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On Quotient INK-Algebras

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Abstract. This paper introduces the notion of quotient INK-algebra $(X, *, 0)$. We formalize the notion of congruence relations in INK-algebras, proving that the set of equivalence classes forms a partition. Additionally, we define a normal subset N of X and show that the quotient set $X/N = \{[x]_N | x \in X\}$ where $[x]_N = \{y \in X | x \underset{N}{\sim} y\}$ and a binary operation $\odot$ on X/N such that $[x]_N \odot [y]_N = [x * y]_N$ for all $[x]_N, [y]_N \in X/N$ forms a quotient INK-algebra.

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1. Introduction

Algebraic structures play a crucial role in various branches of mathematics, including logic, topology, and theoretical computer science. Among these, BE-algebras and INK-algebras have been extensively studied due to their applicability in non-classical logics and algebraic systems. BE-algebras, introduced as a generalization of BCK/BCI-algebras, provide a framework for handling abstract algebraic operations with specific axiomatic structures. Meanwhile, INK-algebras have been recently developed as an extension of BE-algebras with additional constraints and properties.

The study of INK-algebras has garnered interest due to their unique structural properties and potential applications in algebraic logic. Understanding congruence relations, ideals, and quotient structures within these algebras is fundamental in advancing the theoretical framework of algebraic systems. The notion of congruences provides insight into the partitioning of algebraic structures, while ideals and quotient algebras help in forming new algebraic systems from existing ones.

In 2002, J. Neggers and H. S. Kim ([1]) introduced the notion of B-algebras and some properties of exponents on them. One such structure is the INK-algebras, introduced in 2017 ([2]) by Kaviyarasu and Indhira, who introduced a new notion called INK-algebras to generalize and extend the properties of B-algebras. INK-algebras have been widely studied in relation to their ideal structures, congruences, and algebraic operations. In

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particular, the concept of quotient INK-algebras arises naturally when considering the role of congruences and normal subsets in these algebras. A quotient INK-algebra is constructed by partitioning an INK-algebra using a congruence relation induced by an ideal. This process mirrors the formation of quotient groups in group theory and quotient rings in ring theory. The primary objective of forming a quotient INK-algebra is to simplify the structure while preserving essential algebraic properties. Given an INK-algebra $(X, *, 0)$, an ideal I of X induces a congruence relation $\underset{I}{\sim}$ on X as $x \underset{I}{\sim} y$ iff $x*y \in I$ and $y * x \in I$ for any $x, y \in X$. Define an equivalence classes $[x]_I = \{y \in X | x \underset{I}{\sim} y\}$ and hence the set of all equivalence classes forms a partition of X. Moreover, this paper aims to explore the construction and properties of quotient INK-algebras $(X/N, \odot, [0]_I)$, including the role of normal subsets defining these structures. We establish key theorems that demonstrate how the quotient operation preserves the axioms of an INK-algebra. Furthermore, we provide illustrative examples to highlight the behavior of quotient INK-algebras and their applications within algebraic systems.

This paper aims to explore fundamental properties of INK-algebras, particularly focusing on congruence relations and their impact on the structure of INK-algebras. We establish necessary definitions, investigate the conditions for congruences, and present results related to the formation of quotient INK-algebras. The organization of the paper is as follows: Section 2 introduces preliminaries, including essential definitions and fundamental results on INK-algebras. Section 3 discusses congruence relations and their characterization in INK-algebras. Finally, Section 4 explores the construction of quotient INK-algebras and their properties.

2. Preliminaries

First, we will review some essential notations and definitions of INK-algebras and ordinary senses that are needed for this study in this section. Throughout this paper, X will denote the INK-algebra $(X, *, 0)$ unless otherwise specified.

Definition 1. *[2] A BE-algebras $(X, *, 0)$ is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:*

(BE1) $x * x = 0$,
(BE2) $x * 0 = 0$,
(BE3) $0 * x = x$,
(BE4) $x * (y * z) = y * (x * z)$ *for all* $x, y, z \in X$.

Definition 2. *[3] A INK-algebras $(X, *, 0)$ is a non-empty set X with a constant 0 and a binary operation $*$ satisfies the following axioms for all $x, y, z \in X$.*

(INK1) $((x * y) * (x * z)) * (z * y) = 0$,

(INK2) $((x * z) * (y * z)) * (x * y) = 0$,

(INK3) $x * 0 = x$,

(INK4) $x * y = 0$ *and* $y * x = 0$ *implies* $x = y$.

A non-empty subset S of X is said to be a subalgebra of X if $x * y \in S$ for all $x, y \in S$.

Example 1. *[4]: Let* $X = \{0, 1, 2, 3\}$ *and a binary operations* $*$ *on* X *defined by the following table:*

$*$	0	1	2	3
0	0	0	2	2
1	1	0	2	2
2	2	2	0	0
3	3	2	1	0

Then $(X, *, 0)$ *is a INK-algebra but not a BE-algebra, since* $0 * 1 = 0 \neq 1$ *and* $0 * 3 = 2 \neq 3$.

Example 2. *[5] Let* $X = \{1, a, b, c, d\}$ *with a binary operations* $*$ *on* X *defined by the following table:*

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then $(X, *, 0)$ *is a BE-algebra but not an INK-algebra, since* $a * 1 = 1 \neq a$.

Definition 3. *[3] Let* $(X, *, 0)$ *be a* INK*-algebra. A nonempty subset* I *of* X *is called an ideal of* X *if it satisfies the following conditions:*

(i) $0 \in I$,

(ii) $x * y \in I$ *and* $y \in I$ *imply that* $x \in I$ *for all* $x, y \in X$.

Theorem 1. *[2] Let* $(X, *, 0)$ *be an INK-algebra. Then the following conditions hold for all* $x, y, z \in X$;

(i) $x * 0 = 0$ *implies* $x = 0$,

(ii) $(x * (x * y) * y) = 0$,

(iii) $0 * (x * y) = (0 * x) * (0 * y)$,

(iv) $x * y = 0$ *and* $y * x = 0$ *implies* $x = y$,

(v) $(x * y) * z = (x * z) * y$,

3. Congruence relations and Partitions on INK-algebras

In the study of algebraic structures, congruence relations play a fundamental role in constructing quotient structures and analyzing their properties. For INK-algebras, the characterization of congruence relations provides a pathway to understanding the internal symmetry and decomposability of such algebras via partitions induced by these relations.

Definition 4. *Let* $\sim$ *be a binary relation on a set* X*. Then* $\sim$ *is called:*

(i) *reflexive if* $x \sim x$ *for all* $x \in X$*;*

(ii) *symmetric if* $x \sim y$ *implies* $y \sim x$ *for all* $x, y \in X$*;*

(iii) *transitive if* $x \sim y$ *and* $y \sim z$ *implies* $x \sim z$ *for all* $x, y, z \in X$*;*

(iv) *compatible if* $a \sim b$ *and* $x \sim y$ *implies* $a * x \sim b * y$ *for all* $a, b, x, y \in X$*.*

A relation $\sim$ is said to be an equivalence relation if $\sim$ is reflexive, symmetric and transitive. An equivalence relation on X that is compatible with the operation $*$ is called a congruence on X.

Example 3. *[3] Let* $X = \{0, 1, a, b\}$ *with a binary operations* $*$ *on* X *defined by the following table:*

$*$	0	1	a	b
0	0	0	a	a
1	1	0	b	a
a	a	a	0	0
b	b	a	1	0

Then $(X, *, 0)$ *is a* INK*-algebra. The set* $I = \{0, a\}$ *is an ideal of* X *and the relation* $\underset{I}{\sim} = \{(0,0),(1,1),(a,a),(b,b),(0,a),(a,0),(1,b),(b,1)\}$ *is a congruence on* X*.*

Theorem 2. *Let* $(X, *, 0)$ *be a INK-algebra and* I *is an ideal of* X*. Then the relation* $\underset{I}{\sim} = \{(x, y) \in X \times X |\ x * y \in I$ *and* $y * x \in I\}$ *is a congruence on* X*.*

Proof. Since $x * x = 0 \in I$ *for all* $x \in X$*, that is* $x \underset{I}{\sim} x$*. If* $x, y \in X$ *and* $x \underset{I}{\sim} y$*, clearly* $y \underset{I}{\sim} x$*. Next if* $x \underset{I}{\sim} y$ *and* $y \underset{I}{\sim} z$*, then* $x * y \in I$*,* $y * x \in I$*,* $y * z \in I$ *and* $z * y \in I$*. By*

(INK2), $((x*z)*(y*z))*(x*y) = 0 \in I$ *and* I *is ideal, implies that* $(x*z)*(y*z) \in I$, *and hence* $x*z \in I$. *Similarly, we can prove that* $((z*x)*(y*x))*(z*y) = 0 \in I$, *then* $(z*x)*(y*x) \in I$, *thus* $z*x \in I$ *and hence* $x \underset{I}{\sim} z$. *Next to show* $\underset{I}{\sim}$ *is compatible, let* $w, x, y, z \in X$ *such that* $w \underset{I}{\sim} y$ *and* $x \underset{I}{\sim} z$, *then* $w*y \in I$, $y*w \in I$, $x*z \in I$ *and* $z*x \in I$. *Since (INK1) satisfying the identity* $((w*x)*(w*z))*(z*x) = 0 \in I$, *so* $(w*x)*(w*z) \in I$ *and consider* $((w*z)*(w*x))*(x*z) = 0 \in I$, *we have* $(w*z)*(w*x) \in I$, *implies that* $(w*x) \underset{I}{\sim} (w*z)$. *Because* $((w*z)*(w*y))*(y*z) = ((w*z)*(y*z))*(w*y) = 0 \in I$ *by theorem 1(iv) and (INK1), implies that* $(y*z)*(w*z) \in I$ *and hence* $(w*z) \underset{I}{\sim} (y*z)$. *Since* $\underset{I}{\sim}$ *is transitive ,so* $(w*x) \underset{I}{\sim} (y*z)$. *Therefore* $\underset{I}{\sim}$ *is a congruence relation on* X.

Lemma 1. *Let* I *be an ideal of a INK-algebra* $(X, *, 0)$ *and* $\underset{I}{\sim}$ *is a congruence relation on* X. *Then the following conditions hold;*

(i) $x \in [x]_I$ *for all* $x \in X$,

(ii) $x \underset{I}{\sim} y$ *if and only if* $[x]_I = [y]_I$ *for all* $x, y \in X$.

Proof. (i) *Clearly, for any* $x \in X$, $x*x = 0 \in I$ *that is* $x \underset{I}{\sim} x$, *and hence* $x \in [x]_I$.
(ii) *Assume that* $x \underset{I}{\sim} y$, *then* $y \underset{I}{\sim} x$. *Let* $a \in [x]_I$, *then* $x \underset{I}{\sim} a$ *and hence* $a \underset{I}{\sim} x$, *thus* $a \underset{I}{\sim} y$, *so* $a \in [y]_I$. *Therefore* $[x]_I \subseteq [y]_I$. *Simmilary, to prove* $[y]_I \subseteq [x]_I$. *Conversely, suppose that* $[x]_I = [y]_I$, *so* $x \in [x]_I = [y]_I$ *by (i), and hence* $y \underset{I}{\sim} x$, *thus* $x \underset{I}{\sim} y$.

Theorem 3. *Let* $(X, *, 0)$ *be a INK-algebra and* I *is an ideal of* X. *Then the set* $P = \{[x]_I | x \in X\}$ *forms a partition of* X.

Proof. *Let* $[x]_I, [y]_I \in P$ *such that* $[x]_I \neq [y]_I$. *Suppose that* $[x]_I \cap [y]_I \neq \emptyset$, *there is a* $b \in [x]_I \cap [y]_I$, *then* $x \underset{I}{\sim} b$, $y \underset{I}{\sim} b$ *and hence* $b \underset{I}{\sim} x$, *so we have* $y \underset{I}{\sim} x$, *implies that* $[x]_I = [y]_I$ *that's a contradiction, thus* $[x]_I \cap [y]_I = \emptyset$. *Next to show* $\bigcup_{x \in X} [x]_I = X$. *Clearly for any* $x \in X$, $[x]_I \subseteq X$ *and hence* $\bigcup_{x \in X} [x]_I \subseteq X$. *And for any* $x \in X$, $x \in [x]_I \subseteq \bigcup_{x \in X} [x]_I$. *Thus* $P = \{[x]_I | x \in X\}$ *is a partition of* X.

Example 4. *From example 3, we see that* $I = \{0, a\}$ *is an ideal of* X *and relation* $\underset{I}{\sim} = \{(0,0), (1,1), (a,a), (b,b), (0,a), (a,0), (1,b), (b,1)\}$, *is a congruence.*

Consider $[0]_I = \{0, a\} = [a]_I$ *and* $[1]_I = \{1, b\} = [b]_I$, *and hence the set* $P = \{[0]_I, [1]_I\}$ *is a partition of* X.

4. Quotient INK-algebras

The notion of quotient structures is fundamental in universal algebra, serving as a tool for classifying and analyzing algebraic systems via congruence relations. In this section, we introduce the concept of quotient INK-algebras, establish their basic properties, and explore structural implications arising from the induced congruences.

Definition 5. *Let $(X, *, 0)$ be a INK-algebra. A nonempty subset N of X is said to be normal of X if $(x * a) * (y * b) \in N$ for any $x * y,\ a * b \in N$.*

Theorem 4. *Every normal subset of an INK-algebra is a subalgebra.*

Proof. *Assume that N is a normal subset of a INK-algebra $(X, *, 0)$ and $x, y \in N$. By (INK3), $x * 0 = x \in N$ and $y * 0 = y \in N$, hence $x * y = (x * y) * 0 = (x * y) * (0 * 0) \in N$.*

The next example is shown that the converse of theorem 4 is not true.

Example 5. *Consider the example 3,*

$I = \{0, a\}$ is a subalgebra of X and so is ideal but not normal, because $1 * b = a \in I$ and $a * b = 0 \in I$, while $(1 * a) * (b * b) = b * 0 = b \notin I$.

$N = \{0, 1\}$ is a normal subset of X and so is a subalgebra.

Lemma 2. *If S is a subalgebra of INK-algebra $(X, *, 0)$, then $0 \in S$.*

Proof. *Let $x \in S$, By INK-3, $x * 0 = x \in S$ and hence by INK-2,*

$$0 = ((0 * x) * (0 * x) * (x * x)) \in S.$$

Lemma 3. *Let $(X, *, 0)$ be a INK-algebra and I is an ideal of X such that $0 * a \in I$ for all $a \in I$, then I is a subalgebra of X.*

Proof. *Let $x, y \in I$, then $0 * y = (x * x) * y = (x * y) * x \in I$ implies that $x * y \in I$.*

Definition 6. *Let $(X, *, 0)$ be an INK-algebra and let I be a subset of X with an associated congruence relation $\underset{I}{\sim}$. Define the set $X/I = \{[x]_I \mid x \in X\}$, where $[x]_I = \{y \in X | x \underset{I}{\sim} y\}$, and a binary operation $\odot$ on X/I defined by $[x]_I \odot [y]_I = [x * y]_I$ for all $[x]_I, [y]_I \in X/I$. We say that $(X/N, \odot, [0]_N)$ is a quotient INK-algebra if it satisfies Definition 2.*

Theorem 5. *Let $(X, *, 0)$ be a INK-algebra and N is a normal subset of X, then $(X/N, \odot, [0]_I)$ is a quotient INK-algebra.*

Proof. *Clearly the operation $\odot$ is well-defined. That is, for any $[a]_N, [b]_N, [x]_N, [y]_N \in X/N$, if $([a]_N, [b]_N) = ([x]_N, [y]_N)$, then $[a]_N = [x]_N$ and $[b]_N = [y]_N$, which implies that*

$a \underset{N}{\sim} x$ and $b \underset{N}{\sim} y$, thus $(a * b) \underset{N}{\sim} (x * y)$, and hence $[a * b]_N = [x * y]_N$, implies that $[a]_N \odot [b]_N = [x]_N \odot [y]_N$, and satisfies the following;

1) $[([x]_N \odot [y]_N) \odot ([x]_N \odot [z]_N)] \odot ([z]_N \odot [y]_N) = [([x * y]_N) \odot ([x * z]_N)] \odot ([z * y]_N)$
$= [(x * y) * (x * z)]_N \odot [z * y]_N$
$= [((x * y) * (x * z)) * (z * y)]_N$
$= [0]_N,$
2) $[([x]_N \odot [z]_N) \odot ([y]_N \odot [z]_N)] \odot ([z]_N \odot [y]_N) = [([x * y]_N) \odot ([x * z]_N)] \odot ([x * y]_N)$
$= [(x * z) * (y * z)]_N \odot [x * y]_N$
$= [((x * z) * (y * z))_N * (x * y)]_N$
$= [0]_N,$
3) $[x]_N \odot [0]_N = [x * 0]_N = [x]_N$
4) *Assume that* $[x]_N \odot [y]_N = [0]_N$, *then* $[x * y]_N = [0]_N$, *implies that* $x * y = 0$, *so* $x = y$.

Therefore $(X/N, \odot, [0]_N)$ is a quotient INK-algebra.

5. Conclusion

In this paper, we investigated the structure of congruence relations induced by ideals in INK-algebras. We began by reviewing the foundational definitions and properties of INK-algebras, a class of non-classical algebras characterized by specific axioms on binary operation axioms. Notably, we established the necessary and sufficient conditions under which a binary relation derived from an ideal constitutes a congruence relation. The main results demonstrated that for any ideal I of an INK-algebra $(X, *, 0)$, the relation $\underset{I}{\sim}$, defined by $x \underset{I}{\sim} y$ if and only if $x * y \in I$ and $y * x \in I$, is a congruence relation on X. Furthermore, it was shown that the equivalence classes under this congruence relation form a partition of the underlying set X, enabling a quotient-like decomposition of the algebraic structure. In addition, we introduced the notion of normal subsets and demonstrated how they play a central role in defining well-structured quotient INK-algebras. Our construction ensures that the quotient algebra X/N inherits the fundamental axioms of INK-algebras. The provided theorems and examples confirm that the operations on the quotient set are well-defined and consistent with the original algebra. These results not only extend the theory of INK-algebras but also pave the way for further exploration of categorical, topological, or fuzzy extensions of quotient structures in algebraic logic and non-classical algebraic systems.

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