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Estimates for the Coefficients of Subclasses Defined by the $\mathfrak{q}$-Babalola Convolution Operator of Bi-Univalent Functions Subordinate to the $\mathfrak{q}$-Fibonacci Analogue

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Abstract. In this work, we introduce and investigate a new subclass of bi-univalent functions defined via the q-Babalola operator and the q-Fibonacci analogue. The q-Babalola operator generalizes classical convolution-type operators in the context of q-calculus, enabling the analysis of geometric properties of analytic functions under quantum calculus frameworks. Meanwhile, the q-Fibonacci analogue extends the classical Fibonacci sequence into the realm of q-theory, offering new structural insights and recursive behavior in analytic function theory. For functions in this subclass, we derive sharp coefficient bounds for the initial Taylor coefficients $|a_2|$ and $|a_3|$. Furthermore, we address the Fekete-Szegö functional problem associated with this class. The interplay between q-calculus and bi-univalent function theory revealed through our approach yields several novel and significant results, enriching the geometric function theory literature with new analytical tools and perspectives.

2020 Mathematics Subject Classifications: 30A36, 30C45, 81P68, 11B37

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1. Introduction

We begin by considering the collection $\mathcal{A}$ of functions that are complex analytic within the open unit disk $\mathbb{U}$. This domain is defined as

$$\mathbb{U} = \{z = \mathfrak{a} + i\ \mathfrak{b} \in \mathbb{C} \ \text{ where } \ \mathfrak{a},\ \mathfrak{b} \in \mathbb{R},\ \text{and } |z| < 1\},$$

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which geometrically corresponds to the interior of the unit circle in the complex plane, centered at the origin and excluding its boundary.

All functions $f \in \mathcal{A}$ are subject to a standard normalization, namely:

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

These initial conditions eliminate translational and scaling ambiguities, ensuring each function is uniquely defined at the origin with a prescribed rate of change. This allows for coherent structural analysis and comparison of such functions under common geometric constraints, see [1].

Each member $f \in \mathcal{A}$ possesses a Maclaurin series representation about the origin, which can be written as:

$$f(z) = z + \sum_{n=2}^{\infty} a_n\, z^n, \quad \text{for } z \in \mathbb{U}, \tag{1}$$

where the coefficients a_n determine the nonlinear components of f. The leading term z arises from the derivative condition $f'(0) = 1$, and subsequent terms capture the analytic structure beyond linearity.

A function f is called a *Schwarz function* if it is analytic throughout $\mathbb{U}$, satisfies $f(0) = 0$, and its modulus remains strictly less than one within the disk, i.e., $|f(z)| < 1$ for all $z \in \mathbb{U}$. These functions are of central importance in geometric function theory, particularly in the context of conformal and univalent mappings.

Furthermore, for any two functions $f_1, f_2 \in \mathcal{A}$, the function f_1 is said to be *subordinate* to f_2, denoted $f_1 \prec f_2$, if there exists a Schwarz function η such that

$$f_1(z) = f_2(\eta(z)) \quad \text{for all} \quad z \in \mathbb{U}.$$

This relation implies that f_1 is functionally dependent on f_2 through composition with η, preserving analyticity while embedding geometric structure, see [2]. The notion of subordination is a key analytical tool for examining inclusion relations, growth estimates, and mapping behavior in complex analysis.

In addition, let us consider the subclass S, $\mathsf{S} \subset \mathcal{A}$, which comprises all functions that are univalent (i.e., one-to-one) within the unit disk $\mathbb{U}$. We also introduce the class P, defined as the family of functions in $\mathcal{A}$ whose real parts are strictly positive throughout $\mathbb{U}$. A typical function $\varphi \in \mathsf{P}$ admits the following power series expansion:

$$\mathsf{p}(z) = 1 + \sum_{n=1}^{\infty} \mathsf{p}_n z^n = 1 + \mathsf{p}_1 z + \mathsf{p}_2 z^2 + \mathsf{p}_3 z^3 + \ldots, \quad (z \in \mathbb{U}). \tag{2}$$

where the coefficients satisfy the sharp bound

$$|\mathsf{p}_n| \leq 2, \quad \text{for all} \quad n \geq 1. \tag{3}$$

In accordance with the classical Caratheodory lemma (refer to [1] for further details). Furthermore, a function $\varphi \in \mathsf{P}$ if and only if it is subordinate to the Mobius transformation $\frac{1+z}{1-z}$, i.e.,

$$\varphi(z) \prec \frac{1+z}{1-z}, \quad z \in \mathbb{U}.$$

The class of starlike functions, denoted S^*, can be characterized in various ways using subordination techniques. A notable generalization was proposed by Ma and Minda [3], who defined the following class:

$$\mathsf{S}^*(\Omega) = \left\{ f \in \mathcal{A} : \frac{z\, f'(z)}{f(z)} \prec \Omega(z), \quad \text{where} \quad \Omega \in \mathsf{P} \ \text{ and } \ z \in \mathbb{U} \right\}.$$

In this formulation, Ω is assumed to be analytic in $\mathbb{U}$ and possess a positive real part throughout the disk. Table 1 provides a variety of subclasses of S^*, arising from specific choices of the function Ω, reflecting the diversity of approaches adopted in the literature for constructing refined categories of starlike mappings. The class P forms the

Table 1: Enumerates various starlike function classes characterized via the principle of subordination.

	The subclasses of starlike functions	**Ref.**	**Author**
1	$\mathsf{S}^*\left(\frac{1+z}{1-z}\right) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} \right\}$	[4]	Janowsk
2	$\mathsf{S}^*(\vartheta) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\vartheta)z}{1-z} \right\}, \quad \text{where } 0 \leq \vartheta < 1$	[5]	Robertson
3	$\mathsf{SL}(\vartheta) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+\vartheta^2 z^2}{1-\vartheta z-\vartheta^2 z^2} \right\}, \quad \text{where } \vartheta = \frac{1-\sqrt{5}}{2}$	[6]	Sokol

cornerstone for the development of numerous significant subclasses of analytic functions, making it a key object of study in complex analysis. For any function f in the subclass $\mathsf{S} \subset \mathcal{A}$, there exists an inverse function, denoted f^{-1}, which is defined as

$$z = f^{-1}(f(z)) \ \text{ and } \ z = f(f^{-1}(z)), \quad (\mathsf{r}_0(f) \geq 0.25; \ \ |z| < \mathsf{r}_0(f); z \in \mathbb{U}) \,. \tag{4}$$

where

$$\hbar(\xi) = f^{-1}(\xi) = \xi - a_2\xi^2 + \left(2a_2^2 - a_3\right)\xi^3 - \left(5a_2^3 + a_4 - 5a_3a_2\right)\xi^4 + \cdots \,. \tag{5}$$

function $f \in \mathsf{S}$ is said to be bi-univalent if its inverse function $f^{-1} \in \mathsf{S}$. The subclass of S denoted by Σ contains all bi-univalent functions in $\mathbb{U}$. A table illustrating certain functions within the class Σ and their inverse functions is provided below.

Table 2: Representative examples of bi-univalent functions along with their corresponding inverse functions.

f	f^{-1}
$f_1(z) = \frac{z}{1+z}$	$f_1^{-1}(\xi) = \frac{\xi}{1-\xi}$
$f_2 = -\log(1-z)$	$f_2^{-1}(\xi) = \frac{e^{2\xi}-1}{e^{2\xi}+1}$
$f_3 = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$	$f_3^{-1}(\xi) = \frac{e^{\xi}-1}{e^{\xi}}$

Quantum calculus, or q-calculus, has become an influential tool in geometric function theory, offering an alternative to classical calculus that avoids the traditional limit process. Initiated by Jackson [7, 8] through the introduction of the q-difference operator and its integral, this framework was later expanded by Aral and Gupta [9–11] to include q-analogues of classical operators relevant to analytic function theory. Central to q-calculus is the deformation parameter $q \in (0, 1)$, which ensures convergence and structural integrity of function classes, particularly those defined by geometric properties such as starlikeness and convexity. This approach has opened new directions for both theoretical exploration and application in complex analysis.

Definition 1. *[12] The q-bracket $\lceil\lambda\rfloor_q$ is defined as follows:*

$$\lceil\lambda\rfloor_q = \begin{cases} \frac{1-q^{\lambda}}{1-q}, & 0<q<1,\ \lambda \in \mathbb{C}^* = \mathbb{C}\setminus\{0\} \\ 1, & q \mapsto 0^+,\ \lambda \in \mathbb{C}^* \\ \lambda, & q \mapsto 1^-,\ \lambda \in \mathbb{C}^* \\ q^{\gamma-1} + q^{\gamma-2} + \cdots + q + 1 = \sum\limits_{n=0}^{\gamma-1} q^n, & 0<q<1,\ \lambda = \gamma \in \mathbb{N}, \end{cases}$$

with the useful identity $\lceil\lambda+1\rfloor_q = \lceil\lambda\rfloor_q + q^{\lambda}$. Resided that the q-factorial $[\lambda]_q!$ is defined by

$$\lceil\lambda\rfloor_q! = \begin{cases} \prod\limits_{n=1}^{\lambda} \lceil n\rfloor_q = \lceil\lambda\rfloor_q . \lceil\lambda-1\rfloor_q \cdots \lceil 3\rfloor_q . \lceil 2\rfloor_q . \lceil 1\rfloor_q & ,(\lambda \in \mathbb{N}) \\ 1 & ,\quad \lambda = 0, \end{cases}$$

Definition 2. *[12] The $q-$derivative, also known as the $q-$difference operator, of a function f is defined by*

$$\eth_q\langle f(z)\rangle = \begin{cases} (f(z) - f(q\,z))(z - q\,z)^{-1}, & \text{if} \quad 0<q<1,\ z \neq 0, \\ f'(0), & \text{if} \quad z = 0, \\ f'(z), & \text{if} \quad q \mapsto 1^-,\ z \neq 0. \end{cases} .$$

Remark: For $f \in \mathcal{A}$ of the form (1), it is straightforward to verify that

$$\eth_q\langle f(z)\rangle = \eth_q \left\langle z + \sum_{n=2}^{\infty} a_n\, z^n \right\rangle = 1 + \sum_{n=2}^{\infty} \lceil n\rfloor_q a_n\, z^{n-1}, \quad (z \in \mathbb{U}),$$

and for the inverse function $\hbar = f^{-1}$ of the form (4), we have

$$\eth_q\langle \hbar(\xi)\rangle = 1 - \lceil 2\rfloor_q a_2\xi + \lceil 3\rfloor_q \left(2a_2^2 - a_3\right)\xi^2 - \lceil 4\rfloor_q \left(5a_2^3 + a_4 - 5a_3a_2\right)\xi^3 + \cdots .$$

More recently, Al-shbeil et al. [13] defined the q-Babalola convolution operator $\mathscr{D}_z^{a,\alpha,\beta} : \mathcal{A} \to \mathcal{A}$ by

$$\mathscr{D}_q^{\chi} f(z) = z + \sum_{n=2}^{\infty} (\lambda]_q^c\, a_n z^n, \tag{6}$$

where

$$(\lambda]_q^c = \frac{1 + q(a) + q^2(a) + \cdots + q^{n-1}(a)}{1 + q(\chi) + q^2(\chi) + \cdots + q^{n-1}(\chi)},\ (\chi = a - t > -1).$$

$$q^{n-1}(a) = \frac{q^{n-1}\,(a+n-2)!}{(n-1)!\,(a-1)!} \quad \text{and} \quad q^{n-1}(\chi) = \frac{q^{n-1}\,(\chi+n-2)!}{(n-1)!\,(\chi-1)!}.$$

In a more recent advancement, Alsoboh et al. [14] introduced a noteworthy class of functions known as q-starlike functions, denoted by SL_q, which were defined using the q-Jackson difference operators. The formal definition of this class is given by

$$\mathsf{SL}_q = \left\{ f \in \mathcal{A} : \quad \frac{z\,\eth_q\langle f(z)\rangle}{f(z)} \prec \Upsilon(z;q), \quad z \in \mathbb{U} \right\}, \tag{7}$$

where the function $\Upsilon(z;q)$ is expressed explicitly as

$$\Upsilon(z;q) = \frac{1 + q\vartheta_q^2 z^2}{1 - \vartheta_q z - q\vartheta_q^2 z^2}, \tag{8}$$

and

$$\vartheta_q = \frac{1 - \sqrt{4q+1}}{2q} \tag{9}$$

represents the q-analog of the Fibonacci numbers. Additionally, Alsoboh et al. [14] established a significant connection between these q-Fibonacci numbers, denoted as ϑ_q, and the related Fibonacci polynomials $\varphi_n(q)$. Specifically, they demonstrated that if

$$\Upsilon(z;q) = 1 + \sum_{n=1}^{\infty} \widetilde{\mathsf{p}_n} z^n,$$

the coefficients $\widetilde{\mathsf{p}_n}$ satisfy the following recurrence relation:

$$\widetilde{\mathsf{p}_n} = \begin{cases} \vartheta_q, & \text{for } n = 1, \\ (2q+1)\vartheta_q^2, & \text{for } n = 2, \\ (3q+1)\vartheta_q^3, & \text{for } n = 3, \\ \left(\varphi_{n+1}(q) + q\varphi_{n-1}(q)\right)\vartheta_q^n, & \text{for } n \geq 4. \end{cases} \tag{10}$$

Here, the q-Fibonacci polynomials $\varphi_{\mathsf{s}}(q)$ are defined as

$$\varphi_{\mathsf{s}}(q) = \frac{(1 - q\vartheta_q)^{\mathsf{s}} - (\vartheta_q)^{\mathsf{s}}}{\sqrt{4q+1}}, \quad \mathsf{s} \in \mathbb{N}. \tag{11}$$

This research presents a thorough framework for examining the relationship between the q-modified Fibonacci numbers and their corresponding polynomial representations.

The initial terms of the q-Fibonacci sequence, which constitutes a natural generalization of the classical Fibonacci numbers and converges to them as $q \to 1^-$, are enumerated in Table 3.

Table 3: Comparison of the classical Fibonacci numbers with their corresponding q-analogue terms from the q-Fibonacci sequence.

The classical Fibonacci numbers	The q-analogue of Fibonacci numbers
$\varphi_0 = 0$	$\varphi_0(q) = 0$
$\varphi_1 = 1$	$\varphi_1(q) = 1$
$\varphi_2 = 1$	$\varphi_2(q) = 1$
$\varphi_3 = 2$	$\varphi_3(q) = 1 + q$
$\varphi_4 = 3$	$\varphi_4(q) = 1 + 2q$

It is noteworthy that the function $\Upsilon(z;q)$ is not injective over the domain $\mathbb{U}$. Specifically, there exist distinct points in $\mathbb{U}$ at which $\Upsilon(z;q)$ attains the same value. For instance,

$$\Upsilon(0;q) = 1 \quad \text{and} \quad \Upsilon\left(-\frac{1}{2q\vartheta_q};q\right) = 1.$$

In the following example, we explore the behavior of the q-starlike functions as the parameter q approaches 1 from below. This transition leads to the classical case of starlike functions, often referred to as the class SL. By taking the limit as $q \to 1^-$, we observe how the q-starlike functions generalize to the traditional starlike functions, and the associated function $\Upsilon(z)$ simplifies to a form that connects directly with the classical Fibonacci numbers. This example illustrates the connection between the q-starlike functions and their classical counterparts.

Example 1. *[6] To illustrate the asymptotic behavior of the q-starlike functions as $q \to 1^-$, we examine the limiting case of the class SL_q. In the limit, this class converges to the classical starlike function class associated with the Fibonacci generating function, namely*

$$\mathsf{SL} = \lim_{q \to 1^-} \mathsf{SL}_q = \left\{ f \in \mathcal{A} \;:\; \frac{z\,f'(z)}{f(z)} \prec \Upsilon(z) \right\},$$

where the function $\Upsilon(z)$ *is given by*

$$\Upsilon(z;1) = \Upsilon(z) = \frac{1+\vartheta^2 z^2}{1-\vartheta z-\vartheta^2 z^2}, \tag{12}$$

and $\vartheta = \frac{1-\sqrt{5}}{2}$ *denotes the classical Fibonacci constant.*

In addition to introducing the class of q-starlike functions, Alsoboh et al. [14] further extended the framework by defining a novel class of analytic functions termed the q-convex class, denoted by KSL_q. This class is characterized by a subordination condition analogous to that of the q-starlike class, but involves the application of a second-order q-difference operator, thereby capturing a more nuanced geometric structure. Specifically, a function f is said to belong to the class KSL_q if and only if the following subordination condition is satisfied:

$$1 + \frac{z\,\eth_q^2\langle f(z)\rangle}{\eth_q\langle f(z)\rangle} \prec \Upsilon(z;q), \qquad (z \in \mathbb{U}), \tag{13}$$

where the function $\Upsilon(z;q)$ is defined by the rational expression in (8), and the parameter ϑ_q is specified in (9).

Fractional calculus (FC) operators have been extensively utilized in a wide array of applied scientific disciplines, including but not limited to the study of geometric function theory, as highlighted in [15]. One notable extension of classical fractional calculus is fractional q-calculus, which integrates the discrete framework of q-analysis with the principles of fractional differentiation and integration. This fusion has led to significant advancements and applications across various mathematical and engineering fields. Specifically, fractional q-calculus has been employed to address complex problems such as optimal control systems, q-difference equations, q-integral equations, and more traditional areas of fractional calculus. Comprehensive treatments of these applications can be found in specialized literature, including the foundational text [16] and recent scholarly contributions [17–19]. The emergence of q-calculus has profoundly enriched the study of analytic function theory by facilitating the development and exploration of new subclasses of functions with rich geometric, algebraic, and topological properties. This analytical framework reveals the inherent flexibility of q-calculus, showcasing its potential to generalize classical theories, uncover novel mathematical structures, and provide deep theoretical insights. The implications of these developments are far-reaching, influencing both abstract theory and practical applications. As such, q-calculus serves not only as a bridge between classical analysis and modern mathematical techniques but also as a fertile ground for further research and innovation in the field [20–34].

2. Definition and examples

This section begins by defining the subclass $\mathsf{SLM}_\Sigma(\beta,\chi,\psi;q)$, which is specifically linked to the q-Babalola convolution operator and q-Fibonacci sequence.

Definition 3. *For* $0 < q < 1$, $\beta \in \mathbb{C} \setminus \{0\}$ *and* $\psi \geq 0$. *A function* $f \in \Sigma$ *given by (1) is said to be in the class* $\mathsf{SLM}_{\Sigma}(\beta, \chi, \psi; q)$ *if the following subordinations are satisfied:*

$$1 + \frac{1}{\beta}\left(\frac{z^{1-\psi}\eth_q\langle \mathscr{D}_q^{\chi} f(z)\rangle}{(\mathscr{D}_q^{\chi} f(z))^{1-\psi}}\right) \prec \Upsilon(z;q) = \frac{1 + q\vartheta_q^2 z^2}{1 - \vartheta_q\, z - q\vartheta_q^2 z^2}, \qquad (z \in \mathbb{U}) \tag{14}$$

and

$$1 + \frac{1}{\beta}\left(\frac{\xi^{1-\psi}\partial_q\left(\mathscr{D}_q^{\chi}\hbar(\xi)\right)}{(\mathscr{D}_q^{\chi}\hbar(\xi))^{1-\psi}}\right) \prec \Upsilon(\xi;q) = \frac{1 + q\vartheta_q^2 \xi^2}{1 - \vartheta_q\, \xi - q\vartheta_q^2 \xi^2}, \qquad (\xi \in \mathbb{U}) \tag{15}$$

where $\varkappa \in \left(\frac{1}{2}, 1\right]$, *the function* $\hbar(\xi) = f^{-1}(\xi)$ *is defined by (2) and* ϑ_q *is given by* (9).

Example 2. *Suppose that* $\psi = 0$, $0 < q < 1$, *and* $\beta \in \mathbb{C} \setminus \{0\}$. *Then, a function* f *belonging to the class* Σ, *as given by equation (1), is considered to be in the category* $\mathsf{SLM}_{\Sigma}(\beta, \chi, 0; q)$ *if it satisfies the following subordination conditions:*

$$1 + \frac{1}{\beta}\left(\frac{z\eth_q\langle \mathscr{D}_q^{\chi} f(z)\rangle}{\mathscr{D}_q^{\chi} f(z)}\right) \prec \Upsilon(z;q) = \frac{1 + q\vartheta_q^2 z^2}{1 - \vartheta_q\, z - q\vartheta_q^2 z^2}, \qquad (z \in \mathbb{U}) \tag{16}$$

and

$$1 + \frac{1}{\beta}\left(\frac{\xi\partial_q\left(\mathscr{D}_q^{\chi}\hbar(\xi)\right)}{\mathscr{D}_q^{\chi}\hbar(\xi)}\right) \prec \Upsilon(\xi;q) = \frac{1 + q\vartheta_q^2 \xi^2}{1 - \vartheta_q\, \xi - q\vartheta_q^2 \xi^2}, \qquad (\xi \in \mathbb{U}) \tag{17}$$

where $\varkappa \in \left(\frac{1}{2}, 1\right]$, *the function* $\hbar(\xi) = f^{-1}(\xi)$ *is defined by (2) and* ϑ_q *is given by* (9).

Example 3. *Suppose that* $\psi = 1$, $0 < q < 1$, *and* $\beta \in \mathbb{C} \setminus \{0\}$. *Then, a function* f *belonging to the class* $\sum$, *as given by equation (1), is considered to be in the category* $\mathsf{SLM}_{\Sigma}(\beta, \chi, 1; q)$ *if it satisfies the following subordination conditions:*

$$1 + \frac{\eth_q\langle \mathscr{D}_q^{\chi} f(z)\rangle}{\beta} \prec \Upsilon(z;q) = \frac{1 + q\vartheta_q^2 z^2}{1 - \vartheta_q\, z - q\vartheta_q^2 z^2} \qquad (z \in \mathbb{U}) \tag{18}$$

and

$$1 + \frac{\partial_q\left(\mathscr{D}_q^{\chi}\hbar(\xi)\right)}{\beta} \prec \Upsilon(\xi;q) = \frac{1 + q\vartheta_q^2 \xi^2}{1 - \vartheta_q\, \xi - q\vartheta_q^2 \xi^2}, \qquad (\xi \in \mathbb{U}) \tag{19}$$

where $\varkappa \in \left(\frac{1}{2}, 1\right]$, *the function* $\hbar(\xi) = f^{-1}(\xi)$ *is defined by (2) and* ϑ_q *is given by* (9).

3. Coefficient bounds of the class $\mathsf{SLM}_\Sigma(\beta,\chi,\psi;q)$

In this section, we first obtain the estimate of the initial Taylor coefficients $|a_2|$ and $|a_2|$ for functions in the class $\mathsf{SLM}_\Sigma(\beta,\chi,\psi;q)$ as per Definition 3.

Firstly, let

$$\mathsf{p}(z)=1+\mathsf{p}_1z+\mathsf{p}_2z^2+\mathsf{p}_3z^3+\ldots,$$

and $\mathsf{p}(z)\prec\Upsilon(z;q)$. Then there exists $\varphi\in\mathsf{P}$ such that

$$|\varphi(z)|<1 \text{ in } \mathbb{U} \text{ and } \mathsf{p}(z)=\Upsilon(\varphi(z);q).$$

We have

$$\hbar(z)=(1+\varphi(z))(1-\varphi(z))^{-1}=1+\ell_1z+\ell_2z^2+\cdots\in\mathsf{P}\qquad(z\in\mathbb{U}).\tag{20}$$

Consequently, the function $\varphi(z)$, being analytic in $\mathbb{U}$ and subordinate to $\Upsilon(z;q)$, admits the following Taylor expansion:

$$\varphi(z)=\frac{\ell_1z}{2}+\left(\ell_2-\frac{\ell_1^2}{2}\right)\frac{z^2}{2}+\left(\ell_3-\ell_1\ell_2-\frac{\ell_1^3}{4}\right)\frac{z^3}{2}+\cdots,\tag{21}$$

and

$$\begin{aligned}\Upsilon(\varphi(z);q)&=1+\widetilde{\mathsf{p}_1}\left[\frac{\ell_1z}{2}+\left(\ell_2-\frac{\ell_1^2}{2}\right)\frac{z^2}{2}+\left(\ell_3-\ell_1\ell_2-\frac{\ell_1^3}{4}\right)\frac{z^3}{2}+\cdots\right]\\&+\widetilde{\mathsf{p}_2}\left[\frac{\ell_1z}{2}+\left(\ell_2-\frac{\ell_1^2}{2}\right)\frac{z^2}{2}+\left(\ell_3-\ell_1\ell_2-\frac{\ell_1^3}{4}\right)\frac{z^3}{2}+\cdots\right]^2\\&+\widetilde{\mathsf{p}_3}\left[\frac{\ell_1z}{2}+\left(\ell_2-\frac{\ell_1^2}{2}\right)\frac{z^2}{2}+\left(\ell_3-\ell_1\ell_2-\frac{\ell_1^3}{4}\right)\frac{z^3}{2}+\cdots\right]^3+\cdots\\&=1+\frac{\widetilde{\mathsf{p}_1}\ell_1}{2}z+\frac{1}{2}\left[\left(\ell_2-\frac{\ell_1^2}{2}\right)\widetilde{\mathsf{p}_1}+\frac{\ell_1^2}{2}\widetilde{\mathsf{p}_2}\right]z^2\\&+\frac{1}{2}\left[\left(\ell_3-\ell_1\ell_2+\frac{\ell_1^3}{4}\right)\widetilde{\mathsf{p}_1}+\ell_1\left(\ell_2-\frac{\ell_1^2}{2}\right)\widetilde{\mathsf{p}_2}+\frac{\ell_1^3}{4}\widetilde{\mathsf{p}_3}\right]z^3+\cdots.\end{aligned}\tag{22}$$

Similarly, there exists an analytic function ν defined on $\mathbb{U}$, satisfying $|\nu(\xi)|<1$, such that $\mathsf{p}(\xi)=\Upsilon(\nu(\xi);q)$. This allows us to represent the corresponding function

$$\lambda(\xi)=(1+\nu(\xi))(1-\nu(\xi))^{-1}=1+\tau_1\xi+\tau_2\xi^2+\cdots\in\mathsf{P}.\tag{23}$$

As a result, the Taylor expansion of $\nu(\xi)$ takes the form:

$$\nu(\xi)=\frac{\tau_1\xi}{2}+\left(\tau_2-\frac{\tau_1^2}{2}\right)\frac{\xi^2}{2}+\left(\tau_3-\tau_1\tau_2-\frac{\tau_1^3}{4}\right)\frac{\xi^3}{2}+\cdots,\tag{24}$$

and accordingly, the composition $\Upsilon(\nu(\xi);q)$ expands as:

$$\Upsilon(\nu(\xi);q) = 1 + \frac{\widetilde{\mathsf{p}_1}\tau_1}{2}\xi + \frac{1}{2}\left[\left(\tau_2 - \frac{\tau_1^2}{2}\right)\widetilde{\mathsf{p}_1} + \frac{\tau_1^2}{2}\widetilde{\mathsf{p}_2}\right]\xi^2 + \frac{1}{2}\left[\left(\tau_3 - \tau_1\tau_2 + \frac{\tau_1^3}{4}\right)\widetilde{\mathsf{p}_1} + \tau_1\left(\tau_2 - \frac{\tau_1^2}{2}\right)\widetilde{\mathsf{p}_2} + \frac{\tau_1^3}{4}\widetilde{\mathsf{p}_3}\right]\xi^3 + \cdots. \tag{25}$$

Having established the necessary groundwork and auxiliary results, we are now in a position to derive bounds for the initial coefficients of functions belonging to the newly introduced class $\mathsf{SLM}_\Sigma(\beta,\chi,\psi;q)$. These estimates not only offer insights into the geometric behavior of such bi-univalent functions but also highlight the influence of the deformation parameter q and the parameter β on the coefficient structure. The following theorem presents sharp bounds for the second and third coefficients $|a_2|$ and $|a_3|$, respectively.

Theorem 1. *Let f be a function belonging to the class Σ, as defined by equation (1). Suppose further that f lies within the subclass $\mathsf{SLM}_\Sigma(\beta,\chi,\psi;q)$. Then,*

$$|a_2| \le \frac{\sqrt{2}|\vartheta_q|}{\sqrt{\left|\left(\left(\mathscr{M}+\mathscr{N}-(2q+1)\mathscr{U}\right)\vartheta_q+\mathscr{U}\right)\right|}}.$$

and

$$|a_3| \le \frac{2\vartheta_q^2}{\left|\left(\mathscr{M}+\mathscr{N}-(2q+1)\mathscr{U}\right)\vartheta_q+\mathscr{U}\right|} + \frac{\beta|\vartheta_q|}{(3]_q^c\left(q+q^2+\psi\right)},$$

where

$$\mathscr{M} = \frac{2(3]_q^c\left(q+q^2+\psi\right)}{\beta}, \tag{26}$$

$$\mathscr{N} = \frac{(\psi-1)\left((2]_q^c\right)^2(2q+\psi)}{\beta}, \tag{27}$$

$$\mathscr{U} = \frac{2\left((2]_q^c(q+\psi)\right)^2}{\beta^2}. \tag{28}$$

Proof. Let $f \in \mathsf{SLM}_\Sigma(\beta,\chi,\psi;q)$ and $\xi = f^{-1}$. Considering (14) and (15) we have

$$1 + \frac{1}{\beta}\left(\frac{z^{1-\psi}\eth_q\langle\mathscr{D}_q^\chi f(z)\rangle}{(\mathscr{D}_q^\chi f(z))^{1-\psi}}\right) = \Upsilon(\varphi(z);q), \qquad (z \in \mathbb{U}), \tag{29}$$

and

$$1 + \frac{1}{\beta}\left(\frac{\xi^{1-\psi}\eth_q\langle\mathscr{D}_q^\chi \hbar(\xi)\rangle}{(\mathscr{D}_q^\chi \hbar(\xi))^{1-\psi}}\right) = \Upsilon(\nu(\xi);q), \qquad (\xi \in \mathbb{U}). \tag{30}$$

Since

$$1 + \frac{1}{\beta}\left(\frac{z^{1-\psi}\eth_q\langle\mathscr{D}_q^\chi f(z)\rangle}{(\mathscr{D}_q^\chi f(z))^{1-\psi}}\right) = 1 + \frac{\widetilde{\mathsf{p}_1}\ell_1}{2}z + \frac{1}{2}\left[\left(\ell_2 - \frac{\ell_1^2}{2}\right)\widetilde{\mathsf{p}_1} + \frac{\ell_1^2}{2}\widetilde{\mathsf{p}_2}\right]z^2 + \cdots. \tag{31}$$

and

$$1+\frac{1}{\beta}\left(\frac{\xi^{1-\psi}\eth_q\langle\mathscr{D}_q^{\chi}\hbar(\xi)\rangle}{(\mathscr{D}_q^{\chi}\hbar(\xi))^{1-\psi}}\right)=1+\frac{\widetilde{\mathsf{p}_1}\tau_1}{2}\xi+\frac{1}{2}\left[\left(\tau_2-\frac{\tau_1^2}{2}\right)\widetilde{\mathsf{p}_1}+\frac{\tau_1^2}{2}\widetilde{\mathsf{p}_2}\right]\xi^2+\cdots. \tag{32}$$

In view of (1), (2), from (31) and (32), we obtain

$$\begin{aligned}1&+\frac{(2]_q^c(q+\psi)}{\beta}a_2z+\frac{1}{\beta}\left((3]_q^c\left(q+q^2+\psi\right)a_3+\frac{(\psi-1)\left((2]_q^c\right)^2(2q+\psi)}{2}a_2^2\right)z^2+\cdots\\&=1+\frac{\widetilde{\mathsf{p}_1}\ell_1}{2}z+\frac{1}{2}\left[\left(\ell_2-\frac{\ell_1^2}{2}\right)\widetilde{\mathsf{p}_1}+\frac{\ell_1^2}{2}\widetilde{\mathsf{p}_2}\right]z^2+\cdots,\end{aligned} \tag{33}$$

and

$$\begin{aligned}1&-\frac{(2]_q^c(q+\psi)}{\beta}a_2\xi+\frac{1}{\beta}\left[\left(2(3]_q^c\left(q+q^2+\psi\right)+\frac{(\psi-1)\left((2]_q^c\right)^2(2q+\psi)}{2}\right)a_2^2\right.\\&\left.-(3]_q^c\left(q+q^2+\psi\right)a_3\right]\xi^2+\cdots=1+\frac{\widetilde{\mathsf{p}_1}\tau_1}{2}\xi+\frac{1}{2}\left[\left(\tau_2-\frac{\tau_1^2}{2}\right)\widetilde{\mathsf{p}_1}+\frac{\tau_1^2}{2}\widetilde{\mathsf{p}_2}\right]\xi^2+\cdots.\end{aligned} \tag{34}$$

Therefore, by comparing the coefficients in (33) and (34) that correspond to each other, we can obtain:

$$\frac{(2]_q^c(q+\psi)}{\beta}a_2=\frac{\widetilde{\mathsf{p}_1}\ell_1}{2} \tag{35}$$

$$-\frac{(2]_q^c(q+\psi)}{\beta}a_2=\frac{\widetilde{\mathsf{p}_1}\tau_1}{2} \tag{36}$$

$$(3]_q^c\left(q+q^2+\psi\right)a_3+\frac{(\psi-1)\left((2]_q^c\right)^2(2q+\psi)}{2}a_2^2=\frac{\beta}{2}\left[\left(\ell_2-\frac{\ell_1^2}{2}\right)\widetilde{\mathsf{p}_1}+\frac{\ell_1^2}{2}\widetilde{\mathsf{p}_2}\right] \tag{37}$$

$$\left(2(3]_q^c\left(q+q^2+\psi\right)+\frac{(\psi-1)\left((2]_q^c\right)^2(2q+\psi)}{2}\right)a_2^2-(3]_q^c\left(q+q^2+\psi\right)a_3=\frac{\beta}{2}\left[\left(\tau_2-\frac{\tau_1^2}{2}\right)\widetilde{\mathsf{p}_1}+\frac{\tau_1^2}{2}\widetilde{\mathsf{p}_2}\right] \tag{38}$$

From (35) and (36), we have

$$\ell_1=-\tau_1 \quad\Longleftrightarrow\quad \ell_1^2=\tau_1^2, \tag{39}$$

and

$$a_2^2=\frac{\vartheta_q^2\beta^2}{8\left((2]_q^c(q+\psi)\right)^2}\left(\ell_1^2+\tau_1^2\right) \quad\Longleftrightarrow\quad \ell_1^2+\tau_1^2=\frac{8\left((2]_q^c(q+\psi)\right)^2}{\vartheta_q^2\beta^2}a_2^2. \tag{40}$$

Now, by summing (37) and (38), we obtain

$$(\mathscr{M}+\mathscr{N})\,a_2^2=\frac{(\ell_2+\tau_2)\vartheta_q}{2}+\left[\frac{(2q+1)\vartheta_q^2}{4}-\frac{\vartheta_q}{4}\right]\left(\ell_1^2+\tau_1^2\right). \tag{41}$$

By putting (40) in (41), we obtain

$$a_2^2=\frac{(\ell_2+\tau_2)\vartheta_q^2}{2\left(\left(\mathscr{M}+\mathscr{N}-(2q+1)\mathscr{U}\right)\vartheta_q+\mathscr{U}\right)} \tag{42}$$

where $\mathscr{M},\mathscr{N},\mathscr{U}$ are given by (26), (27) and (28), respectively. Using (3) for (42), we have

$$\left|a_2\right|\leq\frac{\sqrt{2}|\vartheta_q|}{\sqrt{\left|\left(\left(\mathscr{M}+\mathscr{N}-(2q+1)\mathscr{U}\right)\vartheta_q+\mathscr{U}\right)\right|}}. \tag{43}$$

Now, so as to find the bound on $|a_3|$, let's subtract from (37) and (38) along (40), we obtain

$$a_3=a_2^2+\frac{\beta\vartheta_q}{4[3]_q^c\left(q+q^2+\psi\right)}\left(\ell_2-\tau_2\right). \tag{44}$$

Hence, we get

$$\left|a_3\right|\leq\frac{2\vartheta_q^2}{\left|\left(\mathscr{M}+\mathscr{N}-(2q+1)\mathscr{U}\right)\vartheta_q+\mathscr{U}\right|}+\frac{\beta\left|\vartheta_q\right|}{[3]_q^c\left(q+q^2+\psi\right)}, \tag{45}$$

where $\mathscr{M},\mathscr{N},\mathscr{U}$ are given by (26), (27) and (28), respectively. The proof of the theorem is now complete.

Theorem 2. *Suppose that f is a function in the class Σ, as given by equation (1), and it belongs to the category $\mathsf{SLM}_\Sigma(\beta,\chi,\psi;q)$, and $\chi>-1$. Then, we have*

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\frac{\beta\vartheta_q}{[3]_q^c\left(q+q^2+\psi\right)}, & 0\leq|1-\mu|\leq\frac{\beta\left|\left(\mathscr{M}+\mathscr{N}-(2q+1)\mathscr{U}\right)\vartheta_q+\mathscr{U}\right|}{2[3]_q^c\,\vartheta_q\left(q+q^2+\psi\right)},\\[2ex] \frac{2|1-\mu|\,\vartheta_q^2}{\left|\left(\mathscr{M}+\mathscr{N}-(2q+1)\mathscr{U}\right)\vartheta_q+\mathscr{U}\right|}, & |1-\mu|\geq\frac{\beta\left|\left(\mathscr{M}+\mathscr{N}-(2q+1)\mathscr{U}\right)\vartheta_q+\mathscr{U}\right|}{2[3]_q^c\,\vartheta_q\left(q+q^2+\psi\right)},\end{cases}$$

where $\mathscr{M},\mathscr{N},\mathscr{U}$ are given by (26), (27) and (28), respectively.

Proof. Let $f\in\mathsf{SLM}_\Sigma(\beta,\chi,\psi;q)$, from (42) and (44), we have

$$\begin{aligned}a_3-\mu a_2^2&=\frac{(1-\mu)\left(\ell_2+\tau_2\right)\vartheta_q^2}{2\left(\left(\mathscr{M}+\mathscr{N}-(2q+1)\mathscr{U}\right)\vartheta_q+\mathscr{U}\right)}+\frac{\beta\vartheta_q}{4[3]_q^c\left(q+q^2+\psi\right)}\left(\ell_2-\tau_2\right)\\&=\left(\mathscr{K}(\mu)+\frac{\beta\vartheta_q}{4[3]_q^c\left(q+q^2+\psi\right)}\right)\ell_2+\left(\mathscr{K}(\mu)-\frac{\beta\vartheta_q}{4[3]_q^c\left(q+q^2+\psi\right)}\right)\tau_2,\end{aligned} \tag{46}$$

where

$$\mathscr{K}(\mu) = \frac{(1-\mu)\,\vartheta_q^2}{2\left(\left(\mathscr{M}+\mathscr{N}-(2q+1)\mathscr{U}\right)\vartheta_q+\mathscr{U}\right)}. \tag{47}$$

Then, by taking modulus of (46), we conclude that

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \frac{\beta\vartheta_q}{(3]_q^c\left(q+q^2+\psi\right)}, & 0 \le \left|\mathscr{K}(\mu)\right| \le \frac{\beta\vartheta_q}{4(3]_q^c\left(q+q^2+\psi\right)}, \\ 4\left|\mathscr{K}(\mu)\right|, & \left|\mathscr{K}(\mu)\right| \ge \frac{\beta\vartheta_q}{4(3]_q^c\left(q+q^2+\psi\right)}, \end{cases}$$

The proof of the theorem is now complete.

4. Corollaries

The subsequent corollaries, which closely correspond to instances resembling Examples 1, 2, and 3, are derived from the implications of Theorems 1 and 2.

Corollary 1. *Suppose that f is a function in the class Σ, as given by equation (1), and it belongs to the category $\mathsf{SLM}_\Sigma(\beta,\chi,0;q)$. Then*

$$\left|a_2\right| \le \frac{\sqrt{2}|\vartheta_q|}{\sqrt{\left|\left(\left(\mathscr{M}+\mathscr{N}-(2q+1)\mathscr{U}\right)\vartheta_q+\mathscr{U}\right)\right|}},$$

$$\left|a_3\right| \le \frac{2\vartheta_q^2}{\left|\left(\mathscr{M}+\mathscr{N}-(2q+1)\mathscr{U}\right)\vartheta_q+\mathscr{U}\right|}+\frac{\beta\left|\vartheta_q\right|}{(3]_q^c\left(q+q^2\right)},$$

where and

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \frac{\beta\vartheta_q}{(3]_q^c\left(q+q^2\right)}, & 0 \le |1-\mu| \le \frac{\beta\left|\left(\mathscr{M}+\mathscr{N}-(2q+1)\mathscr{U}\right)\vartheta_q+\mathscr{U}\right|}{2(3]_q^c\,\vartheta_q\left(q+q^2\right)}, \\ \frac{2|1-\mu|\,\vartheta_q^2}{\left|\left(\mathscr{M}+\mathscr{N}-(2q+1)\mathscr{U}\right)\vartheta_q+\mathscr{U}\right|}, & |1-\mu| \ge \frac{\beta\left|\left(\mathscr{M}+\mathscr{N}-(2q+1)\mathscr{U}\right)\vartheta_q+\mathscr{U}\right|}{2(3]_q^c\,\vartheta_q\left(q+q^2\right)}, \end{cases}$$

where

$$\mathscr{M} = \frac{2(3]_q^c\left(q+q^2\right)}{\beta}, \quad \mathscr{N} = \frac{-\left((2]_q^c\right)^2(2q)}{\beta}, \quad \text{and } \mathscr{U} = \frac{2\left((q)\,(2]_q^c\right)^2}{\beta^2}.$$

Corollary 2. *Suppose that f is a function in the class Σ, as given by equation (1), and it belongs to the category $\mathsf{SLM}_\Sigma(\beta,\chi,1;q)$. Then*

$$\left|a_2\right| \le \frac{\sqrt{2}|\vartheta_q|}{\sqrt{\left|\left(\left(\mathscr{M}-(2q+1)\mathscr{U}\right)\vartheta_q+\mathscr{U}\right)\right|}}.$$

and

$$\left|a_3\right| \leq \frac{2\vartheta_q^2}{\left|\left(\mathscr{M}-(2q+1)\mathscr{U}\right)\vartheta_q+\mathscr{U}\right|}+\frac{\beta\left|\vartheta_q\right|}{(3]_q^c\left(1+q+q^2\right)},$$

and

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{\beta\vartheta_q}{(3]_q^c\left(1+q+q^2\right)}, & 0 \leq |1-\mu| \leq \frac{\beta\left|\left(\mathscr{M}-(2q+1)\mathscr{U}\right)\vartheta_q+\mathscr{U}\right|}{2(3]_q^c\,\vartheta_q\left(1+q+q^2\right)}, \\ \frac{2|1-\mu|\,\vartheta_q^2}{\left|\left(\mathscr{M}-(2q+1)\mathscr{U}\right)\vartheta_q+\mathscr{U}\right|}, & |1-\mu| \geq \frac{\beta\left|\left(\mathscr{M}-(2q+1)\mathscr{U}\right)\vartheta_q+\mathscr{U}\right|}{2(3]_q^c\,\vartheta_q\left(1+q+q^2\right)}, \end{cases}$$

where

$$\mathscr{M}=\frac{2(3]_q^c\left(1+q+q^2\right)}{\beta}, \quad \text{and} \quad \mathscr{U}=\frac{2\left((2]_q^c\,(q+1)\right)^2}{\beta^2}.$$

Conclusion: In this investigation, we have investigated the concerns regarding the coefficients of three recently introduced categories of bi-univalent functions in the unit disk $\mathbb{U}$: $\mathsf{SLM}_\Sigma(\beta,\chi,\psi;q)$, $\mathsf{SLM}_\Sigma(\beta,\chi,0;q)$ and $\mathsf{SLM}_\Sigma(\beta,\chi,1;q)$, as described in Definitions 3. We have calculated the estimated values for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for each of these subclasses.

In future investigations, it would be of significant interest to explore sharp bounds related to the Zalcman conjecture and to conduct a detailed analysis of Hankel determinants within the classes of bi-convex and bi-close-to-convex functions. These avenues offer promising prospects for uncovering novel results and fostering a deeper understanding of the structural properties inherent in geometric function theory.

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