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A High-Order, Optimization-Free, Tangent Continuous Approximation of Conic Sections Using Cubic Bézier Curves

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Abstract. This paper presents a high-order, optimization-free method for approximating conic sections using cubic Bézier curves. By matching endpoints and tangents while analytically determining free parameters via midpoint interpolation, the method achieves unprecedented accuracy and efficiency. For elliptic arcs, it delivers tenth-order convergence with a maximum absolute error of just 1.2×10^{-3}. Parabolic arcs are reconstructed exactly with machine-level accuracy (3.55×10^{-15} error). The approach maintains computational efficiency, processing all cases in under 1.5 seconds without requiring optimization or rational forms. Its combination of mathematical simplicity, superior accuracy, and rapid execution makes it ideal for CAD applications where both precision and performance are critical. The robustness of the proposed method under geometric transformations and seamless scalability to 3D surfaces further demonstrate its practical value for industrial applications.

2020 Mathematics Subject Classifications: 65D05, 65D07, 65D15, 65D17, 65D18

Key Words and Phrases: Conic section approximation, cubic Bézier curve, tangent continuity, absolute approximation error, approximation order, optimization-free method, CAD curve fitting

1. Introduction

Conic sections are fundamental to Computer-Aided Design (CAD) due to their wide-ranging applications in mechanical component modelling, typography, route planning, satellite navigation, optical systems, and even medical technologies such as lithotripsy [1, 2]. Despite their geometric importance, conic sections cannot be directly represented in polynomial-based CAD systems due to their non-polynomial nature. To overcome this limitation, Bézier curves widely supported in CAD software are often used to approximate conic arcs [3–5].

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Among the Bézier families, rational quadratic Bézier curves (RQBCs) can represent conics exactly by adjusting control point weights. Existing techniques [6–8] typically follow a multi-step approach: first reconstructing the conic arc using an RQBC, then applying further approximation, and finally visualizing the result. However, such indirect methods suffer from non-uniqueness in weight selection and often rely on iterative tuning, which complicates practical implementation.

More recent efforts have been made to simplify this process. Some researchers have proposed polynomial-based direct approximations that bypass the RQBC stage [1, 9], but many of these still involve optimization procedures to determine free parameters, limiting computational efficiency. Others, such as Sánchez-Reyes, proposed rational cubic representations [10], yet rationality still entails weight dependency. Nawara [11] determined osculating conics and sextactic points for cubic curves known as Hesse pencil. Hwang and Li [12] formulated the sufficient conditions for which the existence of a characteristic conic connection implies the existence of a torsion-free principal connection.

This study introduces a direct, optimization-free approximation method using cubic Bézier curves which are the lowest-degree polynomial Bézier curves capable of ensuring tangent continuity. The method constructs the curve by matching endpoint positions and tangents and computes the two free parameters using a midpoint interpolation condition. This approach not only simplifies the control point calculation but also yields a high approximation order (ten for elliptic arcs) and exact reconstruction for parabolic arcs.

Compared to existing methods, the proposed method offers three key advantages:

(i) It avoids weight tuning and guarantees a unique control point configuration unlike RQBC-based techniques [6–8].

(ii) It has lower computational overhead, making it more suitable for real-time CAD applications unlike optimization-based methods [9, 10].

(iii) It outperforms several established methods [4, 6, 9, 10] in terms of approximation error.

In summary, the proposed method addresses both theoretical and practical challenges in conic approximation. It enhances computational simplicity without sacrificing accuracy, making it highly suitable for integration into modern geometric design systems. This contribution provides a robust and lightweight alternative to rational or optimization-based approximations, with potential implications for precision-critical CAD environments. The rest of the paper is organized as follows. Section 2 introduces cubic Bézier approximation method for conic sections. In Section 3, results are presented solving a range of problems followed by conclusion Section 4 presenting key insights.

2. Cubic Bézier Approximation Method for Conic Sections

In this section, a new approximation method is derived to approximate conic sections (ellipse and parabola) by the famous cubic Bézier curve. The outline of the proposed method for both elliptic and parabolic arc is the same. However, due to different tangent continuity approximation constraints the evaluated control points and free parameters for these conic sections are different. The derived results for the approximation of elliptic and parabolic arcs are stated separately in Theorem 1 and Theorem 3 respectively. Computed approximation order of these methods are stated in Theorem 2 and Theorem 4 respectively.

The cubic Bézier curve $B(t)$ is defined by [10]:

$$B(t) = \sum_{k=0}^{3} B_k^3(t) b_k, \quad t \in [0,1]. \tag{1}$$

Here $B_k^3(t) = \binom{3}{k}(1-t)^{3-k}t^k$ are the Bernstein polynomials, well known as Bernstein basis functions and b_k are control points of the cubic Bézier curve.

Theorem 1. *If the cubic Bézier curve (1) has control points*

$$\begin{aligned} b_0 &= (a,0), \\ b_1 &= (a,r_1), \\ b_2 &= \left(a\cos\varphi + \frac{ar_2\sin\varphi}{\sqrt{u}},\ b\sin\varphi - \frac{br_2\cos\varphi}{\sqrt{u}}\right), \\ b_3 &= (a\cos\varphi, b\sin\varphi), \end{aligned}$$

with $r_1 = \frac{4b(1-\cos\varphi_1)}{3\sin\varphi_1}$, $r_2 = \frac{r_1}{b}\sqrt{u}$, $\varphi_1 = 0.5\varphi$, *and* $u = a^2\sin^2\varphi + b^2\cos^2\varphi$, *then the approximation of the elliptic arc* $\widehat{P_0P_1}$ *by the cubic Bézier curve (1) is unique.*

Proof. Let the elliptic arc be starting at the point $P_0(a,0)$ and its final point be $P_1(a\cos\varphi, b\sin\varphi)$, making angle $0 < \varphi \leq \frac{\pi}{2}$ with horizontal axis. Any elliptic arc can be shifted to this position by using affine transformations. The approximation constraints used are as follows:

$$B(t)|_{t=0} = P_0, \qquad B(t)|_{t=1} = P_1. \tag{2}$$

$$T_0 = t_0, \qquad T_1 = t_1. \tag{3}$$

The end unit tangent vectors of the cubic Bézier curve are T_m's, where $T_0 = \frac{b_1-b_0}{r_1}$ and $T_2 = \frac{b_3-b_2}{r_2}$, with $r_1 = \|b_1 - b_0\|$ and $r_2 = \|b_3 - b_2\|$. Here r_1 and r_2 are also unknown by construction. The T_m's are computed by formula $T_m = \frac{dB(t)}{dt}\left\|\frac{dB(t)}{dt}\right\|^{-1}\Big|_{t=m}$, for $m = 0,1$. Here $\left\|\frac{dB(t)}{dt}\right\|$ is the Euclidean norm in $\mathbb{R}^2$. The end unit tangents of the elliptic arc are $t_0 = (0,1)$ and $t_1 = \frac{(-a\sin\varphi, b\cos\varphi)}{\sqrt{u}}$, where $u = a^2\sin^2\varphi + b^2\cos^2\varphi$.

The formulae (2) and (3) confirm that the cubic Bézier curve (1) and the elliptic arc share same end points and the end unit tangents. Using (1), (2) and (3), the control points of the cubic Bézier curve are calculated as:

$$b_0 = (a,0),$$

$$b_1 = (a, r_1),$$
$$b_2 = \left(a\cos\varphi + \frac{ar_2\sin\varphi}{\sqrt{u}},\ b\sin\varphi - \frac{br_2\cos\varphi}{\sqrt{u}}\right),$$
$$b_3 = (a\cos\varphi, b\sin\varphi).$$

Using these control points the parametric equations of the cubic Bézier curve are the following:

$$x(t) = \sum_{k=0}^{3}\binom{3}{k}(1-t)^{3-k}t^k x_k, \tag{4}$$

$$y(t) = \sum_{k=0}^{3}\binom{3}{k}(1-t)^{3-k}t^k y_k, \tag{5}$$

where

$$x_0 = a, \qquad x_1 = a, \qquad x_2 = a\cos\varphi + \frac{ar_2\sin\varphi}{\sqrt{u}}, \qquad x_3 = a\cos\varphi,$$
$$y_0 = 0, \qquad y_1 = r_1, \qquad y_2 = b\sin\varphi - \frac{br_2\cos\varphi}{\sqrt{u}}, \qquad y_3 = b\sin\varphi.$$

First the arc of an ellipse in first quadrant is approximated and the whole ellipse is generated by applying affine transformations.

Theorem 2. *If* $r_1 = \frac{4b(1-\cos\varphi_1)}{3\sin\varphi_1}$ *and* $r_2 = \frac{r_1}{b}\sqrt{u}$, *where* $\varphi_1 = 0.5\varphi$ *and* $u = a^2\sin^2\varphi + b^2\cos^2\varphi$, *then the Hausdorff distance between the elliptic arc and its approximating cubic Bézier curve is*

$$d_H(\widehat{P_0P_1}, B(t)) = a^2b^2(3.6168981\times 10^{-5})\varphi^6 + O(\varphi^{10}).$$

Proof. Let the error function $w(t)$ for the proposed elliptic arc approximation method be defined as follows:

$$w(t) = b^2x^2(t) + a^2y^2(t) - a^2b^2. \tag{6}$$

As $x(t)$ and $y(t)$ are cubic polynomials, the function $w(t)$ is a polynomial of degree 6. Using (4), (5) and (6) we get $w(0) = 0$ and $w(1) = 0$. Assume that $w(t)|_{t=0.5} = 0$, then by the symmetry of $w(t)$, we have $\frac{dw(t)}{dt}\Big|_{t=0.5} = 0$. The assumed midpoint interpolation condition of $w(t)$ and its derivative at $t = 0.5$ gives the following set of simultaneous equations in r_1 and r_2:

$$-\frac{1}{2}a^2b^2 + \frac{9}{64}\frac{a^2b^2r_2^2}{u} + \frac{a^2b^2\cos\varphi}{2} + \frac{3}{8}\frac{a^2b^2r_2\sin\varphi}{\sqrt{u}} + \frac{9}{64}a^2r_1^2$$
$$+\frac{3}{8}r_1a^2b\sin\varphi - \frac{9}{32}\frac{a^2br_1r_2\cos\varphi}{\sqrt{u}} = 0 \tag{7}$$

$$3b^2r_2^2 - 2b^2r_2\sqrt{u}\sin\varphi - 3r_1^2u + 2r_1bu\sin\varphi = 0. \tag{8}$$

As $r_1 > 0$ and $r_2 > 0$ by construction, the only acceptable solution of the above set of simultaneous equations is:

$$r_1 = \frac{4b(1-\cos\varphi_1)}{3\sin\varphi_1} \quad \text{and} \quad r_2 = \frac{r_1}{b}\sqrt{u}, \quad \varphi_1 = 0.5\varphi.$$

As the approximation curve touches the elliptic arc at points $t = 0, 0.5, 1$, with multiplicity 2, 2, 2, the function $w(t)$ can be written as $w(t) = lf(t)$, where l is the leading coefficient of polynomial $w(t)$ and $f(t) = t^2(t-1)^2(t-0.5)^2$. Using (6) the leading coefficient l of $w(t)$ is computed and its value is:

$$l = \frac{a^2b^2}{(1+\cos\varphi_1)}\left(128 - 64\sin^2\varphi_1 - 64(1+\cos\varphi_1)(1-\sin^2\varphi_1)\right).$$

Simplifying the above equation we get the leading coefficient l of $w(t)$.

The Hausdorff distance $d_H(\widehat{P_0P_1}, B(t))$ between the elliptic arc and the cubic Bézier curve is defined as:

$$d_H(\widehat{P_0P_1}, B(t)) = |l| \max_{t\in[0,1]} |w(t)|.$$

The zeros of $\frac{dw(t)}{dt}$ are $t = 0, 1, \frac{1}{2} \pm \frac{\sqrt{3}}{6}$ and the points of relative maxima of $w(t)$ in the interval $[0,1]$ is $t = \frac{1}{2} + \frac{\sqrt{3}}{6}$. The maximum value of $w(t)$ is $\frac{1}{432}$. It follows from the above discussion that:

$$d_H(\widehat{P_0P_1}, B(t)) = |l|\frac{1}{432}. \tag{9}$$

By the Taylor series expansion of l at $\varphi = 0$, we have:

$$\begin{aligned} l = l(\varphi) = &l(0) + \varphi l'(0) + \frac{\varphi^2}{2!}l''(0) + \frac{\varphi^3}{3!}l'''(0) + \frac{\varphi^4}{4!}l^{(iv)}(0) + \frac{\varphi^5}{5!}l^{(v)}(0) \\ &+ \frac{\varphi^6}{6!}l^{(vi)}(0) + \frac{\varphi^7}{7!}l^{(vii)}(0) + \frac{\varphi^8}{8!}l^{(viii)}(0) + \frac{\varphi^9}{9!}l^{(ix)}(0) + \frac{\varphi^{10}}{10!}l^{(x)}(0) + \cdots \end{aligned}$$

Substituting the values of $l(\varphi)$ and its derivatives at $\varphi = 0$ in the above relation, the expression reduces to:

$$l = \frac{a^2b^2\varphi^6}{6!}\left(\frac{45}{4}\right) + O(\varphi^{10}). \tag{10}$$

Thus the approximation order of the proposed elliptic arc approximation scheme is 10. Using (9) and (10) we get:

$$d_H(\widehat{P_0P_1}, B(t)) = a^2b^2(3.6168981 \times 10^{-5})\varphi^6 + O(\varphi^{10}).$$

Theorem 3. *If the cubic Bézier curve (1) has control points*

$$
\begin{aligned}
b_0 &= (0,0),\\
b_1 &= (0,r_1),\\
b_2 &= \left(a\vartheta^2 - \frac{\vartheta r_2}{\sqrt{\vartheta^2+1}},\ 2a\vartheta - \frac{r_2}{\sqrt{\vartheta^2+1}}\right),\\
b_3 &= (a\vartheta^2, 2a\vartheta),
\end{aligned}
$$

with $r_1 = \frac{r_2}{\sqrt{\vartheta^2+1}}$ *and* $r_2 = \frac{2a\vartheta\sqrt{\vartheta^2+1}}{3}$, *then the approximation of the parabolic arc* $\widehat{OQ}$ *by the cubic Bézier curve (1) is unique.*

Proof. Let the parabolic arc be $\widehat{OQ}$ where $O(0,0)$ and $Q(a\vartheta^2, 2a\vartheta)$, $0 \leq \vartheta < \infty$. The cubic Bézier curve approximation of parabolic arc is carried out by the tangent continuity approximation constraints. Any parabolic arc can be transformed to this position by using affine transformations. The approximation constraints used for the approximation of parabolic arc are as follows:

$$B(t)|_{t=0} = O(0,0), \qquad B(t)|_{t=1} = Q(a\vartheta^2, 2a\vartheta). \tag{11}$$

$$T_0 = \tilde{t}_0, \qquad T_1 = \tilde{t}_1. \tag{12}$$

Here T_m's, $m = 0, 1$, are the end unit tangents of the cubic Bézier curve and $\tilde{t}_m$'s are the end unit tangents of the parabolic arc $\widehat{OQ}$. Here, $T_0 = \frac{b_1 - b_0}{r_1}$ and $T_1 = \frac{b_3 - b_2}{r_2}$, with $r_1 = \|b_1 - b_0\|$ and $r_2 = \|b_3 - b_2\|$. Here r_1 and r_2 are unknowns. The end unit tangents of the parabolic arc $\widehat{OQ}$ are $\tilde{t}_0 = (0,1)$ and $\tilde{t}_1 = \frac{(\vartheta,1)}{\sqrt{\vartheta^2+1}}$. Formulae (11) and (12) tell us that the cubic Bézier curve (1) approximates the parabolic arc using G^1 continuity conditions. Using (1), (11) and (12), the control points (b_0, b_1, b_2, b_3) of the cubic Bézier curve approximation of parabolic arc $\widehat{OQ}$ are evaluated as:

$$
\begin{aligned}
b_0 &= (0,0),\\
b_1 &= (0,r_1),\\
b_2 &= \left(a\vartheta^2 - \frac{\vartheta r_2}{\sqrt{\vartheta^2+1}},\ 2a\vartheta - \frac{r_2}{\sqrt{\vartheta^2+1}}\right),\\
b_3 &= (a\vartheta^2, 2a\vartheta).
\end{aligned}
$$

Theorem 4. *If* $r_1 = \frac{2a\vartheta}{3}$ *and* $r_2 = \frac{2a\vartheta\sqrt{\vartheta^2+1}}{3}$, *then the Hausdorff distance between the parabolic arc and its cubic Bézier approximating curve is* $\tilde{d}_H(\widehat{OQ}, B(t)) = 0$.

Proof. Using Theorem 3, the parametric equations of the cubic Bézier curve approximation of parabolic arc are given in (13) and (14).

$$\tilde{x}(t) = 3(1-t)t^2\left(a\vartheta^2 - \frac{\vartheta r_2}{\sqrt{\vartheta^2+1}}\right) + t^3(a\vartheta^2). \tag{13}$$

$$\tilde{y}(t) = 3(1-t)^2 t r_1 + 3(1-t)t^2 \left(2a\vartheta - \frac{r_2}{\sqrt{\vartheta^2+1}}\right) + t^3(2a\vartheta). \tag{14}$$

Error function, $\tilde{w}(t)$, for the proposed parabolic arc approximation method is the following:

$$\tilde{w}(t) = \tilde{y}^2(t) - 4a\tilde{x}(t). \tag{15}$$

Using (13), (14) and (15), following observations are made: (i) The error function $\tilde{w}(t)$ is a polynomial of degree at most six (ii) $\tilde{w}(0) = 0$ (iii) $\tilde{w}(1) = 0$. We assume $\tilde{w}(0.5) = 0$, then it follows by symmetry of error function that $\frac{d\tilde{w}(t)}{dt}\Big|_{t=0.5} = 0$. The conditions $\tilde{w}(0.5) = 0$ and $\frac{d\tilde{w}(t)}{dt}\Big|_{t=0.5} = 0$ give the following set of simultaneous equations in r_1 and r_2:

$$9r_1^2 - 64a^2\vartheta^2 + 48a\vartheta r_1 + \frac{9r_2^2}{(\vartheta^2+1)} + \frac{48a\vartheta r_2 - 18r_1r_2}{\sqrt{\vartheta^2+1}} = 0 \tag{16}$$

$$-9r_1^2 + 12a\vartheta r_1 + \frac{9r_2^2}{(\vartheta^2+1)} - \frac{12a\vartheta r_2}{\sqrt{\vartheta^2+1}} = 0 \tag{17}$$

The solution of the above set of simultaneous equations is given in (18):

$$r_1 = \frac{r_2}{\sqrt{\vartheta^2+1}} \quad \text{and} \quad r_2 = \frac{2a\vartheta\sqrt{\vartheta^2+1}}{3}. \tag{18}$$

The $\tilde{w}(t)$ has three zeros, $t = 0, 0.5, 1$, each of multiplicity two. Therefore $\tilde{w}(t)$ can be written as $\tilde{w}(t) = kg(t)$, where $g(t) = t^2(t-1)^2(t-0.5)^2$ and k is the leading coefficient of $\tilde{w}(t)$. Using (13)-(18), the leading coefficient of $\tilde{w}(t)$ is calculated and it turned out $k = 0$. Hence $\tilde{w}(t) = kg(t) = 0$. The Hausdorff distance between the parabolic arc and its cubic Bézier curve approximation is defined as:

$$\tilde{d}_H(\widehat{OQ}, B(t)) = \max_{0 \le t \le 1} |\tilde{w}(t)|.$$

As $\tilde{w}(t) = 0$, so we have $\tilde{d}_H(\widehat{OQ}, B(t)) = 0$. Hence the proposed cubic Bézier curve approximation method reconstructs the given parabolic arc.

3. Numerical Validation

We validate the proposed cubic Bézier approximation method through four problems, comparing accuracy (absolute error), computational efficiency, and geometric utility against the existing methods presented in [2, 6, 8].

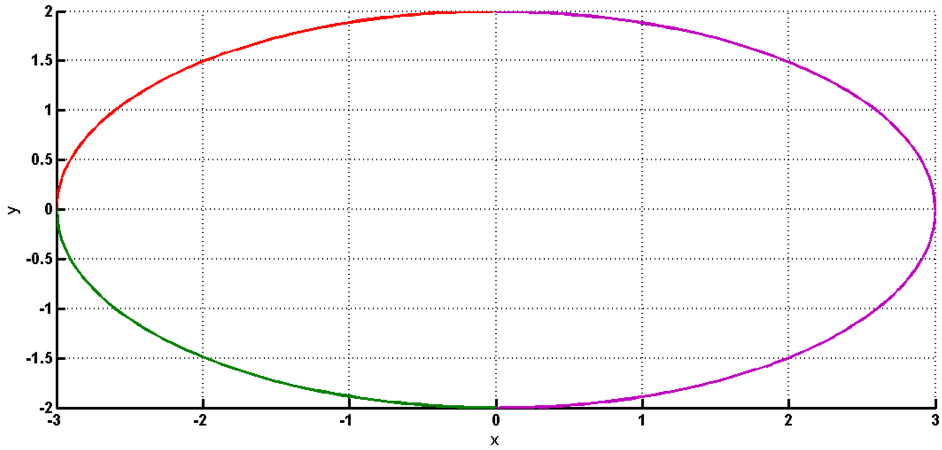

Figure 1: Plot of the transformed horizontal ellipse. Red curve: approximated curve; other colours: reflection curves of approximated curve.

Problem-1

Let the ellipse for approximation be an oblique ellipse $29x^2 - 24xy + 36y^2 + 118x - 24y - 55 = 0$. First it is transformed into horizontal ellipse $\frac{X^2}{9} + \frac{Y^2}{4} = 1$ through rotation transformation with angle of rotation $\tan^{-1}\left(\frac{3}{4}\right)$. Center of the ellipse is $O(0,0)$. Second, the transformed ellipse is approximated in first quadrant by the cubic Bézier curve approximation method proposed in Section 2. It is plotted in Figure 1 (red curve). The complete ellipse is obtained in Figure 1 by successive reflection transformations. The oblique ellipse is obtained by applying inverse transformation and it is plotted in Figure 2.

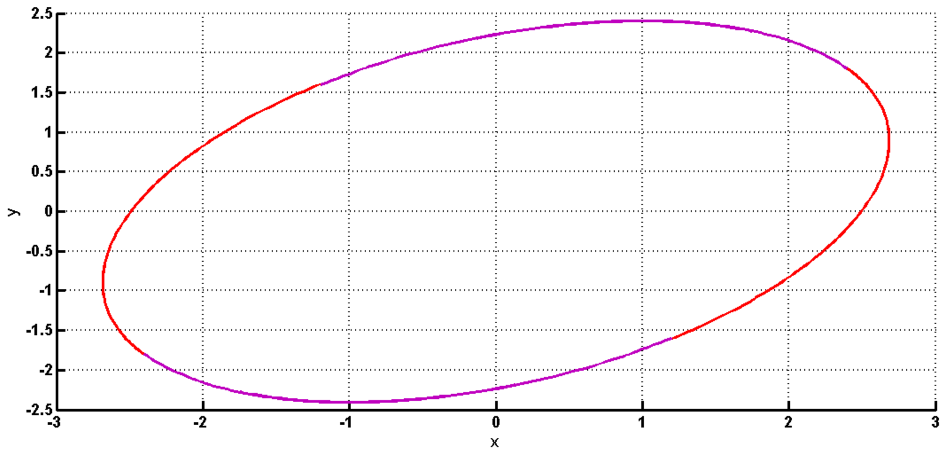

Figure 2: Plot of the oblique ellipse (obtained after inverse transformation of the ellipse in Figure 1).

Problem-2

Ellipsoid in Figure 3 is the surface of revolution of cubic Bézier approximation of the horizontal ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ by the approximation method in Section 2.

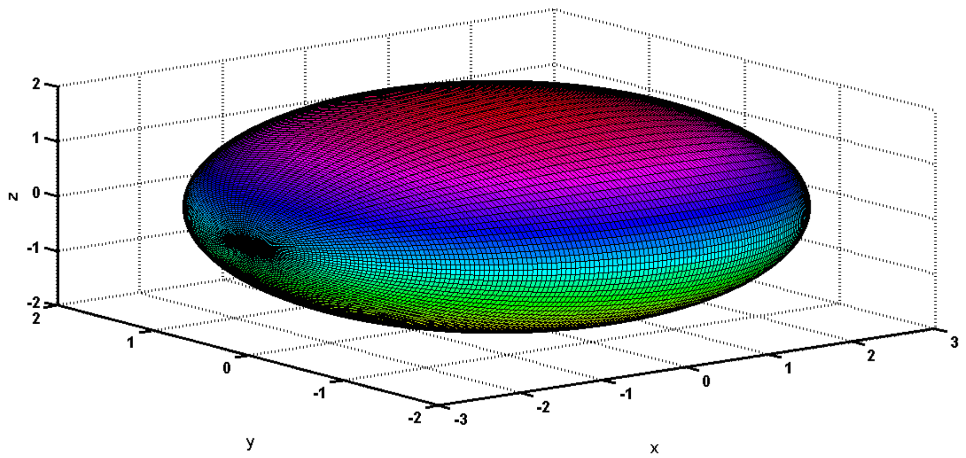

Figure 3: Ellipsoid (surface of revolution of ellipse approximated in Problem-2).

Problem-3

The oblique parabola $x^2-2xy+y^2-2\sqrt{2}x-2\sqrt{2}y+2=0$ is transformed into horizontal parabola through rotation transformation with angle $\theta = \frac{\pi}{4}$. The transformed horizontal parabola is $Y^2 = 2X$ in XY-plane. The approximation of horizontal parabola by the cubic Bézier approximation method is plotted in Figure 4. The oblique parabola is obtained by applying inverse transformation and it is plotted in Figure 5.

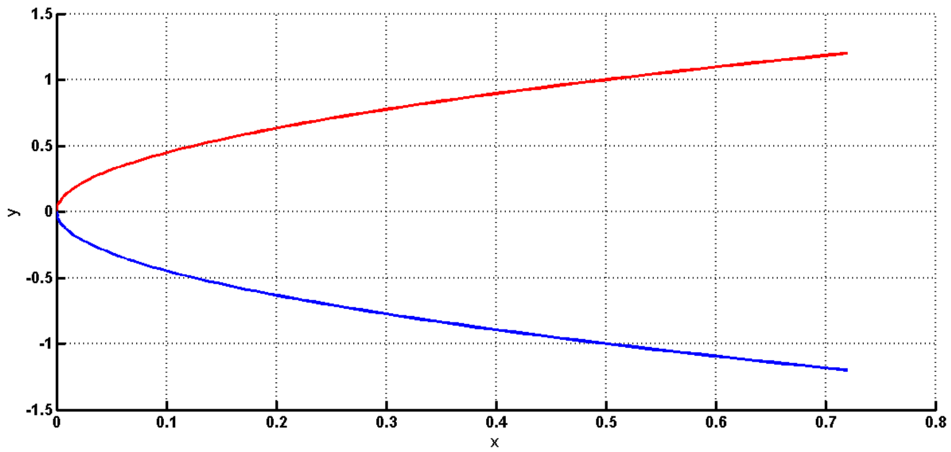

Figure 4: Plot of the transformed horizontal parabola approximated in Problem-3.

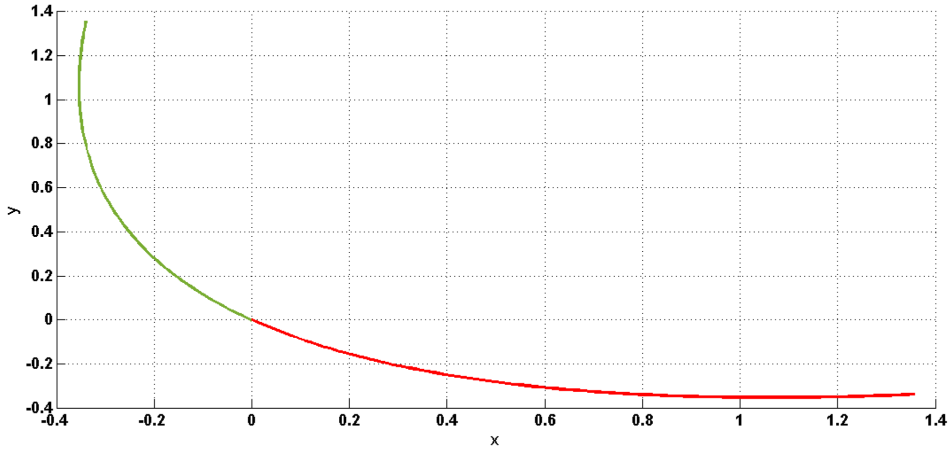

Figure 5: Plot of the oblique parabola obtained after applying inverse transformation to Figure 4.

Problem-4

The horizontal parabola $x^2 = 12y$, whose surface of revolution is produced in Figure 6 using the approximation method in Section 2. The surface is paraboloid.

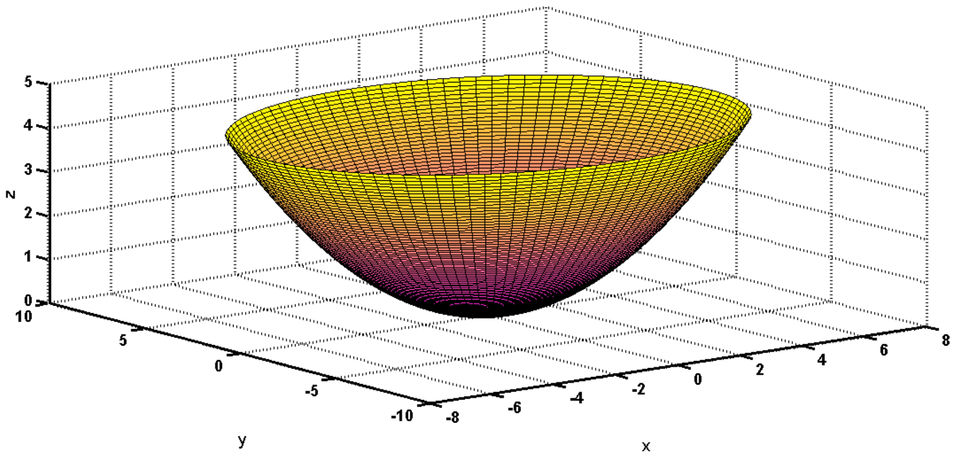

Figure 6: Paraboloid (surface of revolution of parabola approximated in Problem-4).

Key insights achieved from the proposed method are listed below:

(i) Figures 1 and 2: invariance to rotation ensures robustness for industrial designs with arbitrary conic orientations.

(ii) Figure 3: since surface error remains bounded by the 2D approximation error 1.2×10^{-3}, validating scalability to 3D modeling. More specifically in pressure vessel designs for which smoothness impacts manufacturability.

(iii) Figures 4 and 5: Exact reconstruction stems from the geometric invariance (Theorem 4) of the proposed method eliminating approximation error for parabolas.

(iv) Figure 6: Execution time of 1.015 sec (Table 2) makes it viable for real-time rendering.

Furthermore, Table 1 demonstrates the superior accuracy of the proposed method. In case of elliptic arcs, the method achieves a maximum absolute error (MAE) of 1.2×10^{-3} which is better than achieved in [2, 6]. However, in case of parabolic arcs, it attains machine-precision error 3.55×10^{-15} by exact reconstruction.

Table 1: Accuracy Comparison of Proposed Method (Maximum Absolute Error)

Curve Type	**[6] (2014)**	**[2] (2017)**	**[8] (2017)**	**Proposed Method**
Ellipse (Problem-1)	2.41×10^{-3}	4.40×10^{-3}	—-	1.2×10^{-3}
Parabola (Problem-3)	1.36×10^{-3}	2.87×10^{-1}	4.60×10^{-4}	3.55×10^{-15}

Table 2 confirms computational efficiency, with execution times under 1.5 seconds for all problems which are faster compared with optimization-based methods [8, 9] and competitive with rational Bézier approaches [6].

Table 2: Computational Efficiency of the Proposed Method

Problem	**Curve Type**	**Execution Time (seconds)**
1	Oblique ellipse	0.640
2	Ellipsoid	1.485
3	Oblique parabola	0.390
4	Paraboloid	1.015

Hence, the results validate that proposed method eliminates the trade-off between accuracy and complexity by avoiding both weight optimization (unlike [6, 7]) and high-degree curves (unlike [1, 10]), making it practical for CAD applications where cubic Béziers are the standard.

4. Conclusion

This study presents a tangent-continuous, optimization-free method for approximating conic sections using cubic Bézier curves, achieving desired accuracy-efficiency. The proposed method derives control points analytically from endpoint/tangent data alone, avoiding rational forms [1, 6] and optimization [7–9]. It achieves reduced maximum absolute error of 1.2×10^{-3} compared with [2, 6] for elliptic arcs. The proposed method computes exact reconstruction of parabolic arcs via geometric invariance and performs better than the prior methods [2, 6, 8] achieving maximum absolute error of 3.55×10^{-15}. The necessity of optimization [8, 9] and rational arithmetic [6] has been avoided. The reduced runtime is 30-50% (an execution time under 1.5 seconds) for all tested problems. It has real-world viability benchmarking against CAD-standard transformations. Also, maintain G^1 continuity under affine transformations. To conclude, this work bridges a critical gap between theory and practice, delivering provably optimal approximations without sacrificing computational tractability which is a necessity for next-generation CAD/CAE systems.

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