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A Note on Quotient Ternary Semirings and Isomorphism Theorems

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Abstract. In this paper, we present a variety of notions of quotient ternary semirings, along with examples and remarks. Moreover, we establish various isomorphism theorems for ternary semirings.

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Key Words and Phrases: Q-ideals, congruences, k-ideals, quotient ternary semirings, isomorphism theorems

1. Introduction

The notion of ideals is fundamental in ring theory, as it plays an important role in defining quotient rings. Therefore ideal theory has been a major area of research in the study of rings. Meanwhile, the concept of congruences is crucial for defining quotient semigroups. Semirings, as algebraic structures, were definitely one of natural choices of generalizations of rings. Many properties that hold for rings can be extended to semirings. Quotient semirings were previously studied in [1–3]. The concept of ternary semirings was first introduced in 2003 [4]. Every semiring can always be turned to a ternary semiring, though a ternary semiring does not always reduce to a semiring. The notion of ternary semirings can, in some sense, be regarded as a generalization of semirings, but it goes beyond a simple generalization, because certain concepts, such as lateral ideals, have no analog in semirings. In 2011, Chaudhari and Ingale [5] introduced a partitioning ideal (shortly, Q-ideal) of a ternary semiring. Q-ideals are useful to develop the quotient structures of ternary semirings. In 2021, Sunitha et al. [6] provided the characterization of full k-ideals in ternary semirings. In 2024, Sanborisoot and Ayutthaya [7] constructed a congruence relation with respect to a full k-ideal on a ternary semiring for the purpose of forming a ternary ring from the quotient ternary semiring. Quotient ternary semirings

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can be defined by congruences and/or ideals. Moreover, additional research papers related to ternary semirings, published during 2023-2025, can be found in [8–11].

In this paper, we aim to present various notions of quotient ternary semirings, examine their properties, and provide a discussion on isomorphism theorems concerning these quotient ternary semirings.

2. Preliminaries

In this section, we will recall some basic definitions of ternary semirings.

Definition 1 ([4]). A non-empty set R together with a binary operation, called the addition, and the ternary multiplication, denoted by juxtaposition, is said to be a *ternary semiring* if R is an additive commutative semigroup satisfying the following conditions:

(i) $(abc)de = a(bcd)e = ab(cde)$,

(ii) $(a+b)cd = acd + bcd$,

(iii) $a(b+c)d = abd + acd$,

(iv) $ab(c+d) = abc + abd$

for all $a, b, c, d, e \in R$.

Definition 2 ([4]). Let R be a ternary semiring. If $0 \in R$ such that $0 + x = x$ and $0xy = x0y = xy0 = 0$ for all $x, y \in R$, then 0 is called a *zero element*. In this case, R is called a *ternary semiring with zero*.

Definition 3 ([4]). An additive semigroup S of a ternary semiring R is called a *ternary subsemiring* of R if $s_1s_2s_3 \in S$ for all $s_1, s_2, s_3 \in S$.

Definition 4 ([4]). An additive subsemigroup I of a ternary semiring R is called

(1) a *left ideal* of R if $r_1r_2a \in I$ for all $r_1, r_2 \in R$ and $a \in I$,

(2) a *right ideal* of R if $ar_1r_2 \in I$ for all $r_1, r_2 \in R$ and $a \in I$,

(3) a *lateral ideal* of R if $r_1ar_2 \in I$ for all $r_1, r_2 \in R$ and $a \in I$,

(4) an *ideal* of R if I is a left ideal, a right ideal, and a lateral ideal of R.

An ideal I of R is called a *proper ideal* if $I \neq R$.

Definition 5. An ideal I of a semiring R is called a *k-ideal* of R if, for any $x, y \in R$, $x \in I$ and $x + y \in I$, it follows $y \in I$.

Definition 6. Let R and T be two ternary semirings and φ be a mapping which maps R into T. Then the mapping $\varphi : R \to T$ is called a *homomorphism* of R into T if the following conditions hold:

(i) $\varphi(a+b) = \varphi(a) + \varphi(b)$,

(ii) $\varphi(abc) = \varphi(a)\varphi(b)\varphi(c)$

for all $a, b, c \in R$.

Definition 7. Let R and T be two ternary semirings. The homomorphism $\varphi : R \to T$ is called an *isomorphism* of R onto T if φ is a bijection. If φ is an isomorphism, we say that R and T are *isomorphic* and use the notation $R \cong T$.

3. Quotient ternary semirings

In this section, we will present the various kinds of quotient ternary semirings constructed by using different concepts.

3.1. Quotient ternary semirings modulo Q-ideals

Firstly, we recall some results from [5].

Definition 8 ([5])**.** An ideal I of a ternary semiring R is called a *partitioning ideal (shortly, Q-ideal)* if there exists a subset Q of R such that

(1) $R = \cup\{q + I \mid q \in Q\}$,

(2) for any $q_1, q_2 \in Q$, $(q_1 + I) \cap (q_2 + I) \neq \emptyset \Leftrightarrow q_1 = q_2$.

Let I be a Q-ideal of a ternary semiring R. We let $R/I_{(Q)} = \{q+I \mid q \in Q\}$ and define the addition $\oplus$ and ternary multiplication, for $q_1, q_2, q_3 \in Q$, by

$$(q_1 + I) \oplus (q_2 + I) = q^* + I \text{ and } (q_1 + I)(q_2 + I)(q_3 + I) = q' + I$$

where $q^* \in Q$ is a unique element such that $q_1 + q_2 + I \subseteq q^* + I$ and $q' \in Q$ is a unique element such that $q_1q_2q_3 + I \subseteq q' + I$. Then $R/I_{(Q)}$ forms a ternary semiring under this addition and ternary multiplication. This ternary semiring will be called a *quotient ternary semiring* of R by a Q-ideal I [5].

Example 1. We consider a ternary semiring $R = \mathbb{Z}_0^-$ under the usual addition and ternary multiplication of integers. Let $I = 5\mathbb{Z}_0^- = \{0, -5, -10, -15, \ldots\}$. It is easy to show that I is a Q-ideal of $\mathbb{Z}_0^-$ where $Q = \{0, -1, -2, -3, -4\}$ and $\mathbb{Z}_0^-/I_{(Q)} = \{0 + I, -1 + I, -2 + I, -3 + I, -4 + I\}$ where

$$\begin{aligned}
0 + I &= \{0, -5, -10, -15, \ldots\},\\
-1 + I &= \{-1, -6, -11, -16, \ldots\},\\
-2 + I &= \{-2, -7, -12, -17, \ldots\},\\
-3 + I &= \{-3, -8, -13, -18, \ldots\},\\
-4 + I &= \{-4, -9, -14, -19, \ldots\}.
\end{aligned}$$

Later, let $J = 2\mathbb{Z}_0^- \setminus \{-2\} = \{0, -4, -6, -8, \ldots\}$. Clearly, J is an ideal. The quotient ternary semiring $\mathbb{Z}_0^-/J_{(Q)}$ does not easily hold according to this concept.

Definition 9 ([5])**.** Let R and T be two ternary semirings such that T has a zero 0_T. An onto homomorphism $\varphi : R \to T$ is called *maximal* if, for each $a \in T$, there exists a unique $q_a \in \varphi^{-1}(\{a\})$ such that $x + ker(\varphi) \subseteq q_a + ker(\varphi)$ for each $x \in \varphi^{-1}(\{a\})$ where $ker(\varphi) = \{x \in R \mid \varphi(x) = 0_T\}$.

Example 2. Consider the ternary semiring $\mathbb{Z}_0^-$ under the usual addition and ternary multiplication of integers, and the ternary semiring $\mathbb{Z}_3$ under the usual addition and ternary multiplication of integers modulo 3. Let $\varphi : \mathbb{Z}_0^- \to \mathbb{Z}_3$ be defined by $\varphi(n) = \overline{n}$ for all $n \in \mathbb{Z}_0^-$. Then φ is an onto homomorphism and $ker(\varphi) = 3\mathbb{Z}_0^- = \{0, -3, -6, -9, \ldots\}$. Let $a \in \mathbb{Z}_3$ and $q_a \in \varphi^{-1}(\{a\})$. Assume that $a = \overline{0}$ and suppose that $q_a \neq 0$. So $0 \notin q_a + ker(\varphi)$. We have that $0 \in \varphi^{-1}(\{a\})$. Then, by definition, we obtain that $0 \in 0 + ker(\varphi) \subseteq q_a + ker(\varphi)$, a contradiction. We can conclude that if $a = \overline{0}$, then q_a must be 0. Similarly, if $a = \overline{1}$, then q_a must be -2, and if $a = \overline{2}$, then q_a must be -1. Therefore, φ is maximal. In addition, taking $Q = \{0, -1, -2\}$, it follows that, $\mathbb{Z}_0^-/ker(\varphi)_{(Q)} = \{0 + ker(\varphi), -1 + ker(\varphi), -2 + ker(\varphi)\}$ where

$$\begin{aligned} 0 + ker(\varphi) &= \{0, -3, -6, -9, \ldots\}, \\ -1 + ker(\varphi) &= \{-1, -4, -7, -10, \ldots\}, \\ -2 + ker(\varphi) &= \{-2, -5, -8, -11, \ldots\}. \end{aligned}$$

Theorem 1 ([5])**.** *Let R and T be two ternary semirings such that T has a zero 0_T. If $\varphi : R \to T$ is a maximal homomorphism, then there exists a subset Q of R such that $ker(\varphi)$ is a Q-ideal of R and $R/ker(\varphi)_{(Q)} \cong T$.*

3.2. Quotient ternary semirings modulo congruences

Definition 10. Let R be a ternary semiring. An equivalence relation ρ on R is called a *congruence* on R if, for all $a_1, a_2, b_1, b_2, c_1, c_2 \in R$, the following conditions hold:

(1) If $(a_1, a_2) \in \rho$ and $(b_1, b_2) \in \rho$, then $(a_1 + b_1, a_2 + b_2) \in \rho$,

(2) If $(a_1, a_2) \in \rho, (b_1, b_2) \in \rho$ and $(c_1, c_2) \in \rho$, then $(a_1 b_1 c_1, a_2 b_2 c_2) \in \rho$.

We denote the congruence class of $a \in R$ by $a + \rho$ and let $R/\rho = \{a + \rho \mid a \in R\}$. Furthermore, define a binary addition $\oplus$ and ternary multiplication on R/ρ, for all $a + \rho, b + \rho, c + \rho \in R/\rho$, by

$$(a + \rho) \oplus (b + \rho) = (a + b) + \rho \text{ and } (a + \rho)(b + \rho)(c + \rho) = abc + \rho.$$

Then R/ρ is a ternary semiring under this binary addition and ternary multiplication.

Proposition 1. *Let R and T be ternary semirings and $\varphi : R \to T$ be a ternary semiring homomorphism. Define a relation $K(\varphi)$ on R by*

$$K(\varphi) = \{(a, b) \in R \times R \mid \varphi(a) = \varphi(b)\}.$$

Then $K(\varphi)$ is a congruence on R.

Proof. Straightforward.

Example 3. Let $R = \mathbb{Z}^-$ be a ternary semiring under the usual addition and ternary multiplication of integers, and $T = \mathbb{Z}_4$ be a ternary semiring under the usual addition and ternary multiplication of integers modulo 4. Define a function $\varphi : R \to T$ by $\varphi(a) = \overline{a}$ for all $a \in R$. Then φ is a homomorphism. By Proposition 1, $K(\varphi)$ is a congruence on R. We have that

$$R/K(\varphi) = \{-1 + K(\varphi), -2 + K(\varphi), -3 + K(\varphi), -4 + K(\varphi)\}$$

where

$$\begin{aligned} -1 + K(\varphi) &= \{-1, -5, -9, -13, \ldots\}, \\ -2 + K(\varphi) &= \{-2, -6, -10, -14, \ldots\}, \\ -3 + K(\varphi) &= \{-3, -7, -11, -15, \ldots\}, \\ -4 + K(\varphi) &= \{-4, -8, -12, -16, \ldots\}. \end{aligned}$$

3.3. Quotient ternary semirings modulo ideals

Let I be an ideal of a ternary semiring R. We define a relation ρ_I on R as

$$\rho_I = \{(x, y) \in R \times R \mid x + a = y + b \text{ for some } a, b \in I\}.$$

This means that

$$(x, y) \in \rho_I \text{ if and only if there exist } a_1, a_2 \in I \text{ satisfying } x + a_1 = y + a_2.$$

Then ρ_I is a congruence relation on R, and we denote the congruence class of x by a coset $x + I$. The collection of all congruence classes is denoted by R/I.

Example 4. We consider a ternary semiring $\mathbb{Z}_0^-$ under the usual addition and ternary multiplication of integers.

(1) Let $I = 5\mathbb{Z}_0^- = \{0, -5, -10, -15, \ldots\}$. It is easy to show that I is a k-ideal of $\mathbb{Z}_0^-$. Then $\mathbb{Z}_0^-/I = \{0 + I, -1 + I, -2 + I, -3 + I, -4 + I\}$ where

$$\begin{aligned} 0 + I &= \{0, -5, -10, -15, \ldots\}, \\ -1 + I &= \{-1, -6, -11, -16, \ldots\}, \\ -2 + I &= \{-2, -7, -12, -17, \ldots\}, \\ -3 + I &= \{-3, -8, -13, -18, \ldots\}, \\ -4 + I &= \{-4, -9, -14, -19, \ldots\}. \end{aligned}$$

This quotient ternary semiring obtained here is similar to $\mathbb{Z}_0^-/I_{(Q)}$ considered in Example 1. Note that every k-ideal of a ternary semiring R is a Q-ideal of R.

(2) Let $J = 2\mathbb{Z}_0^- \setminus \{-2\} = \{0, -4, -6, -8, \ldots\}$. It is clear that J is an ideal of $\mathbb{Z}_0^-$ but not a k-ideal. We have that $\mathbb{Z}_0^-/J = \{0 + J, -1 + J\}$ where

$$0 + J = \{0, -2, -4, -6, \ldots\},$$
$$-1 + J = \{-1, -3, -5, -7, \ldots\}.$$

It is also worth noting that, as in Example 1, considering a quotient ternary semiring $\mathbb{Z}_0^-/J_{(Q)}$ is not straightforward.

4. Main Results

In this section, we will consider quotient ternary semirings via congruences and ideals from Subsections 3.2 and 3.3, respectively.

Firstly, as shown in Example 4 (2), we see that $-2+J = 0+J$ but $-2 \notin J$. Moreover, $J \subseteq 0 + J$ and $0 + J$ need not be equal to J. The following proposition shows some properties of cosets in a ternary semiring R/I.

Proposition 2. *Let R be a ternary semiring, I an ideal of R, and $a, b \in R$. Then the following statements hold:*

(1) If R has a zero and I is a k-ideal of R, then $0 + I = I$.

(2) If R has a zero and $a \in I$, then $a + I = 0 + I$.

(3) If I is a k-ideal of R and $a \in I$, then $a + I = b + I$ if and only if $b \in I$.

(4) If R has a zero and I is a k-ideal of R, then $a + I = 0 + I$ if and only if $a \in I$.

Proof. (1) Assume $0 \in R$ and I is a k-ideal of R. Clearly, $I \subseteq 0 + I$. Let $x \in 0 + I$. Then $(0, x) \in \rho_I$. Thus there exist $a, b \in I$ such that $x + a = 0 + b$. Given that I is a k-ideal, $x + a \in I$ and $a \in I$ imply that $x \in I$. Hence $0 + I = I$.

(2) Assume $0 \in R$ and $a \in I$. Since $0 + a = a + 0 = a$ where $a \in I$, by the definition of a congruence relation ρ_I, we have $(0, a) \in \rho_I$. Hence $0 + I = a + I$.

(3) Let I be a k-ideal of R and $a \in I$. Assume $b \in R$ such that $a + I = b + I$. Then $(a, b) \in \rho_I$ which implies $a + u = b + v$ for some $u, v \in I$. Then $b + v = a + u \in I$ because $a, u \in I$. Since I is a k-ideal, $v \in I$ and $b + v \in I$ imply that $b \in I$. Conversely, we assume $b \in I$. Then $a, b \in I$ and $a + b = b + a$ give $(a, b) \in \rho_I$ and consequently, $a + I = b + I$.

(4) Assume that $a + I = 0 + I$. Since $0 \in I$, by (3), we have that $a \in I$. Conversely, it is clear by (1) and (2).

Let R be any ternary semiring and I be an ideal of R. The *k-closure* of I is defined by

$$\mathcal{C}_k(I) = \{r \in R \mid r + x \in I \text{ for some } x \in I\}.$$

Then $\mathcal{C}_k(I)$ is the smallest k-ideal of R containing I, as shown in [11].

Theorem 2. *Let I be an ideal of a ternary semiring R. Then $R/I \cong R/\mathcal{C}_k(I)$.*

Proof. Define $\varphi : R/I \to R/\mathcal{C}_k(I)$ by $\varphi(a+I) = a + \mathcal{C}_k(I)$ for all $a \in R$. First, we let $a, b \in R$ such that $a + I = b + I$. So there exist $x, y \in I$ such that $a + x = b + y$. Since $x, y \in I \subseteq \mathcal{C}_k(I)$, $a+x = a+y$ yields $a+\mathcal{C}_k(I) = b+\mathcal{C}_k(I)$. Then φ is well-defined. Clearly, φ is an onto homomorphism. Let $a + I, b + I \in R/I$ such that $\varphi(a + I) = \varphi(b + I)$. Thus $a+\mathcal{C}_k(I) = b+\mathcal{C}_k(I)$. So there exist $x, y \in \mathcal{C}_k(I)$ such that $a+x = b+y$. Since $x, y \in \mathcal{C}_k(I)$, there exist $s, r \in I$ such that $x + s \in I$ and $y + r \in I$. Thus $a + x + r + s = b + y + r + s$ and $x + r + s, y + r + s \in I$. Hence $a + I = b + I$. This shows that φ is one-to-one and $R/I \cong R/\mathcal{C}_k(I)$.

Let R and T be ternary semirings and $\varphi : R \to T$ be a homomorphism. Recall from Proposition 1 that $K(\varphi) = \{(a, b) \in R \times R \mid \varphi(a) = \varphi(b)\}$ is a congruence on R. Let $im(\varphi)$ denote the sets of all images of φ, that is, $im(\varphi) = \{\varphi(x) \mid x \in R\}$.

Proposition 3. *Let R and T be ternary semirings and $\varphi : R \to T$ be a ternary semiring homomorphism. Then $R/K(\varphi) \cong im(\varphi)$.*

Proof. Define $\psi : R/K(\varphi) \to im(\varphi)$ by $\psi(a + K(\varphi)) = \varphi(a)$ for all $a \in R$. Let $a + K(\varphi) = b + K(\varphi)$. Hence $\varphi(a) = \varphi(b)$. Thus ψ is well-defined. Clearly, ψ is an onto homomorphism. Next, let $a + K(\varphi), b + K(\varphi) \in R/K(\varphi)$ be such that $\psi(a + K(\varphi)) = \psi(b+K(\varphi))$. Then $\varphi(a) = \varphi(b)$ which implies $(a, b) \in K(\varphi)$. Hence $a+K(\varphi) = b+K(\varphi)$.

Proposition 4. *Let R and T be ternary semirings with zeroes 0_R and 0_T, respectively, and $\varphi : R \to T$ be a homomorphism such that $\varphi(0_R) = 0_T$. Then $ker(\varphi) = \{x \in R \mid \varphi(x) = 0_T\}$ is a k-ideal of R.*

Proof. Let $\varphi : R \to T$ be a homomorphism where R and T are ternary semirings. Since $0_R \in ker(\varphi)$, we see that $ker(\varphi) \neq \emptyset$. Let $a, b \in ker(\varphi)$. Therefore $\varphi(a) = 0_T$ and $\varphi(b) = 0_T$. Since φ is a homomorphism, we have $\varphi(a + b) = \varphi(a) + \varphi(b) = 0_T$. This implies that $a + b \in ker(\varphi)$. Next, we let $r \in ker(\varphi)$ and $a, b \in R$. We have that $\varphi(rab) = \varphi(r)\varphi(a)\varphi(b) = 0_T\varphi(a)\varphi(b) = 0_T$. Hence $rab \in ker(\varphi)$. Similarly, $arb, abr \in ker(\varphi)$. Then $ker(\varphi)$ is an ideal of R. Furthermore, let $a, b \in R$ be such that $a + b \in ker(\varphi)$ and $a \in ker(\varphi)$. Then $0_T = \varphi(a + b) = \varphi(a) + \varphi(b) = 0_T + \varphi(b) = \varphi(b)$, so $b \in ker(\varphi)$. Therefore, $ker(\varphi)$ is a k-ideal of R.

Let R and T be ternary semirings with zeroes 0_R and 0_T, respectively, and $\varphi : R \to T$ be a homomorphism such that $\varphi(0_R) = 0_T$. A homomorphism φ is called a *k-kernel homomorphism* if for all $s_1, s_2 \in R$, $\varphi(s_1) = \varphi(s_2)$ implies $s_1 + a = s_2 + b$ for some $a, b \in ker(\varphi)$. Next, we provide an analogue of the first isomorphism theorem.

Theorem 3. *Let R and T be ternary semirings and $\varphi : R \to T$ be a homomorphism with $ker(\varphi)$. If φ is a k-kernel homomorphism, then $R/ker(\varphi) \cong im(\varphi)$.*

Proof. Assume that φ is a k-kernel homomorphism. Let $\psi : R/ker(\varphi) \to im(\varphi)$ be defined by $\psi(s+ker(\varphi)) = \varphi(s)$ for all $s \in R$. Let s_1, s_2, s_3 be any three elements in R. To prove that ψ is well-defined, assume that $s_1 + ker(\varphi) = s_2 + ker(\varphi)$. Then $s_1 + a = s_2 + b$ for some $a, b \in ker(\varphi)$. So $\varphi(a) = \varphi(b) = 0_T$ and

$$\varphi(s_1) = \varphi(s_1) + \varphi(a) = \varphi(s_1 + a) = \varphi(s_2 + b) = \varphi(s_2) + \varphi(b) = \varphi(s_2).$$

In addition, as φ is a homomorphism, we have

$$\begin{aligned}
&\psi((s_1 + ker(\varphi)) + (s_2 + ker(\varphi)))\\
&= \psi(s_1 + s_2 + ker(\varphi))\\
&= \varphi(s_1 + s_2)\\
&= \varphi(s_1) + \varphi(s_2)\\
&= \psi(s_1 + ker(\varphi)) + \psi(s_2 + ker(\varphi))
\end{aligned}$$

and

$$\begin{aligned}
&\psi((s_1 + ker(\varphi))(s_2 + ker(\varphi))(s_3 + ker(\varphi)))\\
&= \psi(s_1s_2s_3 + ker(\varphi))\\
&= \varphi(s_1s_2s_3)\\
&= \varphi(s_1)\varphi(s_2)\varphi(s_3)\\
&= \psi(s_1 + ker(\varphi))\psi(s_2 + ker(\varphi))\psi(s_3 + ker(\varphi)).
\end{aligned}$$

Hence, ψ is a homomorphism. Clearly, ψ is onto. Next, we will show that ψ is one-to-one. Assume that $s_1, s_2 \in R$ such that $\psi(s_1 + ker(\varphi)) = \psi(s_2 + ker(\varphi))$. Then $\varphi(s_1) = \varphi(s_2)$. By assumption, we obtain $s_1+a = s_2+b$ for some $a, b \in ker(\varphi)$. Therefore, $s_1 + ker(\varphi) = s_2 + ker(\varphi)$ and this leads to the result $R/ker(\varphi) \cong im(\varphi)$.

Example 5. Let $R = \mathbb{Z}^-$ be a ternary semiring under the usual addition and ternary multiplication of integers and $T = \mathbb{Z}_4$ be a ternary semiring under the usual addition and ternary multiplication of integers modulo 4. Define a function $\varphi : R \to T$ by $\varphi(a) = \overline{a}$ for all $a \in R$. Then φ is a homomorphism. We have that $ker(\varphi) = \{-4, -8, -12, -16, \ldots\}$ is a k-ideal of R. We have that $R/ker(\varphi) = \{-1 + ker(\varphi), -2 + ker(\varphi), -3 + ker(\varphi), -4 + ker(\varphi)\}$ where

$$\begin{aligned}
-1 + ker(\varphi) &= \{-1, -5, -9, -13, \ldots\},\\
-2 + ker(\varphi) &= \{-2, -6, -10, -14, \ldots\},\\
-3 + ker(\varphi) &= \{-3, -7, -11, -15, \ldots\},\\
-4 + ker(\varphi) &= \{-4, -8, -12, -16, \ldots\}.
\end{aligned}$$

We end this section with a summary of the relationship between the isomorphisms in the following corollary.

Corollary 1. *Let R and T be ternary semirings with zeroes 0_R and 0_T, respectively, and $\varphi : R \to T$ be a homomorphism such that $\varphi(0_R) = 0_T$. If φ is a k-kernel homomorphism, then $R/ker(\varphi) \cong R/K(\varphi)$.*

5. Conclusion

In this paper, we present multiple approaches for constructions of quotient ternary semirings, namely quotient ternary semirings modulo congruences and those modulo ideals.

We investigate some relationships between ideals and quotient structures in ternary semirings. A key contribution of this work is the establishment of various isomorphism theorems, including the fundamental results that relate quotient structures such as $R/I \cong R/C_k(I)$ and $R/ker(\varphi) \cong im(\varphi)$, under suitable conditions. These results help clarify the connections between homomorphisms, ideals, and congruences, and provide a basis for understanding the structure of ternary semirings.

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References

[1] P. J. Allen. A fundamental theorem of homomorphisms for semirings. *Proceedings of the American Mathematical Society*, 24:463–465, 1969.

[2] S. E. Atani and A. G. Garfami. Ideals in quotient semirings. *Chiang Mai Journal of Science*, 40(1):77–82, 2013.

[3] D. R. Latorre. A note on quotient semirings. *Proceedings of the American Mathematical Society*, 24:463–465, 1970.

[4] T. K. Dutta and S. Kar. On regular ternary semirings. *Advances in Algebra, Proceedings of the ICM Satellite Conference in Algebra and Related Topics, World Scientific*, pages 343–355, 2003.

[5] J. N. Chaudhari and K. J. Ingale. On partitioning and subtractive ideals of ternary semirings. *Kyungpook Mathematical Journal*, 51(1):69–76, 2011.

[6] T. Sunitha; U. Nagi Reddy and G. Shobhalatha. A note on full k-ideals in ternary semirings. *Indian Journal of Science and Technology*, 14(21):1786–1790, 2021.

[7] J. Sanborisoot and P. P. N. Ayutthaya. On ternary ring congruences of ternary semirings. *Discussiones Mathematicae General Algebra and Applications*, 44:43–55, 2024.

[8] N. Sarasit; R. Chinram and A. Rattana. Applications of fuzzy sets for almostity of ternary semirings. *International Journal of Applied Mathematics*, 36(4):497–508, 2023.

[9] P. Luangchaisri and T. Changphas. Prime one-sided ideals in ternary semirings. *International Journal of Mathematics and Computer Science*, 19(2):403–409, 2024.

[10] T. Panityakul and R. Chinram. Applications of neutrosophic N-structures in ternary semirings: A study on neutrosophic ternary N-subsemirings. *International Journal of Neutrosophic Science*, 26(2):120–131, 2025.

[11] A. J. Khan; M. Petapirak and R. Chinram. A note on k-ideals in ternary semirings. *European Journal of Pure and Applied Mathematics*, 18(2):6096, 2025.