EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6496
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Some Properties of Differentiable and Riemann Integrable Functions via δ-fine Tagged Partitions

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Abstract. The concept of δ-fine tagged partitions is used to simplify and unify proofs of various theorems in elementary real analysis. This paper is a continuation of the author's project to present some properties of differentiable functions via δ-fine tagged partitions. Some basic theorems concerning Riemann integrable functions are also reproved by this concept.

2020 Mathematics Subject Classifications: 26A24, 26A42

Key Words and Phrases: δ-fine tagged partitions, differentiable function, full covers, Riemann integrable function

1. Introduction

The concept of full covers was introduced to simplify and unify the proofs of various theorems in real analysis by Botsko [1] in 1987. Depending upon Thomson's Lemma which ensures that every full cover $\mathcal{C}$ of a closed interval $[a, b]$ must contain a partition of $[a, b]$, this concept bring harmony into the proofs of diverse theorems. Later in 1989, Botsko [2] published the second paper on this topic concerning some harder theorems than the previous study. Moreover, Klaimon [3] as well as Zangara and Marafino [4] also use this concept to prove in a unified style for many other theorems in real analysis. By analyzing proofs in Botsko's works, partitions extracted from full covers should be the keys in establishing the unified treatments of various theorems.

In 1998, Gordon [5] showed alternate approach to several well known theorems in elementary real analysis by using δ fine tagged partitions instead of using the full covering. Prongjit and Sodsiri [6, 7] published in 2014 presented how δ-fine tagged partitions could be used to replace partitions taking from full covers, and these papers reproved almost theorems discussed in [2–4]. Furthermore, Zheng and Shi [8] have provided notion of dyadic partitions in showing alternative unified proofs of theorems in real analysis as distinct from the δ-fine tagged partition concept.

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6496

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This research continue to the results from [6], which shines us to draw some properties of differentiable functions to study via δ-fine tagged partitions. Some of these properties too can be applied to reveal another view of Mean Value Theorem and Cauchy Mean Value Theorem. Besides, we also study some basic properties of Riemann integrable functions via δ-fine tagged partitions that deal with the Cauchy criterion for Riemann integrability version formulated by Gordon (see [5]).

2. Preliminaries

Throughout this paper, we assume that $a, b \in \mathbb{R}$ with $a < b$. We denote $]\,a, b\,[$ for an open interval, while $[\,a, b\,]$ is denoted for a closed interval as usual.

2.1. δ-fine Tagged partitions

Definition 1. [9] *A **partition** of a closed interval $[\,a, b\,]$ is a finite collection of closed intervals $\{[\,x_{i-1}, x_i\,] : i = 1, \ldots, n\}$, where $a = x_0 < x_1 < \cdots < x_n = b$.*

Definition 2. [9] *A **tagged partition** $\mathcal{P}$ of $[\,a, b\,]$ is a finite collection of order pairs*

$$\mathcal{P} = \{(t_i, [\,x_{i-1}, x_i\,]) : i = 1, \ldots, n\},$$

where $a = x_0 < x_1 < \cdots < x_n = b$ and each $t_i \in [\,x_{i-1}, x_i\,]$.

Definition 3. [9] *A **gauge** on $[\,a, b\,]$ is a strictly positive real valued function defined on $[\,a, b\,]$. If δ is a gauge on $[\,a, b\,]$, then a tagged partition $\mathcal{P} = \{(t_i, [\,x_{i-1}, x_i\,]) : i = 1, \ldots, n\}$ is said to be a **δ-fine tagged partition** of $[\,a, b\,]$ if and only if*

$$[\,x_{i-1}, x_i\,] \subset \,]\,t_i - \delta(t_i), t_i + \delta(t_i)\,[$$

for all $i = 1, \ldots, n$.

Figure 1 shows an illustration of a portion of some δ-fine tagged partition.

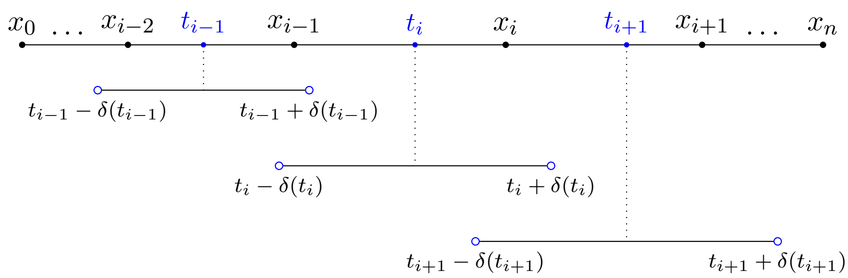

Figure 1: A portion of some δ-fine tagged partition of [a,b]

The following lemma is the backbone of proving several theorems in a unified style. It assures the existence of a δ-fine tagged partition on $[a,b]$ for every given gauge δ on $[a,b]$.

Lemma 1 (Cousin's Lemma). [9] *For every gauge δ on $[a,b]$, there exists a δ-fine tagged partition of $[a,b]$.*

2.2. Basic Knowledge from Real Analysis

Theorem 1 (Maximum-Minimum Theorem). [9] *If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$, then f has an absolute maximum and an absulute minimum on $[a,b]$.*

Theorem 2 (Interior Extremum Theorem). [9] *Let c be an interior point of an interval I at which $f:I\to\mathbb{R}$ has a relative extremum. If the derivative of f at c exists, then $f'(c)=0$.*

Definition 4. [9] *Let $\{[c_{j-1},c_j]:j=1,\dots,l\}$ be a partition of $[a,b]$, and let each $k_j\in\mathbb{R}$. Let $\varphi:[a,b]\to\mathbb{R}$ be such that $\varphi=\sum\limits_{j=1}^{l}k_j\varphi_{J_j}$, for each $j=1,\dots,l$, $J_j:=[c_{j-1},c_j]$ and*

$$\varphi_{J_j}(x)=\begin{cases}1 & \text{if } c_{j-1}\le x<c_j,\\ 0 & \text{elsewhere,}\end{cases}$$

except for the last φ_{J_l} that

$$\varphi_{J_l}(x)=\begin{cases}1 & \text{if } c_{l-1}\le x\le c_l,\\ 0 & \text{elsewhere.}\end{cases}$$

*We say that φ is a **step function** on $[a,b]$.*

An example of a step function on $[1,4]$ is illustrated in Figure 2.

Definition 5. [5] *A function f is **Riemann integrable** on $[a,b]$ if and only if for each $\varepsilon>0$ there exists a partition $\{[x_{i-1},x_i]:i=1,\dots,n\}$ of $[a,b]$ such that*

$$\sum_{i=1}^{n}\omega(f,[x_{i-1},x_i])(x_i-x_{i-1})<\varepsilon,$$

where $\omega(f,[c,d])=\sup\{|f(t)-f(s)|:s,t\in[c,d]\}$.

3. Proving Theorems via δ-fine Tagged Partitions

3.1. Differentiation via δ-fine Tagged Partitions

The following lemma provides important results for defining a gauge δ in proving Theorem 3. Theorem 4 arise from Theorem 3 by extending an open interval $]a,b[$ to a closed interval $[a,b]$. Applying Theorem 4, the reproving of the Mean Value Theorem and the Cauchy Mean Value Theorem are obtained. In addition, Corollary 1 is also presented, for this result and Theorem 3 are rely upon each other.

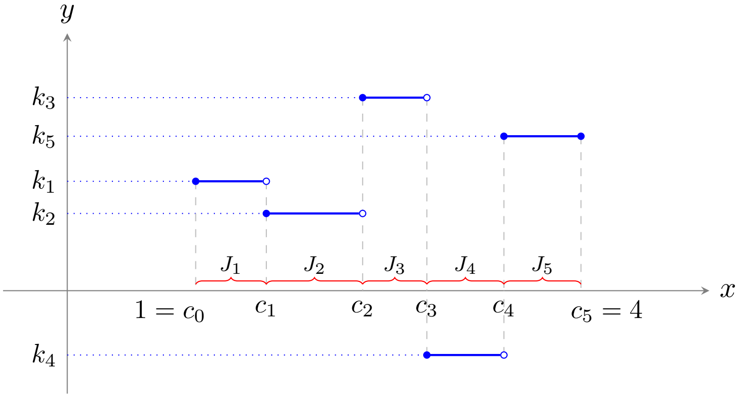

Figure 2: A step function $\varphi = \sum_{j=1}^{5} k_j \varphi_{J_j}$ on the closed interval $[1, 4]$

Lemma 2. *Suppose that f is differentiable on $]\,a, b\,[$ and that $f'(x) \neq 0$ for all $x \in\,]\,a, b\,[$.*

(a) If $f'(x) > 0$, then there exists a real number $\zeta > 0$ such that f is strictly increasing on $]\,x - \zeta, x + \zeta\,[$.

(b) If $f'(x) < 0$, then there exists a real number $\xi > 0$ such that f is strictly decreasing on $]\,x - \xi, x + \xi\,[$.

Proof. (a) Since $f'(x) > 0$, then there exists a real number $\zeta > 0$ such that $]\,x - \zeta, x + \zeta\,[\, \subseteq\,]\,a, b\,[$, and

$$\frac{f(y) - f(x)}{y - x} > 0$$

for all $y \in\,]\,x - \zeta, x + \zeta\,[$ with $y \neq x$. Claim that f is strictly increasing on $]\,x - \zeta, x + \zeta\,[$. Let $s, t \in\,]\,x - \zeta, x + \zeta\,[$ with $s < t$, then we have three cases to consider.

Case 1: $s \leq x < t$ or $s < x \leq t$. It is clear that $f(s) < f(t)$.

Case 2: $s < t < x$. Suppose to the contrary that $f(s) \geq f(t)$. By Theorem 1, the function f has an absolute minimum at some point $c \in\,]\,s, x\,[$. Since c is an interior point of $[\,s, x\,]$ and by Theorem 2, it follows that $f'(c) = 0$ which contradicts by the assumption of the Lemma. Thus, $f(s) < f(t)$.

Case 3: $x < s < t$. The proof is similar to Case 2.

Therefore, f is strictly increasing on $]\,x - \zeta, x + \zeta\,[$ as claimed.

(b) Similar to the proof of (a).

Theorem 3. *Suppose that f is differentiable on $]\,a, b\,[$. If $f'(x) \neq 0$ for all $x \in\,]\,a, b\,[$, then f is strictly monotone on $]\,a, b\,[$.*

Proof. Let $c, d \in\,]\,a, b\,[$ with $c < d$. Define a gauge $\delta : [\,c, d\,] \to \mathbb{R}^+$ as follows: Let $x \in [\,c, d\,]$ be fixed. By Lemma 2, there exists a $\delta_x > 0$ such that f is either strictly increasing or strictly decreasing on $]\,x - \delta_x, x + \delta_x\,[$. We define $\delta(x) := \delta_x$, and let $\mathcal{P} := \{(t_i, [\,x_{i-1}, x_i\,]) : i = 1, \ldots, n\}$ be a δ-fine tagged partition of $[\,c, d\,]$. Note that f is either strictly increasing or strictly decreasing on each $[x_{i-1}, x_i]$, and now we have two cases to consider.

Case 1: f is strictly increasing on $[x_0, x_1]$. To show that the function f is strictly increasing on $[c, d]$. Suppose to the contrary that f is strictly decreasing on $[x_1, x_2]$. This leads f to have a relative maximum at the interior point $x_1 \in]x_0, x_2[$. Hence, $f'(x_1) = 0$ and a contradiction is obtained now. Continuing the process, we can conclude that f is strictly increasing on $[c, d]$.

Case 2: f is strictly decreasing on $[x_0, x_1]$. Similarly, we can prove that f is strictly decreasing on $[c, d]$.

These two cases give the conclusion that f is either strictly increasing or strictly decreasing on $[c, d]$. Therefore, f is strictly monotone on $[c, d]$. Since c and d are arbitrary elements of $]a, b[$, so f is strictly monotone on $]a, b[$. The proof is completed.

Corollary 1. *Suppose that f is differentiable on $]a, b[$. If $f'(x) \neq 0$ for all $x \in]a, b[$, then either $f'(x) > 0$ for all $x \in]a, b[$ or $f'(x) < 0$ for all $x \in]a, b[$.*

Proof. From Theorem 3, we have two cases to consider.

Case 1: f is strictly increasing on $]a, b[$. Let $x \in]a, b[$ be fixed. Since $\frac{f(y)-f(x)}{y-x} > 0$ for every $y \in]a, b[$ with $y \neq x$, then

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x} \geq 0.$$

Hence, we conclude that $f'(x) > 0$.

Case 2: f is strictly decreasing on $]a, b[$. We can prove in the same manner that $f'(x) < 0$ for all $x \in]a, b[$.

Theorem 4. *Suppose that f is continuous on $[a, b]$ and differentiable on $]a, b[$. If $f'(x) \neq 0$ for all $x \in]a, b[$, then f is strictly monotone on $[a, b]$.*

Proof. From Theorem 3, let us consider as follows:

Case 1: f is strictly increasing on $]a, b[$. Let $c, s, d \in]a, b[$ be such that $c < s < d$. Since f is continuous at a and $f(y) < f(c)$ for every $y \in]a, c[$, then

$$f(a) = \lim_{y \to a^+} f(y) \leq f(c) < f(s).$$

Likewise, since f is continuous at b and $f(d) < f(y)$ for all $y \in]d, b[$, then

$$f(b) = \lim_{y \to b^-} f(y) \geq f(d) > f(s).$$

This give us $f(a) < f(s) < f(b)$ for all $s \in]a, b[$. Consequently, we can prove that f is strictly increasing on $[a, b]$.

Case 2: f is strictly decreasing on $]a, b[$. Similar to Case 1, we can conclude that f

is strictly decreasing on $[\,a, b\,]$.

By applying Theorem 4, the proving of two following well known theorems (see [9]) give a different view of these theorems.

Theorem 5 (Mean Value Theorem). *Let f be continuous on $[\,a, b\,]$ and differentiable on $]\,a, b\,[$. Then, there exists an element $c \in\,]\,a, b\,[$ which*

$$f(b) - f(a) = f'(c)(b - a).$$

Proof. Let $g : [\,a, b\,] \to \mathbb{R}$ defined by

$$g(x) := f(x) - \frac{f(b) - f(a)}{b - a}x.$$

Suppose to the contrary that $g'(x) \neq 0$ for all $x \in\,]\,a, b\,[$. Since g satisfies the hypotheses of Theorem 4, then either $g(a) < g(b)$ or $g(a) > g(b)$ which contradicts the fact that $g(a) = g(b)$. Thus, there must be some $c \in\,]\,a, b\,[$ such that $g'(c) = 0$. Actually,

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Hence, $f(b) - f(a) = f'(c)(b - a)$.

Theorem 6 (Cauchy Mean Value Theorem). *Let f and g be continuous on $[\,a, b\,]$ and differentiable on $]\,a, b\,[$. If $g'(x) \neq 0$ for all $]\,a, b\,[$, then there exists $c \in\,]\,a, b\,[$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. Since $g'(x) \neq 0$ for all $x \in\,]\,a, b\,[$, then it follows from Theorem 4 that $g(a) \neq g(b)$. So we can define $h : [\,a, b\,] \to \mathbb{R}$ by

$$h(x) := \frac{f(b) - f(a)}{g(b) - g(a)}g(x) - f(x).$$

Suppose to the contrary that $h'(x) \neq 0$ for all $x \in\,]\,a, b\,[$. We can prove by the same manner as Theorem 5 that the supposition leads to $h(a) \neq h(b)$, which is a contradiction. Hence, there must be some $c \in\,]\,a, b\,[$ such that $h'(c) = 0$. Actually,

$$0 = h'(c) = \frac{f(b) - f(a)}{g(b) - g(a)}g'(c) - f'(c).$$

Therefore, $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$.

3.2. Riemann Integral via δ-fine Tagged Partitions

In this section, we deal with some basic properties of Riemann integrable functions. Definition 5 is a version of Cauchy criterion for Riemann integrability formulated by Gordon [5] that we shall use in this paper.

Theorem 7. *If $f : [a,b] \to \mathbb{R}$ is monotone on $[a,b]$, then f is Riemann integrable on $[a,b]$.*

Proof. Assume that f is increasing on $[a,b]$, and let $\varepsilon > 0$ be fixed. Define a gauge $\delta : [a,b] \to \mathbb{R}^+$ by $\delta(x) := \varepsilon$. Let $\mathcal{P} := \{(t_i, [x_{i-1}, x_i]) : i = 1, \ldots, n\}$ be a δ-fine tagged partition of $[a,b]$. For each $i = 1, \ldots, n$, we have

$$\omega(f, [x_{i-1}, x_i]) = \sup\{|f(s) - f(t)| : s, t \in [x_{i-1}, x_i]\} = f(x_i) - f(x_{i-1}),$$

and $[x_{i-1}, x_i] \subset \,]t_i - \delta(t_i), t_i + \delta(t_i)[$ which implies $x_i - x_{i-1} < 2\varepsilon$. Then, we have a partition $\{[x_{i-1}, x_i] : i = 1, \ldots, n\}$ of $[a,b]$ which is extracted from $\mathcal{P}$ such that

$$\begin{aligned}\sum_{i=1}^{n} \omega(f, [x_{i-1}, x_i])(x_i - x_{i-1}) &< \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))2\varepsilon \\ &= 2\varepsilon(f(b) - f(a)).\end{aligned}$$

Since ε is arbitrary, so f is Riemann integrable on $[a,b]$. We can prove in the same manner whence f is decreasing on $[a,b]$.

Lemma 3. *Let $f : [a,b] \to \mathbb{R}$ be a step function such that $f = \sum_{j=1}^{l} k_j \varphi_{J_j}$. If $P := \{[x_{i-1}, x_i] : i = 1, \ldots, n\}$ is a partition of $[a,b]$, then the number of elements in P which are not contained in any open interval $]c_{j-1}, c_j[$ is not greater than $2l$. (Recall from Definition 4 that $J_j := [c_{j-1}, c_j]$.)*

Proof. Let

$$\mathcal{C} := \{I \in P : I \not\subseteq \,]c_{j-1}, c_j[\text{ for all } j = 1, \ldots, l\}.$$

Note that if $[\alpha, \beta] \in \mathcal{C}$, then there must be at least one point $c_j \in [\alpha, \beta]$ for some $j = 0, 1, \ldots, l$. Since $\mathcal{C}$ is the set of nonoverlapping closed intervals, then each c_j can be attributed to belong in any elements of $\mathcal{C}$ by separating into two cases as follows.

Case 1: $j = 0, l$. It is clear that c_0 and c_l can only be contained in $[x_0, x_1]$ and $[x_{n-1}, x_n]$ respectively. Then we get exactly two elements of $\mathcal{C}$ from this.

Case 2: $j = 1, \ldots, l-1$. Since c_j may be an endpoint of two adjacent elements of $\mathcal{C}$, then c_j can only be contained in at most two elements of $\mathcal{C}$. This give us at most $2(l-1)$ elements of $\mathcal{C}$ to count.

Consequently, $|\mathcal{C}| \leq 2l$.

Theorem 8. *If $f : [a,b] \to \mathbb{R}$ is a step function, then f is Riemann integrable on $[a,b]$.*

Proof. Let f be a step function on $[\,a, b\,]$ such that $f = \sum_{j=1}^{l} k_j \varphi_{J_j}$ and let $\varepsilon > 0$ be given. Define a gauge $\delta : [\,a, b\,] \to \mathbb{R}^+$ by $\delta(x) = \frac{\varepsilon}{k+1}$, where $k = \max\{|k_{j_1} - k_{j_2}| \,:\, j_1, j_2 = 1, \ldots, l\}$. Let $\mathcal{P} := \{(t_i, [\,x_{i-1}, x_i\,]) : i = 1, 2, \ldots, n\}$ be a δ-fine tagged partition of $[\,a, b\,]$. For each $i = 1, \ldots, n$, we separate into two cases.

Case 1: $[\,x_{i-1}, x_i\,] \subseteq \,]\,c_{j-1}, c_j\,[\,$ for some $j = 1, 2, \ldots, l$. Since $|f(t)-f(s)| = |k_j - k_j| = 0$ for all $s, t \in [\,x_{i-1}, x_i\,]$, then $\omega(f, [\,x_{i-1}, x_i\,]) = 0$.

Case 2: $[\,x_{i-1}, x_i\,] \not\subseteq \,]\,c_{j-1}, c_j\,[\,$ for all $j = 1, 2, \ldots, l$. Since $|f(t) - f(s)| \leq k$ for all $s, t \in [\,x_{i-1}, x_i\,]$, then $\omega(f, [\,x_{i-1}, x_i\,]) \leq k$.

Furthermore, since $[\,x_{i-1}, x_i\,] \subset \left]\, t_i - \frac{\varepsilon}{k+1}, t_i + \frac{\varepsilon}{k+1} \,\right[$, so $x_i - x_{i-1} < \frac{2\varepsilon}{k+1}$ for all $i = 1, \ldots, n$. Let

$$\mathcal{I} := \{i : [\,x_{i-1}, x_i\,] \not\subseteq \,]\,c_{j-1}, c_j\,[\quad \text{for all } j = 1, 2, \ldots, l\}.$$

It follows from Lemma 3 that $|\mathcal{I}| \leq 2l$. Consequently,

$$\begin{aligned}
&\sum_{i=1}^{n} \omega(f, [\,x_{i-1}, x_i\,])(x_i - x_{i-1}) \\
&\qquad = \sum_{i \in \mathcal{I}} \omega(f, [\,x_{i-1}, x_i\,])(x_i - x_{i-1}) \\
&\qquad < \sum_{i \in \mathcal{I}} k \left(\frac{2\varepsilon}{k+1} \right) < 2\varepsilon |\mathcal{I}| \leq 4l\varepsilon.
\end{aligned}$$

Therefore, f is Riemann integrable on $[\,a, b\,]$.

Lemma 4. *Let $f : [\,a, b\,] \to \mathbb{R}$ be a function, and let $\{[\,y_{j-1}, y_j\,] : j = 1, \ldots, m\}$ be a partition of $[\,a, b\,]$. If $\sum_{j=1}^{m} \omega(f, [\,y_{j-1}, y_j\,])(y_j - y_{j-1}) < 1$, then $\omega(f, [\,y_{j-1}, y_j\,])$, $j = 1, \ldots, m$, are all nonnegative real numbers.*

Proof. Let $M := \min\{y_j - y_{j-1} : j = 1, \ldots, m\}$. Let us consider,

$$\begin{aligned}
\frac{1}{M} &> \frac{1}{M} \sum_{j=1}^{m} \omega(f, [\,y_{j-1}, y_j\,])(y_j - y_{j-1}) \\
&= \sum_{j=1}^{m} \omega(f, [\,y_{j-1}, y_j\,]) \frac{(y_j - y_{j-1})}{M} \\
&\geq \sum_{j=1}^{m} \omega(f, [\,y_{j-1}, y_j\,]).
\end{aligned}$$

Since $\omega(f, [\,y_{j-1}, y_j\,]) \geq 0$ for all $j = 1, \ldots, m$, then each $\omega(f, [\,y_{j-1}, y_j\,]) < \frac{1}{M}$. Therefore, we conclude that, for each $j = 1, \ldots, m$, there exists $\omega_j \in \mathbb{R}$ with $0 \leq \omega_j < \frac{1}{M}$ such that $\omega(f, [\,y_{j-1}, y_j\,]) = \omega_j$.

Remark 1. *Lemma 4 also holds for the condition* $\sum_{j=1}^{m} \omega(f, [\, y_{j-1}, y_j \,])(y_j - y_{j-1}) < M$, *where* M *is any positive real number. We prove only a special case* $M = 1$ *for using in the next theorem.*

Theorem 9. *If* f *is Riemann integrable on* $[\, a, b \,]$, *then* f *is bounded on* $[\, a, b \,]$.

Proof. By assumption, there exists a partition $\{[\, y_{j-1}, y_j \,] : j = 1, \ldots, m\}$ of $[\, a, b \,]$ such that

$$\sum_{j=1}^{m} \omega(f, [\, y_{j-1}, y_j \,])(y_j - y_{j-1}) < 1.$$

It follows from Lemma 4 that, for each $j = 1, \ldots, m$, $\omega(f, [\, y_{j-1}, y_j \,])$ is a nonnegative real number. We denote $\omega_j := \omega(f, [\, y_{j-1}, y_j \,])$ for convenience.

To show that f is bounded on $[\, y_0, y_1 \,]$, let $c, d \in \,]\, y_0, y_1 \,[$ with $c < d$. Define a gauge δ on $[\, c, d \,]$ as follows: Let $x \in [\, c, d \,]$ be fixed, then there exists a number $\delta_x > 0$ such that $]\, x - \delta_x, x + \delta_x \,[\, \subset [\, y_0, y_1 \,]$. We define $\delta(x) := \delta_x$, and let $\mathcal{P} := \{(t_i, [\, x_{i-1}, x_i \,]) : i = 1, 2, \ldots, n\}$ be a δ-fine tagged partition of $[\, c, d \,]$. Set $m := \max\{|f(t_i)| : i = 1, \ldots, n\}$. Let x be any element of $[\, c, d \,]$, then $x \in [\, x_{i-1}, x_i \,]$ for some $i = 1, 2, \ldots, n$, and hence

$$|f(x)| \leq |f(x) - f(t_i)| + |f(t_i)| \leq \omega_j + m.$$

For this reason, f is bounded on $[\, c, d \,]$. Since c and d are arbitrary elements in $]\, y_0, y_1 \,[$, so f is bounded on $]\, y_0, y_1 \,[$. We assume further that f is bounded on $]\, y_0, y_1 \,[$ by a positive real number $\hat{m}$, and let $M_1 := \max\{\hat{m}, |f(y_0)|, |f(y_1)|\}$. Clearly, f is bounded on $[\, y_0, y_1 \,]$ by M_1 as desired.

Moreover, for each $j = 2, \ldots, m$, we can prove in the same way that f is bounded on $[\, y_{j-1}, y_j \,]$ by some positive real number M_j. Finally, the conclusion of this theorem is attained by letting $M := \max\{M_j : j = 1, \ldots, m\}$, so that $|f(y)| \leq M$ for all $y \in [\, a, b \,]$.

4. Conclusions

The concept of δ-fine tagged partitions allows us to perceive knowledge of elementary real analysis in a different facet. This research has shown a little new property of differentiable functions in Lemma 2. By this property, we can define the gauge δ to reach the conclusion of Theorem 3 more elegant than the author's previous some work which similar to this (see [6]). The proofs of Mean Value Theorem and Cauchy Mean Value Theorem have been adjusted a little bit through Theorem 4 to give another views of these two well known theorems. In the part of Riemann integrable function, we have presented new proofs of three basic theorems by applying this concept. Sincerely, we are awe in Gordon' work for his version of criterion for Riemann integrability that is suitable to δ-fine tagged partitions technique resulting in simplicity and unity of these three proofs, at least in our opinion. There are many results in real analysis that have not yet been studied via δ-fine tagged partitions. In the next project, we would extend our research to some other interesting theorems or properties through this idea.

Acknowledgements

We would like to express our sincere appreciation to Research and Academic Service Division and Faculty of Sciences and Engineering, Kasetsart University, Chalermprakiat Sakon Nakhon Province Campus, Sakon Nakhon, Thailand for their generous financial support for this research. Additionally, the authors wish to thank the reviewers of European Journal of Pure and Applied Mathematics, for making our journal successful.

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