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Product Difference Fibonacci Identities Revisited: Quaternionic Generalizations of Everman and Koshy

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Abstract. In this paper, we investigate new identities involving generalized Fibonacci and Lucas sequences, as well as their associated quaternions. After establishing the fundamental properties of these generalized number sequences, we derive quaternionic extensions of product-difference identities originally introduced by Everman and Koshy for Fibonacci numbers. These results not only generalize classical identities, but also reveal new algebraic structures within the framework of generalized quaternion sequences.

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1. Introduction

Fibonacci and Lucas number sequences are used in many areas of mathematics due to their repetitive structures. They arise in number theory with their divisibility properties, in analysis with their generator functions, in linear algebra with their matrix representations, in combinatorics with tiling problems, and in geometry with their fractal structures, etc. Because of such a wide range of applications, many generalized versions of these number sequences have been defined; some by changing the recurrence relation, and others by changing the initial conditions. These sequences offer more complex and richer mathematical structures, especially when combined with quaternion structures. Furthermore, the extension of these numbers to quaternions is being studied by many authors in both theoretical and applied fields.

Fibonacci sequences are seen in many structures in nature, such as the arrangement of sunflower seeds, the structure of a pine cone, the crystallization of a snowflake, and the spiral pattern in seashells. In [1], Sharma explores the evolution of the Fibonacci sequence and its modern applications, especially in fractal geometry. Generalized sequences can be

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used to explain more complex models of these natural patterns, for example the structure of DNA [2]. In addition to being used to model patterns in quantum physics and other physical systems, quarks, which we know from physics theory, can also be given as an example[3].

Fibonacci and Lucas number sequences are notable for their mathematical structure as well as their applications in various disciplines. The properties of these number sequences are utilized in computer science, algorithm development, cryptography, nature science, art, engineering and design, finance and economics [4]. Among the algorithms used in computer science, the Fibonacci search algorithm and the golden ratio search algorithm offer particularly good solutions for searching and sorting. The increasing number of different generalizations of these sequences of numbers is being used to develop new algorithms [5]. Thus, more flexible and adaptable algorithm versions can be developed that reduce data processing costs, such as the running time of the algorithm used, and how close the results obtained are to the true value with less error. In addition, these sequences of numbers are used for key exchange and encryption in secure communication systems. Moreover, Fibonacci and Lucas sequences are used in engineering for signal processing and structural optimization, as well as architecture and design [6]. The golden ratio is also used in determining aesthetic proportions in art and architecture, rhythmic structures and compositions in music. In addition, Fibonacci sequences are used in technical analysis in finance and economics to determine support and resistance levels [7]. Generalized sequences can be used to build more complex models of market movements.

There are many studies on the mathematical properties and applications of quaternions whose elements are generalized Fibonacci and Lucas numbers. Research in this area not only provides new results in theoretical mathematics, but also new solutions in optimization, cryptography, and especially in transformation and rotation problems in physics and engineering.

Halıcı obtained numerous new equations by generalizing various Fibonacci and Lucas quaternions and deriving their Binet formulas, generator functions, and matrix representations [8–10]. Kesim obtained exponential generator functions for the generalized Fibonacci and Lucas quaternions and obtained binomial sums of these quaternions [11]. Kome et al. introduced the modified generalized Fibonacci and Lucas quaternions and gave the generator functions, Binet formulas and matrix representations for these quaternions [12]. Moreover, Aydinyüz and Asci introduced the generalized k-order Fibonacci and Lucas quaternions and obtained their generating functions and matrix representations [13]. Kızılateş et al. define higher-order generalized Fibonacci quaternions with q-integer components using q-integers and higher order generalized Fibonacci numbers and obtain Binet-like formulas, generator functions, recurrence relations, and matrix representations for these new quaternions in [14]. In [15], Shpakivskyi investigates some properties of generalized Fibonacci quaternions and Fibonacci-Narayana quaternions.

In this paper, we rederive the classical results of Koshy and Everman using generalized Fibonacci and Lucas numbers, and subsequently apply these results to quaternions. In this context, generalized Fibonacci and Lucas quaternions will be defined, and their properties will be studied in detail. In addition, generalized versions of some equations in the literature will also be obtained in this study. The aim of this study is to understand the mathematical structures of quaternions based on generalized Fibonacci and Lucas numbers and to show their connection to classical results through the derivation of new identities.

In this section some literature overwiev is given about Fibonacci and Lucas sequences, their generalizations, and quaternion generalizations. The rest of this paper is organized as follows. In the next section, the basic properties of generalized Fibonacci and Lucas sequences will be discussed and new identities related to these sequences will be obtained. Then in the third section, the algebraic properties of quaternions whose terms are composed of generalized Fibonacci and Lucas numbers will be given, and new identities related to these quaternions will be obtained by using Binet formulas. Finally, we will give quaternion generalizations of product differences of Everman and Koshy based Fibonacci identities'.

2. Properties and Identities of Generalized Fibonacci and Lucas Sequences

Definition 1. *Let H_n be a sequence defined by the recurrence relation*

$$H_n = H_{n-1} + H_{n-2}, \ for \quad n \geq 3, \tag{1}$$

with the initial conditions $H_1 = p, \quad H_2 = p + q$, where p and q are arbitrary integers. The terms of this sequence are

$$p, \ p+q, \ 2p+q, \ 3p+2q, \ 5p+3q, \ 8p+5q, \ 13p+8q, \ \ldots.$$

This sequence is called Horadam's Generalized Fibonacci Sequence [16, 17].

(F_n) is the Fibonacci sequence with the recurrence relation

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2,$$

defined by taking $p = 1$ and $q = 0$ in (1). Similarly, taking $p = 1$ and $q = 2$ in (1), the Lucas sequence (L_n) is defined with the recurrence relation

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \quad \text{for } n \geq 2, [18], [19].$$

Kalman and Mena, in [20], defined generalized Fibonacci and Lucas numbers with the recurrence relation

$$A_{n+2} = aA_{n+1} + bA_n,$$

for all $n \geq 0$ where the sequences depend on initial conditions A_0 and A_1. They identified several number sequences corresponding to different values of $R(a, b)$, as illustrated below.

Table 1: Generalized Fibonacci and Lucas-Type Sequences [20]

Sequence	**Initial Conditions**	**First Terms**
Fibonacci numbers	$R(1,1),\ A_0 = 0,\ A_1 = 1$	$\{0,1,1,2,3,5,8,13,21,34,55,\dots\}$
Lucas numbers	$R(1,1),\ A_0 = 2,\ A_1 = a$	$\{2,1,3,4,7,11,18,29,47,76,\dots\}$
Pell numbers	$R(2,1),\ A_0 = 0,\ A_1 = 1$	$\{0,1,2,5,12,29,70,169,\dots\}$
Pell-Lucas numbers	$R(2,1),\ A_0 = 2,\ A_1 = a$	$\{2,2,6,14,34,82,\dots\}$
Natural numbers	$R(2,-1),\ A_0 = 0,\ A_1 = 1$	$\{0,1,2,3,4,5,6,\dots\}$
Constant sequence	$R(2,-1),\ A_0 = 2,\ A_1 = a$	$\{2,2,2,2,2,2,2,\dots\}$
Mersenne sequence	$R(3,-2),\ A_0 = 0,\ A_1 = 1$	$\{0,1,3,7,15,31,\dots\}$
Fermat sequence	$R(3,-2),\ A_0 = 2,\ A_1 = a$	$\{2,3,5,9,16,33,\dots\}$
Periodic (with period=6)	$R(1,-1),\ A_0 = 0,\ A_1 = 1$	$\{0,1,1,0,-1,-1,0,1,\dots\}$
Periodic (with period=6)	$R(1,-1),\ A_0 = 2,\ A_1 = a$	$\{2,1,-1,-2,-1,1,2,1,\dots\}$
Even-indexed Fibonacci	$R(3,-1),\ A_0 = 0,\ A_1 = 1$	$\{0,1,3,8,21,55,\dots\}$
Even-indexed Lucas	$R(3,-1),\ A_0 = 2,\ A_1 = a$	$\{2,3,7,18,47,\dots\}$

Koshy, in [19], introduced the notion of (p,q) generalized Fibonacci numbers.

In this study, we present several properties of the (p,q)-generalized Fibonacci and Lucas numbers. Now let us give the definition of a (p,q) generalization of Fibonacci and Lucas sequences.

Definition 2. *([16, 17]) Let $p,q \in \mathbb{Z}$. The sequence (U_n) defined by the relation*

$$U_n = pU_{n-1} + qU_{n-2},$$

for all $n \geq 2$ with initial conditions $U_0 = 0, \quad U_1 = 1$ is called the generalized Fibonacci sequence, and the number U_n is called the $n-th$ generalized Fibonacci number.

Definition 3. *([16, 17]) Let $p,q \in \mathbb{Z}$.The sequence (V_n) defined by the relation*

$$V_n = pV_{n-1} + qV_{n-2},$$

for all $n \geq 2$ with initial conditions $V_0 = 2, \quad V_1 = p$, is called the generalized Lucas sequence, and the number V_n is called the $n-th$ generalized Lucas number.

The characteristic equation of these generalized sequences are $x^2 - px - q = 0$. Let $\Delta = p^2 + 4q > 0$; the roots of this characteristic equation are given by

$$\alpha = \frac{p+\sqrt{\Delta}}{2} \quad \text{and} \quad \beta = \frac{p-\sqrt{\Delta}}{2},$$

which satisfy $\alpha + \beta = p$ and $\alpha\beta = -q$. Moreover, $\alpha^2 = p\alpha + q \quad$ and $\quad \beta^2 = p\beta + q$. By induction, it can be shown that for every $n \in \mathbb{N}$,

$$\alpha^n = \alpha U_n + qU_{n-1}, \tag{2}$$

$$\beta^n = \beta U_n + qU_{n-1}. \tag{3}$$

Similarly, it can be shown by induction that for every $n \in \mathbb{N}$, the elements of the generalized Lucas sequence satisfy the identities

$$\sqrt{\Delta}\,\alpha^n = \alpha V_n + qV_{n-1}, \tag{4}$$

$$-\sqrt{\Delta}\,\beta^n = \beta V_n + qV_{n-1}. \tag{5}$$

Moreover, the Binet formulas for generalized Fibonacci and Lucas numbers are given by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = \alpha^n + \beta^n \tag{6}$$

for all $n \in \mathbb{Z}$ [20, 21].

Since each term is a linear combination of the two preceding terms, we have

$$U_{-1} = \frac{1}{q}, \quad U_{-2} = -\frac{p}{q^2}, \quad U_{-3} = \frac{p^2+q}{q^3}, \ldots$$

and

$$V_{-1} = -\frac{p}{q}, \quad V_{-2} = \frac{p^2+2q}{q^2}, \quad V_{-3} = -\frac{p^3+3pq}{q^3}, \ldots$$

From these results, it was shown in [22], using the Binet formulas, that the generalized Fibonacci and Lucas numbers with negative indices satisfy

$$U_{-n} = -(-q)^n U_n, \quad V_{-n} = (-q)^n V_n.$$

Many authors have used the properties of these sequences and have proved numerious identities, sum formulas, and matrix structures, while many others have investigated the product differences of Fibonacci and Lucas numbers. Some of these classical Fibonacci identities are such as Cassini's identity

$$F_n^2 - F_{n-1} F_{n+1} = (-1)^{n-1},$$

which was proved in 1680 and Catalan's identity

$$F_n^2 - F_{n-m} F_{n+m} = (-1)^{n-m} F_m^2,$$

which was proved 1879, and the d'Ocagne's identity

$$F_{m+n} = F_{m-1} F_n + F_m F_{n+1},$$

was proved in 1882.

Since we will focus on the product differences of generalized Everman and Koshy based identities, let us first recall the term "Product Difference Fibonacci Identity" with the following definition.

Definition 4. *(Product Difference Fibonacci Identity) Let $s \geq 1$, and the a_i and b_i be specified integers and D_n be of some interesting form for all integers n. Then the product of the form*

$$\prod_{i=1}^{s} F_{n+a_i} - \prod_{i=1}^{s} F_{n+b_i} = D_n(a_i, b_i;\, s) = D_n. \tag{7}$$

is introduced by Fairgrieve and Gould in [23].

It can be seen that Cassini's and Catalan's identities are some versions of the equation (7). There are many identities related to (7). For example, Morgado used the Catalan identity to prove the following equation

$$F_{n-2}\, F_{n-1}\, F_{n+1}\, F_{n+2} - F_n^4 = -1.$$

in [24]. Recently, in [25], Melham discovered the following formula

$$F_{n+1}F_{n+2}F_{n+6} - F_{n+3}^3 = (-1)^n F_n.$$

More generally the following equation

$$F_{n+a}\, F_{n+b} - F_n\, F_{n+a+b} = (-1)^n\, F_a\, F_b\,. \tag{8}$$

was stated by Everman at al. as a problem in The American Mathematical Monthly [26] and appears in Vajda [21, p. 177, Eq. (20a)]. (8) can be called the extended version of these classical Fibonacci identities. Actually, Horadam had already expressed some identities similar to equality (8) with some generalizations and he had stated the following equation

$$H_n\, H_{n+r+1} - H_{n-s}\, H_{n+r+s+1} = (-1)^{n+s}\, [p^2 - p\, q - q^2]\, F_{r+s+1} \tag{9}$$

in [16].

Then in [25] Melham proved the identity

$$F_{n+a+b-c}\, F_{n-a+c}\, F_{n-b+c} - F_{n-a-b+c}\, F_{n+a}\, F_{n+b} = (-1)^{n+a+b+c}\, F_{a+b-c}(F_c\, F_{n+a+b-c} + (-1)^c\, F_{a-c}\, F_{b-c}\, L_n).$$

Since Cassini's, Catalan's, d'Ocagne's identities, and all of the identities given in equation (7), (8), (9) and their variations are also given in Koshy's book [19], we refer to the new identities that we prove in this article as "Quaternionic Generalizations of Everman and Koshy".

Using the Binet formulas given in (6), Cassini's and Catalan's identities can also be generalized.

Theorem 1. *(Cassini's Identity) For all $n \in \mathbb{Z}$, the following identity holds*

$$U_{n-1}U_{n+1} - U_n^2 = -(-q)^{n-1}.$$

Proof. Using the Binet formula and the relation $\alpha\beta = -q$, we have

$$\begin{aligned}
U_{n-1}U_{n+1} - U_n^2 &= \frac{(\alpha^{n-1} - \beta^{n-1})(\alpha^{n+1} - \beta^{n+1})}{(\alpha-\beta)^2} - \frac{(\alpha^n - \beta^n)^2}{(\alpha-\beta)^2} \\
&= \frac{\alpha^{2n} + \beta^{2n} - \alpha^{n-1}\beta^{n+1} - \alpha^{n+1}\beta^{n-1}}{(\alpha-\beta)^2} - \frac{\alpha^{2n} + \beta^{2n} - 2\alpha^n\beta^n}{(\alpha-\beta)^2} \\
&= \frac{-\alpha^{n+1}\beta^{n-1} - \alpha^{n-1}\beta^{n+1} + 2\alpha^n\beta^n}{(\alpha-\beta)^2} \\
&= -(\alpha\beta)^{n-1}\frac{\alpha^2 + \beta^2 - 2\alpha\beta}{(\alpha-\beta)^2} = -(\alpha\beta)^{n-1} = -(-q)^{n-1}.
\end{aligned}$$

Theorem 2. *For all $n \in \mathbb{Z}$, the following identity holds*

$$V_{n-1}V_{n+1} - V_n^2 = (-q)^{n-1}\Delta.$$

Proof. Using the Binet formula and identities $\alpha\beta = -q$, $\alpha - \beta = \sqrt{\Delta}$, we obtain

$$\begin{aligned}
V_{n-1}V_{n+1} - V_n^2 &= (\alpha^{n-1} + \beta^{n-1})(\alpha^{n+1} + \beta^{n+1}) - (\alpha^n + \beta^n)^2 \\
&= (\alpha^{2n} + \beta^{2n} + \alpha^{n-1}\beta^{n+1} + \alpha^{n+1}\beta^{n-1}) - (\alpha^{2n} + \beta^{2n} + 2\alpha^n\beta^n) \\
&= \alpha^{n+1}\beta^{n-1} + \alpha^{n-1}\beta^{n+1} - 2\alpha^n\beta^n \\
&= (\alpha\beta)^{n-1}(\alpha^2 + \beta^2 - 2\alpha\beta) \\
&= (\alpha\beta)^{n-1}(\alpha-\beta)^2 \\
&= (-q)^{n-1}\Delta.
\end{aligned}$$

Theorem 3. *(Catalan's Identity) For all $n, r \in \mathbb{Z}$, it follows that*

$$U_{n-r}U_{n+r} - U_n^2 = -(-q)^{n-r}U_r^2.$$

Proof. Using the Binet formula and $\alpha\beta = -q$, we have

$$\begin{aligned}
U_{n-r}U_{n+r} - U_n^2 &= \frac{(\alpha^{n-r} - \beta^{n-r})(\alpha^{n+r} - \beta^{n+r})}{(\alpha-\beta)^2} - \frac{(\alpha^n - \beta^n)^2}{(\alpha-\beta)^2} \\
&= \frac{\alpha^{2n} + \beta^{2n} - \alpha^{n-r}\beta^{n+r} - \alpha^{n+r}\beta^{n-r}}{(\alpha-\beta)^2} - \frac{\alpha^{2n} + \beta^{2n} - 2\alpha^n\beta^n}{(\alpha-\beta)^2} \\
&= \frac{-\alpha^{n+r}\beta^{n-r} - \alpha^{n-r}\beta^{n+r} + 2\alpha^n\beta^n}{(\alpha-\beta)^2} \\
&= -(\alpha\beta)^{n-r}\frac{\alpha^{2r} + \beta^{2r} - 2\alpha^r\beta^r}{(\alpha-\beta)^2} \\
&= -(\alpha\beta)^{n-r}\left(\frac{\alpha^r - \beta^r}{\alpha-\beta}\right)^2 \\
&= -(\alpha\beta)^{n-r}(U_r)^2 \\
&= -(-q)^{n-r}U_r^2.
\end{aligned}$$

Theorem 4. *For all $n, r \in \mathbb{Z}$,*

$$V_{n-r}V_{n+r} - V_n^2 = (-q)^{n-r}\Delta U_r^2.$$

Proof. Considering the Binet formula, identities $\alpha\beta = -q$ and $\alpha - \beta = \sqrt{\Delta}$, we have

$$\begin{aligned}
V_{n-r}V_{n+r} - V_n^2 &= (\alpha^{n-r} + \beta^{n-r})(\alpha^{n+r} + \beta^{n+r}) - (\alpha^n + \beta^n)^2 \\
&= (\alpha^{2n} + \beta^{2n} + \alpha^{n-r}\beta^{n+r} + \alpha^{n+r}\beta^{n-r}) - (\alpha^{2n} + \beta^{2n} + 2\alpha^n\beta^n) \\
&= \alpha^{n+r}\beta^{n-r} + \alpha^{n-r}\beta^{n+r} - 2\alpha^n\beta^n \\
&= (\alpha\beta)^{n-r}(\alpha^{2r} + \beta^{2r} - 2\alpha^r\beta^r) \\
&= (\alpha\beta)^{n-r}(\alpha^r - \beta^r)^2 \\
&= (\alpha\beta)^{n-r}\left(\frac{\alpha^r - \beta^r}{\alpha - \beta}\right)^2(\alpha - \beta)^2 \\
&= (\alpha\beta)^{n-r}U_r^2(\alpha - \beta)^2 \\
&= (-q)^{n-r}\Delta U_r^2.
\end{aligned}$$

Corollary 1. *For all $n, r \in \mathbb{Z}$, we have $V_{n-r}V_{n+r} - V_n^2 = -\Delta(U_{n-r}U_{n+r} - U_n^2)$.*

Proof. Follows directly from Theorem 3 and Theorem 4.

Şiar and Keskin [27] established several identities using the matrices $\begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} U_{n+1} & qU_n \\ U_n & qU_{n-1} \end{bmatrix}$, and proved the following results

$$\begin{aligned}
V_n^2 - (p^2 + 4q)U_n^2 &= 4(-q)^n, \\
\Delta U_m U_n &= V_{m+n} - (-q)^n V_{m-n}, \\
(-q)^n V_{m-n} &= U_{m+1}V_n - V_{n+1}U_m, \\
U_r U_{m+n+r} &= U_{m+r}U_{n+r} - (-q)^r U_m U_n, \\
U_r U_{m+n-r} &= U_m U_n - (-q)^r U_{m-r}U_{n-r}, \\
U_r U_{m+n} &= U_m U_{n+r} - (-q)^r U_{m-r}U_n, \\
V_r V_{m+n+r} &= V_{m+r}V_{n+r} + (-q)^r \Delta U_m U_n, \\
V_r V_{m+n-r} &= (-q)^r V_{m-r}V_{n-r} + \Delta U_m U_n, \\
V_r U_{m+n} &= U_n V_{m+r} + (-q)^r V_{n-r}U_m,
\end{aligned}$$

and more. Now we will prove other identities as follows.

Theorem 5. *For all $n, r \in \mathbb{Z}$, we have $U_{n+r}^2 - q^{2r}U_{n-r}^2 = U_{2n}U_{2r}$.*

Proof.

$$U_{n+r}^2 - q^{2r}U_{n-r}^2 = \left(\frac{\alpha^{n+r}-\beta^{n+r}}{\alpha-\beta}\right)^2 - (-q)^{2r}\left(\frac{\alpha^{n-r}-\beta^{n-r}}{\alpha-\beta}\right)^2$$

$$= \frac{\alpha^{2n+2r}+\beta^{2n+2r}-2(\alpha\beta)^{n+r}}{(\alpha-\beta)^2} - (\alpha\beta)^{2r}\frac{\alpha^{2n-2r}+\beta^{2n-2r}-2(\alpha\beta)^{n-r}}{(\alpha-\beta)^2}$$

$$= \frac{\left[\alpha^{2n+2r}+\beta^{2n+2r}-2(\alpha\beta)^{n+r}\right] - (\alpha\beta)^{2r}\left[\alpha^{2n-2r}+\beta^{2n-2r}-2(\alpha\beta)^{n-r}\right]}{(\alpha-\beta)^2}$$

$$= \frac{\alpha^{2n+2r}+\beta^{2n+2r}-\alpha^{2n}\beta^{2r}-\alpha^{2r}\beta^{2n}}{(\alpha-\beta)^2} = \frac{\alpha^{2n+2r}-\alpha^{2n}\beta^{2r}+\beta^{2n+2r}-\alpha^{2r}\beta^{2n}}{(\alpha-\beta)^2}$$

$$= \frac{\alpha^{2n}\left(\alpha^{2r}-\beta^{2r}\right)+\beta^{2n}\left(\beta^{2r}-\alpha^{2r}\right)}{(\alpha-\beta)^2} = \frac{\alpha^{2n}\left(\alpha^{2r}-\beta^{2r}\right)-\beta^{2n}\left(\alpha^{2r}-\beta^{2r}\right)}{(\alpha-\beta)^2}$$

$$\frac{\left(\alpha^{2n}-\beta^{2n}\right)\left(\alpha^{2r}-\beta^{2r}\right)}{(\alpha-\beta)^2} = \frac{\left(\alpha^{2n}-\beta^{2n}\right)}{\alpha-\beta}\frac{\left(\alpha^{2r}-\beta^{2r}\right)}{\alpha-\beta} = U_{2n}U_{2r}.$$

Theorem 6. *For all $n, r \in \mathbb{Z}$, it follows that $V_{n+r}^2 - q^{2r}V_{n-r}^2 = \Delta U_{2n}U_{2r}$.*

Proof.

$$V_{n+r}^2 - q^{2r}V_{n-r}^2 = \left(\alpha^{n+r}+\beta^{n+r}\right)^2 - (-q)^{2r}\left(\alpha^{n-r}+\beta^{n-r}\right)^2$$

$$= \left[\alpha^{2n+2r}+\beta^{2n+2r}+2(\alpha\beta)^{n+r}\right] - (\alpha\beta)^{2r}\left[\alpha^{2n-2r}+\beta^{2n-2r}+2(\alpha\beta)^{n-r}\right]$$

$$= \alpha^{2n+2r}+\beta^{2n+2r}-\alpha^{2n}\beta^{2r}-\alpha^{2r}\beta^{2n} = \alpha^{2n}\left(\alpha^{2r}-\beta^{2r}\right)+\beta^{2n}\left(\beta^{2r}-\alpha^{2r}\right)$$

$$= \alpha^{2n}\left(\alpha^{2r}-\beta^{2r}\right)-\beta^{2n}\left(\alpha^{2r}-\beta^{2r}\right) = \left(\alpha^{2n}-\beta^{2n}\right)\left(\alpha^{2r}-\beta^{2r}\right)$$

$$= \frac{\left(\alpha^{2n}-\beta^{2n}\right)}{\alpha-\beta}\frac{\left(\alpha^{2r}-\beta^{2r}\right)}{\alpha-\beta}(\alpha-\beta)^2 = \Delta U_{2n}U_{2r}.$$

The following result is a direct consequence of Theorem 5 and Theorem 6.

Corollary 2. *For all $n, r \in \mathbb{Z}$, it follows that $V_{n+r}^2 - q^{2r}V_{n-r}^2 = \Delta\left[U_{n+r}^2 - q^{2r}U_{n-r}^2\right]$.*

3. Generalized Fibonacci and Lucas Quaternions

Quaternions are four-dimensional hypercomplex numbers introduced by William Rowan Hamilton in the 19th century and used to represent transformations in three-dimensional space [28]. Quaternions, whose coefficients are Fibonacci and Lucas numbers, have also become an interesting sequence with many generalizations over time. Let us start by giving the fundamental definition of a quaternion.

Definition 5. *Let $a, b, c, d \in \mathbb{R}$ and $i^2 = -1$, $j^2 = -1$, $k^2 = -1$, with the multiplication rules $ij = k = -ki$, $jk = i = -kj$, and $ki = j = -ik$. A hypercomplex number of the form*

$$q = a + bi + cj + dk$$

is called a quaternion [28]. The elements 1, i, j, and k are called the basis or characteristic elements of the quaternion.

Definition 6. *Let $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in \mathbb{R}$. For two quaternions*

$$a = a_1 + a_2 i + a_3 j + a_4 k, \quad b = b_1 + b_2 i + b_3 j + b_4 k,$$

the addition, subtraction, and multiplication operations in the quaternion algebra are defined as follows:

$$\begin{aligned} a + b &= (a_1 + a_2 i + a_3 j + a_4 k) + (b_1 + b_2 i + b_3 j + b_4 k) \\ &= (a_1 + b_1) + (a_2 + b_2)i + (a_3 + b_3)j + (a_4 + b_4)k \end{aligned}$$

$$\begin{aligned} a - b &= (a_1 + a_2 i + a_3 j + a_4 k) - (b_1 + b_2 i + b_3 j + b_4 k) \\ &= (a_1 - b_1) + (a_2 - b_2)i + (a_3 - b_3)j + (a_4 - b_4)k \end{aligned}$$

$$\begin{aligned} ab &= (a_1 + a_2 i + a_3 j + a_4 k)(b_1 + b_2 i + b_3 j + b_4 k) \\ &= a_1(b_1 + b_2 i + b_3 j + b_4 k) + a_2 i(b_1 + b_2 i + b_3 j + b_4 k) \\ &\quad + a_3 j(b_1 + b_2 i + b_3 j + b_4 k) + a_4 k(b_1 + b_2 i + b_3 j + b_4 k) \\ &= (a_1 b_1 + a_1 b_2 i + a_1 b_3 j + a_1 b_4 k) + (a_2 b_1 i + a_2 b_2 ii + a_2 b_3 ij + a_2 b_4 ik) \\ &\quad + (a_3 b_1 j + a_3 b_2 ji + a_3 b_3 jj + a_3 b_4 jk) + (a_4 b_1 k + a_4 b_2 ki + a_4 b_3 kj + a_4 b_4 kk) \\ &= (a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4) + i(a_1 b_2 + a_2 b_1 + a_3 b_4 - a_4 b_3) \\ &\quad + j(a_1 b_3 + a_3 b_1 - a_2 b_4 + a_4 b_2) + k(a_1 b_4 + a_4 b_1 + a_2 b_3 - a_3 b_2). \end{aligned}$$

The quaternion

$$\bar{q} = a - bi - cj - dk$$

is called the conjugate of the quaternion $q = a + bi + cj + dk$. The norm of q is defined by $N(q) = \|q\| = \sqrt{q\bar{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}$, [29–31].

Horadam, in [32], defined Fibonacci and Lucas quaternions as

$$Q_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k, \quad K_n = L_n + L_{n+1}i + L_{n+2}j + L_{n+3}k,$$

for all $n \in \mathbb{Z}$, where F_n and L_n are Fibonacci and Lucas numbers, respectively.

In this section, we define the (p, q) generalizations of the Fibonacci and Lucas quaternions and examine their properties.

Definition 7. *Let U_n be the n-th generalized Fibonacci number. For all $n \in \mathbb{N}$, quaternions of the form*

$$Q_n = U_n + U_{n+1}i + U_{n+2}j + U_{n+3}k$$

are called generalized Fibonacci quaternions. Let V_n be the n-th generalized Lucas number. For all $n \in \mathbb{N}$, quaternions of the form

$$K_n = V_n + V_{n+1}i + V_{n+2}j + V_{n+3}k$$

are called generalized Lucas quaternions [28, 29, 31, 33].

Binet Fomulas for generalized Fibonacci and Lucas quaternions will be given in Theorem 7.

Theorem 7. *([33]) Let $\hat{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$ and $\hat{\beta} = 1 + \beta i + \beta^2 j + \beta^3 k$, then the generalized Fibonacci and Lucas quaternions are given by*

$$Q_n = \frac{\alpha^n \hat{\alpha} - \beta^n \hat{\beta}}{\alpha - \beta} \quad \text{and} \quad K_n = \alpha^n \hat{\alpha} + \beta^n \hat{\beta},$$

for all $n \in \mathbb{N}$.

Proof. Using (6) in the expression $Q_n = U_n + U_{n+1}i + U_{n+2}j + U_{n+3}k$, we obtain

$$\begin{aligned} Q_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} i + \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} j + \frac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta} k \\ &= \frac{(\alpha^n + \alpha^{n+1}i + \alpha^{n+2}j + \alpha^{n+3}k) - (\beta^n + \beta^{n+1}i + \beta^{n+2}j + \beta^{n+3}k)}{\alpha - \beta} \\ &= \frac{\alpha^n(1 + \alpha i + \alpha^2 j + \alpha^3 k) - \beta^n(1 + \beta i + \beta^2 j + \beta^3 k)}{\alpha - \beta} \\ &= \frac{\alpha^n \hat{\alpha} - \beta^n \hat{\beta}}{\alpha - \beta}. \end{aligned}$$

Similarly, using (6) in the equation $K_n = V_n + V_{n+1}i + V_{n+2}j + V_{n+3}k$, we get

$$\begin{aligned} K_n &= (\alpha^n + \beta^n) + (\alpha^{n+1} + \beta^{n+1})i + (\alpha^{n+2} + \beta^{n+2})j + (\alpha^{n+3} + \beta^{n+3})k \\ &= \alpha^n(1 + \alpha i + \alpha^2 j + \alpha^3 k) + \beta^n(1 + \beta i + \beta^2 j + \beta^3 k) \end{aligned}$$

$$= \alpha^n \hat{\alpha} + \beta^n \hat{\beta}.$$

The generalized negative-indexed Fibonacci and Lucas quaternions were defined by Iakin in [31] as:

$$Q_{-n} = U_{-n} + U_{-n+1}i + U_{-n+2}j + U_{-n+3}k \quad \textit{and} \quad K_{-n} = V_{-n} + V_{-n+1}i + V_{-n+2}j + V_{-n+3}k$$

for all $n \in \mathbb{N}$. For example,

$$Q_{-1} = U_{-1} + U_0 i + U_1 j + U_2 k = \frac{1}{q} + 0i + j + pk, K_{-1} = V_{-1} + V_0 i + V_1 j + V_2 k = -\frac{p}{q} + 2i + pj + (p^2 + 2q)k.$$

According to Iyer [34], the following properties involving the Fibonacci and Lucas quaternions hold. Here, Q_n and K_n are the Fibonacci and Lucas quaternions, not their generalizations.

$$\begin{aligned}
Q_n - iQ_{n+1} - jQ_{n+2} - kQ_{n+3} &= L_{n+3}, \\
Q_{n-1}^2 + Q_n^2 &= 2Q_{2n-1} - 3L_{2n+2}, \\
Q_{n+1}^2 - Q_{n-1}^2 &= Q_n K_n = (2Q_{2n} - 3L_{2n+3}) + 2(-1)^{n+1}(Q_0 - 3k), \\
Q_{n-2}Q_{n-1} + Q_n Q_{n+1} &= 6F_n Q_{n-1} - 9F_{2n+2} + 2(-1)^{n+1}(Q_{-1} - 3k), \\
Q_{n-1}Q_{n+3} - Q_{n+1}^2 &= (-1)^n[2 + 4i + 3j + k], \\
Q_{n-1}Q_{n+1} - Q_{n-2}Q_{n+2} &= (-1)^n[2K_0 - k] + 4(-1)^{n+1}[Q_0 - 2k], \\
Q_{n-3}Q_{n-2} + Q_n Q_{n+1} &= 4Q_{2n-2} - 6L_{2n+1}, \\
Q_{n-1}^2 + Q_{n+1}^2 &= 6F_{n+1}Q_{n-1} - 9F_{2n+3} + 2(-1)^n Q_{-2}, \\
Q_{n+r} + (-1)^r Q_{n-r} &= Q_n L_r, \\
Q_{n+1-r}Q_{n+1+r} - Q_{n+1}^2 &= (-1)^{n-r}[F_r^2 K_0 + F_{2r}(Q_0 - 3r)], \\
Q_{n+r}L_{n+r} &= Q_{2n+2r} + (-1)^{n+r}Q_0, \\
Q_{n-r}L_{n-r} &= Q_{2n-2r} + (-1)^{n+r}Q_0, \\
Q_{n+r}L_{n+r} + Q_{n-r}L_{n-r} &= Q_{2n}L_{2r} + 2(-1)^{n+r}Q_0, \\
Q_{n+r}L_{n+r} - Q_{n-r}L_{n-r} &= F_{2r}K_{2n}, \\
Q_{n+r}L_{n-r} &= Q_{2n} + (-1)^{n-r}Q_{2r}.
\end{aligned}$$

Before presenting the Catalan's identity for generalized Fibonacci and Lucas quaternions, we will first prove some preliminary lemmas.

Lemma 1. *Let* $\hat{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$, $\hat{\beta} = 1 + \beta i + \beta^2 j + \beta^3 k$, *and define*

$$A = K_0 - (1 - q)(1 + q^2), \quad B = (-q)i + (-p)j + k,$$

then the following identities hold

$$\hat{\alpha}\hat{\beta} = A + qB\sqrt{\Delta}, \tag{10}$$

$$\hat{\beta}\hat{\alpha} = A - qB\sqrt{\Delta}. \tag{11}$$

Proof. Using quaternion multiplication and the identity $\alpha\beta = -q$, we compute

$$\hat{\alpha}\hat{\beta} = (1 + \alpha i + \alpha^2 j + \alpha^3 k)(1 + \beta i + \beta^2 j + \beta^3 k)$$
$$= [2+(\alpha+\beta)i+(\alpha^2+\beta^2)j+(\alpha^3+\beta^3)k]-[1+\alpha\beta+\alpha^2\beta^2+\alpha^3\beta^3]-\alpha\beta(\alpha-\beta)[\alpha\beta i-(\alpha+\beta)j+k]$$
$$= K_0 - (1-q)(1+q^2) + q\sqrt{\Delta}[(-q)i + (-p)j + k] = A + qB\sqrt{\Delta},$$

which completes the proof. Similarly, it can be shown that $\hat{\beta}\hat{\alpha} = A - qB\sqrt{\Delta}$.

Lemma 2. *Let $\hat{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$, $\hat{\beta} = 1 + \beta i + \beta^2 j + \beta^3 k$, and $r \in \mathbb{N}$. Then the following identities hold:*

$$\frac{\hat{\alpha}\hat{\beta}\beta^r - \hat{\beta}\hat{\alpha}\alpha^r}{\alpha - \beta} = qBV_r - AU_r, \tag{12}$$
$$\hat{\alpha}\hat{\beta}\beta^r + \hat{\beta}\hat{\alpha}\alpha^r = AV_r - q\Delta BU_r, \tag{13}$$

where $A = K_0 - (1-q)(1+q^2)$ and $B = (-q)i + (-p)j + k$.

Proof. Using equations (10) and (11), we get

$$\frac{\hat{\alpha}\hat{\beta}\beta^r - \hat{\beta}\hat{\alpha}\alpha^r}{\alpha - \beta} = \frac{(A + qB\sqrt{\Delta})\beta^r - (A - qB\sqrt{\Delta})\alpha^r}{\alpha - \beta}$$
$$= \frac{qB\sqrt{\Delta}(\alpha^r + \beta^r) - A(\alpha^r - \beta^r)}{\alpha - \beta} = qBV_r - AU_r,$$

which proves the equation (12).

Now considering the equations (10) and (11) we have

$$\hat{\alpha}\hat{\beta}\beta^r + \hat{\beta}\hat{\alpha}\alpha^r = (A + qB\sqrt{\Delta})\beta^r + (A - qB\sqrt{\Delta})\alpha^r$$
$$= A(\alpha^r + \beta^r) - qB\sqrt{\Delta}(\alpha^r - \beta^r) = AV_r - q\Delta BU_r,$$

which proves the equation (13).

Lemma 3. *Let $\hat{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 k$, $\hat{\beta} = 1 + \beta i + \beta^2 j + \beta^3 k$. Then it follows that*

$$\hat{\alpha}\beta - \hat{\beta}\alpha = (-q)\sqrt{\Delta}Q_{-r}, \tag{14}$$
$$\hat{\alpha}\beta^r - \hat{\beta}\alpha^r = \sqrt{\Delta}(-q)^r Q_{-r}, \tag{15}$$

for all $r \in \mathbb{Z}$.

Proof. Using the Binet formula, we achive that

$$\hat{\alpha}\beta - \hat{\beta}\alpha = \alpha\beta\left(\frac{\hat{\alpha}}{\alpha} - \frac{\hat{\beta}}{\beta}\right) = (-q)\sqrt{\Delta}Q_{-r},$$

$$\hat{\alpha}\beta^r - \hat{\beta}\alpha^r = (\alpha\beta)^r\left(\frac{\hat{\alpha}}{\alpha^r} - \frac{\hat{\beta}}{\beta^r}\right) = \sqrt{\Delta}(-q)^r Q_{-r}.$$

Now we can prove the generalized product differences of Fibonacci and Lucas quaternions in Theorem 8 and Theorem 9.

Theorem 8. *For all $n, r \in \mathbb{Z}$, it follows that*

$$Q_{n-r}Q_{n+r} - Q_n^2 = (-q)^{n-r}U_r[qBV_r - AU_r],$$

where $A = K_0 - (1-q)(1+q^2)$ and $B = (-q)i + (-p)j + k$.

Proof. Using the Binet formula and (12), we obtain

$$\begin{aligned}
Q_{n-r}Q_{n+r} - Q_n^2 &= \left(\frac{\hat{\alpha}\alpha^{n-r} - \hat{\beta}\beta^{n-r}}{\alpha-\beta}\right)\left(\frac{\hat{\alpha}\alpha^{n+r} - \hat{\beta}\beta^{n+r}}{\alpha-\beta}\right) - \left(\frac{\hat{\alpha}\alpha^{n} - \hat{\beta}\beta^{n}}{\alpha-\beta}\right)^2 \\
&= \frac{\hat{\alpha}^2\alpha^{2n} + \hat{\beta}^2\beta^{2n} - \hat{\alpha}\hat{\beta}\alpha^{n-r}\beta^{n+r} - \hat{\beta}\hat{\alpha}\alpha^{n+r}\beta^{n-r}}{(\alpha-\beta)^2} - \frac{\hat{\alpha}^2\alpha^{2n} + \hat{\beta}^2\beta^{2n} - 2\hat{\alpha}\hat{\beta}\alpha^n\beta^n}{(\alpha-\beta)^2} \\
&= \frac{-\hat{\alpha}\hat{\beta}\alpha^{n-r}\beta^{n+r} - \hat{\beta}\hat{\alpha}\alpha^{n+r}\beta^{n-r} + 2\hat{\alpha}\hat{\beta}\alpha^n\beta^n}{(\alpha-\beta)^2} \\
&= \frac{(\alpha\beta)^n}{(\alpha-\beta)^2}\left[\hat{\alpha}\hat{\beta}\left(\frac{\alpha^r - \beta^r}{\alpha^r}\right) + \hat{\beta}\hat{\alpha}\left(\frac{\beta^r - \alpha^r}{\beta^r}\right)\right] \\
&= \frac{(\alpha\beta)^n(\alpha^r - \beta^r)}{(\alpha-\beta)^2}\left(\frac{\hat{\alpha}\hat{\beta}}{\alpha^r} - \frac{\hat{\beta}\hat{\alpha}}{\beta^r}\right) \\
&= \frac{(\alpha\beta)^n(\alpha^r - \beta^r)}{(\alpha-\beta)^2}\left(\frac{\hat{\alpha}\hat{\beta}\beta^r - \hat{\beta}\hat{\alpha}\beta^r}{(\alpha\beta)^r}\right) = (\alpha\beta)^{n-r}\cdot\frac{(\alpha^r - \beta^r)}{(\alpha-\beta)}\cdot\frac{\hat{\alpha}\hat{\beta}\beta^r - \hat{\beta}\hat{\alpha}\beta^r}{(\alpha-\beta)} \\
&= (\alpha\beta)^{n-r}\cdot\frac{\alpha^r - \beta^r}{\alpha-\beta}\cdot\left(\frac{\hat{\alpha}\hat{\beta}\beta^r - \hat{\beta}\hat{\alpha}\beta^r}{\alpha-\beta}\right) \\
&= (\alpha\beta)^{n-r}U_r\cdot(qBV_r - AU_r) = (-q)^{n-r}U_r[qBV_r - AU_r],
\end{aligned}$$

as desired.

Theorem 9. *For all $n, r \in \mathbb{Z}$, it follows that*

$$K_{n-r}K_{n+r} - K_n^2 = -(-q)^{n-r}U_r\Delta[qBV_r - AU_r],$$

where $A = K_0 - (1-q)(1+q^2)$ and $B = (-q)i + (-p)j + k$.

Proof. Using the Binet formula and equation (14), we have:

$$\begin{aligned}
K_{n-r}K_{n+r} - K_n^2 &= (\hat{\alpha}\alpha^{n-r} + \hat{\beta}\beta^{n-r})(\hat{\alpha}\alpha^{n+r} + \hat{\beta}\beta^{n+r}) - (\hat{\alpha}\alpha^n + \hat{\beta}\beta^n)^2 \\
&= \hat{\alpha}^2\alpha^{2n} + \hat{\beta}^2\beta^{2n} + \hat{\alpha}\hat{\beta}\alpha^{n-r}\beta^{n+r} + \hat{\beta}\hat{\alpha}\alpha^{n+r}\beta^{n-r} - \left(\hat{\alpha}^2\alpha^{2n} + \hat{\beta}^2\beta^{2n} + 2\hat{\alpha}\hat{\beta}\alpha^n\beta^n\right) \\
&= -\hat{\alpha}\hat{\beta}\alpha^{n-r}\beta^{n+r} - \hat{\beta}\hat{\alpha}\alpha^{n+r}\beta^{n-r} + 2\hat{\alpha}\hat{\beta}\alpha^n\beta^n \\
&= -(-q)^{n-r}U_r\Delta[qBV_r - AU_r].
\end{aligned}$$

As a consequence of Theorem 8 and Theorem 9, we can give Corollary 3.

Corollary 3. *For all $n, r \in \mathbb{Z}$, it follows that*

$$K_{n-r}K_{n+r} - K_n^2 = -\Delta[Q_{n-r}Q_{n+r} - Q_n^2].$$

Theorem 10. *For all $n, r \in \mathbb{Z}$, it follows that*

$$Q_{n+r}^2 - q^{2r}Q_{n-r}^2 = U_{2r}\left[2Q_{2n} - (U_{2n} + U_{2n+2} + U_{2n+4} + U_{2n+6})\right].$$

Proof.

$$Q_{n+r}^2 - q^{2r}Q_{n-r}^2 = \left(\frac{\hat{\alpha}\alpha^{n+r} - \hat{\beta}\beta^{n+r}}{\alpha - \beta}\right)^2 - (-q)^{2r}\left(\frac{\hat{\alpha}\alpha^{n-r} - \hat{\beta}\beta^{n-r}}{\alpha - \beta}\right)^2$$

$$= \frac{\hat{\alpha}\hat{\alpha}\alpha^{2n+2r} + \hat{\beta}\hat{\beta}\beta^{2n+2r} - (\alpha\beta)^{n+r}\left(\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\alpha}\right)}{(\alpha - \beta)^2} - (\alpha\beta)^{2r}\frac{\hat{\alpha}\hat{\alpha}\alpha^{2n-2r} + \hat{\beta}\hat{\beta}\beta^{2n-2r} - (\alpha\beta)^{n-r}\left(\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\alpha}\right)}{(\alpha - \beta)^2}$$

$$= \frac{\left[\hat{\alpha}\hat{\alpha}\alpha^{2n+2r} + \hat{\beta}\hat{\beta}\beta^{2n+2r} - (\alpha\beta)^{n+r}\left(\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\alpha}\right)\right] - (\alpha\beta)^{2r}\left[\hat{\alpha}\hat{\alpha}\alpha^{2n-2r} + \hat{\beta}\hat{\beta}\beta^{2n-2r} - (\alpha\beta)^{n-r}\left(\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\alpha}\right)\right]}{(\alpha - \beta)^2}$$

$$= \frac{\hat{\alpha}\hat{\alpha}\alpha^{2n+2r} + \hat{\beta}\hat{\beta}\beta^{2n+2r} - \hat{\alpha}\hat{\alpha}\alpha^{2n}\beta^{2r} - \hat{\beta}\hat{\beta}\alpha^{2r}\beta^{2n}}{(\alpha - \beta)^2} = \frac{\hat{\alpha}\hat{\alpha}\left(\alpha^{2n+2r} - \alpha^{2n}\beta^{2r}\right) + \hat{\beta}\hat{\beta}\left(\beta^{2n+2r} - \alpha^{2r}\beta^{2n}\right)}{(\alpha - \beta)^2}$$

$$= \frac{\hat{\alpha}\hat{\alpha}\alpha^{2n}\left(\alpha^{2r} - \beta^{2r}\right) + \hat{\beta}\hat{\beta}\beta^{2n}\left(\beta^{2r} - \alpha^{2r}\right)}{(\alpha - \beta)^2} = \frac{\hat{\alpha}\hat{\alpha}\alpha^{2n}\left(\alpha^{2r} - \beta^{2r}\right) - \hat{\beta}\hat{\beta}\beta^{2n}\left(\alpha^{2r} - \beta^{2r}\right)}{(\alpha - \beta)^2}$$

$$= \frac{\alpha^{2r} - \beta^{2r}}{\alpha - \beta}\frac{\hat{\alpha}\hat{\alpha}\alpha^{2n} - \hat{\beta}\hat{\beta}\beta^{2n}}{\alpha - \beta} = U_{2r}\frac{\hat{\alpha}\hat{\alpha}\alpha^{2n} - \hat{\beta}\hat{\beta}\beta^{2n}}{\alpha - \beta}$$

$$= U_{2r}\frac{\alpha^{2n}\left(2\hat{\alpha} - 1 - \alpha^2 - \alpha^4 - \alpha^6\right) - \beta^{2n}\left(2\hat{\beta} - 1 - \beta^2 - \beta^4 - \beta^6\right)}{\alpha - \beta}$$

$$= U_{2r}\frac{2\left(\hat{\alpha}\alpha^{2n} - \hat{\beta}\beta^{2n}\right) - \left(\alpha^{2n} - \beta^{2n}\right) - \left(\alpha^{2n+2} - \beta^{2n+2}\right) - \left(\alpha^{2n+4} - \beta^{2n+4}\right) - \left(\alpha^{2n+6} - \beta^{2n+6}\right)}{\alpha - \beta}$$

$$= U_{2r}\left[2\frac{\hat{\alpha}\alpha^{2n} - \hat{\beta}\beta^{2n}}{\alpha - \beta} - \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} - \frac{\alpha^{2n+2} - \beta^{2n+2}}{\alpha - \beta} - \frac{\alpha^{2n+4} - \beta^{2n+4}}{\alpha - \beta} - \frac{\alpha^{2n+6} - \beta^{2n+6}}{\alpha - \beta}\right]$$

$$= U_{2r}\left[2Q_{2n} - (U_{2n} + U_{2n+2} + U_{2n+4} + U_{2n+6})\right].$$

Theorem 11. *For all $n, r \in \mathbb{Z}$, it follows that*

$$K_{n+r}^2 - q^{2r} K_{n-r}^2 = \Delta U_{2r} \left[2Q_{2n} - (U_{2n} + U_{2n+2} + U_{2n+4} + U_{2n+6})\right].$$

Proof.

$$K_{n+r}^2 - q^{2r} K_{n-r}^2 = \left(\hat{\alpha}\alpha^{n+r} + \hat{\beta}\beta^{n+r}\right)^2 - (-q)^{2r}\left(\hat{\alpha}\alpha^{n-r} + \hat{\beta}\beta^{n-r}\right)^2$$

$$= \left[\hat{\alpha}\hat{\alpha}\alpha^{2n+2r} + \hat{\beta}\hat{\beta}\beta^{2n+2r} + (\alpha\beta)^{n+r}\left(\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\alpha}\right)\right] - (\alpha\beta)^{2r}\left[\hat{\alpha}\hat{\alpha}\alpha^{2n-2r} + \hat{\beta}\hat{\beta}\beta^{2n-2r} + (\alpha\beta)^{n-r}\left(\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\alpha}\right)\right]$$

$$= \hat{\alpha}\hat{\alpha}\alpha^{2n+2r} + \hat{\beta}\hat{\beta}\beta^{2n+2r} - \hat{\alpha}\hat{\alpha}\alpha^{2n}\beta^{2r} - \hat{\beta}\hat{\beta}\alpha^{2r}\beta^{2n} = \hat{\alpha}\hat{\alpha}\left(\alpha^{2n+2r} - \alpha^{2n}\beta^{2r}\right) + \hat{\beta}\hat{\beta}\left(\beta^{2n+2r} - \alpha^{2r}\beta^{2n}\right)$$

$$= \hat{\alpha}\hat{\alpha}\alpha^{2n}\left(\alpha^{2r} - \beta^{2r}\right) + \hat{\beta}\hat{\beta}\beta^{2n}\left(\beta^{2r} - \alpha^{2r}\right) = \hat{\alpha}\hat{\alpha}\alpha^{2n}\left(\alpha^{2r} - \beta^{2r}\right) - \hat{\beta}\hat{\beta}\beta^{2n}\left(\alpha^{2r} - \beta^{2r}\right)$$

$$= \left[\alpha^{2r} - \beta^{2r}\right]\left[\hat{\alpha}\hat{\alpha}\alpha^{2n} - \hat{\beta}\hat{\beta}\beta^{2n}\right]$$

$$= \left[\alpha^{2r} - \beta^{2r}\right]\left[\alpha^{2n}\left(2\hat{\alpha} - 1 - \alpha^2 - \alpha^4 - \alpha^6\right) - \beta^{2n}\left(2\hat{\beta} - 1 - \beta^2 - \beta^4 - \beta^6\right)\right]$$

$$= \left[\alpha^{2r} - \beta^{2r}\right]\left[2\left(\hat{\alpha}\alpha^{2n} - \hat{\beta}\beta^{2n}\right) - \left(\alpha^{2n} - \beta^{2n}\right) - \left(\alpha^{2n+2} - \beta^{2n+2}\right) - \left(\alpha^{2n+4} - \beta^{2n+4}\right) - \left(\alpha^{2n+6} - \beta^{2n+6}\right)\right]$$

$$= (\alpha - \beta)^2\left[\frac{\alpha^{2r} - \beta^{2r}}{\alpha - \beta}\right]\left[2\frac{\hat{\alpha}\alpha^{2n} - \hat{\beta}\beta^{2n}}{\alpha - \beta} - \frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} - \frac{\alpha^{2n+2} - \beta^{2n+2}}{\alpha - \beta} - \frac{\alpha^{2n+4} - \beta^{2n+4}}{\alpha - \beta} - \frac{\alpha^{2n+6} - \beta^{2n+6}}{\alpha - \beta}\right]$$

$$= \Delta U_{2r}\left[2Q_{2n} - (U_{2n} + U_{2n+2} + U_{2n+4} + U_{2n+6})\right]$$

As a consequence of Theorem 10 and Theorem 11, we can give Corollary 4.

Corollary 4. *For all $n, r \in \mathbb{Z}$, it follows that* $K_{n+r}^2 - q^{2r} K_{n-r}^2 = \Delta\left[Q_{n+r}^2 - q^{2r} Q_{n-r}^2\right].$

Theorem 12. *For all $n, r \in \mathbb{Z}$, it follows that*

$$K_n^2 - \Delta Q_n^2 = 4(-q)^n A,$$

where $A = K_0 - (1-q)(1+q^2)$ *and* $B = (-q)i + (-p)j + k$.

Proof. $K_n^2 - \Delta Q_n^2 = \left(\hat{\alpha}\alpha^n + \hat{\beta}\beta^n\right)^2 - \Delta\left(\frac{\hat{\alpha}\alpha^n - \hat{\beta}\beta^n}{\alpha-\beta}\right)^2$

$$= \left[\hat{\alpha}\hat{\alpha}\alpha^{2n} + \hat{\beta}\hat{\beta}\beta^{2n} + (\alpha\beta)^n\left(\hat{\alpha}\hat{\beta}+\hat{\beta}\hat{\alpha}\right)\right] - \Delta\left[\frac{\hat{\alpha}\hat{\alpha}\alpha^{2n} + \hat{\beta}\hat{\beta}\beta^{2n} - (\alpha\beta)^n\left(\hat{\alpha}\hat{\beta}+\hat{\beta}\hat{\alpha}\right)}{(\alpha-\beta)^2}\right]$$
$$= \left[\hat{\alpha}\hat{\alpha}\alpha^{2n} + \hat{\beta}\hat{\beta}\beta^{2n} + (\alpha\beta)^n\left(\hat{\alpha}\hat{\beta}+\hat{\beta}\hat{\alpha}\right)\right] - \left[\hat{\alpha}\hat{\alpha}\alpha^{2n} + \hat{\beta}\hat{\beta}\beta^{2n} - (\alpha\beta)^n\left(\hat{\alpha}\hat{\beta}+\hat{\beta}\hat{\alpha}\right)\right]$$
$$= 2(\alpha\beta)^n(2A) = 4(-q)^n A.$$

Theorem 13. *For all $m, n \in \mathbb{Z}$, it follows that*

$$K_{2m}K_{2n} - \Delta Q_{m+n}^2 = (-q)^{2n} V_{m-n}\left[AV_{m-n} + Bq\Delta U_{m-n}\right],$$

where $A = K_0 - (1-q)(1+q^2)$ and $B = (-q)i + (-p)j + k$.

Proof. $K_{2m}K_{2n} - \Delta Q_{m+n}^2 = \left(\hat{\alpha}\alpha^{2m} + \hat{\beta}\beta^{2m}\right)\left(\hat{\alpha}\alpha^{2n} + \hat{\beta}\beta^{2n}\right) - \Delta\left(\frac{\hat{\alpha}\alpha^{m+n} - \hat{\beta}\beta^{m+n}}{\alpha-\beta}\right)^2$

$$= \left[\hat{\alpha}\hat{\alpha}\alpha^{2m+2n} + \hat{\beta}\hat{\beta}\beta^{2m+2n} + \hat{\alpha}\hat{\beta}\alpha^{2m}\beta^{2n} + \hat{\beta}\hat{\alpha}\alpha^{2n}\beta^{2m}\right] - \left[\hat{\alpha}\hat{\alpha}\alpha^{2m+2n} + \hat{\beta}\hat{\beta}\beta^{2m+2n} - (\alpha\beta)^{m+n}\left(\hat{\alpha}\hat{\beta}+\hat{\beta}\hat{\alpha}\right)\right]$$

$$= \hat{\alpha}\hat{\beta}\left(\alpha^{2m}\beta^{2n} + \alpha^{m+n}\beta^{m+n}\right) + \hat{\beta}\hat{\alpha}\left(\alpha^{2n}\beta^{2m} + \alpha^{m+n}\beta^{m+n}\right)$$
$$= \alpha^{m+n}\beta^{2n}\hat{\alpha}\hat{\beta}\left(\alpha^{m-n} + \beta^{m-n}\right) + \alpha^{2n}\beta^{m+n}\hat{\beta}\hat{\alpha}\left(\alpha^{m-n} + \beta^{m-n}\right)$$
$$= \left(\alpha^{m-n} + \beta^{m-n}\right)\left[\alpha^{m+n}\beta^{2n}\hat{\alpha}\hat{\beta} + \alpha^{2n}\beta^{m+n}\hat{\beta}\hat{\alpha}\right]$$

$$= (\alpha\beta)^{2n}\left(\alpha^{m-n} + \beta^{m-n}\right)\left[\alpha^{m-n}\hat{\alpha}\hat{\beta} + \beta^{m-n}\hat{\beta}\hat{\alpha}\right]$$
$$= (\alpha\beta)^{2n}\left(\alpha^{m-n} + \beta^{m-n}\right)\left[\alpha^{m-n}\left(A + Bq\sqrt{\Delta}\right) + \beta^{m-n}\left(A - Bq\sqrt{\Delta}\right)\right]$$
$$= (\alpha\beta)^{2n}\left(\alpha^{m-n} + \beta^{m-n}\right)\left[A\left(\alpha^{m-n} + \beta^{m-n}\right) + Bq\sqrt{\Delta}\left(\alpha^{m-n} - \beta^{m-n}\right)\right]$$
$$= (-q)^{2n} V_{m-n}\left[AV_{m-n} + Bq\Delta U_{m-n}\right].$$

Theorem 14. *For all $m, n \in \mathbb{Z}$, it follows that*

$$K_{2m}K_{2n} - K_{m+n}^2 = (-q)^{2n}\Delta U_{m-n}\left[AU_{m-n} + BqV_{m-n}\right],$$

where $A = K_0 - (1-q)(1+q^2)$ and $B = (-q)i + (-p)j + k$.

Proof. $K_{2m}K_{2n} - K_{m+n}^2 = \left(\hat{\alpha}\alpha^{2m} + \hat{\beta}\beta^{2m}\right)\left(\hat{\alpha}\alpha^{2n} + \hat{\beta}\beta^{2n}\right) - \left(\hat{\alpha}\alpha^{m+n} + \hat{\beta}\beta^{m+n}\right)^2$

$$= \left[\hat{\alpha}\hat{\alpha}\alpha^{2m+2n} + \hat{\beta}\hat{\beta}\beta^{2m+2n} + \hat{\alpha}\hat{\beta}\alpha^{2m}\beta^{2n} + \hat{\beta}\hat{\alpha}\alpha^{2n}\beta^{2m}\right] - \left[\hat{\alpha}\hat{\alpha}\alpha^{2m+2n} + \hat{\beta}\hat{\beta}\beta^{2m+2n} + (\alpha\beta)^{m+n}\left(\hat{\alpha}\hat{\beta}+\hat{\beta}\hat{\alpha}\right)\right]$$

$$= \hat{\alpha}\hat{\beta}\left(\alpha^{2m}\beta^{2n} - \alpha^{m+n}\beta^{m+n}\right) + \hat{\beta}\hat{\alpha}\left(\alpha^{2n}\beta^{2m} - \alpha^{m+n}\beta^{m+n}\right)$$

$$
\begin{aligned}
&= \alpha^{m+n}\beta^{2n}\hat{\alpha}\hat{\beta}\left(\alpha^{m-n}-\beta^{m-n}\right)+\alpha^{2n}\beta^{m+n}\hat{\beta}\hat{\alpha}\left(\beta^{m-n}-\alpha^{m-n}\right)\\
&= \left(\alpha^{m-n}-\beta^{m-n}\right)\left[\alpha^{m+n}\beta^{2n}\hat{\alpha}\hat{\beta}-\alpha^{2n}\beta^{m+n}\hat{\beta}\hat{\alpha}\right]\\
&= (\alpha\beta)^{2n}\left(\alpha^{m-n}-\beta^{m-n}\right)\left[\alpha^{m-n}\hat{\alpha}\hat{\beta}-\beta^{m-n}\hat{\beta}\hat{\alpha}\right]\\
&= (\alpha\beta)^{2n}\left(\alpha^{m-n}-\beta^{m-n}\right)\left[\alpha^{m-n}\left(A+Bq\sqrt{\Delta}\right)-\beta^{m-n}\left(A-Bq\sqrt{\Delta}\right)\right]\\
&= (\alpha\beta)^{2n}\left(\alpha^{m-n}-\beta^{m-n}\right)\left[A\left(\alpha^{m-n}-\beta^{m-n}\right)+Bq\sqrt{\Delta}\left(\alpha^{m-n}+\beta^{m-n}\right)\right]\\
&= (-q)^{2n}\sqrt{\Delta}U_{m-n}\left[A\sqrt{\Delta}U_{m-n}+Bq\sqrt{\Delta}V_{m-n}\right]\\
&= (-q)^{2n}\Delta U_{m-n}\left[AU_{m-n}+BqV_{m-n}\right].
\end{aligned}
$$

Let us generalize Everman's product difference of Fibonacci identities to generalized Fibonacci quaternions in Theorem 15.

Theorem 15. *For all n, k and $h \in \mathbb{Z}$, it follows that*

$$
Q_{n+h}Q_{n+k}-Q_nQ_{n+k+h}=(-q)^n U_h\left[AU_k-qBV_k\right],
$$

where $A = K_0-(1-q)(1+q^2)$ and $B=(-q)i+(-p)j+k$.

Proof.

$$
Q_{n+h}Q_{n+k}-Q_nQ_{n+k+h}=\frac{\alpha^{n+h}\hat{\alpha}-\beta^{n+h}\hat{\beta}}{\alpha-\beta}\frac{\alpha^{n+k}\hat{\alpha}-\beta^{n+k}\hat{\beta}}{\alpha-\beta}-\frac{\alpha^{n}\hat{\alpha}-\beta^{n}\hat{\beta}}{\alpha-\beta}\frac{\alpha^{n+h+k}\hat{\alpha}-\beta^{n+h+k}\hat{\beta}}{\alpha-\beta}
$$

$$
=\frac{\left[\alpha^{n+h}\hat{\alpha}-\beta^{n+h}\hat{\beta}\right]\left[\alpha^{n+k}\hat{\alpha}-\beta^{n+k}\hat{\beta}\right]-\left[\alpha^{n}\hat{\alpha}-\beta^{n}\hat{\beta}\right]\left[\alpha^{n+h+k}\hat{\alpha}-\beta^{n+h+k}\hat{\beta}\right]}{(\alpha-\beta)^2}
$$

$$
=\frac{\left[\alpha^{2n+h+k}\hat{\alpha}\hat{\alpha}-\alpha^{n+h}\beta^{n+k}\hat{\alpha}\hat{\beta}-\alpha^{n+k}\beta^{n+h}\hat{\beta}\hat{\alpha}+\beta^{2n+h+k}\hat{\beta}\hat{\beta}\right]-\left[\alpha^{2n+h+k}\hat{\alpha}\hat{\alpha}-\alpha^{n}\beta^{n+h+k}\hat{\alpha}\hat{\beta}-\alpha^{n+h+k}\beta^{n}\hat{\beta}\hat{\alpha}+\beta^{2n+h+k}\hat{\beta}\hat{\beta}\right]}{(\alpha-\beta)^2}
$$

$$
=\frac{\left[-\alpha^{n+h}\beta^{n+k}\hat{\alpha}\hat{\beta}-\alpha^{n+k}\beta^{n+h}\hat{\beta}\hat{\alpha}\right]-\left[-\alpha^{n}\beta^{n+h+k}\hat{\alpha}\hat{\beta}-\alpha^{n+h+k}\beta^{n}\hat{\beta}\hat{\alpha}\right]}{(\alpha-\beta)^2}
$$

$$
=\frac{-\alpha^{n+h}\beta^{n+k}\hat{\alpha}\hat{\beta}-\alpha^{n+k}\beta^{n+h}\hat{\beta}\hat{\alpha}+\alpha^{n}\beta^{n+h+k}\hat{\alpha}\hat{\beta}+\alpha^{n+h+k}\beta^{n}\hat{\beta}\hat{\alpha}}{(\alpha-\beta)^2}
$$

$$
=\frac{\alpha^n\beta^n\left[\hat{\alpha}\hat{\beta}\left(\beta^{h+k}-\alpha^h\beta^k\right)+\hat{\beta}\hat{\alpha}\left(\alpha^{h+k}-\alpha^k\beta^h\right)\right]}{(\alpha-\beta)^2}=\frac{\alpha^n\beta^n\left[-\beta^k\hat{\alpha}\hat{\beta}\left(\alpha^h-\beta^h\right)+\alpha^k\hat{\beta}\hat{\alpha}\left(\alpha^h-\beta^h\right)\right]}{(\alpha-\beta)^2}
$$

$$
=\frac{\alpha^n\beta^n\left(\alpha^h-\beta^h\right)\left[\alpha^k\hat{\beta}\hat{\alpha}-\beta^k\hat{\alpha}\hat{\beta}\right]}{(\alpha-\beta)^2}=\frac{\alpha^n\beta^n\left(\alpha^h-\beta^h\right)\left[\alpha^k\left(A-qB\sqrt{\Delta}\right)-\beta^k\left(A+qB\sqrt{\Delta}\right)\right]}{(\alpha-\beta)^2}
$$

$$= (\alpha\beta)^n \frac{\alpha^h - \beta^h}{\alpha - \beta}\left[\frac{A\left(\alpha^k - \beta^k\right)}{\alpha - \beta} - q\sqrt{\Delta}B\frac{\alpha^k + \beta^k}{\alpha - \beta}\right] = (-q)^n U_h \left[AU_k - qBV_k\right].$$

Now we will generalize Everman's product difference of Fibonacci identity to generalized Lucas quaternions in Theorem 16.

Theorem 16. *For all n, k and $h \in \mathbb{Z}$, it follows that*

$$K_{n+h}K_{n+k} - K_nK_{n+k+h} = \Delta\left(-q\right)^n U_h \left[qBV_k - AU_k\right],$$

where $A = K_0 - (1-q)(1+q^2)$ and $B = (-q)i + (-p)j + k$.

Proof.

$$K_{n+h}K_{n+k} - K_nK_{n+k+h} = \left[\alpha^{n+h}\hat{\alpha} + \beta^{n+h}\hat{\beta}\right]\left[\alpha^{n+k}\hat{\alpha} + \beta^{n+k}\hat{\beta}\right] - \left[\alpha^n\hat{\alpha} + \beta^n\hat{\beta}\right]\left[\alpha^{n+h+k}\hat{\alpha} + \beta^{n+h+k}\hat{\beta}\right]$$

$$= \left[\alpha^{2n+h+k}\hat{\alpha}\hat{\alpha} + \alpha^{n+h}\beta^{n+k}\hat{\alpha}\hat{\beta} + \alpha^{n+k}\beta^{n+h}\hat{\beta}\hat{\alpha} + \beta^{2n+h+k}\hat{\beta}\hat{\beta}\right]$$

$$- \left[\alpha^{2n+h+k}\hat{\alpha}\hat{\alpha} + \alpha^n\beta^{n+h+k}\hat{\alpha}\hat{\beta} + \alpha^{n+h+k}\beta^n\hat{\beta}\hat{\alpha} + \beta^{2n+h+k}\hat{\beta}\hat{\beta}\right]$$

$$= \left[\alpha^{n+h}\beta^{n+k}\hat{\alpha}\hat{\beta} + \alpha^{n+k}\beta^{n+h}\hat{\beta}\hat{\alpha}\right] - \left[\alpha^n\beta^{n+h+k}\hat{\alpha}\hat{\beta} + \alpha^{n+h+k}\beta^n\hat{\beta}\hat{\alpha}\right]$$

$$= \alpha^{n+h}\beta^{n+k}\hat{\alpha}\hat{\beta} + \alpha^{n+k}\beta^{n+h}\hat{\beta}\hat{\alpha} - \alpha^n\beta^{n+h+k}\hat{\alpha}\hat{\beta} - \alpha^{n+h+k}\beta^n\hat{\beta}\hat{\alpha}$$

$$= \alpha^n\beta^n\left[\hat{\alpha}\hat{\beta}\left(\alpha^h\beta^k - \beta^{h+k}\right) + \hat{\beta}\hat{\alpha}\left(\alpha^k\beta^h - \alpha^{h+k}\right)\right] = \alpha^n\beta^n\left[\hat{\alpha}\hat{\beta}\beta^k\left(\alpha^h - \beta^h\right) + \hat{\beta}\hat{\alpha}\alpha^k\left(\beta^h - \alpha^h\right)\right]$$

$$= (\alpha\beta)^n\left(\alpha^h - \beta^h\right)\left[\hat{\alpha}\hat{\beta}\beta^k - \hat{\beta}\hat{\alpha}\alpha^k\right] = (\alpha\beta)^n\left(\alpha^h - \beta^h\right)\left[\beta^k\left(A + qB\sqrt{\Delta}\right) - \alpha^k\left(A - qB\sqrt{\Delta}\right)\right]$$

$$= (\alpha\beta)^n\left(\alpha^h - \beta^h\right)\left[qB\sqrt{\Delta}\left(\alpha^k + \beta^k\right) - A\left(\alpha^k - \beta^k\right)\right]$$

$$= \Delta\left(\alpha\beta\right)^n\left(\frac{\alpha^h - \beta^h}{\alpha - \beta}\right)\left[\frac{qB\sqrt{\Delta}\left(\alpha^k + \beta^k\right) - A\left(\alpha^k - \beta^k\right)}{\alpha - \beta}\right]$$

$$= \Delta\left(-q\right)^n U_h\left[qBV_k - AU_k\right].$$

As a consequence of Theorem 15 and Theorem 16, we can give Corollary 5.

Corollary 5. *For all n, k and $h \in \mathbb{Z}$, it follows that*
$$K_{n+h}K_{n+k} - K_nK_{n+k+h} = -\Delta\left[Q_{n+h}Q_{n+k} - Q_nQ_{n+k+h}\right].$$

4. Conclusion

This paper presents new identities for both theoretical mathematical theories and various applied fields, by integrating generalized Fibonacci and Lucas sequences with quaternion structures. Known identities of Fibonacci and Lucas sequences, such as the classical Binet formulas, Cassini's, Catalan's and d'Ocagne's identities are generalized, as well as new identities, are rederived through quaternion extensions. Therefore, this paper demonstrates the interaction between number sequences and quaternions by extending some fundamental equations in the existing literature to high-dimensional hypercomplex structures. In particular, the original part of the paper is the reconstruction of the classical results of researchers such as Koshy and Everman by means of generalized Fibonacci and Lucas sequences and the systematic presentation of the quaternion forms of these equations and product differences generalizations. Generalized quaternions provide algebraic models that can be used in many fields such as computer graphics, cryptography, signal processing, modeling of physical systems and bioinformatics. In particular, quaternion-based structures provide natural solutions to transformation and rotation problems in a three-dimensional space, and the use of generalized number sequences in these structures is important both theoretically and practically. In this respect, this paper provides an extension of algebraic properties in the intersection set of quaternions and generalized number sequences.

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