**EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS**
2025, Vol. 18, Issue 3, Article Number 6497
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

# Almost, Weakly, and Nearly Lindelöf Ideal Topological Spaces

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**Abstract.** The concepts of almost, weakly and nearly Lindelöf closed ideal topological spaces are introduced in this work. We examine their subspaces and the connection between the subspaces and their topological characteristics and explain how countable covers affect Almost Lindelöf spaces and concentrate on the significance of these covers. These covers are made up of countable subfamilies whose closures cover the ideal spaces. Definitions, claims, characterizations, and observations pertaining to the recently presented concepts of almost and weakly Lindelöf ideal topological spaces are stated, examined, and discussed. Additionally, the connections among various ideal topological spaces are analyzed and explored. We provide examples of the consequences of these novel ideal spaces.

**2020 Mathematics Subject Classifications**: 54A05, 54A10

**Key Words and Phrases**: Almost Lindelöf, Weakly Lindelöf, Nearly Lindelöf, Ideal spaces

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## 1. Introduction

The ideal $\mathcal{J}$ on the $(\mathsf{X}, \tau)$ is a non-empty collection of $\wp(\mathsf{X})$ subsets satisfying:
 i) If $Z \in \mathcal{J}$ and $W \subset Z$, then $W \in \mathcal{J}$;
 ii) If $Z \in \mathcal{J}$ and $W \in \mathcal{J}$, then $Z \cup W \in \mathcal{J}$.
Hamlett and Jankovic [1, 2] created the first generalization of several important features of general topology via topological ideals in ideal topological spaces. Features like separation axioms, decomposition of continuity, connectedness, compactness, and resolvability have all been generalized using the concept of ideals [3, 4]. Historically, there have been two

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6497

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primary developments in the application of ideals in general topology. The study of topological spaces' local properties, which can be extended to their global properties, is the focus of the first line [5, 6].

The compatibility of an ideal with a topology is the main idea in these studies. An example of an immediate corollary of the $\sigma$-Extension Theorem is the well-known Banach Category Theorem [5]. The second-line works employ ideals to generalize some topological space features, like the separation axioms [7] and compactness [8]. In 1996, Santoro and Cammaroto [9] presented the concept of weakly Lindelöf spaces ($\mathbb{S}_{\mathrm{L}}$). The well-known Lindelöf property naturally deteriorates into the weak Lindelöf property. The fact that we only need the countable set $\hat{W}$ to cover a dense subset of the space $\mathsf{X}$ naturally weakens the Lindelöf criterion. In particular, if there exists a countable $\hat{W}\hat{U}$ such that $\cup\hat{W}$ is dense in $\mathsf{X}$ for each open cover $\hat{U}$ of $\mathsf{X}$, then $\mathsf{X}$ is a weakly $\mathbb{S}_{\mathrm{L}}$ [10].

The traditional idea of $\mathbb{S}_{\mathrm{L}}$s in topology is generalized by the concept of almost $\mathbb{S}_{\mathrm{L}}$s. An almost $\mathbb{S}_{\mathrm{L}}$ is a topological space where a little weaker requirement is satisfied: we just need that each open cover has a "countable refinement" that covers the space, rather than that each open cover has a countable subcover.

**Definition 1.** *The space* $\mathsf{X}$ *is almost Lindelöf if for each family of open subsets*

$$\Big\{U_\alpha\,:\,\alpha\in\Lambda\Big\},$$

*covering it, there exists* $\big\{U'_n\,:\,n\in\mathbb{N}\big\}$ *a countable collection of open sets:*

i) *There exists* $\alpha\in\Lambda\,:\,U'_n\subset U_\alpha$*;*

ii) $\mathsf{X}=\cup_{n\in\mathbb{N}}U'_n$*.*

**Remark 1.** *Important differences between almost Lindelöf and Lindelöf spaces*

a) *Countable subcovers are directly extracted from the original open cover in a* $\mathbb{S}_{\mathrm{L}}$*;*

b) *Before extracting a countable subcover, we decrease the sets (while keeping them open) in an almost* $\mathbb{S}_{\mathrm{L}}$*, allowing us to modify the original open cover.*

As a result, all $\mathbb{S}_{\mathrm{L}}$s are almost Lindelöf, albeit this is not always the case. Moreover, compact spaces are Lindelöf and so, they are almost Lindelöf. $([0,1],\tau_u)$ and $(\mathbb{R},\tau_u)$ are Lindelöf and hence almost $\mathbb{S}_{\mathrm{L}}$s. Examine $\mathbb{R}$ that has the lower limit topology, which is produced by intervals of the $[a,b)$ type $a,b\in\mathbb{R}$. Because some open covers cannot have a countable subcover extracted, this space is not Lindelöf. Nonetheless, it is almost Lindelöf since a countable subfamily spanning $\mathbb{R}$ may be created using the closures of the intervals $[a,b]$.

If each open cover in the space $\mathsf{X}$ has a countable subcollection whose union is dense in $\mathsf{X}$, then the space is weakly Lindelöf. This trait is stronger than separability but weaker than the Lindelöf property. Understanding how compactness, separability, and covering qualities interact in topology requires an understanding of weakly $\mathbb{S}_{\mathrm{L}}$s. The study of functional analysis and general topology naturally leads to weakly $\mathbb{S}_{\mathrm{L}}$s. They offer a helpful compromise between the Lindelöf characteristic and separability. They are especially pertinent in areas like measure theory and function space analysis where density arguments are crucial. Comparison with $\mathbb{S}_{\mathrm{L}}$s: If a countable subcover exists, its union is the entire space (and hence dense), making every $\mathbb{S}_{\mathrm{L}}$, weakly Lindelöf. The opposite isn't true, though. Weakly Lindelöf areas that are not Lindelöf do exist [11].

**Example 1.** i) *Weakly Lindelöf is any separable space. This is due to the fact that a countable subcollection of an open cover whose union is dense may always be constructed from a countable dense subset.*
ii) *With the conventional topology, the real line $\mathbb{R}$ is both Lindelöf and weakly Lindelöf.*
iii) *The true line with the lower limit topology, the Sorgenfrey line, is not Lindelöf but faintly so.*

**Remark 2.** *Relation to Further Topological Features:*

a) *Separability: While the opposite is not true, any separable space is weakly Lindelöf. For instance, the one-point compactification of a discrete uncountable space is weakly Lindelöf but not separable;*

b) *First Countability: Neither first countability nor weak Lindelöfness are a prerequisite for the other;*

c) *Normality: A weakly $\mathbb{S}_{\mathrm{L}}$ that is normal is not always Lindelöf. The Moore (Niemytzki) plane, for instance, is normal and weakly Lindelöf but not Lindelöf.*

Typically, the extent $e(X)$ of $X$ represents the smallest cardinal number $\kappa$ (The smallest cardinal number in set theory is represented by the formula $\kappa = \aleph_\circ = |\mathbb{N}|$ [12]) for which the cardinality of each discrete closed subset of $\mathsf{X}$ is less than or equal $\kappa$. If $A \subset \mathsf{X}$, then $|A|$ denotes the cardinality of $A$. Definitely, $\omega$ is the first infinite cardinal number, $\omega_1$is the first uncountable cardinal number and $c$ represents the cardinality of $\mathbb{R}$. Additionally, Cammaroto and Santoro [9] created an example (see [9, Example 3.11]) demonstrating that there is a Tychonoff weakly $\mathbb{S}_{\mathrm{L}}$ that is not nearly Lindelöf. A generalization or relaxation of the concept is a nearly $\mathbb{S}_{\mathrm{L}}$, which is frequently described in terms of specific "almost" requirements pertaining to Lindelöfness. The exact concept of nearly $\mathbb{S}_{\mathrm{L}}$s depends on the context and the specific generalization being studied.

## 2. Main results

**Definition 2.** *A topological space $\mathsf{X}$ is an almost Lindelöf ideal space modulo $\mathcal{J}$ if each*

$$\ddot{U} = \left\{U_\alpha \,:\, \alpha \in \Lambda\right\},$$

*an open cover of $\mathsf{X}$ admits countable subfamily: $\mathsf{X} = \cup_{n\in\mathbb{N}} \mathrm{Cl}\left(U_{\alpha_n}\right)$ [13].*

**Definition 3.** *The space $\mathsf{X}$ is an almost ideal space modulo $\mathcal{J}$ if every almost Lindelöf subset of $\mathsf{X}$ modulo $\mathcal{J}$ is closed.*

**Theorem 1.** *A topological space* $\mathsf{X}$ *is almost* $\mathbb{S}_{\mathrm{L}}$ *modulo* $\mathcal{J}$ *if and only if*

$$\hat{H} = \Big\{H_\alpha : \alpha \in \Lambda\Big\},$$

*the family of closed subsets of* $\mathsf{X} : \cap_{\alpha\in\Lambda} H_\alpha = \varnothing$ *admits the countable subfamily*

$$\underset{n\in\mathbb{N}}{\cap} H_{\alpha_n} = \varnothing.$$

*Proof.* Let $\hat{H} = \{H_\alpha : \alpha \in \Lambda\}$ be a family consisting of closed subsets of $\mathsf{X} : \cap_{\alpha\in\Lambda} H_\alpha = \varnothing$. So, $\{\mathsf{X} \setminus H_\alpha : \alpha \in \Lambda\}$ covers $\mathsf{X}$. Since $\mathsf{X}$ is almost Lindelöf modulo $\mathcal{J}$, there exists $\{\alpha_1, \alpha_2, \dots\}$ a countable subfamily: $\mathsf{X} = \cup_{n\in\mathbb{N}} \mathrm{Cl}\left(\mathsf{X} \setminus H_{\alpha_n}\right)$. Hence,

$$\underset{n\in\mathbb{N}}{\cap} \mathrm{int}\left(H_{\alpha_n}\right) = \varnothing.$$

On the other hand, if $\ddot{U} = \{U_\alpha : \alpha \in \Lambda\}$ forms an open cover of $X$ modulo $\mathcal{J}$. Therefore, $\{\mathsf{X} \setminus U_\alpha : \alpha \in \Lambda\}$ forms a family of closed subsets of $\mathsf{X} : \cap_{\alpha\in\Lambda}\big(\mathsf{X} \setminus U_\alpha\big) = \varnothing$. Thus, there is $\{\alpha_1, \alpha_2, \dots\}$ a countable subfamily:

$$\underset{n\in\mathbb{N}}{\cap}\big(\mathsf{X} \setminus U_{\alpha_n}\big) = \varnothing,$$

i.e $\mathsf{X} = \cup_{n\in\mathbb{N}} \mathrm{Cl}\left(U_{\alpha_n}\right)$.

**Theorem 2.** *If* $\mathsf{X}$ *is an almost regular* $\mathbb{S}_{\mathrm{L}}$ *modulo* $\mathcal{J}$*, then it is Lindelöf.*

*Proof.* Suppose that $\tilde{U} = \{U_\alpha : \alpha \in \Lambda\}$ forms an open cover of $\mathsf{X}$ for each $U_\alpha$ a regularly open subset. But $\mathsf{X}$ is almost regular Lindelöf modulo $\mathcal{J}$, then for every $a \in \mathsf{X}$, there exists $\alpha_a \in \Lambda : a \in U\alpha_a$. For the regularly open $a$-neighborhood $V_{\alpha_a}$,

$$a \in V_{\alpha_a} \subset \mathrm{Cl}(V_{\alpha_a}) \subset U_{\alpha_a}.$$

For $\mathsf{X}$, being a nearly $L$-closed, there is $\{a_1, a_2, \dots\}$ a countable subfamily :

$$\mathsf{X} = \underset{n\in\mathbb{N}}{\cup} \mathrm{Cl}(V_{\alpha_a}).$$

Thus, $\widetilde{V} = \{V_{\alpha_a} : \alpha \in \Lambda\}$ covers $\mathsf{X}$ regularly because $\mathsf{X}$ is almost modulo $\mathcal{J}$. Therefore, $\mathsf{X}$ is Lindelöf modulo $\mathcal{J}$.

**Corollary 1.** $\mathsf{X}$ *is an almost* $\mathbb{S}_{\mathrm{L}}$ *modulo* $\mathcal{J}$ *if and only if it is nearly Lindelöf.*

**Theorem 3.** $\mathsf{X}$ *is an almost* $\mathbb{S}_{\mathrm{L}}$ *modulo* $\mathcal{J}$ *if and only if for every family* $\tilde{M} = \{M_\alpha : \alpha \in \Lambda\}$ *of closed subsets of* $\mathsf{X}$ *modulo* $\mathcal{J}$*, there is* $\tilde{U} = \{U_\alpha : \alpha \in \Lambda\}$ *a family of open subsets with*

$$\underset{\alpha\in\Lambda}{\cap} \mathrm{Cl}(U_\alpha) = \varnothing : \quad M_\alpha \subset U_\alpha,$$

*and there exists* $\{\alpha_1, \alpha_2, \dots\}$ *a countable subfamily such that* $\cap_{n\in\mathbb{N}} \mathrm{int}(M_{\alpha_n}) = \varnothing$.

*Proof.* `Direct part:` Assume that $\widetilde{M} = \{M_\alpha : \alpha \in \Lambda\}$ is a family of closed subsets of $\mathsf{X}$ modulo $\mathcal{J}$. If $\widetilde{U} = \{U_\alpha : \alpha \in \Lambda\}$ is a family of open subsets with $\cap_{\alpha\in\Lambda} \mathrm{Cl}(U_{\alpha\in\Lambda}) = \varnothing$, then $M_\alpha \in \tilde{U}$. Since $\mathsf{X}$ is regular modulo $\mathcal{J}$, then $M_\alpha \subset U_\alpha \subset \mathrm{Cl}(U_\alpha)$. So,

$$\mathsf{X} \setminus U_\alpha \subset \mathsf{X} \setminus M_\alpha,$$

and $\mathsf{X} = \cup_{\alpha\in\Lambda}\big(\mathsf{X} \setminus M_\alpha\big)$, because of being an almost regular space modulo $\mathcal{J}$. But $\mathsf{X}$ is a $\mathbb{S}_{\mathrm{L}}$ modulo $\mathcal{J}$, there exists a countable subset $\{\alpha_1, \alpha_2, \dots\}$:

$$\mathsf{X} = \bigcup_{n\in\mathbb{N}} \mathrm{Cl}\left(\mathsf{X} \setminus M_{\alpha_n}\right) = \bigcup_{n\in\mathbb{N}}\big(\mathsf{X} \setminus \mathrm{int}(M_{\alpha_n})\big) = \mathsf{X} \setminus \bigcap_{n\in\mathbb{N}} \mathrm{int}(M_{\alpha_n}).$$

Thus, $\cap_{n\in\mathbb{N}} \mathrm{int}(M_{\alpha_n}) = \varnothing$.
`Converse part:` If the family $\widetilde{U} = \{U_\alpha : \alpha \in \Lambda\}$ comprises regular open subsets of $\mathsf{X}$ modulo $\mathcal{J}$ with the empty set $\cap_{n\in\mathbb{N}} \mathrm{int}(M_{\alpha_n})$. Then, for every $\alpha \in \Lambda$, there is $M_\alpha \subset X$ : $M_\alpha \subset U_\alpha$ where $M_\alpha$ is regular and closed modulo $\mathcal{J}$ with the empty set $\cup_{n\in\mathbb{N}} \mathrm{int}(M_{\alpha_n})$. Thus,

$$\bigcap_{n\in\mathbb{N}} \mathrm{Cl}(\mathsf{X} \setminus M_{\alpha_n}) = \bigcap_{n\in\mathbb{N}}\big(\mathsf{X} \setminus \mathrm{int}(M_{\alpha_n})\big) = \mathsf{X} \setminus \bigcap_{n\in\mathbb{N}} \mathrm{int}(M_{\alpha_n}) = \mathsf{X} \setminus \mathsf{X} = \varnothing.$$

Hence, there exists $\{\alpha_1, \alpha_2, \dots\}$ : $\cap_{n\in\mathbb{N}} \mathrm{int}(\mathsf{X} \setminus U_{\alpha_n})$ is empty. Consequently,

$$\mathsf{X} \setminus \bigcup_{n\in\mathbb{N}} \mathrm{Cl}(U_{\alpha_n}) = \varnothing,$$

too and $\mathsf{X} = \cup_{n\in\mathbb{N}} \mathrm{Cl}(U_{\alpha_n})$.

**Definition 4.** $\mathsf{X}$ *is a hereditarily almost* $\mathbb{S}_{\mathrm{L}}$ *modulo* $\mathcal{J}$ *if each of its subspaces is almost* $\mathbb{S}_{\mathrm{L}}$ *modulo* $\mathcal{J}$*.*

**Remark 3.** *If* $\mathsf{X}$ *is a hereditarily almost* $\mathbb{S}_{\mathrm{L}}$ *modulo* $\mathcal{J}$*, then* $\mathsf{X}$ *is countable discrete if and only if* $\mathsf{X}$ *is hereditarily Lindelöf.*

**Definition 5.** $\mathsf{X}$ *is an almost* $\mathbb{S}_{\mathrm{L}}$ *modulo* $\mathcal{J}$ *if each almost regular Lindelöf subset of* $\mathsf{X}$ *modulo* $\mathcal{J}$ *is closed.*

**Definition 6.** *If* $\mathsf{X}$ *is an almost regular space modulo* $\mathcal{J}$*, then a subspace* $\mathsf{Y}$ *is almost regular relative to* $\mathsf{X}$ *modulo* $\mathcal{J}'$ *if for each* $\breve{O} = \{O_\alpha : \alpha \in \Lambda\}$ *the collection of open subsets of* $\mathsf{X}$ *modulo* $\mathcal{J} :\subset \cup_{\alpha\in\Lambda} O_\alpha$*, there exists* $K_\alpha \subset \mathsf{X}$ *a regularly closed subset moulo* $\mathcal{J} : K_\alpha \subset O_\alpha$ *and* $\mathsf{Y} \subset \cup_{\alpha\in\Lambda} \mathrm{int}(K_\alpha)$*. In addition, there exists* $\{\alpha_n : n \in \mathbb{N}\}$ *the countable subset of* $\mathsf{X}$ *modulo* $\mathcal{J} : \mathsf{Y} \subset \cup_{n\in\mathbb{N}} \mathrm{Cl}(O_\alpha)$*.*

**Remark 4.** *If* $\mathsf{X}$ *is almost regular modulo* $\mathcal{J}$*, then* $\mathsf{X}$ $\mathbb{S}_{\mathrm{L}}$ *modulo* $\mathcal{J}$*.*

The remark above is not true for the Urysohn space [14].

**Example 2.** *Let* $\aleph = \{n_\alpha : \alpha < \omega_1\}$, $\grave{\aleph} = \{m_i : i \in \omega\}$ *and* $\mathsf{Y} = \{(n_\alpha, m_i)\}_{\alpha<\omega_1}$ *for each* $i \in \omega$. *If* $\mathsf{X} = \mathsf{Y} \cup \aleph \cup \{n\}$ *for each* $n \notin \mathsf{Y} \cup \aleph$, *then* $\mathsf{X}$ *modulo* $\mathcal{J}$ *is topologized as follows:*

*Each* $y \in \mathsf{Y}$ *is an isolated point. The neighborhood of* $n_\alpha \in \aleph$ *has the form:*

$$U_{n_\alpha}(i) = \{n_\alpha\} \cup \{(n_\alpha, m_i)\}_{\alpha<\omega_1}, \qquad \forall i \in \omega.$$

*The neighborhood of* $n$ *is*

$$U_n(\alpha) = \{n_\alpha\} \cup \{(n_\gamma, m_i) : \gamma > \alpha\}_{\alpha<\omega_1}, \qquad \forall i \in \omega.$$

*Now,* $\mathsf{X}$ *is a not regular Urysohn space modulo* $\mathcal{J}$ *because* $n$ *cannot be separated from the uncountable discrete closed set* $\aleph$. *Hence,* $\mathsf{X}$ *is not a* $\mathbb{S}_{\mathrm{L}}$ *modulo* $\mathcal{J}$. *If* $U$ *covers* $\mathsf{X}$ *modulo* $\mathcal{J}$, *so there exists* $U_n \in U$ *for each* $n \in U_n$ *and there is* $\gamma < \omega_1 : U_n(\gamma) \subset U_n$. *Now,*

$$\{n_\alpha : \alpha > \gamma\} \cup \{n\} \cup \{(n_\gamma, m_i) : \gamma > \alpha\}_{\alpha<\omega_1} \subset \overline{U}_n, \qquad \forall\, i \in \omega,$$

*and* $\mathsf{X} \setminus \overline{U}_n$ *is countable. So there is* $\hat{W} \subset U$ *countable such that* $\mathsf{X} \setminus \overline{U}_n \subset \cup\hat{W}$. *If* $\tilde{O} = \hat{W} \cup \{U_n\}$, *then* $\tilde{O} \subset U$ *is a countable subfamily.*

**Theorem 4.** *If* $\mathsf{X}$ *is almost regular modulo* $\mathcal{J}$, *and* $\mathsf{Y}$ *is an almost regular subspace of* $\mathsf{X}$ *modulo* $\mathcal{J}'$, *then* $\mathsf{Y}$ *is almost regular relative to* $\mathsf{X}$ *modulo* $\mathcal{J}'$.

*Proof.* If $\breve{O} = \{O_\alpha : \alpha \in \Lambda\}$ forms an open cover of $\mathsf{Y}$ modulo $\mathcal{J}'$, then there exists $F_\alpha \subset \mathsf{X}$ regularly closed subset $F_\alpha \subset O_\alpha$, $\mathsf{Y} \subset \cup_{\alpha\in\Lambda} \operatorname{int}(F_\alpha)$, and there is $\{\alpha_n : n \in \mathbb{N}\}$ the countable subset of $\mathsf{X}$ modulo $\mathcal{J}$: $\mathsf{Y} \subset \cup_{n\in\mathbb{N}} \mathrm{Cl}(O_\alpha)$. If

$$U_\alpha = \operatorname{int}(F_\alpha) \cap \mathsf{Y}, \qquad W_\alpha = O_\alpha \cap \mathsf{Y},$$

both are open subsets of $\mathsf{Y}$: $\mathrm{Cl}(U_\alpha)$ is regularly closed subset of $\mathsf{Y}$ modulo $\mathcal{J}'$. So,

$$\mathrm{Cl}(U_\alpha) \subset F_\alpha \cap \mathsf{Y} \subset O_\alpha \cap \mathsf{Y}, \qquad \forall\, \alpha \in \Lambda.$$

Now, $\mathsf{X} = \cap_{\alpha\in\Lambda} U_\alpha$ and $U_\alpha \subset \operatorname{int}(\mathrm{Cl}\,(U_\alpha))$, hence $\mathsf{Y} = \cup_{\alpha\in\Lambda} \operatorname{int}\,(\mathrm{Cl}(U_\alpha))$. Since $\mathsf{Y}$ is almost regular $L$-closed modulo $\mathcal{J}'$,

$$\exists\{\alpha_n : n \in \mathbb{N}\} : \qquad \mathsf{X} = \bigcup_{n\in\mathbb{N}} \mathrm{Cl}(W_{\alpha_n}).$$

But $\mathrm{Cl}(W_{\alpha_n}) \subset \mathrm{Cl}(U_\alpha)$, so $\mathsf{Y} = \cup_{n\in\mathbb{N}} \mathrm{Cl}(O_{\alpha_n})$. As a consequence, $\mathsf{Y}$ is almost regular subspace relative to $\mathsf{X}$ modulo $\mathcal{J}'$.

**Corollary 2.** *If* $\mathsf{X}$ *is an almost regular space modulo* $\mathcal{J}$ *and each of its proper regularly closed subset is almost regular* $L$*-closed modulo* $\mathcal{J}$, *then* $\mathsf{X}$ *is almost regular modulo* $\mathcal{J}$.

**Theorem 5.** *If* $\mathsf{X}$ *is almost regular modulo* $\mathcal{J}$, *then each of its proper clopen subsets is almost regular relative to* $\mathsf{X}$ *modulo* $\mathcal{J}$.

*Proof.* If $K \subset \mathsf{X}$ and $\ddot{U} = \{U_\alpha : \alpha \in \Lambda\}$ covers $K : \forall \alpha \in \Lambda$, there is $A_\alpha$ regularly closed and $A_\alpha \subset U_\alpha$. Since

$$K \subset \bigcup_{\alpha \in \Lambda} \operatorname{int}(A_\alpha),$$

there exists $\{\alpha_1, \alpha_2, \dots\}$ a countable subset such that $K \subset \cup_{n \in \mathbb{N}} \operatorname{Cl}(U_{\alpha_n})$. But,

$$\{U_\alpha\}_{\alpha \in \Lambda} \cup (\mathsf{X} \setminus K),$$

covers $\mathsf{X}$ regularly, and $\mathsf{X}$ is almost regular $L$-closed modulo $\mathcal{J}$. Hence, $\mathsf{X} = \cup_{n \in \mathbb{N}} \big(\mathsf{X} \setminus K\big)$.

**Definition 7.** *The space* $\mathsf{X}$ *modulo* $\mathcal{J}$ *is weakly Lindelöf if for each cover*

$$\tilde{O} = \Big\{O_\beta : \beta \in \Gamma\Big\},$$

*of its subsets, there is* $\{\alpha_1, \alpha_2, \dots\}$ *a countable subset such that* $\mathsf{X} = \operatorname{Cl}\big(\cup_{n \in \mathbb{N}} (O_{\beta_n})\big)$.

**Definition 8.** *The space* $\mathsf{X}$ *modulo* $\mathcal{J}$ *is weakly regular if for each of its covers by the regular open subsets* $\{U_\beta : \beta \in \Gamma\}$, *there exists* $\{\alpha_1, \alpha_2, \dots\}$ *a countable subset:*

$$\mathsf{X} = \operatorname{Cl}(\cup_{n \in \mathbb{N}} (U_{\beta_n})).$$

**Theorem 6.** *The space* $\mathsf{X}$ *modulo* $\mathcal{J}$ *is weakly if and only if for the family*

$$\underset{\sim}{K} = \Big\{K_\beta : \beta \in \Gamma\Big\},$$

*of closed subsets:* $\cap_{\beta \in \Gamma}(K_\beta)$ *is empty, there exists* $\{\alpha_1, \alpha_2, \dots\}$ *a countable subset such that* $\operatorname{int}\big(\cap_{n \in \mathbb{N}} (K_{\beta_n})\big)$ *is empty.*

*Proof.* Let $\underset{\sim}{K} = \{K_\beta : \beta \in \Gamma\}$ consists of closed subsets of $\mathsf{X}$ modulo $\mathcal{J} : \cap_{\beta \in \Gamma}(K_\beta)$. Now, $\mathsf{X} = \cup_{\beta \in \Gamma}(\mathsf{X} \setminus K_\beta)$, there exists $\{\alpha_1, \alpha_2, \dots\}$ a countable subset such that $\mathsf{X} = \operatorname{Cl}(\mathsf{X} \setminus K_{\beta_n})$. Hence, $\mathsf{X} \setminus \operatorname{Cl}(\mathsf{X} \setminus K_{\beta_n}) = \varnothing$. Thus,

$$\operatorname{int}\Big(\mathsf{X} \setminus \bigcup_{n \in \mathbb{N}} (\mathsf{X} \setminus K_{\beta_n})\Big) = \operatorname{int}\Big(\bigcap_{n \in \mathbb{N}} (K_{\beta_n})\Big),$$

is empty.

**Theorem 7.** *There is a Tychonoff weakly* $\mathbb{S}_{\mathrm{L}}$ *modulo* $\mathcal{J}$ *that is not almost Lindelöf.*

*Proof.* If $\mathsf{Y}$ is a discrete space modulo $\mathcal{J} : |\mathsf{Y}| = \omega_1$,

$$\mathsf{X} = \big(\beta\mathsf{Y} \times (\omega + 1)\big) - \big((\beta\mathsf{Y} \setminus \mathsf{Y}) \times \{\omega\}\big),$$

is a subspace of $\beta\mathsf{Y} \times (\omega+1)$ modulo $\mathcal{J}$, and $\ddot{U}$ is an open cover of $\mathsf{X}$ modulo $\mathcal{J}$. $\beta\mathsf{Y} \times \omega$ is a $\sigma$-compact dense in $\mathsf{X}$ modulo $\mathcal{J}$, there exists $\hat{W}$ a countable subset of $\ddot{U} : \beta\mathsf{Y} \times \omega \subset \cup\hat{W}$.

Hence, $\mathsf{X} = \overline{\cup \hat{W}}$ and so $\mathsf{X}$ is a weakly $\mathbb{S}_{\mathrm{L}}$ modulo $\mathcal{J}$. Since $|\mathsf{Y}| = \omega_1$, $\mathsf{Y}$ can be enumerated by $\{y_\gamma : \gamma < \omega_1\}$ and for every $n \in \omega$. Let $U_\gamma = \{y_\gamma\} \times (\omega + 1)$ and $W_n = \beta\mathsf{Y} \times \{n\}$. If

$$\ddot{U} = \Big\{U_\gamma : \gamma < \omega_1\Big\} \cup \Big\{W_n : n \in \omega\Big\},$$

then $\cup \hat{W} = \cup\{\overline{w} : w \in \hat{W}\}$. If

$$\gamma_\circ = \sup\Big\{\gamma : U_\gamma \in \hat{W}\Big\},$$

then $\gamma_\circ < \omega_1$ because $\hat{W}$ is countable. Considering $\gamma' > \gamma_\circ$, then

$$(y_{\gamma'}, \omega) \notin \Big\{\overline{w} : w \in \hat{W}\Big\}.$$

But $(y_{\gamma'}, \omega) \in U_{\gamma'} \subset \ddot{U}$, hence $\mathsf{X}$ is not an almost $\mathbb{S}_{\mathrm{L}}$ modulo $\mathcal{J}$.

**Theorem 8.** *For each infinite cardinal $\kappa$, there is a Tychonoff weakly $\mathbb{S}_{\mathrm{L}}$, $\mathsf{X}$ modulo $\mathcal{J}$ with $e(\mathsf{X}) > \kappa$.*

Although there are other methods for defining nearly Lindelöf ideal spaces, weakening the Lindelöf condition is a popular strategy. The following are two common definitions.

i) Dense Subsets Definition: The ideal space $\mathsf{X}$ is nearly Lindelöf if, for each open cover $\hat{U}$ of $\mathsf{X}$ modulo $\mathcal{J}$, there is a dense subset $D \subseteq \mathsf{X}$ such that a countable subcollection of $\hat{U}$ may cover $D$.

ii) Definition via Nearly Covers: If there exists a countable subcollection $\tilde{O} \subseteq \hat{U}$ where the union of the sets in $\tilde{O}$ is dense in $\mathsf{X}$ modulo $\mathcal{J}$ for each open cover $\hat{U}$ of $\mathsf{X}$, then the topological space $\mathsf{X}$ is nearly $\mathbb{S}_{\mathrm{L}}$ modulo $\mathcal{J}$.

Since any open cover has a countable subcover if $\mathsf{X}$ is Lindelöf ideal space, this trivially satisfies the nearly Lindelöf condition, making any Lindelöf ideal space nearly Lindelöf [15]. According to some definitions, any ideal space that has a countable dense subset— such as separable spaces — is nearly Lindelöf because the dense subset is frequently covered by a countable subcollection of any open cover. Furthermore, if and only if a discrete ideal space is countable, it is Lindelöf. Depending on the definition, anything that is uncountable may nevertheless meet certain weakened kinds of Lindelöfness even though it is not Lindelöf [16]. The study of covering qualities and their generalizations in general topology inevitably leads to the emergence of nearly Lindelöf ideal spaces. They are especially helpful in situations when density-based covering or a weaker type of countability is still preferred but Lindelöfness is an excessively strong condition [17, 18].

## Applications

I) Functional Analysis: The study of function spaces and their topological characteristics involves the use of nearly $L$-closed ideal $\mathbb{S}_{\mathrm{L}}$s.

II) Set-Theoretic Topology: Cardinal invariants and combinatorial principles are examined in relation to these spaces.

III) Generalized Metric Spaces: Nearly Lindelöf ideal spaces serve as a link between more generic ideal topological spaces and metric-like spaces.

**Theorem 9.** *If $\mathsf{X}$ is Lindelöf ideal and $A \subseteq \mathsf{X}$, then the following are equivalent:*
a) *$A$ is an almost Lindelöf subset modulo $\mathcal{J}$.*
b) *$A$ is weakly Lindelöf subset modulo $\mathcal{J}$.*

*Proof.* (a $\Rightarrow$ b) Clear. (b $\Rightarrow$a) If $\mathsf{X}$ is not a $\mathbb{S}_{\mathrm{L}}$ modulo $\mathcal{J}$ and $\tilde{O}$ covers $\mathsf{X}$. Then, for every $O \in \tilde{O}$, $\big\{O \times \{0,1\}\big\}$ covers $A$. Hence, $A$ is not a weakly Lindelöf subset modulo $\mathcal{J}$ because all points of $\mathsf{X} \times \big\{1\big\}$ are isolated in $A$.

## 3. Conclusion

This paper introduces the notions of virtually, weakly, and almost Lindelöf closed ideal topological spaces. In addition to explaining how countable covers impact Almost Lindelöf spaces and focusing on their importance, we also look at their subspaces and the relationship between the subspaces and their topological properties. The ideal spaces are covered by the closures of countable subfamilies that make up these covers. We begin, analyze, and discuss definitions, assertions, characterizations, and observations related to the recently introduced notions of almost and weakly Lindelöf ideal topological spaces. The relationships between different ideal topological spaces are also examined and studied. We illustrate the implications of these new ideal environments.

## Declarations

### Availability of Data and Materials

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

### Competing interests

The authors declare that they have no competing interests.

### Conflict of interest

No conflicts of interest are disclosed by the authors.

### Funding

No particular grant for this study was provided by funding agencies from the commercial, nonprofit sectors, or public.

**Authors' contributions**

All authors have equally contributed, read, and approved the final manuscript.

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