EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6500
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Generalized Fuzzy Subalgebras of Sheffer Stroke Hilbert Algebras

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Abstract. This paper introduces new generalized fuzzy subalgebras and investigates their important properties within the framework of Sheffer stroke Hilbert algebras. We characterize these generalized subalgebras through their level subsets and establish key properties that define their structure. The Sheffer stroke operation, known for its ability to construct logical systems independently of other operators, plays a central role in our study. Using fuzzy set theory, we adapt the traditional ideas of subalgebras to fit fuzzy contexts, giving a detailed look at $(\in, \in \vee q_m)$-fuzzy subalgebras. Our results include necessary and sufficient conditions for a fuzzy set to qualify as such a subalgebra, along with theorems addressing their intersections, unions, and homomorphic invariance. This work contributes to the broader understanding of algebraic structures in fuzzy logic and their applications in logical systems.

2020 Mathematics Subject Classifications: 03G25, 03E72

Key Words and Phrases: Sheffer stroke Hilbert algebra, subalgebra, fuzzy set, fuzzy subalgebra

1. Introduction

The study of algebraic structures in logic has been profoundly influenced by the discovery of universal operations, among which the Sheffer stroke (also known as the NAND operator) stands out as a cornerstone. Introduced by Sheffer in 1913 [1], this operation possesses the remarkable property of functional completeness: it can express all other logical connectives independently, thereby simplifying the axiomatization of logical systems.

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DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6500

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This universality extends to Boolean algebras, where the Sheffer stroke alone suffices to formulate their axioms, offering a minimalist yet powerful framework for algebraic and logical investigations [2]. Inspired by its efficiency and expressive power, recent studies by Oner et al. [3] introduced Sheffer stroke Hilbert algebras, merging the conceptual simplicity of the NAND operation with the structured implications of Hilbert algebras. Their work revealed new algebraic properties and paved the way for exploring substructures within these algebras, particularly in fuzzy settings.

In parallel, Hilbert algebras—initially studied in the 1930s as algebraic models for the implicative fragment of intuitionistic logic [4–7]—emerged as pivotal objects in algebraic logic due to their ability to model logical implication in a purely algebraic setting. Over time, researchers have extended Hilbert algebras to enriched systems, incorporating operations such as infimum and supremum to enhance their expressive power [7]. However, it was not until the synthesis with Sheffer stroke operations that a minimalist yet expressive representation of logical structures was achieved [3].

The introduction of fuzzy set theory by Zadeh in 1965 [8] revolutionized the modeling of vagueness and imprecision in mathematical systems. This extension inspired the fuzzification of various algebraic structures, beginning with fuzzy subgroups [9] and evolving into fuzzy ideals in rings [10]. Building on these foundations, Dudek and Jun [11] extended the concept of ideals in Hilbert algebras to fuzzy ideals, introducing foundational results on closure properties and their connections to deductive systems. Borzooei et al. [12] introduced fuzzy weak filters in Sheffer stroke Hilbert algebras, exploring their definitions and key properties. The recent incorporation of fuzzy principles into Sheffer stroke Hilbert algebras by Oner et al. [3] marks a critical step forward, demonstrating that the Sheffer stroke's functional completeness persists even in fuzzy settings.

Despite these advancements, the study of fuzzy subalgebras within Sheffer stroke Hilbert algebras, particularly those characterized by $(\in, \in \vee q_m)$-fuzzy substructures, remains largely uncharted. The generalization of subalgebras to fuzzy contexts introduces graded membership, which allows for varying degrees of inclusion, thereby reflecting real-world imprecision more effectively. This gap presents a promising opportunity for deepening the theoretical understanding of fuzzy algebraic systems and expanding their applications in logic and information processing.

The exploration of Sheffer stroke Hilbert algebras has witnessed increasing depth and diversity, highlighting their central role in algebraic logic and fuzzy systems. Foundational work has addressed the algebraic basis of these structures, particularly the interplay between Sheffer stroke and Hilbert algebras [3], and has been extended through the study of fuzzy ideals [13] and fuzzy filters [14]. The introduction of fuzzy weak filters [12] and bipolar-valued fuzzy deductive systems [15] further enriched the theoretical framework, allowing nuanced treatment of uncertainty and graded reasoning. More recent developments have focused on structural generalizations, such as the incorporation of $\mathcal{N}$-based soft subalgebras and ideals [16], and the formulation of length and mean-fuzzy ideals [17] and subalgebras [18], revealing new perspectives on the quantitative dimensions of fuzzy membership.

Building upon this progression, the present paper introduces a new class of fuzzy substructures—namely, $(\in, \in \vee q_m)$-fuzzy subalgebras—within Sheffer stroke Hilbert algebras. These subalgebras extend existing frameworks by incorporating graded membership with tolerance for ambiguity, providing refined algebraic tools to model fuzzy logic under Sheffer stroke operations. Our investigation establishes necessary and sufficient conditions, level set characterizations, and structural invariance under algebraic operations and homomorphisms, thereby contributing to the ongoing evolution of fuzzy algebraic theory.

In this paper, we leverage this interaction to introduce generalized fuzzy subalgebras within Sheffer stroke Hilbert algebras. Specifically, we define and analyze $(\in, \in \vee q_m)$-fuzzy subalgebras, a class of fuzzy substructures that extend traditional subalgebras by incorporating graded membership and tolerance for ambiguity. These subalgebras are characterized by their interaction with level subsets and their behavior under algebraic operations, offering a nuanced perspective on the interplay between logic and fuzziness.

The primary contributions of this paper include:

- The definition and characterization of $(\in, \in \vee q_m)$-fuzzy subalgebras in Sheffer stroke Hilbert algebras.
- The establishment of necessary and sufficient conditions for a fuzzy set to be an $(\in, \in \vee q_m)$-fuzzy subalgebra.
- The investigation of level subsets and their role in characterizing these subalgebras.
- The study of algebraic properties such as intersections, unions, and homomorphic invariance in the context of fuzzy subalgebras.

Our results not only extend the theoretical understanding of Sheffer stroke Hilbert algebras but also provide a foundation for future applications in fuzzy logic and algebraic structures.

2. Preliminaries

Sheffer stroke Hilbert algebras form a significant algebraic system bridging logic and lattice theory. These structures incorporate the Sheffer stroke (NAND) operation—an essential logical connective in Boolean algebra—into the classical Hilbert algebra framework. By extending Hilbert algebras with this operation, they provide a robust tool for analyzing logical structures and addressing applications in fuzzy logic, decision-making, and computational frameworks. Their study deepens the theoretical foundation of algebraic systems and offers practical models for uncertainty and vagueness.

Definition 1. *[1] Let $H = \langle H, | \rangle$ be a groupoid. The operation $|$ is said to be a Sheffer stroke operation if it satisfies the following conditions:*

$(S1)$ $(x|(y|y))|(x|(y|y)) = y|x$
$(S2)$ $(x|x)|((x|(y|y))|(x|(y|y))) = x$
$(S3)$ $x|((y|z)|(y|z)) = ((((x|(y|y))|(x|(y|y))))|((x|(y|y))|(x|(y|y))))|z$
$(S4)$ $(x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x.$

Definition 2. *[3] A Sheffer stroke Hilbert algebra is a structure* $H = \langle H, |, 0\rangle$ *of type* $(2, 0)$*, in which* H *is a non-empty set,* $|$ *is a Sheffer stroke operation on* H*, and* 0 *is the fixed element in* H *such that the following identities are satisfied for all* $x, y, z \in H$*:*

(1) $(x|((y|(z|z))|(y|(z|z))))|(((x|(y|y))|((x|(z|z))|(x|(z|z))))|$
$((x|(y|y))|((x|(z|z))|(x|(z|z))))) = x|(x|x)$

(2) $x|(y|y) = y|(x|x) \Rightarrow x = y$.

Proposition 1. *[3] Let* $H = \langle H, |, 0\rangle$ *be a Sheffer stroke Hilbert algebra. Then the binary relation* $x \leq y$ *if and only if* $(x|(y|y)) = 0$ *is a partial order on* H*.*

Definition 3. *[3] A nonempty subset* G *of a Sheffer stroke Hilbert algebra* $H = \langle H, |, 0\rangle$ *is called a subalgebra of* H *if* $(x|(y|y))|(x|(y|y)) \in G$ *for all* $x, y \in G$*.*

Definition 4. *A fuzzy set* μ *in a non-empty set* X *of the form*

$$\mu = \begin{cases} t \in (0, 1] & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a fuzzy point with support x *and value* t *and is denoted by* x_t*.*

The general form of the symbol $x_t\ q\ \lambda$ as follows: for an arbitrary element k of $[0, 1)$, we say that

- $x_t\ q_k\ \lambda$ if $\lambda(x) + t + k > 1$.
- $x_t \in \vee q_k \lambda$ if $x_t \in \lambda$ or $x_t q_k \lambda$.

3. New fuzzy subalgebras of Sheffer stroke Hilbert algebras

In this section, let $H = \langle H, |, 0\rangle$ denote the Sheffer stroke Hilbert algebra unless otherwise specified.

Definition 5. *A fuzzy set* μ *in* H *is called an* $(\in, \in \vee q)$*-fuzzy subalgebra of* $H = \langle H, |, 0\rangle$*, if it satisfies*

$$(\forall x, y \in H, t_1, t_2 \in (0, 1])(x_{t_1}, y_{t_2} \in \mu \Rightarrow ((x|(y|y))|(x|(y|y)))_{\min\{t_1,t_2\}} \in \vee q\mu). \quad (1)$$

Remark 1. *Let* m *be an element of* $[0, 1)$ *unless otherwise specified. By* $x_t q_m \mu$*, we mean* $\mu(x) + t + m > 1$, $t \in (0, \frac{1-m}{2}]$*. The notation* $x_t \in \vee q_m \mu$ *means that* $x_t \in \mu$ *or* $x_t q_m \mu$*.*

Definition 6. *A fuzzy set* μ *in* H *is called an* $(\in, \in \vee q_m)$*-fuzzy subalgebra of* H *if*

$$(\forall x, y \in H, t_1, t_2 \in (0, 1])(x_{t_1}, y_{t_2} \in \mu \Rightarrow ((x|(y|y))|(x|(y|y)))_{\min\{t_1,t_2\}} \in \vee q_m\mu). \quad (2)$$

We note that different types of fuzzy subalgebras can be constructed for different values of $m \in [0, 1)$. Hence, an $(\in, \in \vee q_m)$-fuzzy subalgebra with $m = 0$ is called an $(\in, \in \vee q)$-fuzzy subalgebra.

Proposition 2. *Every* $(\in, \in)$*-fuzzy subalgebra is an* $(\in, \in \vee q_m)$*-fuzzy subalgebra.*

Proof. Straightforward.

Theorem 1. *A fuzzy set* μ *in* H *is an* $(\in, \in \vee q_m)$*-fuzzy subalgebra of* H *if and only if* $\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\}$ *holds for all* $x, y \in H$.

Proof. Let μ be an $(\in, \in \vee q_m)$-fuzzy subalgebra of H. Assume that

$$\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\}$$

is not true. Then there exist $x', y' \in H$ such that

$$\mu((x'|(y'|y'))|(x'|(y'|y'))) < \min\{\mu(x'), \mu(y'), \frac{1-m}{2}\}.$$

If $\min\{\mu(x'), \mu(y')\} < \frac{1-m}{2}$, then $\mu((x'|(y'|y'))|(x'|(y'|y'))) < \min\{\mu(x'), \mu(y')\}$. Thus, $\mu((x'|(y'|y'))|(x'|(y'|y'))) < t \leq \min\{\mu(x'), \mu(y')\}$ for some $t \in (0, 1]$. It follows that $x'_t \in \mu$ and $y'_t \in \mu$, but $((x'|(y'|y'))|(x'|(y'|y')))\overline{\in}\mu$, a contradiction. Moreover,

$$\mu((x'|(y'|y'))|(x'|(y'|y'))) + t < 2t < 1 - m,$$

and so $((x'|(y'|y'))|(x'|(y'|y')))_t\overline{q_m}\mu$. Hence, $((x'|(y'|y'))|(x'|(y'|y')))_t\overline{\in \vee q_m}\mu$, a contradiction. On the other hand, if $\min\{\mu(x'), \mu(y')\} \geq \frac{1-m}{2}$, then $\mu(x') \geq \frac{1-m}{2}$, $\mu(y') \geq \frac{1-m}{2}$ and $\mu((x'|(y'|y'))|(x'|(y'|y'))) < \frac{1-m}{2}$. Thus, $x'_{\frac{1-m}{2}} \in \mu$ and $y'_{\frac{1-m}{2}} \in \mu$, but

$$((x'|(y'|y'))|(x'|(y'|y')))_{\frac{1-m}{2}}\overline{\in}\mu.$$

Also, $\mu((x'|(y'|y'))|(x'|(y'|y'))) + \frac{1-m}{2} < \frac{1-m}{2} + \frac{1-m}{2} = 1 - m$, that is,

$$((x'|(y'|y'))|(x'|(y'|y')))_{\frac{1-m}{2}}\overline{q_m}\mu.$$

Hence, $((x'|(y'|y'))|(x'|(y'|y')))_{\frac{1-m}{2}}\overline{\in \vee q_m \mu}$, a contradiction. Hence, $\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\}$ holds for all $x, y \in H$.

Conversely, assume that $\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\}$ holds for all $x, y \in H$. Let $x, y \in H$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$. Then $\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\} \geq \min\{t_1, t_2, \frac{1-m}{2}\}$. Assume that $t_1 \leq \frac{1-m}{2}$ or $t_2 \leq \frac{1-m}{2}$. Then $\mu((x|(y|y))|(x|(y|y))) \geq \min\{t_1, t_2\}$, which implies that

$$((x|(y|y))|(x|(y|y)))_{\min\{t_1,t_2\}} \in \mu.$$

Now, suppose that $t_1 > \frac{1-m}{2}$ and $t_2 > \frac{1-m}{2}$. Then $\mu((x|(y|y))|(x|(y|y))) \geq \frac{1-m}{2}$, and thus $\mu((x|(y|y))|(x|(y|y))) + \min\{t_1, t_2\} > \frac{1-m}{2} + \frac{1-m}{2} = 1 - m$, that is,

$$((x|(y|y))|(x|(y|y)))_{\min\{t_1,t_2\}} q_m \mu.$$

Hence, $((x|(y|y))|(x|(y|y)))_{\min\{t_1,t_2\}} \in \vee q_m \mu$, and consequently, μ is an $(\in, \in \vee q_m)$-fuzzy subalgebra of H.

Theorem 2. *A fuzzy set μ of H is an $(\in, \in \vee q_m)$-fuzzy subalgebra of H if and only if each nonempty level set $U(\mu, t) = \{x \in H : \mu(x) \geq t\}$, $t \in (0, \frac{1-m}{2}]$, is a subalgebra of X.*

Proof. Assume that a fuzzy set μ is an $(\in, \in \vee q)$-fuzzy subalgebra of H. Let $t \in (0, \frac{1-m}{2}]$ and $x, y \in U(\mu, t)$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. It follows from

$$\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\}$$

holds for all $x, y \in H$ that $\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\} \geq \min\{t, \frac{1-m}{2}\} = t$, so that $(x|(y|y))|(x|(y|y)) \in U(\mu, t)$. Hence, $U(\mu, t)$ is an $(\in, \in \vee q_m)$-fuzzy subalgebra of H.

Conversely, suppose that the nonempty set $U(\mu, t)$ is a subalgebra of H for all $t \in (0, \frac{1-m}{2}]$. If the condition $\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\}$ holds for all $x, y \in H$ is not true, then there exist $a, b \in H$ such that

$$\mu((a|(b|b))|(a|(b|b))) < \min\{\mu(a), \mu(b), \frac{1-m}{2}\}.$$

Hence, we can take $t \in (0, 1]$ such that $\mu((a|(b|b))|(a|(b|b))) < t_1 < \min\{\mu(a), \mu(b), \frac{1-m}{2}\}$. Then $t \in (0, \frac{1-m}{2}]$ and $a, b \in U(\mu, t)$. Since $U(\mu, t)$ is a subalgebra of H, we have $(a|(b|b))|(a|(b|b)) \in U(\mu, t)$, so $\mu((a|(b|b))|(a|(b|b))) \geq t$. This is a contradiction. Therefore,

$$\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\}$$

holds for all $x, y \in H$, and so μ is an $(\in, \in \vee q_m)$-fuzzy subalgebra of H.

Theorem 3. *Let μ be a fuzzy set of H. Then the nonempty level set $U(\mu, t)$ is a subalgebra of H for all $t \in (\frac{1-m}{2}, 1]$ if and only if $\max\{\mu((x|(y|y))|(x|(y|y))), \frac{1-m}{2}\} \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in H$.*

Proof. Suppose that $U(\mu, t) \neq \emptyset$ is a subalgebra of H. Assume that

$$\max\{\mu((x|(y|y))|(x|(y|y))), \frac{1-m}{2}\} < \min\{\mu(x), \mu(y)\} = t$$

for some $x, y \in H$. Then $t \in (\frac{1-m}{2}, 1], \mu((x|(y|y))|(x|(y|y))) < t$, $x \in U(\mu, t)$ and $y \in U(\mu, t)$. Since $x, y \in U(\mu, t)$, we have $U(\mu, t)$ is a subalgebra of H, so $(x|(y|y))|(x|(y|y)) \in U(\mu, t)$, a contradiction. The proof of the second part of the theorem is straightforward.

Theorem 4. *Let μ be an $(\in, \in \vee q_m)$-fuzzy subalgebra of H. If it satisfies $\mu(x) < \frac{1-m}{2}$ for all $x \in H$, then it is a fuzzy subalgebra of H.*

Proof. Let $x, y \in H$ and $t_1, t_2 \in (0, 1]$ be such that $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$. Then $\mu(x) \geq t_1$ and $\mu(y) \geq t_2$. It follows from Theorem 1 that $\mu((x|(y|y))|(x|(y|y))) > \min\{\mu(x), \mu(y), \frac{1-m}{2}\} = \min\{\mu(x), \mu(y)\} = \min\{t_1, t_2\}$, so $((x|(y|y))|(x|(y|y)))_{\min\{t_1,t_2\}} \in \mu$. Hence, μ is a fuzzy subalgebra of H.

Theorem 5. *If $0 \leq m < n < 1$, then each $(\in, \in \vee q_m)$-fuzzy subalgebra of H is an $(\in, \in \vee q_n)$-fuzzy subalgebra of H.*

Proof. Let μ be an $(\in, \in \vee q_m)$-fuzzy subalgebra of H and let $x, y \in H$. Then $\mu((x|(y|y))|(x|(y|y))) > \min\{\mu(x), \mu(y), \frac{1-m}{2}\} \geq \min\{\mu(x), \mu(y), \frac{1-n}{2}\}$. Thus, from Theorem 1, we have μ is an $(\in, \in \vee q_n)$-fuzzy subalgebra of H.

Note that an $(\in, \in \vee q_n)$-fuzzy subalgebra may not be an $(\in, \in \vee q_m)$-fuzzy subalgebra for $0 \leq m < n < 1$.

Theorem 6. *A nonempty subset M of H is a subalgebra of H if and only if its characteristic function is an $(\in, \in \vee q_m)$-fuzzy subalgebra of H.*

Proof. Let M be a subalgebra of H. Then $\chi_M(x) = 1$ for $x \in M$ and $\chi_M(x) = 0$ for $x \notin M$. Thus, $U(\mu_M, t) = M$ for all $t \in (0, \frac{1-m}{2}]$. Hence, by Theorem 2, we have χ_M is an $(\in, \in \vee q_m)$-fuzzy subalgebra of H.

Conversely, suppose that μ_M is an $(\in, \in \vee q_m)$-fuzzy subalgebra of H. Then

$$\mu((x|(y|y))|(x|(y|y))) > \min\{\chi_M(x), \chi_M(y), \frac{1-m}{2}\} = \min\{1, \frac{1-m}{2}\} = \frac{1-m}{2}$$

for all $x, y \in H$. Since $m \in [0, 1)$, $\chi_M((x|(y|y))|(x|(y|y))) = 1$, so $(x|(y|y))|(x|(y|y)) \in M$. Hence, M is a subalgebra of H.

Theorem 7. *For every subalgebra M of H and every $t \in (0, \frac{1-m}{2}]$ there exists an $(\in, \in \vee q_m)$-fuzzy subalgebra μ of H such that $U(\mu, t) = M$.*

Proof. Let μ be a fuzzy set in H defined by

$$\mu(x) = \begin{cases} t & \text{if } x \in M \\ 0 & \text{otherwise,} \end{cases}$$

where $t \in (0, \frac{1-m}{2}]$. Obviously, $U(\mu, t) = M$. Assume that $\mu((x|(y|y))|(x|(y|y))) < \min\{\mu(x), \mu(y), \frac{1-m}{2}\}$ for some $x, y \in H$. Since $|Im(\mu)| = 2$, $\mu((x|(y|y))|(x|(y|y))) = 0$ and $\min\{\mu(x), \mu(y), \frac{1-m}{2}\} = t$. Hence, $\mu(x) = \mu(y) = t$, and so $x, y \in M$. Since M is a subalgebra of H, $(x|(y|y))|(x|(y|y)) \in M$. Thus, $\mu((x|(y|y))|(x|(y|y))) = t$, which is a contradiction. Therefore, $\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\}$ for all $x, y \in H$. By Theorem 1, we have μ is an $(\in, \in \vee q_m)$-fuzzy subalgebra of H.

Theorem 8. *The intersection of any family of $(\in, \in \vee q_m)$-fuzzy subalgebras of H is an $(\in, \in \vee q_m)$-fuzzy subalgebra of H.*

Proof. Let $\mu = \bigcap_{i\in\Delta} \mu_i$, where μ_i is an $(\in, \in \vee q_m)$-fuzzy subalgebras of H. Then

$$\begin{aligned} \mu((x|(y|y))|(x|(y|y))) &= \inf_{i\in\Delta} \mu_i((x|(y|y))|(x|(y|y))) \\ &\geq \inf_{i\in\Delta} \min\{\mu_i(x), \mu_i(y), \tfrac{1-m}{2}\} \\ &\geq \min\left\{\inf_{i\in\Delta} \mu_i(x), \inf_{i\in\Delta} \mu_i(y), \tfrac{1-m}{2}\right\} \\ &= \min\left\{\bigcap_{i\in\Delta} \mu_i(x), \bigcap_{i\in\Delta} \mu_i(y), \tfrac{1-m}{2}\right\} \\ &= \min\{\mu(x), \mu(y), \tfrac{1-m}{2}\}. \end{aligned}$$

Hence, by Theorem 1, we have μ is an $(\in, \in \vee q_m)$-fuzzy subalgebra of H.

Theorem 9. *For any finite strictly increasing chain of subalgebras of H there exists an $(\in, \in \vee q_m)$-fuzzy subalgebra μ of H whose level subalgebras are precisely the members of the chain with $\mu_{\frac{1-m}{2}} = H_0 \subset H_1 \subset \ldots \subset H_n = H$.*

Proof. Let $\{t_i : t_i \in (0, \frac{1-m}{2}], i = 1, 2, \ldots, n\}$ be such that $\frac{1-m}{2} > t_1 > t_2 > t_3 > \ldots > t_n$. Consider the fuzzy set μ defined by

$$\mu(x) = \begin{cases} \frac{1-m}{2} & \text{if} \quad x \in H_0 \\ t_k & \text{if} \quad x \in H_k \backslash H_{k-1}, k = 1, 2, \ldots, n. \end{cases}$$

Let $x, y \in H$ be such that $x \in H_i \backslash H_{i-1}$ and $y \in H_j \backslash H_{j-1}$, where $1 \leq i, j \leq n$. If $i \geq j$, then $x \in H_i$ and $y \in H_i$, so $(x|(y|y))|(x|(y|y)) \in H_i$. Thus,

$$\mu((x|(y|y))|(x|(y|y))) \geq t_i = \min\{t_i, t_j\} = \min\{\mu(x), \mu(y), \frac{1-m}{2}\}.$$

If $i < j$, then $x \in H_j$ and $y \in H_j$, so $(x|(y|y))|(x|(y|y)) \in H_j$. Thus,

$$\mu((x|(y|y))|(x|(y|y))) \leq t_j = \min\{t_i, t_j\} = \min\{\mu(x), \mu(y), \frac{1-m}{2}\}.$$

Hence, μ is an $(\in, \in \vee q_m)$-fuzzy subalgebra of H.

Definition 7. *For any fuzzy set μ in H and $t \in (0, 1]$, we define the sets $[\mu]_t = \{x \in H : x_t \in \vee q_m \mu\}$ and $Q(\mu, t) = \{x \in H : x_t q_m \mu\}$. It is clear that $[\mu]_t = U(\mu, t) \cup Q(\mu, t)$.*

Theorem 10. *Let μ be a fuzzy set in H. Then μ is an $(\in, \in \vee q_m)$-fuzzy subalgebra of H if and only if $[\mu]_t$ is a subalgebra of H for all $t \in (0, 1]$. We call $[\mu]_t$ an $(\in \vee q_m)$-level subalgebra of μ.*

Proof. Assume that μ is an $(\in, \in \vee q_m)$-fuzzy subalgebra of $H$and let $x, y \in [\mu]_t$ for $t \in (0, 1]$. Then $(x, t) \in \vee q_m \mu$ and $(y, t) \in \vee q_m \mu$, that is, $\mu(x) > 1$ or $\mu(x) + t > 1 - m$, and $\mu(y) > 1$ or $\mu(y) + t > 1 - m$. By Theorem 1, we have $\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\}$.

Case 1: If $\mu(x) \geq t$ and $\mu(y) \geq t$, then $\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\} = \frac{1-m}{2}$. Hence, $\mu((x|(y|y))|(x|(y|y))) + t > \frac{1-m}{2} + \frac{1-m}{2} = 1 - m$, and so

$$((x|(y|y))|(x|(y|y)), t) q_m \mu.$$

If $t \leq \frac{1-m}{2}$, then $\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\} = \frac{1-m}{2} \geq t$, and thus $((x|(y|y))|(x|(y|y)), t) \in \mu$. Hence,

$$((x|(y|y))|(x|(y|y)), t) \in \vee q_m \mu.$$

Therefore, $(x|(y|y))|(x|(y|y)) \in [\mu]_t$.

Case 2: If $\mu(x) \geq t$ and $\mu(y) + t > 1 - m$. If $t > \frac{1-m}{2}$, then $\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\} = \mu(y) \wedge \frac{1-m}{2} > (1 - m - t) \wedge \frac{1-m}{2} = 1 - m - t$, and so

$$((x|(y|y))|(x|(y|y)), t) q_m \mu.$$

If $t \leq \frac{1-m}{2}$, then $\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\} \geq \min\{t, 1-m-t, \frac{1-m}{2}\} = t$. Hence, $((x|(y|y))|(x|(y|y)), t) \in \mu$, and hence

$$((x|(y|y))|(x|(y|y)), t) \in \vee q_m \mu.$$

Therefore, $(x|(y|y))|(x|(y|y)) \in [\mu]_t$.

Case 3: If $\mu(x) + t > 1 - m$ and $\mu(y) \geq t$. If $t > \frac{1-m}{2}$, then $\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\} = \mu(x) \wedge \frac{1-m}{2} > (1 - m - t) \wedge \frac{1-m}{2} = 1 - m - t$, and so

$$((x|(y|y))|(x|(y|y)), t) q_m \mu.$$

If $t \leq \frac{1-m}{2}$, then $\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\} \geq \min\{1-m-t, t, \frac{1-m}{2}\} = t$. Hence, $((x|(y|y))|(x|(y|y)), t) \in \mu$, and hence

$$((x|(y|y))|(x|(y|y)), t) \in \vee q_m \mu.$$

Therefore, $(x|(y|y))|(x|(y|y)) \in [\mu]_t$.

Case 4: If $\mu(x)+t > 1-m$ and $\mu(y)+t > 1-m$. If $t > \frac{1-m}{2}$, then $\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\} > (1 - m - t) \wedge \frac{1-m}{2} = 1 - m - t$, and so

$$((x|(y|y))|(x|(y|y)), t) q_m \mu.$$

If $t \leq \frac{1-m}{2}$, then $\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\} \geq \min\{1-m-t, t, \frac{1-m}{2}\} \geq (1 - m - t) \wedge \frac{1-m}{2} = \frac{1-m}{2} \geq t$. Hence, $((x|(y|y))|(x|(y|y)), t) \in \mu$, and hence

$$((x|(y|y))|(x|(y|y)), t) \in \vee q_m \mu.$$

Therefore, $(x|(y|y))|(x|(y|y)) \in [\mu]_t$.

Consequently, $[\mu]_t$ is a subalgebra of H.

Conversely, let μ be a fuzzy set in H and $t \in (0, 1]$ be such that $[\mu]_t$ is a subalgebra of H. If it is possible, let $\mu((x|(y|y))|(x|(y|y))) < t \leq \min\{\mu(x), \mu(y), \frac{1-m}{2}\}$ for some $t \in (0, 1)$. Then $x, y \in U(\mu, t) \subseteq [\mu]_t$, which implies that $(x|(y|y))|(x|(y|y)) \in [\mu]_t$. Hence, $\mu((x|(y|y))|(x|(y|y))) \in [\mu]_t$ or $\mu((x|(y|y))|(x|(y|y))) + t + m > 1$, a contradiction. Therefore, $\mu((x|(y|y))|(x|(y|y))) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\}$ for all $x, y \in H$. By Theorem 1, we conclude that μ is an $(\in, \in \vee q_m)$-fuzzy subalgebra of H.

Theorem 11. *Let μ be a proper $(\in, \in \vee q_m)$-fuzzy subalgebra of H having at least two values $t_1, t_2 < \frac{1-m}{2}$. If all $[\mu]_t, t \in (0, \frac{1-m}{2}]$, are subalgebras, then μ can be decomposed into the union of two proper nonequivalent $(\in, \in \vee q_m)$-fuzzy subalgebras of H.*

Proof. Let μ be a proper $(\in, \in \vee q_m)$-fuzzy subalgebra of H with values $\frac{1-m}{2} > t_1 > t_2 > \ldots > t_n$, where $n > 2$. Let $H_0 = [\mu]_{\frac{1-m}{2}}$ and $H_k = [\mu]_{t_k}$ for $k = 1, 2, \ldots, n$. Then $\mu_{\frac{1-m}{2}} = H_0 \subset H_1 \subset \ldots \subset H_n = H$ is the chain of $(\in, \in \vee q_m)$-subalgebras of H. Consider the fuzzy sets $\lambda_1, \lambda_2 \leq \mu$ defined by

$$\lambda_1(x) = \begin{cases} t_1 & \text{if} \quad x \in H_1 \\ t_k & \text{if} \quad x \in H_k \backslash H_{k-1}, k = 2, \ldots, n, \end{cases}$$

$$\lambda_2(x) = \begin{cases} \mu(x) & \text{if} \quad x \in H_0 \\ t_2 & \text{if} \quad x \in H_2 \backslash H_0 \\ t_k & \text{if} \quad x \in H_k \backslash H_{k-1}, k = 3, \ldots, n. \end{cases}$$

Then λ_1 and λ_2 are $(\in, \in \vee q_m)$-fuzzy subalgebras of H with $H_0 \subset H_1 \subset \ldots \subset H_n$ and $H_0 \subset H_1 \subset \ldots \subset H_n$ being respectively chains of $(\in, \in \vee q_m)$-fuzzy subalgebras of H. Obviously, $\mu = \lambda_1 \vee \lambda_2$. Moreover, λ_1 and λ_2 are non-equivalent since $H_0 \neq H_1$.

Definition 8. *Let $\langle A, |_A, 0_A \rangle$ and $\langle B, |_B, 0_B \rangle$ be Sheffer stroke Hilbert algebras. Then a mapping $f : A \to B$ is called a homomorphism if $f(x|_A y) = f(x)|_B f(y)$ for all $x, y \in A$ and $f(0_A) = 0_B$.*

Theorem 12. *Let $A = \langle A, |_A, 0_A \rangle$ and $B = \langle B, |_B, 0_B \rangle$ be Sheffer stroke Hilbert algebras, $f : A \to B$ be a surjective homomorphism. If μ is an $(\in, \in \vee q_m)$-fuzzy subalgebra of B for $m \in (0, 1)$, then $f^{-1}(\mathcal{B})$ is an $(\in, \in \vee q_m)$-fuzzy subalgebra of A.*

Proof. Let μ be an $(\in, \in \vee q_m)$-fuzzy subalgebra of B for $m \in (0, 1)$ and $x, y \in A$. Then

$$\begin{aligned} & f^{-1}(\mu)((x|_A(y|_A y))|_A(x|_A(y|_A y))) \\ &= \mu(f((x|_A(y|_A y))|_A(x|_A(y|_A y)))) \\ &= \mu((f(x)|_B(f(y)|_B f(y)))|_B(f(x)|_B(f(y)|_B f(y)))) \\ &\geq \min\{\mu(f(x)), \mu(f(y)), \tfrac{1-m}{2}\} \\ &= \min\{f^{-1}(\mu(x)), f^{-1}(\mu(y)), \tfrac{1-m}{2}\}. \end{aligned}$$

Hence, $f^{-1}(\mu)$ is an $(\in, \in \vee q_m)$-fuzzy subalgebra of A.

Definition 9. *Let $f : X \to Y$ be a function. An $(\in, \in \vee q_m)$-fuzzy subalgebra μ is said to be f-invariant if $f(x) = f(y)$ implies that $\mu(x) = \mu(y)$ for all $x, y \in X$.*

Theorem 13. *Let $A = \langle A, |_A, 0_A \rangle$ and $B = \langle B, |_B, 0_B \rangle$ be Sheffer stroke Hilbert algebras, $f : A \to B$ be a homomorphism and μ an $(\in, \in \vee q_m)$-fuzzy subalgebra of A. If μ is f-invariant, then $f(\mu)$ is an $(\in, \in \vee q_m)$-fuzzy subalgebra of B, where*

$$f(\mu)(x) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x) & \textit{if} \quad f^{-1}(y) \neq \emptyset \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $y_1, y_2 \in B$. If $f^{-1}(y_1) = \emptyset$ or $f^{-1}(y_2) = \emptyset$, then the proof is obvious. Otherwise, let $f^{-1}(y_1) \neq \emptyset$ or $f^{-1}(y_2) \neq \emptyset$. Then there exist $x_1, x_2 \in A$ such that and $f(x_1) = y_1$ and $f(x_2) = y_2$. Thus,

$$\begin{aligned}
&f(\mu)((y_1|_B(y_2|_By_2))|_B(y_1|_B(y_2|_By_2))) \\
&= \sup_{x\in f^{-1}((y_1|_B(y_2|_By_2))|_B(y_1|_B(y_2|_By_2)))} \mu(x) \\
&= \sup_{x\in f^{-1}((f(x_1)|_B(f(x_2)|_Bf(x_2)))|_B(f(x_1)|_B(f(x_2)|_Bf(x_2))))} \mu(x) \\
&= \sup_{x\in f^{-1}(f((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2))))} \mu(x) \\
&= \mu((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2))) \\
&\geq \min\{\mu(x_1), \mu(x_2), \tfrac{1-m}{2}\} \\
&= \min\left\{\sup_{x\in f^{-1}(f(x_1))} \mu(x), \sup_{x\in f^{-1}(f(x_2))} \mu(x), \tfrac{1-m}{2}\right\} \\
&= \min\left\{\sup_{x\in f^{-1}(y_1)} \mu(x), \sup_{x\in f^{-1}(y_2)} \mu(x), \tfrac{1-m}{2}\right\} \\
&= \min\left\{f(\mu)(y_1), f(\mu)(y_2), \tfrac{1-m}{2}\right\}.
\end{aligned}$$

Hence, $f(\mu)$ is an $(\in, \in \vee q_m)$-fuzzy subalgebra of B.

Remark 2. *Let $A = \langle A, |_A, 0_A\rangle$ and $B = \langle B, |_B, 0_B\rangle$ be Sheffer stroke Hilbert algebras. Then $A\times B = \langle A\times B, |_{A\times B}, 0_{A\times B}\rangle$ is a Sheffer stroke Hilbert algebra, where the set $A\times B$ is the Cartesian product of A and B and the operation $|_{A\times B}$ on this set is defined by $(a_1, b_1)|_{A\times B}(a_2, b_2) = (a_1|_Aa_2, b_1|_Bb_2)$, and the fixed element is $0_{A\times B} = (0_A, 0_B)$.*

Let μ_A and μ_B be $(\in, \in \vee q_m)$-fuzzy subalgebras of Sheffer stroke Hilbert algebras $A = \langle A, |_A, 0_A\rangle$ and $B = \langle B, |_B, 0_B\rangle$, respectively, for $m \in (0, 1)$. The cartesian product of μ_A and μ_B is defined by $\mu = \mu_A \times \mu_B$, where $\mu(x, y) = \min\{\mu_A(x), \mu_B(y)\}$.

Theorem 14. *If μ_A and μ_B are $(\in, \in \vee q_m)$-fuzzy subalgebras of Sheffer stroke Hilbert algebras $A = \langle A, |_A, 0_A\rangle$ and $B = \langle B, |_B, 0_B\rangle$, respectively, then μ is an $(\in, \in \vee q_m)$-fuzzy subalgebra of $A \times B = \langle A \times B, |_{A\times B}, 0_{A\times B}\rangle$.*

Proof. For any $(x_1, y_1), (x_2, y_2) \in A \times B$, we have

$$\begin{aligned}
&\mu(((x_1, y_1)|_{A\times B}((x_2, y_2)|_{A\times B}(x_2, y_2)))|_{A\times B}((x_1, y_1)|_{A\times B}((x_2, y_2)|_{A\times B}(x_2, y_2)))) \\
&= \mu((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2)), (y_1|_B(y_2|_By_2))|_B(y_1|_B(y_2|_By_2))) \\
&= \min\{\mu_A((x_1|_A(x_2|_Ax_2))|_A(x_1|_A(x_2|_Ax_2))), \mu_B((y_1|_B(y_2|_By_2))|_B(y_1|_B(y_2|_By_2)))\} \\
&\geq \min\left\{\min\left\{\mu_A(x_1), \mu_A(x_2), \tfrac{1-m}{2}\right\}, \min\left\{\mu_B(y_1), \mu_B(y_2), \tfrac{1-m}{2}\right\}\right\} \\
&= \min\left\{\min\left\{\mu_A(x_1), \mu_B(y_1), \tfrac{1-m}{2}\right\}, \min\left\{\mu_A(x_2), \mu_B(y_2), \tfrac{1-m}{2}\right\}\right\} \\
&= \min\left\{\mu(x_1, y_1), \mu(x_2, y_2), \tfrac{1-m}{2}\right\}.
\end{aligned}$$

Hence, $\langle A \times B, |_{A\times B}\rangle$ is an $(\in, \in \vee q_m)$-fuzzy subalgebra of $A \times B$.

4. Conclusion

In this paper, we introduce and study generalized fuzzy subalgebras within the framework of Sheffer stroke Hilbert algebras. By defining $(\in, \in \vee q_m)$-fuzzy subalgebras, we have extended classical subalgebra concepts to the fuzzy setting, providing a robust theoretical foundation for further research. Our key results include the characterization of these subalgebras through their level subsets, as well as necessary and sufficient conditions for their existence. Additionally, we have demonstrated the algebraic properties of these subalgebras, including their behavior under intersection, union, and homomorphism.

The implications of this work are twofold. First, it enriches the theoretical understanding of Sheffer stroke Hilbert algebras by incorporating fuzzy set theory. Second, it opens new avenues for applications in logical systems and algebraic structures where uncertainty and vagueness are inherent. Future research directions could explore the applications of these generalized fuzzy subalgebras in automated reasoning, artificial intelligence, and other areas of computer science.

Acknowledgements

This work was supported by the revenue budget in 2025, School of Science, University of Phayao (Grant No. PBTSC68018).

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