbb{C}\Omega^2+\gamma\aleph\right)\mathcal{R}''+\mathcal{R}^{(4)}=0. \tag{7}$$

Thus, the exact solutions for Eq. (7) can be generated in the following manner by using the principle of balance that was mentioned in Section 2 to Eq. (7), the balance is performed between the terms include $\mathcal{R}^{(4)}$ and $\mathcal{R}\mathcal{R}''$, determining the balance constant $\mathbb{N}=2$. Hence, Eq. (5) becomes

$$\mathcal{R}(\eta)=\mathbf{d}_0+\mathbf{d}_1\mathcal{W}(\eta)+\mathbf{d}_2\mathcal{W}^2(\eta)+\frac{\mathbf{f}_1}{\mathcal{W}(\eta)}+\frac{\mathbf{f}_2}{\mathcal{W}^2(\eta)}. \tag{8}$$

Using Eq. (8) and the constraint in Eq. (6), Eq. (7) yields a polynomial in $\mathcal{W}(\eta)$. An algebraic non-linear equations system is created when all terms with the same power are combined and set to equal zero. The Wolfram Mathematica software tool is used to solve these equations and generate the potential solutions. It is specified that $\mathbf{d}_2$ and $\mathbf{f}_2$ cannot both be zero simultaneously.

Family (1): If $\tau_0=\tau_1=\tau_3=0$, the below set of solutions is resulted:

$$\mathbf{d}_1=\mathbf{f}_1=\mathbf{f}_2=0,\ \mathbf{d}_0=-\frac{\alpha+\gamma\aleph-\Omega(\vartheta-\mathbb{C}\Omega)+k(\beta+k\mu)+4\tau_2}{2\varrho},\ \mathbf{d}_2=-\frac{6\tau_4}{\varrho}.$$

The following exact solutions are obtained for Eq. (1), as per the above set of solutions:

(1.1) If $\tau_2>0$, $\tau_4<0$ and $\varrho\neq 0$, the following bright soliton solution is obatined:

$$\phi_{1.1}=-\frac{\alpha+\gamma\aleph-\Omega(\vartheta-\mathbb{C}\Omega)+k(\beta+k\mu)+4\tau_2}{2\varrho}+\frac{6\tau_2}{\varrho}\operatorname{sech}^2\left[(x+ky+\aleph z-\Omega t)\sqrt{\tau_2}\right]. \tag{9}$$

(1.2) If $\tau_2<0$, $\tau_4>0$ and $\varrho\neq 0$, the following singular periodic solution is obtained:

$$\phi_{1.2}=-\frac{\alpha+\gamma\aleph-\Omega(\vartheta-\mathbb{C}\Omega)+k(\beta+k\mu)+4\tau_2}{2\varrho}+\frac{6\tau_2}{\varrho}\sec^2\left[(x+ky+\aleph z-\Omega t)\sqrt{-\tau_2}\right]. \tag{10}$$

(1.3) If $\tau_2=0$ and $\tau_4>0$, the following rational solution; $(\varrho)(x+ky+\aleph z-\Omega t)\neq 0$:

$$\phi_{1.3}=-\frac{\alpha+\gamma\aleph-\Omega(\vartheta-\mathbb{C}\Omega)+k(\beta+k\mu)}{2\varrho}-\frac{6}{\varrho(x+ky+\aleph z-\Omega t)^2}. \tag{11}$$

Family (2): If $\tau_1=\tau_3=0$, the sets of solutions listed below are:

(2.1) $\mathbf{d}_1=\mathbf{f}_1=\mathbf{f}_2=0,\ \mathbf{d}_0=-\frac{\alpha+\gamma\aleph-\Omega(\vartheta-\mathbb{C}\Omega)+k(\beta+k\mu)+4\tau_2}{2\varrho},\ \mathbf{d}_2=-\frac{6\tau_4}{\varrho}.$

(2.2) $\mathbf{d}_1 = \mathbf{d}_2 = \mathbf{f}_1 = 0,\ \mathbf{d}_0 = -\frac{\alpha+\gamma\aleph-\Omega(\vartheta-\mathbb{C}\Omega)+k(\beta+k\mu)+4\tau_2}{2\varrho},\ \mathbf{f}_2 = -\frac{6\tau_0}{\varrho}.$

(2.3) $\mathbf{d}_1 = \mathbf{f}_1 = 0,\ \mathbf{d}_0 = -\frac{\alpha+\gamma\aleph-\Omega(\vartheta-\mathbb{C}\Omega)+k(\beta+k\mu)+4\tau_2}{2\varrho},\ \mathbf{d}_2 = -\frac{6\tau_4}{\varrho},\ \mathbf{f}_2 = -\frac{6\tau_0}{\varrho}.$

By considering the solutions' set (2.1), the corresponding exact solutions of Eq. (1) are produced as:

(2.1,1) If $\tau_0 = \frac{\tau_2^2}{4\tau_4}$, $\tau_2 < 0$, $\tau_4 > 0$ and $\varrho \neq 0$, the following dark soliton solution is obtained:

$$\phi_{2.1,1} = -\frac{\alpha+\gamma\aleph-\Omega(\vartheta-\mathbb{C}\Omega)+k(\beta+k\mu)+4\tau_2}{2\varrho} + \frac{3\tau_2}{\varrho}\ \tanh^2\left[(x+ky+\aleph z-\Omega t)\sqrt{-\frac{\tau_2}{2}}\right]. \tag{12}$$

(2.1,2) If $\tau_0 = \frac{\tau_2^2}{4\tau_4}$, $\tau_2 > 0$, $\tau_4 > 0$ and $\varrho \neq 0$, the following singular periodic solution is formed:

$$\phi_{2.1,2} = -\frac{\alpha+\gamma\aleph-\Omega(\vartheta-\mathbb{C}\Omega)+k(\beta+k\mu)+4\tau_2}{2\varrho} - \frac{3\tau_2}{\varrho}\tan^2\left[(x+ky+\aleph z-\Omega t)\sqrt{\frac{\tau_2}{2}}\right]. \tag{13}$$

(2.1,3) If $\tau_0 = \frac{m^2(1-m^2)\tau_2^2}{(2m^2-1)^2\tau_4}$, $\tau_2 > 0$, $\tau_4 < 0$, $\frac{1}{\sqrt{2}} < m \leq 1$ and $\varrho \neq 0$, then the below Jacobi elliptic function solution (JEFS) is obtained:

$$\phi_{2.1,3} = -\frac{\alpha+\gamma\aleph-\Omega(\vartheta-\mathbb{C}\Omega)+k(\beta+k\mu)+4\tau_2}{2\varrho} + \frac{6m^2\tau_2\ \mathrm{cn}^2\left[(x+ky+\aleph z-\Omega t)\sqrt{\frac{\tau_2}{2m^2-1}}\right]}{\varrho(2m^2-1)}. \tag{14}$$

Set $m = 1$ in Eq. (14) as a special case, a bright soliton solution is produced as follows:

$$\phi_{2.1,4} = -\frac{\alpha+\gamma\aleph-\Omega(\vartheta-\mathbb{C}\Omega)+k(\beta+k\mu)+4\tau_2}{2\varrho} + \frac{6\tau_2}{\varrho}\mathrm{sech}^2\left[(x+ky+\aleph z-\Omega t)\sqrt{\tau_2}\right]. \tag{15}$$

(2.1,4) If $\tau_0 = \frac{(1-m^2)\tau_2^2}{(2-m^2)^2\tau_4}$, $\tau_2 > 0$, $\tau_4 < 0$, $0 < m \leq 1$ and $\varrho \neq 0$, the below JEFS is raised:

$$\phi_{2.1,5} = -\frac{\alpha+\gamma\aleph-\Omega(\vartheta-\mathbb{C}\Omega)+k(\beta+k\mu)+4\tau_2}{2\varrho} + \frac{6m^2\ \mathrm{dn}^2\left[(x+ky+\aleph z-\Omega t)\sqrt{\frac{\tau_2}{2-m^2}}\right]}{\varrho(2-m^2)}. \tag{16}$$

Set $m = 1$ in Eq. (16) as a special case, its solution resulted as a bright soliton in the following form:

$$\phi_{2.1,6} = -\frac{\alpha+\gamma\aleph-\Omega(\vartheta-\mathbb{C}\Omega)+k(\beta+k\mu)+4\tau_2}{2\varrho} + \frac{6}{\varrho}\mathrm{sech}^2\left[(x+ky+\aleph z-\Omega t)\sqrt{\tau_2}\right]. \tag{17}$$

(2.1,5) If $\tau_0 = \frac{m^2\tau_2^2}{(m^2+1)^2\tau_4}$, $\tau_2 < 0$, $\tau_4 > 0$, $0 < m \leq 1$ and $\varrho \neq 0$, then a JEFS is resulted as below:

$$\phi_{2.1,7} = -\frac{\alpha + \gamma\aleph - \Omega(\vartheta - \mathbb{C}\Omega) + k(\beta + k\mu) + 4\tau_2}{2\varrho} + \frac{6m^2 \operatorname{sn}^2\left[(x + ky + \aleph z - \Omega t)\sqrt{-\frac{\tau_2}{m^2+1}}\right]}{\varrho(m^2+1)}. \tag{18}$$

Special case, when setting $m = 1$ in Eq. (18), the following dark soliton solution can be obtained:

$$\phi_{2.1,8} = -\frac{\alpha + \gamma\aleph - \Omega(\vartheta - \mathbb{C}\Omega) + k(\beta + k\mu) + 4\tau_2}{2\varrho} + \frac{3\tau_2}{\varrho}\tanh^2\left[(x + ky + \aleph z - \Omega t)\sqrt{-\frac{\tau_2}{2}}\right]. \tag{19}$$

The mentioned set of solutions (2.2) indicates that Eq. (1) has exact solutions, which can be phrased as:

(2.2,1) If $\tau_0 = \frac{\tau_2^2}{4\tau_4}$, $\tau_2 < 0$, $\tau_4 > 0$ and $\varrho \neq 0$, the below singular soliton form is appeared in the obtained solution:

$$\phi_{2.2,1} = -\frac{\alpha + \beta + k\mu + \gamma\aleph - \Omega(\vartheta - \mathbb{C}\Omega) + 4\tau_2}{2\varrho} + \frac{3\tau_2}{\varrho}\coth^2\left[(x + ky + \aleph z - \Omega t)\sqrt{-\frac{\tau_2}{2}}\right]. \tag{20}$$

(2.2,2) If $\tau_0 = \frac{\tau_2^2}{4\tau_4}$, $\tau_2 > 0$, $\tau_4 > 0$ and $\varrho \neq 0$, following singular periodic solution is obtained:

$$\phi_{2.2,2} = -\frac{\alpha + \beta + k\mu + \gamma\aleph - \Omega(\vartheta - \mathbb{C}\Omega) + 4\tau_2}{2\varrho} - \frac{3\tau_2}{\varrho}\cot^2\left[(x + ky + \aleph z - \Omega t)\sqrt{\frac{\tau_2}{2}}\right]. \tag{21}$$

(2.2,3) If $\tau_0 = \frac{m^2(1-m^2)\tau_2^2}{(2m^2-1)^2\tau_4}$, $\tau_2 > 0$, $\tau_4 < 0$, $\frac{1}{\sqrt{2}} < m < 1$ and $\varrho \neq 0$, the below JEFS is obtained:

$$\phi_{2.2,3} = -\frac{\alpha + \beta + k\mu + \gamma\aleph - \Omega(\vartheta - \mathbb{C}\Omega)}{2\varrho} - \frac{2\tau_2}{\varrho}\left(1 + \frac{3\left(m^2-1\right)}{(2m^2-1)\operatorname{cn}^2\left[(x + ky + \aleph z - \Omega t)\sqrt{\frac{\tau_2}{2m^2-1}}\right]}\right). \tag{22}$$

(2.2,4) If $\tau_0 = \frac{(1-m^2)\tau_2^2}{(2-m^2)^2\tau_4}$, $\tau_2 > 0$, $\tau_4 < 0$, $0 < m < 1$ and $\varrho \neq 0$, the below JEFS is raised:

$$\phi_{2.2,4} = -\frac{\alpha + \beta + k\mu + \gamma\aleph - \Omega(\vartheta - \mathbb{C}\Omega) + 4\tau_2}{2\varrho} + \frac{6\left(m^2-1\right)\tau_2^2}{m^2\left(m^2-2\right)\varrho\operatorname{dn}^2\left[(x + ky + \aleph z - \Omega t)\sqrt{\frac{\tau_2}{2-m^2}}\right]}. \tag{23}$$

(2.2,5) If $\tau_0 = \frac{m^2\tau_2^2}{(m^2+1)^2\tau_4}$, $\tau_2 < 0$, $\tau_4 > 0$, $0 \leq m \leq 1$ and $\varrho \neq 0$, the obtained solution is resulted as the below JEFS:

$$\phi_{2.2,5} = -\frac{\alpha + \beta + k\mu + \gamma\aleph - \Omega(\vartheta - \mathbb{C}\Omega)}{2\varrho} - \frac{2\tau_2}{\varrho}\left(1 - \frac{3}{(1+m^2)\,\mathrm{sn}^2\left[(x + ky + \aleph z - \Omega t)\sqrt{-\frac{\tau_2}{1+m^2}}\right]}\right). \tag{24}$$

Set $m = 0$ in Eq. (24) as a special case, then the solution resulted as the below singular periodic solution:

$$\phi_{2.2,6} = -\frac{\alpha + \beta + k\mu + \gamma\aleph - \Omega(\vartheta - \mathbb{C}\Omega) + 4\tau_2}{2\varrho} + \frac{6\tau_2}{\varrho}\csc^2\left[(x + ky + \aleph z - \Omega t)\sqrt{-\tau_2}\right]. \tag{25}$$

Set $m = 1$ in Eq. (24) as a special case, then the solution resulted as the following singular soliton solution:

$$\phi_{2.2,7} = -\frac{\alpha + \beta + k\mu + \gamma\aleph - \Omega(\vartheta - \mathbb{C}\Omega) + 4\tau_2}{2\varrho} + \frac{3\tau_2}{\varrho}\coth^2\left[(x + ky + \aleph z - \Omega t)\sqrt{-\frac{\tau_2}{2}}\right]. \tag{26}$$

Considering the set of solutions (2.3), then Eq. (1) has exact solutions, which can be phrased as:

(2.3,1) If $\tau_0 = \frac{\tau_2^2}{4\tau_4}$, $\tau_2 < 0$, $\tau_4 > 0$, and $\varrho \neq 0$, the solution is obtained in a singular soliton form given as follows:

$$\phi_{2.3,1} = -\frac{\alpha + \gamma\aleph - \Omega(\vartheta - \mathbb{C}\Omega) + k(\beta + k\mu) + 16\tau_2}{2\varrho} + \frac{12\tau_2}{\varrho}\coth^2\left[(x + ky + \aleph z - \Omega t)\sqrt{-2\tau_2}\right]. \tag{27}$$

(2.3,2) If $\tau_0 = \frac{\tau_2^2}{4\tau_4}$, $\tau_2 > 0$, $\tau_4 > 0$ and $\varrho \neq 0$, the following singular periodic solution is obtained:

$$\phi_{2.3,2} = -\frac{\alpha + \gamma\aleph - \Omega(\vartheta - \mathbb{C}\Omega) + k(\beta + k\mu) - 8\tau_2}{2\varrho} - \frac{12\tau_2}{\varrho}\csc^2\left[(x + ky + \aleph z - \Omega t)\sqrt{2\tau_2}\right]. \tag{28}$$

(2.3,3) If $\tau_0 = \frac{m^2(1-m^2)\tau_2^2}{(2m^2-1)^2\tau_4}$, $\tau_2 > 0$, $\tau_4 < 0$, $\frac{1}{\sqrt{2}} < m \leq 1$ and $\varrho \neq 0$, a JEFS is obtained as follows:

$$\phi_{2.3,3} = -\frac{\alpha + \gamma\aleph - \Omega(\vartheta - \mathbb{C}\Omega) + k(\beta + k\mu) + 4\tau_2}{2\varrho} - \frac{6\tau_2}{\varrho}\left(\frac{(m^2 - 1)}{(2m^2 - 1)\,\mathrm{cn}^2\left[(x + ky + \aleph z - \Omega t)\sqrt{\frac{\tau_2}{2m^2-1}}\right]} - \frac{m^2\mathrm{cn}^2\left[(x + ky + \aleph z - \Omega t)\sqrt{\frac{\tau_2}{2m^2-1}}\right]}{2m^2 - 1}\right). \tag{29}$$

Set $m = 1$ in Eq. (29) as a special case, its solution is obtained as the below bright soliton solution:

$$\phi_{2.3,4} = -\frac{\alpha + \gamma\aleph - \Omega(\vartheta - \mathbb{C}\Omega) + k(\beta + k\mu) + 4\tau_2}{2\varrho} + \frac{6\tau_2}{\varrho}\mathrm{sech}^2\left[(x + ky + \aleph z - \Omega t)\sqrt{\tau_2}\right]. \tag{30}$$

(2.3,4) If $\tau_0 = \frac{(1-m^2)\tau_2^2}{(2-m^2)^2\tau_4}$, $\tau_2 > 0$, $\tau_4 < 0$, $0 < m \leq 1$ and $\varrho \neq 0$, a JEFS is obtained as below:

$$\phi_{2.3,5} = -\frac{\alpha + \gamma\aleph - \Omega(\vartheta - \mathbb{C}\Omega) + k(\beta + k\mu) + 4\tau_2}{2\varrho} + \frac{6}{\varrho}\left(\frac{m^2 \mathrm{dn}^2\left[(x + ky + \aleph z - \Omega t)\sqrt{\frac{\tau_2}{2-m^2}}\right]}{2 - m^2} - \frac{\left(m^2 - 1\right)\tau_2^2}{m^2\left(2 - m^2\right)\mathrm{dn}^2\left[(x + ky + \aleph z - \Omega t)\sqrt{\frac{\tau_2}{2-m^2}}\right]}\right). \tag{31}$$

Set $m = 1$ in Eq. (31) as a special case, the solution is obtained as a bright soliton type as below:

$$\phi_{2.3,6} = -\frac{\alpha + \gamma\aleph - \Omega(\vartheta - \mathbb{C}\Omega) + k(\beta + k\mu) + 4\tau_2}{2\varrho} + \frac{6}{\varrho}\mathrm{sech}^2\left[(x + ky + \aleph z - \Omega t)\sqrt{\tau_2}\right]. \tag{32}$$

(2.3,5) If $\tau_0 = \frac{m^2\tau_2^2}{(m^2+1)^2\tau_4}$, $\tau_2 < 0$, $\tau_4 > 0$, $\varrho \neq 0$ and $0 \leq m \leq 1$, the following JEFS is resulted:

$$\phi_{2.3,7} = -\frac{\alpha + \gamma\aleph - \Omega(\vartheta - \mathbb{C}\Omega) + k(\beta + k\mu) + 4\tau_2}{2\varrho} + \frac{6\tau_2}{\varrho}\left(\frac{1}{(m^2 + 1)\,\mathrm{sn}^2\left[(x + ky + \aleph z - \Omega t)\sqrt{-\frac{\tau_2}{m^2+1}}\right]} + \frac{m^2 \mathrm{sn}^2\left(x + ky + \aleph z - \Omega t)\sqrt{-\frac{\tau_2}{m^2+1}}\right]}{m^2 + 1}\right). \tag{33}$$

Special case, when setting $m = 0$ in Eq. (33), the following singular periodic solution can be obtained:

$$\phi_{2.3,8} = -\frac{\alpha + \gamma\aleph - \Omega(\vartheta - \mathbb{C}\Omega) + k(\beta + k\mu) + 4\tau_2}{2\varrho} + \frac{6\tau_2}{\varrho}\csc^2\left[(x + ky + \aleph z - \Omega t)\sqrt{-\tau_2}\right]. \tag{34}$$

Special case, when setting $m = 1$ in Eq. (33), the following singular soliton solution can be obtained:

$$\phi_{2.3,9} = -\frac{\alpha + \gamma\aleph - \Omega(\vartheta - \mathbb{C}\Omega) + k(\beta + k\mu) + 16\tau_2}{2\varrho} + \frac{12\tau_2}{\varrho}\coth^2\left[(x + ky + \aleph z - \Omega t)\sqrt{-2\tau_2}\right]. \tag{35}$$

Family (3): If $\tau_2 = \tau_4 = 0$, the resulted set of solutions is mentioned below:

$$\mathbf{d}_1 = \mathbf{d}_2 = 0,\ \mathbf{d}_0 = -\frac{\alpha + \gamma\aleph - \Omega(\vartheta - \mathbb{C}\Omega) + k(\beta + k\mu) - 3\sqrt[3]{\tau_0\tau_3^2}}{2\varrho},\ \mathbf{f}_1 = \frac{6\sqrt[3]{\tau_0^2\tau_3}}{\varrho},\ \mathbf{f}_2 = -\frac{6\tau_0}{\varrho},\ \tau_1 = -2\sqrt[3]{\tau_0^2\tau_3}.$$

The exact solution to Eq. (1) that arises from this set of solutions is as follows:
If $\tau_0 \neq 0$, $\tau_1 \neq 0$, $\tau_3 > 0$ and $\varrho \neq 0$, a Weierstrass elliptic doubly periodic function is produced in the form of solution:

$$\phi_{3.1} = -\frac{\alpha - \Omega(\vartheta - \mathbb{C}\Omega) + k(\beta + k\mu) - 3\sqrt[3]{\tau_0\tau_3^2}}{2\varrho} + \frac{6\left(\tau_0 - \sqrt[3]{\tau_0^2\tau_3}\ \wp\left[(x + ky + \aleph z - \Omega t)\sqrt{\frac{\tau_3}{4}};\left(\frac{8\sqrt[3]{\tau_0^2\tau_3}}{\tau_3}, -\frac{4\tau_0}{\tau_3}\right)\right]\right)}{\varrho\ \wp^2\left[(x + ky + \aleph z - \Omega t)\sqrt{\frac{\tau_3}{4}};\left(\frac{8\sqrt[3]{\tau_0^2\tau_3}}{\tau_3}, -\frac{4\tau_0}{\tau_3}\right)\right]}. \tag{36}$$

Family (4): If $\tau_3 = \tau_4 = 0$, the raised sets of solutions are generated as:

(4.1) $\mathbf{d}_1 = \mathbf{d}_2 = 0,\ \mathbf{d}_0 = -\frac{\alpha+\gamma\aleph-\Omega(\mathbb{C}\Omega+\vartheta)+k(\beta+k\mu)+\tau_2}{2\varrho},\ \mathbf{f}_1 = \frac{6\sqrt{\tau_0\tau_2}}{\varrho},\ \mathbf{f}_2 = -\frac{6\tau_0}{\varrho},\ \tau_0 = \frac{\tau_1^2}{4\tau_2}.$

(4.2) $\mathbf{d}_1 = \mathbf{f}_1 = \mathbf{d}_2 = \tau_1 = 0,\ \mathbf{d}_0 = -\frac{\alpha+\gamma\aleph-\Omega(\mathbb{C}\Omega+\vartheta)+k(\beta+k\mu)+4\tau_2}{2\varrho},\ \mathbf{f}_2 = -\frac{6\tau_0}{\varrho}$.

Using the revealed set of solutions (4.1), Eq. (1) gives its exact solution that can be expressed as:

(4.1) If $\tau_0 > 0$, $\tau_1 > 0$, $\tau_2 > 0$ and $\varrho \neq 0$, the following exponential solution is produced such that $\tau_1 - 2\tau_2 e^{\sqrt{\tau_2}(x+ky+\aleph z-\Omega t)} \neq 0$:

$$\phi_{4.1} = -\frac{\alpha+\gamma\aleph-\Omega(\mathbb{C}\Omega+\vartheta)+k(\beta+k\mu)+\tau_2}{2\varrho} - \frac{12\tau_1\tau_2}{\varrho}\left[\frac{\tau_1 - \tau_2 e^{(x+ky+\aleph z-\Omega t)\sqrt{\tau_2}}}{\left(\tau_1 - 2\tau_2 e^{(x+ky+\aleph z-\Omega t)\sqrt{\tau_2}}\right)^2}\right]. \tag{37}$$

By inserting the above set of solutions (4.2) to Eq. (1), the following solutions are obtained:

(4.2,1) If $\tau_0 > 0$, $\tau_2 < 0$ and $\varrho \neq 0$, a singular periodic solution is produced in the following form:

$$\phi_{4.2,1} = -\frac{\alpha+\gamma\aleph-\Omega(\mathbb{C}\Omega+\vartheta)+k(\beta+k\mu)+4\tau_2}{2\varrho} - \frac{6\tau_2}{\varrho}\csc^2\left[(x+ky+\aleph z-\Omega t)\sqrt{-\tau_2}\right]. \tag{38}$$

(4.2,2) If $\tau_0 > 0$, $\tau_2 > 0$ and $\varrho \neq 0$, a singular soliton solution is produced in the following form:

$$\phi_{4.2,2} = -\frac{\alpha+\gamma\aleph-\Omega(\mathbb{C}\Omega+\vartheta)+k(\beta+k\mu)+4\tau_2}{2\varrho} - \frac{6\tau_2}{\varrho}\operatorname{csch}^2\left[(x+ky+\aleph z-\Omega t)\sqrt{\tau_2}\right]. \tag{39}$$

Family (5): If $\tau_0 = \tau_1 = \mathbf{d}_0 = 0$ and $\tau_4 > 0$, the sets of the resulted solutions are listed below

(5.1) $\mathbf{d}_1 = \mathbf{f}_1 = \mathbf{f}_2 = \tau_3 = 0,\ \mathbf{d}_2 = -\frac{6\tau_4}{\varrho},\ \vartheta = \frac{\alpha+\gamma\aleph+k(\beta+k\mu)+\mathbb{C}\Omega^2+4\tau_2}{\Omega}$.

(5.2) $\mathbf{f}_1 = \mathbf{f}_2 = 0,\ \mathbf{d}_1 = \frac{6\sqrt{\tau_2\tau_4}}{\varrho},\ \mathbf{d}_2 = -\frac{6\tau_4}{\varrho},\ \tau_2 = \frac{\tau_3^2}{4\tau_4},\ \vartheta = \frac{\alpha+\gamma\aleph+k(\beta+k\mu)+\mathbb{C}\Omega^2+\tau_2}{\Omega}$.

Applying the set (5.1) of solutions, some analytical solutions for Eq. (1) are obtained as follows:

(5.1,1) If $\tau_2 < 0$ and $\varrho \neq 0$, the below singular periodic solution is raised:

$$\phi_{5.1,1} = \frac{6\tau_2}{\varrho}\csc^2\left[(x+ky+\aleph z-\Omega t)\sqrt{-\tau_2}\right]. \tag{40}$$

(5.1,2) If $\tau_2 > 0$, $\tau_4 > 0$, $\tau_3 \neq 2\varepsilon\sqrt{\tau_2\tau_4}$ where $\varepsilon = \pm 1$ and $\varrho \neq 0$, a singular soliton type of solution is obtained as follows:

$$\phi_{5.1,2} = -\frac{6\tau_2}{\varrho}\operatorname{csch}^2\left[(x+ky+\aleph z-\Omega t)\sqrt{\tau_2}\right]. \tag{41}$$

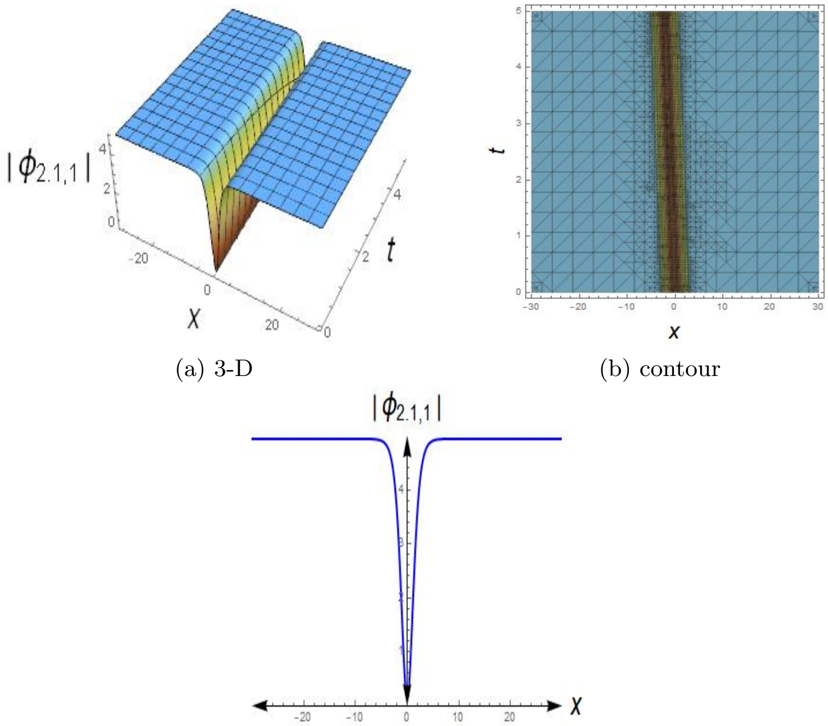
By inserting the set of solutions (5.2) in Eq. (1, the following solution can be obtained:

(5.2) If $\tau_2 > 0$, $\tau_4 > 0$, $\tau_3 = -2\sqrt{\tau_2\tau_4}$ and $\varrho \neq 0$, the following bright soliton solution:

$$\phi_{5.2} = \frac{3\tau_2}{2\varrho}\operatorname{sech}^2\left[(x+ky+\aleph z-\Omega t)\sqrt{\frac{\tau_2}{4}}\right]. \tag{42}$$

4. Results and Discussion

By changing the parameters in the model under investigation, several sets of values that had been documented or reached for Eq. (1) were found. This section includes a variety of graph forms, such as contour plots, 3-D, and 2-D plots of several individual solutions, to help illuminate the mathematical and physical characteristics of the retrieved solutions. Figure (2) shows the dark soliton solution representation of Eq. (12), when selecting $\tau_2 = -0.8,\ k = 0.7,\ \aleph = 0.6,\ \Omega = -0.45,\ \alpha = 0.9,\ \beta = 1.9,\ \mu = 0.9,\ \vartheta = 0.6,\ \mathbb{C} = 0.5,\ \gamma = 0.5,\ \varrho = -0.5,\ y = z = 0,\ 0 \le t \le 5$ and $-30 \le x \le 30$. Localized areas of reduced amplitudes within a surrounding medium are known as dark solitons [30]. In fluid dynamics, they could be connected to regions of decreased fluid density or pressure [31]. The dark soliton solutions, which usually appear in defocusing nonlinear media, are localized intensity dips contained in a continuous wave backdrop. They are important in the context of Bose-Einstein condensates and optical communications, and they simulate phase-shifted energy voids. By settings the values of $\tau_2 = -0.8,\ k = 0.7,\ \aleph = 0.6,\ \Omega = -0.45,\ \varrho = 3,\ y = z = 0,\ \ 0 \le t \le 10$ and $-15 \le x \le 15$. This yields the singular periodic solution of Eq. (40) as shown in Figure (3). In the context of physical phenomena, a system is referred to as having a unique periodic solution when there exist both periodic activity and sudden changes or extreme occurrences. These discontinuities or singularities might be brought on by external stimuli, boundary conditions, or non-linearities. Figure (4) displays the bright soliton solution of Eq. (42) with setting the parameters to be as $\tau_2 = 0.8,\ k = 0.9,\ \aleph = 0.8,\ \Omega = -0.6,\ \varrho = 0.9,\ y = z = 0,\ 0 \le t \le 5$ and $-15 \le x \le 15$. The bright soliton solution has a localized intensity peak above a continuous wave background, thus, it can exist on a finite-depth fluid in the water depth range where the carrier waves have a stable modulation.Because of a careful balancing act between dispersive spreading and nonlinear self-focusing, the resulting dazzling soliton solutions correspond to localized wave packets that maintain their form. These structures are essential for energy localization and distortion-free transmission and are typical of focusing nonlinear media, including optical fibers and certain plasma environments. Figure (5) illustrates the singular soliton solution of Eq. (41) with parameters $\tau_2 = 0.8,\ k = 0.9,\ \aleph = 0.8,\ \Omega = -0.6,\ \varrho = 2.9,\ y = z = 0,\ 0 \le t \le 5$ and $-15 \le x \le 15$. The NPDE solutions that exhibit singular behavior are known as singular solitons, which are mainly characterized by a narrow region due to an infinity approach peak. They may serve as symbolic representations of local events. Because they are so severe, they are less prevalent in physical systems. A divergence in wave amplitude is shown by the singular soliton solutions' unique behavior at particular times or locations in space. These solutions frequently simulate wave breaking in very nonlinear dispersive media, energy collapse occurrences, and shock-like structures.



(a) 3-D (b) contour

(c) 2-D

Figure 2: Visualization of the solution in Eq. (12).

5. Comparison with literature

In Table 1, we present a novel comparative analysis, supported by recent references, from multiple perspectives, including the employed methods, the nature of the obtained solutions, and the specific advantages of each approach when applied to the same class of models investigated in this work.

6. Linear Stability Analysis

In this section, we investigate the linear stability of the steady-state solution of the governing equation.

6.1. Steady-State and Perturbed Solution

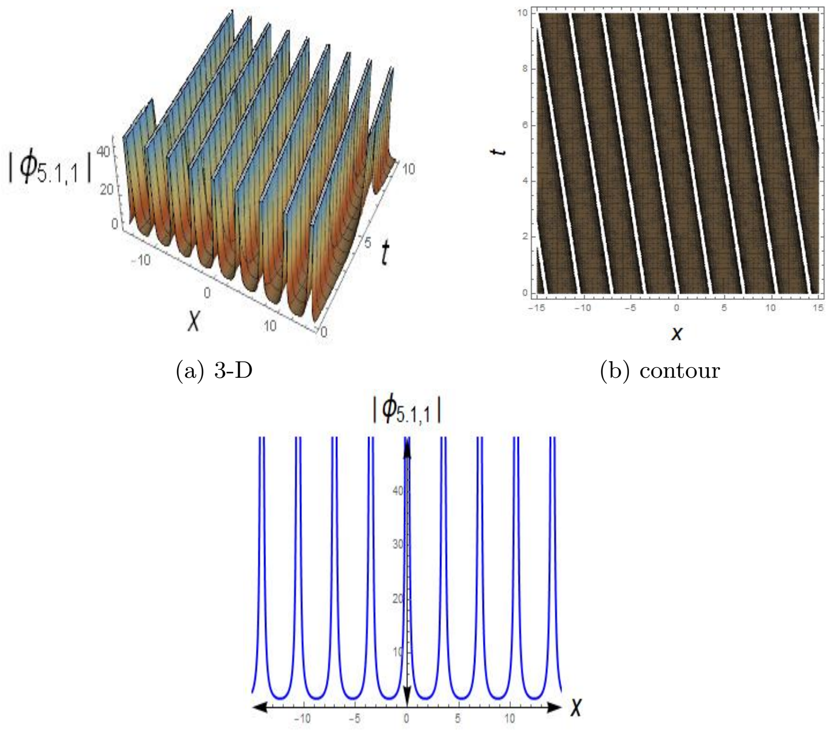
The steady-state solution is one that exhibits no variation in time [36]. For simplicity, we assume it to be a constant, denoted by λ. We then consider a perturbed solution of the form:

$$\phi(x, y, z, t) = \lambda + \rho\,\omega(x, y, z, t),$$

where $\omega(x, y, z, t)$ is a small perturbation function, and ρ is a small parameter.

Substituting this form into the original equation (Eq. 1) and neglecting nonlinear terms yields the linearized equation:

$$\mathbb{C}\,\omega_{tt} + \omega_{xxxx} + \alpha\,\omega_{xx} + \beta\,\omega_{xy} + \gamma\,\omega_{xz} + \vartheta\,\omega_{xt} + \mu\,\omega_{yy} + 2\lambda\varrho\omega_{xx} = 0. \tag{43}$$



(a) 3-D (b) contour

(c) 2-D

Figure 3: Visualization of the solution in Eq. (40).

6.2. Normal Mode Analysis

To analyze Eq. (43), we assume a solution of the form [37]:

$$\omega(x, y, z, t) = \eta \, e^{i(Mx+Ry+Fz+Wt)},$$

where:

- M, R, and F are the wave numbers in the x, y, and z directions, respectively,
- W is the temporal frequency,
- η is the amplitude.

Substituting this ansatz into Eq. (43) gives the algebraic dispersion relation:

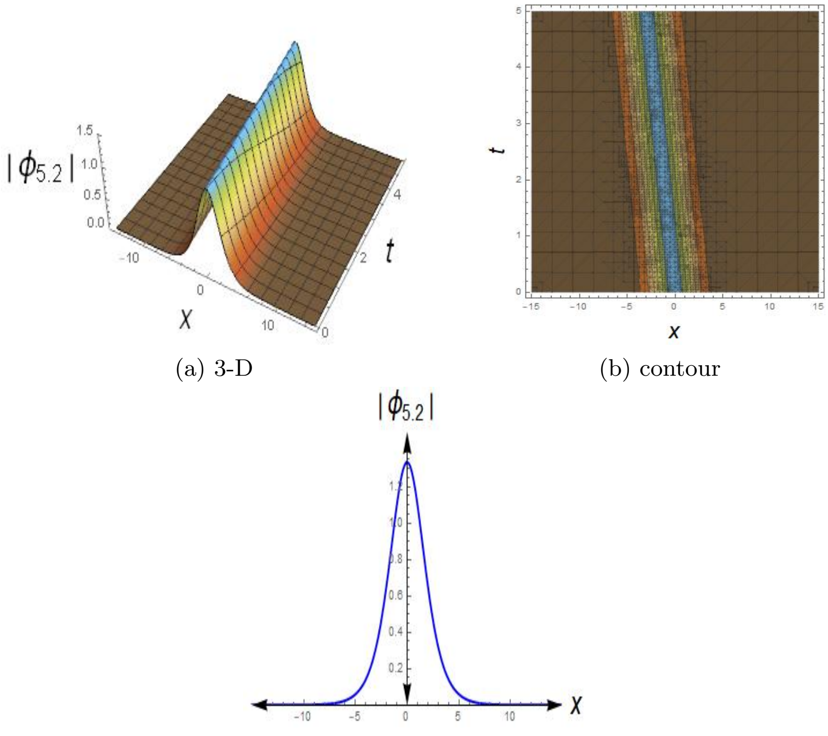
$$M^4 - \mathbb{C}W^2 - \alpha M^2 - \mu R^2 - M(\gamma F + \beta R + \vartheta W) - 2M^2\lambda\varrho = 0. \quad (44)$$

6.3. Dispersion Relation

Solving for W, we obtain the frequency as a function of the wave numbers:

$$W = \frac{-\vartheta M \pm \sqrt{-4\mathbb{C}M(\gamma F + \beta R) + 4\mathbb{C}M^4 + M^2\left(\vartheta^2 - 4\mathbb{C}(\alpha + 2\lambda\varrho)\right) - 4\mathbb{C}\mu R^2}}{2\mathbb{C}} \quad (45)$$

This relation characterizes the temporal evolution of the perturbations and can be used to assess the stability of the steady-state solution.



(a) 3-D (b) contour

(c) 2-D

Figure 4: Visualization of the solution in Eq. (42).

6.4. Compact Stability Criterion

Recall the dispersion relation for mode (M, R, F):

$$W = \frac{-\vartheta M \pm \sqrt{\Delta(M,R,F)}}{2\mathbb{C}}, \qquad \Delta(M,R,F) = -4\mathbb{C}M(\gamma F+\beta R)+4\mathbb{C}M^4+M^2\left(\vartheta^2 - 4\mathbb{C}(\alpha+2\lambda\varrho)\right)-4\mathbb{C}\mu R^2.$$

Write $W = W_{\rm re} + iW_{\rm im}$. The time dependence is $e^{iWt} = e^{iW_{\rm re}t}e^{-W_{\rm im}t}$. Thus:

(i) If $W_{\rm im} > 0$, mode decays $\Rightarrow$ stable.

(ii) If $W_{\rm im} < 0$, mode grows $\Rightarrow$ unstable.

(iii) If $W_{\rm im} = 0$, neutrally stable (pure oscillation).

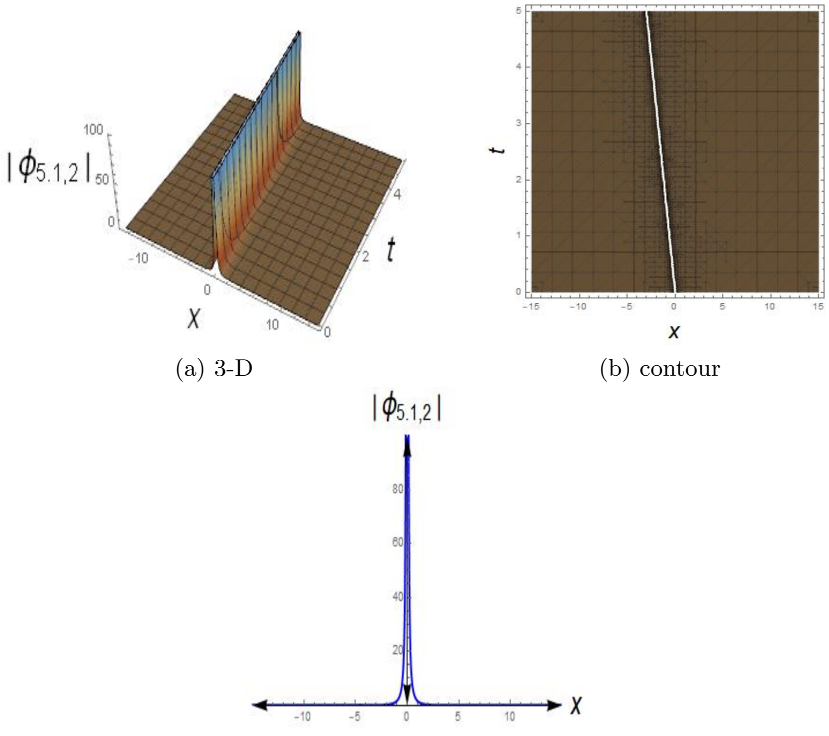
Since for real (M, R, F), Δ is real:

- **Case** $\Delta \geq 0$**:** $\sqrt{\Delta} \in \mathbb{R} \Rightarrow W \in \mathbb{R} \Rightarrow W_{\rm im} = 0$. Modes are neutrally stable.
- **Case** $\Delta < 0$**:** In this case, we write $\Delta = -|\Delta|$, so the square root becomes imaginary: $\sqrt{\Delta} = i\sqrt{|\Delta|}$. Then the frequency becomes

$$W = \frac{-\vartheta M}{2\mathbb{C}} \pm i\,\frac{\sqrt{|\Delta|}}{2\mathbb{C}}.$$

Hence, the imaginary part is

$$W_{\rm im} = \pm\frac{\sqrt{|\Delta|}}{2\mathbb{C}}.$$



(a) 3-D (b) contour

(c) 2-D

Figure 5: Visualization of the solution in Eq. (41).

Therefore:

- If $\mathbb{C} > 0$, the solution to the linearized equation (43) can be expressed as a superposition of two complex conjugate modes corresponding to the roots of the dispersion equation. That is,

$$\omega(x, y, z, t) = e^{i\left(Mx+Ry+Fz+\left(\frac{-\vartheta M}{2\mathbb{C}}+i\,\frac{\sqrt{|\Delta|}}{2\mathbb{C}}\right)t\right)} + e^{i\left(Mx+Ry+Fz+\left(\frac{-\vartheta M}{2\mathbb{C}}-i\,\frac{\sqrt{|\Delta|}}{2\mathbb{C}}\right)t\right)}.$$

  Even though one of the branches corresponds to a decaying mode, the other represents an exponentially growing mode. As time evolves, this growing mode will dominate the dynamics, rendering the steady-state solution linearly unstable. Therefore, the instability is invitable.

- The existence of even a single wave mode (M, R, F) for which $\Delta < 0$ is sufficient to conclude **linear instability** of the steady state.

In the following, we present a set of plots illustrating both **instability** and **neutral stability** cases based on different parameter settings.

For the **neutrally stable case**, we adopt the parameter set:

$$\alpha = 1, \quad \beta = 5, \quad \gamma = 5, \quad \mathbb{C} = 1, \quad F = 0, \quad \mu = 1, \quad \vartheta = 1, \quad R = 2, \quad \varrho = 5.$$

These values ensure that the discriminant remains non-negative, resulting in a purely real dispersion relation and therefore neutral stability.

Article Ref. No.	Method	Outcomes	Advantages
[32]	Bilinear method	Multi-soliton solutions, bright and dark solitons, soliton interactions	Highly effective for integrable systems; systematic and elegant for constructing multi-soliton solutions
[33]	Sine–cosine method	Periodic solutions, bright/dark solitons in trigonometric or hyperbolic form	Simple implementation; effective for constructing wave-type solutions with few parameters
[34]	Lie symmetry method	Similarity reductions, invariant solutions, symmetry classification	Powerful for finding reduction to ODEs, conserved quantities, and hidden symmetries
[35]	Darboux transformation	Bright/dark solitons, rogue and breather solutions	Constructs exact solutions recursively; excellent for integrable models with Lax pairs
In this work	Improved Modified Extended Tanh-Function Method	Bright, dark, and singular solitons; periodic, rational, JEFSs, and Weierstrass doubly elliptic solutions	Unifies diverse solution types; handles nonlinearities systematically; suitable for higher-dimensional models

Table 1: Comparison of classical methods with the IME tanh-function technique used in this study for (3+1)-dimensional models

For the **instability scenario**, we consider the parameters:

$$\alpha = 3, \quad \beta = 3, \quad \gamma = -3, \quad \mathbb{C} = 1, \quad F = 1, \quad \mu = 2, \quad \vartheta = 1, \quad R = 1, \quad \varrho = 1.$$

In this case, the discriminant $\Delta < 0$, leading to a nonzero imaginary part of the dispersion relation. We therefore plot the imaginary part of $\omega(M)$ versus the wavenumber M to identify the regions of instability.

6.5. Conclusion on Stability

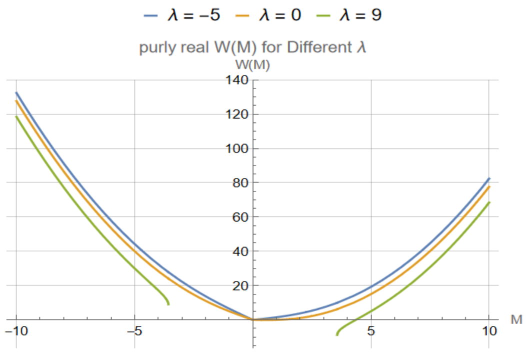
The steady-state solution λ is:

- **Linearly unstable** if there exists any real mode (M, R, F) such that $\Delta(M, R, F) < 0$. In this case, the corresponding perturbation grows exponentially with time for one branch of the root, and the other branch of the root implying decaying, resulting in overall instability of the steady state.
- **Linearly (neutrally) stable** if $\Delta(M, R, F) \geq 0$ for all real wave numbers (M, R, F). Then, all perturbations are purely oscillatory and remain bounded for all time, but they do not decay.

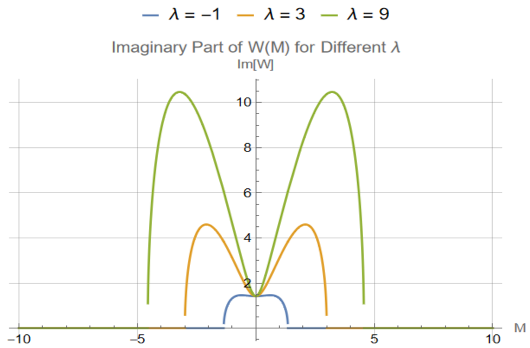
This analysis shows that for linear stability (i.e., decay of perturbations), one would require $W_{\text{im}} < 0$ for all modes, which does not occur in the current conservative system. Therefore, at best, the steady state is marginally (neutrally) stable when $\Delta \geq 0$ for all modes.

7. Conclusions

This investigation was conducted of the (3+1)-dimensional Boussinesq-KP-type equation, which is commonly employed in models of physical phenomena such as fluid dynamics, physics, hydrodynamic models, and optical mathematical models. This equation is more accurate than either the



(a) neutrally stable case



(b) instability case

Figure 6: Representing the neutrally stable and instability cases for different values of λ

KPE or Boussinesq-type equation, each separately, for the traveling waves. A well-known analytical method, the IME tanh-function algorithm, was applied, for which multi-soliton solutions and other solutions were obtained. The resulting solutions included dark, bright, and singular solitons, singular periodic solutions, JEFSs, rational solutions, exponential solutions, moreover Weierstrass elliptic doubly periodic solutions. This study offered a comprehensive explanation for numerous intricate and nonlinear phenomena that emerge and spread throughout a variety of nonlinear physical and engineering systems, particularly the nonlinear phenomena found in fluid mechanics, seas, and oceans, not to mention the abundance of nonlinear phenomena found in plasma physics. In contrast to other approaches, the strategy of this study provided new insights into the issues with this model's problems. This study was potentially expanded to investigate soliton dynamics in multi-dimensional systems, as it became easier to examine through the offered 2D, 3D, and contour graphs for some of the retrieved solutions. Furthermore, a linear stability assessment is conducted to examine the robustness and persistence of the derived solutions under small perturbations.

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Appendix A

In this part, we show all types of solutions for Eq. (6) as shown in [28, 29].

Family (1): $\tau_0 = \tau_1 = \tau_3 = 0$,

$$\mathcal{W}(\eta) = \sqrt{-\frac{\tau_2}{\tau_4}}\operatorname{sech}(\sqrt{\tau_2}\,\eta), \qquad \tau_2 > 0, \tau_4 < 0,$$

$$\mathcal{W}(\eta) = \sqrt{-\frac{\tau_2}{\tau_4}}\sec(\sqrt{-\tau_2}\,\eta), \qquad \tau_2 < 0, \tau_4 > 0,$$

$$\mathcal{W}(\eta) = \frac{-\varepsilon}{\sqrt{\tau_4}\,\eta}, \qquad \tau_2 = 0, \tau_4 > 0.$$

Family (2): $\tau_1 = \tau_3 = 0$,

$$\mathcal{W}(\eta) = \varepsilon\sqrt{-\frac{\tau_2}{2\tau_4}}\tanh\left(\sqrt{-\frac{\tau_2}{2}}\,\eta\right), \qquad \tau_2 < 0, \tau_4 > 0, \tau_0 = \frac{\tau_2^2}{4\tau_4},$$

$$\mathcal{W}(\eta) = \varepsilon\sqrt{\frac{\tau_2}{2\tau_4}}\tan\left(\sqrt{\frac{\tau_2}{2}}\,\eta\right), \qquad \tau_2 > 0, \tau_4 > 0, \tau_0 = \frac{\tau_2^2}{4\tau_4},$$

$$\mathcal{W}(\eta) = \sqrt{\frac{-\tau_2 m^2}{\tau_4(2m^2-1)}}\operatorname{cn}\left(\sqrt{\frac{\tau_2}{2m^2-1}}\,\eta\right), \qquad \tau_2 > 0, \tau_4 < 0, \tau_0 = \frac{\tau_2^2 m^2(1-m^2)}{\tau_4(2m^2-1)^2},$$

$$\mathcal{W}(\eta) = \sqrt{\frac{-m^2}{\tau_4(2-m^2)}}\operatorname{dn}\left(\sqrt{\frac{\tau_2}{2-m^2}}\,\eta\right), \qquad \tau_2 > 0, \tau_4 < 0, \tau_0 = \frac{\tau_2^2(1-m^2)}{\tau_4(2-m^2)^2},$$

$$\mathcal{W}(\eta) = \varepsilon\sqrt{-\frac{\tau_2 m^2}{\tau_4(1+m^2)}}\operatorname{sn}\left(\sqrt{-\frac{\tau_2}{1+m^2}}\,\eta\right), \qquad \tau_2 < 0, \tau_4 > 0, \tau_0 = \frac{\tau_2^2 m^2}{\tau_4(m^2+1)^2},$$

where m is the modulus of the Jacobi elliptic functions, $0 \le m \le 1$.

Family (3): $\tau_2 = \tau_4 = 0,\ \tau_0 \neq 0,\ \tau_1 \neq 0,\ \tau_3 > 0$,
A Weierstrass elliptic doubly periodic type solution is obtained:

$$\mathcal{W}(\eta) = \wp\left[\frac{\sqrt{\tau_3}}{2}\eta,\ \mathcal{A}_2,\ \mathcal{A}_3\right],$$

where $\mathcal{A}_2 = -\dfrac{4f_1}{f_3}$ and $\mathcal{A}_3 = -\dfrac{4f_0}{f_3}$ are called Weierstrass elliptic function in-variants.

Family (4): $\tau_3 = \tau_4 = 0$,

$$\mathcal{W}(\eta) = -\frac{\tau_1}{2\tau_2} + \exp\left(\varepsilon\sqrt{\tau_2}\,\eta\right), \qquad \tau_2 > 0, \tau_0 = \frac{\tau_1^2}{4\tau_2},$$

$$\mathcal{W}(\eta) = -\frac{\tau_1}{2\tau_2} + \frac{\varepsilon\tau_1}{2\tau_2}\sin\left(\sqrt{-\tau_2}\,\eta\right), \qquad \tau_0 = 0, \tau_2 < 0,$$

$$\mathcal{W}(\eta) = -\frac{\tau_1}{2\tau_2} + \frac{\varepsilon\tau_1}{2\tau_2}\sinh\left(2\sqrt{\tau_2}\,\eta\right), \qquad \tau_0 = 0, \tau_2 > 0,$$

$$\mathcal{W}(\eta) = \varepsilon\sqrt{-\frac{\tau_0}{\tau_2}}\sin\left(\sqrt{-\tau_2}\,\eta\right), \qquad \tau_1 = 0, \tau_0 > 0, \tau_2 < 0,$$

$$\mathcal{W}(\eta) = \varepsilon\sqrt{\frac{\tau_0}{\tau_2}}\sinh\left(\sqrt{\tau_2}\,\eta\right), \qquad \tau_1 = 0, \tau_0 > 0, \tau_2 > 0.$$

Family (5): $\tau_0 = \tau_1 = 0$, $\tau_4 > 0$,

$$\mathcal{W}(\eta) = -\frac{\tau_2\sec^2\left(\frac{\sqrt{-\tau_2}}{2}\,\eta\right)}{2\varepsilon\sqrt{-\tau_2\tau_4}\tan\left(\frac{\sqrt{-\tau_2}}{2}\,\eta\right) + \tau_3}, \qquad \tau_2 < 0,$$

$$\mathcal{W}(\eta) = \frac{\tau_2\operatorname{sech}^2\left(\frac{\sqrt{\tau_2}}{2}\,\eta\right)}{2\varepsilon\sqrt{\tau_2\tau_4}\tanh\left(\frac{\sqrt{\tau_2}}{2}\,\eta\right) - \tau_3}, \qquad \tau_2 > 0, \tau_3 \neq 2\varepsilon\sqrt{\tau_2\tau_4},$$

$$\mathcal{W}(\eta) = \frac{1}{2}\varepsilon\sqrt{\frac{\tau_2}{\tau_4}}\left[1 + \tanh\left(\frac{\sqrt{\tau_2}}{2}\,\eta\right)\right], \qquad \tau_2 > 0, \tau_3 = 2\varepsilon\sqrt{\tau_2\tau_4}.$$