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A Generalization of Similarity Measure in Collection of Intuitionistic Fuzzy Sets

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Abstract. A collection of intuitionistic fuzzy sets is a new approach to intuitionistic fuzzy set theory. In collections of intuitionistic fuzzy sets, a similarity measure can determine the degree of similarity based on the information carried by the collections. However, an existing similarity measure is limited to evaluating similarity between two collections defined over the same universal set. To overcome this limitation, thus, in this paper, we propose a generalized similarity measure that can be applied to collections defined over different universal sets. To construct the generalization, we first introduce the concept of inferior and equivalent relations in the collection of intuitionistic fuzzy sets. Then, we present a new formula for the similarity measure. Finally, the proposed measure is illustrated through a pattern recognition problem to demonstrate its effectiveness and practical value.
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1. Introduction

Zadeh [1] first introduced the concept of fuzzy sets to solve the limitations of classical set theory. In fuzzy set theory, each element in a universal set is associated with a membership degree that ranges in the interval [0, 1]. Research related to fuzzy sets has been further developed by many researchers, such as [2], [3]. To extend the concept, Atanassov [4] introduced the concept of intuitionistic fuzzy sets, in which each element of the universal set is assigned a membership degree and a non-membership degree, both in the interval [0, 1], such that the sum of these degrees does not exceed 1. Research related to intuitionistic fuzzy sets has been further developed by many researchers, such as [5], [6], [7].

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Similarity measures between objects based on attributes such as shape, color, size, and texture are critical in various scientific and engineering applications. Many researchers have developed techniques and tools to measure the similarity between objects that relate to scientific developments, including intuitionistic fuzzy sets. Several studies focusing on the development of similarity measures for intuitionistic fuzzy sets have been developed by [8], [9], [10], [11], [12], [13], [14], [15].

The existing similarity measures are limited to determining similarity between two intuitionistic fuzzy sets. This becomes a problem when we want to compare complex objects, each represented not by a single set, but by a collection or union of multiple intuitionistic fuzzy sets. In many real- world applications, such as pattern recognition or decision-making, representing objects as collections provides a flexible model. To overcome this limitation, Yunianti et al.[16] were the first to define a collection of intuitionistic fuzzy sets as $\mathcal{A} = \{(A_j, \mu_{\mathcal{A}}(A_j), v_{\mathcal{A}}(A_j)) : A_j \in \mathcal{X}\}$ where A_j is the intuitionistic fuzzy set in the universe X, and $\mu_{\mathcal{A}}(A_j), v_{\mathcal{A}}(A_j)$ denote the membership and non-membership degree of A_j in the collection $\mathcal{A}$, respectively.

Furthermore, a similarity measure for collections of intuitionistic fuzzy sets has been introduced by Yunianti et al. [17]. An existing similarity measure for collections of intuitionistic fuzzy sets assumes that the two collections being compared are defined over the same universe of discourse. This limits their use when the collections come from different universes, which often happens in real-world cases. To overcome the limitations, we propose a generalized similarity measure that can be applied to collections defined over different universes. This new measure aims to overcome the limitations of existing methods for comparing the similarity of collections when the universes of discourse are not identical and allows for more flexible and realistic similarity comparisons in cases involving heterogeneous data.

Before presenting the generalization, we introduce the inferior and equivalent relations, as both are used to show that the proposed similarity measure satisfies the axioms of similarity measures. Then, we present formula for the generalization. In addition, this paper provides an example of the application of the proposed measure to a pattern recognition problem.

2. Preliminaries

In this section, we review some basic theories related to intuitionistic fuzzy sets, distance of intuitionistic fuzzy sets, collection of intuitionistic fuzzy sets, and similarity measure for collection of intuitionistic fuzzy sets.

Definition 1. *[4] Let X be a non empty and universal set. An intuitionistic fuzzy set A in X is written as*

$$A = \{(x, \mu_A(x), v_A(x)) : x \in X\}$$

where $\mu_A(x)$ and $v_A(x)$ respectively are the degree of membership and the degree of non-membership of x in A and both belong to $[0, 1]$, with $0 \leq \mu_A(x) + v_A(x) \leq 1$. Moreover, the hesitant degree of x in A is $\pi_A(x) = 1 - \mu_A(x) - v_A(x)$.

Next, we describe relations between intuitionistic fuzzy sets.

Definition 2. *[4] Let A and B are intuitionistic fuzzy sets on X where*
$A = \{(x, \mu_A(x), v_A(x)) : x \in X\}$ *and* $B = \{(x, \mu_B(x), v_B(x)) : x \in X\}$
so we have

1. $A \subseteq B$ *if* $\mu_A(x) \leq \mu_B(x)$ *and* $v_A(x) \geq v_B(x), \forall x \in X$.
2. $A = B$ *if* $\mu_A(x) = \mu_B(x)$ *and* $v_A(x) = v_B(x), \forall x \in X$.

Because a similarity measure can be constructed based on distance measure, so we review the definition of a distance measure between two intuitionistic fuzzy sets.

Definition 3. *A function $d : \mathcal{E} \times \mathcal{E} \to [0,1]$ is said distance measure between two intuitionistic fuzzy sets if it satisfies the following*

1. $0 \leq d(A, B) \leq 1$
2. $d(A, B) = d(B, A)$
3. $d(A, B) = 0$ *iff* $A = B$
4. *If* $A \subseteq B \subseteq C$, *then* $d(A, B) \leq d(A, C)$, *and* $d(B, C) \leq d(A, C)$.

The following example is a distance measure developed based on the measure proposed by Atanassov in [4].

Example 1. *[4] Let* $A_j = \{(x_i, \mu_{A_j}(x_i), v_{A_j}(x_i)) : x_i \in X\}$ *and* $B_k = \{(x_i, \mu_{B_k}(x_i), v_{B_k}(x_i)) : x_i \in X\}$ *are intuitionistic fuzzy sets on the universal set* $X = \{x_1, x_2, \ldots x_n\}$ *with* $j, k = 1, 2, .., m$.

$$d_I(A_j, B_k) = \frac{1}{2n} \sum_{i=1}^{n} |\mu_{A_j}(x_i) - \mu_{B_k}(x_i)| + |v_{A_j}(x_i) - v_{B_k}(x_i)|$$

is a distance measure between two intuitionistic fuzzy sets.

Next, we define a collection of intuitionistic fuzzy sets that was constructed by Yunianti et al [16].

Definition 4. *[16] Let* $A_j = \{(x_i, \mu_{A_j}(x_i), v_{A_j}(x_i)) : x_i \in X\}$ *is intuitionistic fuzzy set on* $X = \{x_i : i = 1, 2, \ldots, n\}$ *with* $j = 1, 2, .., m$.
A collection of intuitionistic fuzzy sets on $\mathcal{X} = \{A_j : j = 1, 2, \ldots, m\}$ *can state as*

$$\mathcal{A} = \{(A_j, \mu_{\mathcal{A}}(A_j), v_{\mathcal{A}}(A_j)) : A_j \in \mathcal{X}\}$$

where $\mu_{\mathcal{A}} : \mathcal{X} \to [0,1]$ *is membership function* $\mathcal{A}$ *on* $\mathcal{X}$ *and* $v_{\mathcal{A}} : \mathcal{X} \to [0,1]$ *is non membership function* $\mathcal{A}$ *on* $\mathcal{X}$.
Moreover, $\mu_{\mathcal{A}}(A_j)$ *can be described as the membership degree of* A_j *on* $\mathcal{A}$ *and* $v_{\mathcal{A}}(A_j)$ *can be described as the non-membership degree* A_j *on* $\mathcal{A}$ *where*

$$0 \leq \mu_{\mathcal{A}}(A_j) + v_{\mathcal{A}}(A_j) \leq 1$$

. The hesitancy degree of A_j *on* $\mathcal{A}$ *is stated as*

$$\pi_{\mathcal{A}}(A_j) = 1 - \mu_{\mathcal{A}}(A_j) - v_{\mathcal{A}}(A_j)$$

For understanding the definition, here, we give an example of collection of intuitionistic fuzzy sets.

Example 2. *A consumer will choose to recommend one restaurant out of three based on three criteria: price* ($p\prime$)*, food variety* ($q\prime$)*, and restaurant facilities* ($r\prime$)*.*
Let $S = \{p\prime, q\prime, r\prime\}$ *as a set of criteria .*
The ratings of the first, second, and third restaurants are represented as intuitionistic fuzzy sets as follows:
$P_1 = \{(p', 0.7, 0.2), (q\prime, 0.8, 0.1), (r\prime, 0.8, 0.1)\}$,
$P_2 = \{(p\prime, 0.1, 0.6), (q\prime, 0.2, 0.5), (r\prime, 0.3, 0.6)\}$,
$P_3 = \{(p\prime, 0.5, 0.2), (q\prime, 0.6, 0.1), (r\prime, 0.7, 0.1)\}$
The consumer provides an overall rating for the three restaurants based on P_1, P_2, P_3 *and this overall rating is represented as a collection of intuitionistic fuzzy sets* $\mathcal{P}$ *.*
Consider $\mathcal{X} = \{P_1, P_2, P_3\}$ *be universal set. A collection of intuitionistic fuzzy sets on* $\mathcal{X}$ *is given by*

$$\mathcal{P} = \{(P_1, 0.8, 0.2), (P_2, 0.2, 0.6), (P_3, 0.6, 0.3)\}$$

. Here, the membership degree of 0.8 *for* P_1 *indicates that the first restaurant is highly recommended by the consumer compared to* P_2 *and* P_3*, while the non-membership degree of* 0.2 *for* P_1 *indicates a low tendency for the first restaurant not to be recommended relative to the others.*

Definition 5. *[17] Given*

$$\mathcal{A} = \{(A_j, \mu_{\mathcal{A}}(A_j), v_{\mathcal{A}}(A_j)) : A_j \in \mathcal{X}\}$$

and

$$\mathcal{B} = \{(A_j, \mu_{\mathcal{B}}(A_j), v_{\mathcal{B}}(A_j)) : A_j \in \mathcal{X}\}$$

respectively collection of intuitionistic fuzzy sets on $\mathcal{X} = \{A_j : j = 1, 2, \ldots, m\}$*.*
We define that

i) $\mathcal{A} \subseteq \mathcal{B}$ *if and if* $\mu_{\mathcal{A}}(A_j) \leq \mu_{\mathcal{B}}(A_j)$ *and* $\nu_{\mathcal{A}}(A_j) \geq \nu_{\mathcal{B}}(A_j)$

ii) $\mathcal{A} = \mathcal{B}$ *if and if* $\mu_{\mathcal{A}}(A_j) = \mu_{\mathcal{B}}(A_j)$ *and* $\nu_{\mathcal{A}}(A_j) = \nu_{\mathcal{B}}(A_j)$

The proposed similarity measure is a generalization of the similarity measure introduced in [17], which determines the similarity between collections of intuitionistic fuzzy sets with identical elements. For completeness, we restate the similarity measure from [17] below.

Theorem 1. *[17] Let* $A_j = \{(x_i, \mu_{A_j}(x_i), v_{A_j}(x_i)) : x_i \in X\}$ *be intuitionistic fuzzy sets on* $X = \{x_1, x_2, \ldots, x_n\}$ *with* $j = 1, 2, .., m$*.* $\mathcal{A} = \{(A_j, \mu_{\mathcal{A}}(A_j), v_{\mathcal{A}}(A_j)) : A_j \in \mathcal{X}\}$ *and* $\mathcal{B} = \{(A_j, \mu_{\mathcal{B}}(A_j), v_{\mathcal{B}}(A_j)) : A_j \in \mathcal{X}\}$ *are collections of intuitionistic fuzzy sets on* $\mathcal{X} = \{A_j : j = 1, 2, \ldots, m\}$*.*

$$S_I(\mathcal{A}, \mathcal{B}) = 1 - \frac{1}{2m} \sum_{j=1}^{m} |\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(A_j)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(A_j)|$$

is a similarity measure between two collections of intuitionistic fuzzy sets.

The following example is example for using Theorem 1.

Example 3. *Let* $\mathcal{A} = \{(A_1, 0.7, 0.2), (A_2, 0.8, 0.1), (A_3, 0.6, 0.4), (A_4, 0.4, 0.3)\}$ *and* $\mathcal{B} = \{(A_1, 0.7, 0.1), (A_2, 0.8, 0.1), (A_3, 0.6, 0.4), (A_4, 0.5, 0.3)\}$ *are collections of intuitionistic fuzzy set on* $\mathcal{X} = \{A_1, A_2, A_3, A_4\}$.
Similarity measure between $\mathcal{A}$ *and* $\mathcal{B}$ *is*

$$S_I(\mathcal{A}, \mathcal{B}) = 1 - \frac{1}{2m} \sum_{j=1}^{4} |\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(A_)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(A_j)| = 0.975$$

3. Results

In this section, we present a similarity measure designed to evaluate the similarity between collections of intuitionistic fuzzy sets defined over distinct universal sets. Before introducing the similarity measure, we first provide definitions of the inferiority and equivalence relations on collections of intuitionistic fuzzy sets.

3.1. A New Approach of Relation on Collection of Intuitionistic Fuzzy Sets

In this sub section, we define inferiority and equivalence relations on collections of intuitionistic fuzzy sets. The inferiority and equivalence relations are a new approach of subset and equality relations on collection of intuitionistic fuzzy sets.

Definition 6. *Consider* $A_j = \{(x_i, \mu_{A_j}(x_i), v_{A_j}(x_i)) : x_i \in X\}$ *and* $B_k = \{(x_i, \mu_{B_k}(x_i), v_{B_k}(x_i)) : x_i \in X\}$ *are intuitionistic fuzzy sets on* $X = \{x_1, x_2, \ldots, x_n\}$ *with* $j, k = 1, 2, .., m$.
$\mathcal{A} = \{(A_j, \mu_{\mathcal{A}}(A_j), v_{\mathcal{A}}(A_j)) : A_j \in \mathcal{X}\}$ *is a collection of intuitionistic fuzzy set on* $\mathcal{X} = \{A_j : j = 1, 2, \ldots, m\}$ *and* $\mathcal{B} = \{(B_k, \mu_{\mathcal{B}}(B_k), v_{\mathcal{B}}(B_k)) : B_k \in \mathcal{Y}\}$ *is a collection of intuitionistic fuzzy set on*

$$\mathcal{Y} = \{B_k : k = 1, 2, \ldots, m\}$$

.
$\mathcal{A}$ *is inferior to* $\mathcal{B}$, *or we write* $\mathcal{A} \tilde{\subseteq} \mathcal{B}$ *if:*

- *i. There exists an injective function* f *from* $\mathcal{X}$ *to* $\mathcal{Y}$ *such that* $A_j \subseteq f(A_j)$
- *ii. For every* j , *there exists* k *such that* $\mu_{\mathcal{A}}(A_j) \leq \mu_{\mathcal{B}}(B_k)$ *and* $v_{\mathcal{A}}(A_j) \geq v_{\mathcal{B}}(B_k)$

Example 4. *We have the following intuitionistic fuzzy sets defined on* $X = \{x_1, x_2, x_3\}$
$A_1 = \{(x_1, 0.4, 0.5), (x_2, 0.6, 0.4), (x_3, 0.3, 0.4)\}$
$A_2 = \{(x_1, 0.3, 0.7), (x_2, 0.5, 0.4), (x_3, 0.5, 0.5)\}$
$B_1 = \{(x_1, 0.3, 0.6), (x_2, 0.5, 0.3), (x_3, 0.5, 0.2)\}$
$B_2 = \{(x_1, 0.5, 0.4), (x_2, 0.7, 0.3), (x_3, 0.3, 0.1)\}$
Also let $\mathcal{A} = \{(A_1, 0.5, 0.4), (A_2, 0.4, 0.6)\}$ *be a collection of intuitionistic fuzzy set on*

$\mathcal{X} = \{A_1, A_2\}$*and*
$\mathcal{B} = \{(B_1, 0.4, 0.5), (B_2, 0.6, 0.2)\}$ *be a collection of intuitionistic fuzzy set on* $\mathcal{Y} = \{B_1, B_2\}$.
Because

i. $A_1 \subseteq B_2$ *and* $A_2 \subseteq B_1$.
So we have there exists an injective function $f : \mathcal{X} \to \mathcal{Y}$ *so* $A_1 \subseteq f(A_1)$ *and* $A_2 \subseteq f(A_2)$.

ii. $\mu_{\mathcal{A}}(A_1) \leq \mu_{\mathcal{B}}(B_2)$ *and* $v_{\mathcal{A}}(A_1) \geq v_{\mathcal{B}}(B_2)$
$\mu_{\mathcal{A}}(A_2) \leq \mu_{\mathcal{B}}(B_1)$ *and* $v_{\mathcal{A}}(A_2) \geq v_{\mathcal{B}}(B_1)$

Thus, we can conclude that $\mathcal{A}$ *is inferior to* $\mathcal{B}$ *or* $\mathcal{A} \widetilde{\subseteq} \mathcal{B}$.

The following definition describes about equivalence relation between two collection of intuitionistic fuzzy sets.

Definition 7. *Consider* $A_j = \{(x_i, \mu_{A_j}(x_i), v_{A_j}(x_i)) : x_i \in X\}$ *and* $B_k = \{(x_i, \mu_{B_k}(x_i), v_{B_k}(x_i)) : x_i \in X\}$ *are intuitionistic fuzzy sets on a non-empty universal set* $X = \{x_1, x_2, \ldots, x_n\}$ *with* $j, k = 1, 2, .., m$.
$\mathcal{A} = \{(A_j, \mu_{\mathcal{A}}(A_j), v_{\mathcal{A}}(A_j)) : A_j \in \mathcal{X}\}$ *is a collection of intuitionistic fuzzy set in the universe of discourse* $\mathcal{X} = \{A_j : j = 1, 2, \ldots, m\}$ *and* $\mathcal{B} = \{(B_k, \mu_{\mathcal{B}}(B_k), v_{\mathcal{B}}(B_k)) : B_k \in \mathcal{Y}\}$ *is a collection of intuitionistic fuzzy set in the universe of discourse*

$$\mathcal{Y} = \{B_k : k = 1, 2, \ldots, m\}$$

. $\mathcal{A}$ *is equivalent to* $\mathcal{B}$, *or we write* $\mathcal{A} \widetilde{\cong} \mathcal{B}$ *if* $\mathcal{A} \widetilde{\subseteq} \mathcal{B}$ *and* $\mathcal{B} \widetilde{\subseteq} \mathcal{A}$.

Definition 8. *Let* $A_j = \{(x_i, \mu_{A_j}(x_i), v_{A_j}(x_i)) : x_i \in X\}$ *and*
$B_k = \{(x_i, \mu_{B_k}(x_i), v_{B_k}(x_i)) : x_i \in X\}$ *be intuitionistic fuzzy sets that defined in*
$X = \{x_i : i = 1, 2, \ldots, n\}$ *respectively with* $j, k = 1, 2, .., m$.
$\mathcal{A} = \{(A_j, \mu_{\mathcal{A}}(A_j), v_{\mathcal{A}}(A_j)) : A_j \in \mathcal{X}\}$ *is a collection of intuitionistic fuzzy set on* $\mathcal{X} = \{A_j : j = 1, 2, \ldots, m\}$ *and* $\mathcal{B} = \{(B_k, \mu_{\mathcal{B}}(B_k), v_{\mathcal{B}}(B_k)) : B_k \in \mathcal{Y}\}$ *is a collection of intuitionistic fuzzy set on* $\mathcal{Y} = \{B_k : k = 1, 2, \ldots, m\}$.
We say that $\mathcal{A} \widetilde{\cong} \mathcal{B}$ *if only if*

i. There exists an injective function f from $\mathcal{X}$ *to* $\mathcal{Y}$ *such that* $f(A_j) = B_k$

ii. For every j, *there exists* k *such that* $\mu_{\mathcal{A}}(A_j) = \mu_{\mathcal{B}}(B_k)$ *and* $v_{\mathcal{A}}(A_j) = v_{\mathcal{B}}(B_k)$

Definitions 7 and 8 are equivalent. By using the concept of inferior relation, we can derive Definition 8 from Definition 7. Next, we provide an example to illustrate the equivalence relation between two collections of intuitionistic fuzzy sets.

Example 5. *We have the following intuitionistic fuzzy sets defined on* $X = \{x_1,\ x_2,\ x_3\}$
$A_1 = \{(x_1, 0.1, 0.8), (x_2, 0.2, 0.5), (x_3, 0.3, 0.4)\}$
$A_2 = \{(x_1, 0.2, 0.7), (x_2, 0.3, 0.6), (x_3, 0.4, 0.5)\}$
$B_1 = \{(x_1, 0.2, 0.7), (x_2, 0.3, 0.6), (x_3, 0.4, 0.5)\}$

$B_2 = \{(x_1, 0.1, 0.8), (x_2, 0.2, 0.5), (x_3, 0.3, 0.4)\}$
Let $\mathcal{X} = \{A_1, A_2\}$, $\mathcal{Y} = \{B_1, B_2\}$ *be universal sets,*
Define collections of intuitionistic fuzzy sets on these universes as
$\mathcal{A} = \{(A_1, 0.1, 0.8), (A_2, 0.3, 0.6)\}$ *be a collection of intuitionistic fuzzy set on* $\mathcal{X}$*,*
$\mathcal{B} = \{(B_1, 0.3, 0.6), (B_2, 0.1, 0.8)\}$ *be a collection of intuitionistic fuzzy set on* $\mathcal{Y}$*.*
Because

i. $A_1 = B_2$ *and* $A_2 = B_1$.
So we have there exists an injective function $f : \mathcal{X} \to \mathcal{Y}$ *so* $f(A_1) = B_2$ *and* $f(A_2) = B_1$

ii. $\mu_{\mathcal{A}}(A_1) = \mu_{\mathcal{B}}(B_2)$ *and* $v_{\mathcal{A}}(A_1) = v_{\mathcal{B}}(B_2)$
$\mu_{\mathcal{A}}(A_2) = \mu_{\mathcal{B}}(B_1)$ *and* $v_{\mathcal{A}}(A_2) = v_{\mathcal{B}}(B_1)$

Thus, we can conclude that $\mathcal{A} \widetilde{\cong} \mathcal{B}$

3.2. Generalization of Distance and Similarity Measure Between Collections of Intuitionistic Fuzzy Sets

In this section, we describe generalization of distance and similarity measure between collections of intuitionistic fuzzy sets that have been constructed. First, we declare the formula of distance measure. Because similairt measure is dual of distance, so we get similarity measure based on the distance's formula .

Definition 9. *A function* $D : \mathcal{E}' \times \mathcal{E}' \to [0, 1]$ *is said distance measure between collections of intuitionistic fuzzy sets, if it satisfies the following :*

1. $0 \leq D(\mathcal{A}, \mathcal{B}) \leq 1$
2. $D(\mathcal{A}, \mathcal{B}) = D(\mathcal{B}, \mathcal{A})$
3. $D(\mathcal{A}, \mathcal{B}) = 0$ *if only if* $\mathcal{A} \widetilde{\cong} \mathcal{B}$
4. *If* $\mathcal{A} \widetilde{\subseteq} \mathcal{B} \widetilde{\subseteq} C$*, then* $D(\mathcal{A}, \mathcal{B}) \leq D(\mathcal{A}, C)$*, and* $D(\mathcal{B}, C) \leq D(\mathcal{A}, C)$*.*

The following theorems explain about the distance measure that we proposed.

Theorem 2. *Let two intuitionitic fuzzy sets as* $A_j = \{(x_i, \mu_{A_j}(x_i), v_{A_j}(x_i)) : x_i \in X\}$ *and* $B_k = \{(x_i, \mu_{B_k}(x_i), v_{B_k}(x_i)) : x_i \in X\}$ *on* $X = \{x_1, x_2, \ldots, x_n\}$ *where* $j, k = 1, 2, .., m$*.* $\mathcal{A} = \{(A_j, \mu_{\mathcal{A}}(A_j), v_{\mathcal{A}}(A_j)) : A_j \in \mathcal{X}\}$ *is a collection of intuitionistic fuzzy sets on* $\mathcal{X} = \{A_j : j = 1, 2, \ldots, m\}$ *and* $\mathcal{B} = \{(B_k, \mu_{\mathcal{B}}(B_k), v_{\mathcal{B}}(B_k)) : B_k \in \mathcal{Y}\}$ *is a collection of intuitionistic fuzzy sets on* $\mathcal{Y} = \{B_k : k = 1, 2, \ldots, m\}$*.*

$$D_I(\mathcal{A}, \mathcal{B}) = \frac{1}{2}(\mathcal{R}_{\mathcal{AB}} + R_{AB})$$

is a distance measure between $\mathcal{A}$ and $\mathcal{B}$, where

$$\mathcal{R}_{\mathcal{AB}} = \frac{1}{4m}\left(\sum_{j=1}^{m} \min_{k=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_k)|\}\right.$$
$$\left. + \sum_{k=1}^{m} \min_{j=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_k)|\}\right)$$

and

$$R_{AB} = \frac{1}{4m}\left(\sum_{j=1}^{m} \min_{k=1,\ldots,m} \left\{\frac{1}{n}\sum_{i=1}^{n} \left|\mu_{A_j}(x_i) - \mu_{B_k}(x_i)\right| + \left|v_{A_j}(x_i) - v_{B_k}(x_i)\right|\right\}\right.$$
$$\left. + \sum_{k=1}^{m} \min_{j=1,\ldots,m} \left\{\frac{1}{n}\sum_{i=1}^{n} \left|\mu_{A_j}(x_i) - \mu_{B_k}(x_i)\right| + \left|v_{A_j}(x_i) - v_{B_k}(x_i)\right|\right\}\right)$$

Proof.

1. First, we want to prove that $0 \le D_I(\mathcal{A}, \mathcal{B}) \le 1$. Thus, we must prove that $0 \le \mathcal{R}_{\mathcal{AB}} \le 1$ and $0 \le R_{AB} \le 1$.
   Because of $0 \le \mu_{\mathcal{A}}(A_j), v_{\mathcal{A}}(A_j) \le 1$ and $0 \le \mu_{\mathcal{B}}(B_k), v_{\mathcal{B}}(B_k) \le 1$, so we have $0 \le |\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_k)| \le 1$ and $0 \le |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_k)| \le 1$.
   Therefore, we have $0 \le |\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_k)| \le 2$.
   For some $p = 1, 2, \ldots, m$, we have

$$\min_{k=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_1) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_1) - v_{\mathcal{B}}(B_k)|\}$$
$$\le |\mu_{\mathcal{A}}(A_1) - \mu_{\mathcal{B}}(B_p)| + |v_{\mathcal{A}}(A_1) - v_{\mathcal{B}}(B_p)| \le 2$$

   It implies that

$$\min_{k=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_1) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_1) - v_{\mathcal{B}}(B_k)|\}$$
$$+ \min_{k=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_2) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_1) - v_{\mathcal{B}}(B_k)|\}$$
$$+ \ldots + \min_{k=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_m) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_m) - v_{\mathcal{B}}(B_k)|\} \le 2m$$

   Or we can say that

$$\sum_{j=1}^{m} \min_{k=1,..,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_k)|\} \le 2m$$

Next, we have for some $q = 1, 2, \ldots, m$

$$\min_{j=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_1)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_1)|\}$$
$$\leq |\mu_{\mathcal{A}}(A_q) - \mu_{\mathcal{B}}(B_1)| + |v_{\mathcal{A}}(A_q) - v_{\mathcal{B}}(B_1)| \leq 2$$

It implies that

$$\min_{j=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_1)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_1)|\}$$
$$+ \min_{j=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_2)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_2)|\}$$
$$+ \ldots + \min_{j=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_m)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_m)|\} \leq 2m$$

Or we can say that

$$\sum_{k=1}^{m} \min_{j=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_k)|\} \leq 2m$$

Analogously, we get

$$\sum_{k=1}^{m} \min_{j=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_k)|\} \leq 2m$$

Based on these results, we have

$$\sum_{j=1}^{m} \min_{k=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_k)|\}$$
$$+ \sum_{k=1}^{m} \min_{j=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_k)|\} \leq 4m$$

So

$$\mathcal{R}_{\mathcal{AB}} = \frac{1}{4m}\left(\sum_{j=1}^{m} \min_{k=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_k)|\}\right.$$
$$\left. + \sum_{k=1}^{m} \min_{j=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_k)|\}\right) \leq 1$$

Based on the fact that $|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_k)| \geq 0$ and in a similar manner, we can conclude that

$$\mathcal{R}_{\mathcal{AB}} = \frac{1}{4m}\left(\sum_{j=1}^{m} \min_{k=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_k)|\}\right.$$
$$\left. + \sum_{k=1}^{m} \min_{j=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_k)|\}\right) \geq 0$$

So, we have $0 \leq \mathcal{R}_{\mathcal{AB}} \leq 1$.
Now, we want to prove that $0 \leq R_{AB} \leq 1$.
By taking
$0 \leq \mu_{A_j}(x_i), v_{A_j}(x_i) \leq 1$ and $0 \leq \mu_{B_k}(x_i), v_{B_k}(x_i) \leq 1$, so we can get

$$0 \leq \left|\mu_{A_j}(x_i) - \mu_{B_k}(x_i)\right| \leq 1 \text{ and } 0 \leq \left|v_{A_j}(x_i) - v_{B_k}(x_i)\right| \leq 1$$

Therefore, $0 \leq \left|\mu_{A_j}(x_i) - \mu_{B_k}(x_i)\right| + \left|v_{A_j}(x_i) - v_{B_k}(x_i)\right| \leq 2$
Hence,

$$\min_{k=1,\ldots,m}\left\{\frac{1}{n}\sum_{i=1}^{n}\left|\mu_{A_j}(x_i) - \mu_{B_k}(x_i)\right| + \left|v_{A_j}(x_i) - v_{B_k}(x_i)\right|\right\}$$
$$\leq \frac{1}{n}\sum_{i=1}^{n}\left|\mu_{A_j}(x_i) - \mu_{B_p}(x_i)\right| + \left|v_{A_j}(x_i) - v_{B_p}(x_i)\right| \leq \frac{1}{n}.2n = 2$$

It implies that

$$\sum_{j=1}^{m}\min_{k=1,\ldots,m}\left\{\frac{1}{n}\sum_{i=1}^{n}\left|\mu_{A_j}(x_i) - \mu_{B_k}(x_i)\right| + \left|v_{A_j}(x_i) - v_{B_k}(x_i)\right|\right\} \leq 2m$$

Analogously, we can obtain that

$$\sum_{k=1}^{m}\min_{j=1,\ldots,m}\left\{\frac{1}{n}\sum_{i=1}^{n}\left|\mu_{A_j}(x_i) - \mu_{B_k}(x_i)\right| + \left|v_{A_j}(x_i) - v_{B_k}(x_i)\right|\right\} \leq 2m$$

Thus,

$$R_{AB} = \frac{1}{4m}\left(\sum_{j=1}^{m}\min_{k=1,\ldots,m}\left\{\frac{1}{n}\sum_{i=1}^{n}\left|\mu_{A_j}(x_i) - \mu_{B_k}(x_i)\right| + \left|v_{A_j}(x_i) - v_{B_k}(x_i)\right|\right\}\right.$$
$$\left. + \sum_{k=1}^{m}\min_{j=1,\ldots,m}\left\{\frac{1}{n}\sum_{i=1}^{n}\left|\mu_{A_j}(x_i) - \mu_{B_k}(x_i)\right| + \left|v_{A_j}(x_i) - v_{B_k}(x_i)\right|\right\}\right) \leq 1$$

By using the same approach, we also obtain that $0 \leq R_{AB} \leq 1$.
Moreover,$0 \leq \frac{1}{2}(\mathcal{R} + R) \leq 1$ or $0 \leq D_I(\mathcal{A}, \mathcal{B}) \leq 1$.

2. Because of

$$\begin{aligned}\mathcal{R}_{\mathcal{AB}} = \frac{1}{4m}\Bigg(&\sum_{j=1}^{m} \min_{k=1,\dots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_k)|\} \\ &+ \sum_{k=1}^{m} \min_{j=1,\dots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_k)|\}\Bigg) \\ = \frac{1}{4m}\Bigg(&\sum_{k=1}^{m} \min_{j=1,\dots,m} \{|\mu_{\mathcal{B}}(B_k) - \mu_{\mathcal{A}}(A_j)| + |v_{\mathcal{B}}(B_k) - v_{\mathcal{A}}(A_j)|\} \\ &+ \sum_{j=1}^{m} \min_{k=1,\dots,m} \{|\mu_{\mathcal{B}}(B_k) - \mu_{\mathcal{A}}(A_j)| + |v_{\mathcal{B}}(B_k) - v_{\mathcal{A}}(A_j)|\}\Bigg)\end{aligned}$$

and

$$\begin{aligned}R_{AB} = \frac{1}{4m}\Bigg(&\sum_{j=1}^{m} \min_{k=1,\dots,m} \left\{\frac{1}{n}\sum_{i=1}^{n} \left|\mu_{A_j}(x_i) - \mu_{B_k}(x_i)\right| + \left|v_{A_j}(x_i) - v_{B_k}(x_i)\right|\right\} \\ &+ \sum_{k=1}^{m} \min_{j=1,\dots,m} \left\{\frac{1}{n}\sum_{i=1}^{n} \left|\mu_{A_j}(x_i) - \mu_{B_k}(x_i)\right| + \left|v_{A_j}(x_i) - v_{B_k}(x_i)\right|\right\}\Bigg) \\ = \frac{1}{4m}\Bigg(&\sum_{k=1}^{m} \min_{j=1,\dots,m} \left\{\frac{1}{n}\sum_{i=1}^{n} \left|\mu_{A_j}(x_i) - \mu_{B_k}(x_i)\right| + \left|v_{A_j}(x_i) - v_{B_k}(x_i)\right|\right\} \\ &+ \sum_{j=1}^{m} \min_{k=1,\dots,m} \left\{\frac{1}{n}\sum_{i=1}^{n} \left|\mu_{A_j}(x_i) - \mu_{B_k}(x_i)\right| + \left|v_{A_j}(x_i) - v_{B_k}(x_i)\right|\right\}\Bigg)\end{aligned}$$

So, we get $D_I(\mathcal{A}, \mathcal{B}) = D_I(\mathcal{B}, \mathcal{A})$.

3. Let $\mathcal{A} \cong \mathcal{B}$, so

i) There exists an injective function f from $\mathcal{X}$ to $\mathcal{Y}$ such that $f(A_j) = B_k$.

ii) For all j, there exits k such that $\mu_{\mathcal{A}}(A_j) = \mu_{\mathcal{B}}(B_k)$ and $v_{\mathcal{A}}(A_j) = v_{\mathcal{B}}(B_k)$.

From i), we know that for all j, there exists a unique k such that $A_j = B_k$.
It implies that $\mu_{A_j}(x_i) = \mu_{B_k}(x_i)$ and $v_{A_j}(x_i) = v_{B_k}(x_i)$, thus

$$R_{AB} = \frac{1}{4m}\Bigg(\sum_{j=1}^{m} \min_{k=1,\dots,m} \left\{\frac{1}{n}\sum_{i=1}^{n} \left|\mu_{A_j}(x_i) - \mu_{B_k}(x_i)\right| + \left|v_{A_j}(x_i) - v_{B_k}(x_i)\right|\right\}$$

$$+\sum_{k=1}^{m}\min_{j=1,\dots,m}\left\{\frac{1}{n}\sum_{i=1}^{n}\left|\mu_{A_j}(x_i)-\mu_{B_k}(x_i)\right|+\left|v_{A_j}(x_i)-v_{B_k}(x_i)\right|\right\}\Bigg)=0$$

From ii), we know that

$$\mathcal{R}_{\mathcal{AB}}=\frac{1}{4m}\Bigg(\sum_{j=1}^{m}\min_{k=1,\dots,m}\{|\mu_{\mathcal{A}}(A_j)-\mu_{\mathcal{B}}(B_k)|+|v_{\mathcal{A}}(A_j)-v_{\mathcal{B}}(B_k)|\}$$

$$+\sum_{k=1}^{m}\min_{j=1,\dots,m}\{|\mu_{\mathcal{A}}(A_j)-\mu_{\mathcal{B}}(B_k)|+|v_{\mathcal{A}}(A_j)-v_{\mathcal{B}}(B_k)|\}\Bigg)=0$$

Thus, $D_I(\mathcal{A},\mathcal{B})=0$.
The other side, if we have $D_I(\mathcal{A},\mathcal{B})=\frac{1}{2}(\mathcal{R}_{\mathcal{AB}}+R_{AB})=0$, then $\mathcal{R}_{\mathcal{AB}}=0$ and $R_{AB}=0$.
For $\mathcal{R}_{\mathcal{AB}}=0$, we get for every j there exists k such that $\mu_{\mathcal{A}}(A_j)=\mu_{\mathcal{B}}(B_k)$ and $v_{\mathcal{A}}(A_j)=v_{\mathcal{B}}(B_k)$.
It satisfies the statement of ii).
For $R_{AB}=0$, we get for every j there exists k such that $\mu_{A_j}(x_i)=\mu_{B_k}(x_i)$ and $v_{B_k}(x_i)=\ v_{A_j}(x_i)$.
And for every k there exists j such that $\mu_{A_j}(x_i)=\mu_{B_k}(x_i)$ and $v_{B_k}(x_i)=v_{A_j}(x_i)$.
Hence, we can conclude that that there exists a one to one correspondence between $\mathcal{X}$ and $\mathcal{Y}$ such that $\mu_{A_j}(x_i)=\mu_{B_k}(x_i)$ and $v_{B_k}(x_i)=v_{A_j}(x_i)$.
Therefore, we have $A_j=B_k$ and we can say that there exists an injective function f from $\mathcal{X}$ to $\mathcal{Y}$ such that $f(A_j)=B_k$.
Based on these results, we can conclude that $\mathcal{A}\widetilde{\cong}\mathcal{B}$

4. For proving that $D_I(\mathcal{A},\mathcal{B})\leq D_I(\mathcal{A},\mathcal{C})$ so we must prove that $\mathcal{R}_{\mathcal{AB}}\leq\mathcal{R}_{\mathcal{AC}}$ and $R_{AB}\leq R_{AC}$.
Let $\mathcal{A}\widetilde{\subseteq}\mathcal{B}\widetilde{\subseteq}C$, by definition we have

$$\mu_{\mathcal{A}}(A_j)\leq\mu_{\mathcal{B}}(B_k)\leq\mu_{\mathcal{C}}(C_r)\ \text{ and } v_{A_j}(x_i)\geq v_{B_k}(x_i)\geq v_{C_r}(x_i),\forall x_i\in X$$

Since
$\mu_{\mathcal{B}}(B_k)\leq\mu_{\mathcal{C}}(C_r)$, then

$$-\mu_{\mathcal{B}}(B_k)\geq-\mu_{\mathcal{C}}(C_r)$$
$$\mu_{\mathcal{A}}(A_j)-\mu_{\mathcal{B}}(B_k)\geq\mu_{\mathcal{A}}(A_j)-\mu_{\mathcal{C}}(C_r)$$
$$-(\mu_{\mathcal{A}}(A_j)-\mu_{\mathcal{B}}(B_k))\leq-(\mu_{\mathcal{A}}(A_j)-\mu_{\mathcal{C}}(C_r))$$
$$|\mu_{\mathcal{A}}(A_j)-\mu_{\mathcal{B}}(B_k)|\leq|\mu_{\mathcal{A}}(A_j)-\mu_{\mathcal{C}}(C_r)|$$

Since $v_{\mathcal{B}}(B_k)\geq v_{\mathcal{C}}(C_r)$, then

$$-v_{\mathcal{B}}(B_k)\leq-v_{\mathcal{C}}(C_r)$$

$$v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_k) \leq v_{\mathcal{A}}(A_j) - v_{\mathcal{C}}(C_r)$$
$$|v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_k)| \leq |v_{\mathcal{A}}(A_j) - v_{\mathcal{C}}(C_r)|$$

Hence, for all j, k, r we can get

$$|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_k)| \leq |\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{C}}(C_r)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{C}}(C_r)|$$

It follows that, for all j, we have

$$\min_{k=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_k)|\}$$
$$\leq \min_{r=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{C}}(C_r)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{C}}(C_r)|\}$$

and for all r, we have

$$\min_{j=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_k)|\}$$
$$\leq \min_{j=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{C}}(C_r)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{C}}(C_r)|\}$$

These results imply that

$$\frac{1}{4m}\left(\sum_{j=1}^{m} \min_{k=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_k)|\}\right.$$
$$\left. + \sum_{k=1}^{m} \min_{j=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{B}}(B_k)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{B}}(B_k)|\}\right)$$
$$\leq \frac{1}{4m}\left(\sum_{j=1}^{m} \min_{r=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{C}}(C_r)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{C}}(C_r)|\}\right.$$
$$\left. + \sum_{r=1}^{m} \min_{j=1,\ldots,m} \{|\mu_{\mathcal{A}}(A_j) - \mu_{\mathcal{C}}(C_r)| + |v_{\mathcal{A}}(A_j) - v_{\mathcal{C}}(C_r)|\}\right)$$

Thus $\mathcal{R}_{\mathcal{AB}} \leq \mathcal{R}_{\mathcal{AC}}$.
By the definition of $\mathcal{A} \widetilde{\subseteq} \mathcal{B} \widetilde{\subseteq} C$, and in a similar manner, we can prove that

$$R_{AB} \leq R_{AC}$$

It is obvious that

$$D_I(\mathcal{A}, \mathcal{B}) = \frac{1}{2}(\mathcal{R}_{\mathcal{AB}} + R_{AB}) \leq \frac{1}{2}(\mathcal{R}_{\mathcal{AC}} + R_{AC}) = D_I(\mathcal{A}, \mathcal{C})$$

.
Analogue for proving that $D_1(\mathcal{B},\mathcal{C}) \le D_1(\mathcal{A},\mathcal{C})$.
So, the distance measure $D_I(\mathcal{A},\mathcal{B})$ satisfies all properties of distance measure.

Next, based on the proposed distance measure, we derive a generalized similarity measure between two collections of intuitionistic fuzzy sets.

Definition 10. *A function $S : \mathcal{E}' \times \mathcal{E}' \to [0,1]$ is said similarity measure between two collections of intuitionistic fuzzy sets, if it satisfies the following :*

1. $0 \le S(\mathcal{A},\mathcal{B}) \le 1$
2. $S(\mathcal{A},\mathcal{B}) = 1$ *if only if* $\mathcal{A}\widetilde{=}\mathcal{B}$.
3. $S(\mathcal{A},\mathcal{B}) = S(\mathcal{B},\mathcal{A})$
4. *If* $\mathcal{A}\widetilde{\subseteq}\mathcal{B}\widetilde{\subseteq}C$*, then* $S(\mathcal{A},\mathcal{B}) \ge S(\mathcal{A},C)$*, and* $S(\mathcal{B},C) \ge S(\mathcal{A},C)$.

Theorem 3.

$$S_I(\mathcal{A},\mathcal{B}) = 1 - D_I(\mathcal{A},\mathcal{B})$$

is a similarity measure between $\mathcal{A}$ and $\mathcal{B}$ where $D_I(\mathcal{A},\mathcal{B})$ is the distance measure defined in Theorem 2.

Proof. $D_I(\mathcal{A},\mathcal{B})$ is the distance measure defined in Theorem 2. Therefore, we have

1. $0 \le D_I(\mathcal{A},\mathcal{B}) \le 1$
2. $D_I(\mathcal{A},\mathcal{B}) = D_I(\mathcal{B},\mathcal{A})$
3. $D_I(\mathcal{A},\mathcal{B}) = 0$ if only if $\mathcal{A}\widetilde{=}\mathcal{B}$
4. If $\mathcal{A}\widetilde{\subseteq}\mathcal{B}\widetilde{\subseteq}C$, then $D_I(\mathcal{A},\mathcal{B}) \le D_I(\mathcal{A},C)$, and $D_I(\mathcal{B},C) \le D_I(\mathcal{A},C)$.

Therefore, based on 1 until 4 we have

a.

$$-1 \le -D_I(\mathcal{A},\mathcal{B}) \le 0$$

so

$$0 \le S_I(\mathcal{A},\mathcal{B}) \le 1$$

b. $\mathcal{A}\widetilde{=}\mathcal{B}$ gives that

$$S_I(\mathcal{A},\mathcal{B}) = 1 - D_I(\mathcal{A},\mathcal{B}) = 1 - 0 = 1$$

and

$$S_I(\mathcal{A},\mathcal{B}) = 1 - D_I(\mathcal{A},\mathcal{B}) = 1$$

gives that $D_I(\mathcal{A},\mathcal{B}) = 0$, so $\mathcal{A}\widetilde{=}\mathcal{B}$.

c. $S_I(\mathcal{A},\mathcal{B}) = 1 - D_I(\mathcal{A},\mathcal{B}) = 1 - D_I(\mathcal{B},\mathcal{A}) = S_I(\mathcal{B},\mathcal{A})$

d. If $\mathcal{A}\widetilde{\subseteq}\mathcal{B}\widetilde{\subseteq}C$ then

$$S_I(\mathcal{A},\mathcal{B}) = 1 - D_I(\mathcal{A},\mathcal{B}) \geq 1 - D_1(\mathcal{A},C) = S_I(\mathcal{A},C)$$

$$S_I(\mathcal{B},C) = 1 - D_1(\mathcal{B},C) \geq 1 - D_1(\mathcal{A},C) = S_I(\mathcal{A},C)$$

Therefore

$$S_I(\mathcal{A},\mathcal{B}) \geq S_I(\mathcal{A},C) \text{ and } S_I(\mathcal{B},C) \geq S_I(\mathcal{A},C)$$

In the following section, we present an illustrative example for solving a pattern recognition problem using the proposed similarity measure.

Example 6. *An expert in the health sector wants to determine which region, between region $\mathcal{B}$ and region $\mathcal{C}$ exhibits malnutrition characteristics similar to those of region $\mathcal{A}$. The region $\mathcal{A}$ is made up of three subregions, indicated by A_1, A_2, A_3. Each subregion represents a specific part of the region $\mathcal{A}$ and is characterized by its own level of nutritional indicators based on fuzzy intuitionistic values. To make a fair comparison, the regions $\mathcal{B}$ and $\mathcal{C}$ are also divided into three subregions, denoted B_1, B_2, B_3 and C_1, C_2, C_3, respectively.*
The goal is to measure the similarity between these regions and region $\mathcal{A}$ using a proposed similarity measure between collections of intuitionistic fuzzy sets.

Let the universal set be $X = \{x_1, x_2, x_3\}$ where x_1 is poverty rate, x_2 is low education rate, and x_3 is the ease of access to health.
Given the following collections of intuitionistic fuzzy sets. These are
$\mathcal{A} = \{(A_1, 0.6, 0.3), (A_2, 0.5, 0.5), (A_3, 0.6, 0.4)\}$ *where*

$$A_1 = \{(x_1, 0.5, 0.2), (x_2, 0.6, 0.3), (x_3, 0.1, 0.6)\}$$
$$A_2 = \{(x_1, 0.55, 0.1), (x_2, 0.6, 0.4), (x_3, 0.4, 0.5)\}$$
$$A_3 = \{(x_1, 0.7, 0.1), (x_2, 0.5, 0.2), (x_3, 0.2, 0.6)\}$$

$\mathcal{B} = \{(B_1, 0.6, 0.3), (B_2, 0.6, 0.2), (B_3, 0.7, 0.2)\}$ *where*

$$B_1 = \{(x_1, 0.55, 0.1), (x_2, 0.7, 0.3), (x_3, 0.4, 0.6)\}$$
$$B_2 = \{(x_1, 0.65, 0.2), (x_2, 0.5, 0.3), (x_3, 0.4, 0.5)\}$$
$$B_3 = \{(x_1, 0.7, 0.3), (x_2, 0.6, 0.4), (x_3, 0.3, 0.5)\}$$

$\mathcal{C} = \{C_1, 0.5, 0.5), (C_2, 0.7, 0.3), (C_3, 0.4, 0.4)$ *where*

$$C_1 = \{(x_1, 0.6, 0.2), (x_2, 0.8, 0.1), (x_3, 0.1, 0.8)\}$$
$$C_2 = \{(x_1, 0.6, 0.2), (x_2, 0.8, 0.2), (x_3, 0.3, 0.6)\}$$
$$C_3 = \{(x_1, 0.5, 0.3), (x_2, 0.6, 0.2), (x_3, 0.3, 0.7)\}$$

Using the proposed distance measure and similarity measure between collections of intuitionistic fuzzy sets, we get

$$D_I(\mathcal{A},\mathcal{B}) = 0.11681$$

$$D_I(\mathcal{A}, \mathcal{C}) = 0.10883$$

Therefore

$$S_I(\mathcal{A}, \mathcal{B}) = 1 - D_I(\mathcal{A}, \mathcal{B}) = 1 - 0.11681 = 0.88319$$

$$S_I(\mathcal{A}, \mathcal{C}) = 1 - D_I(\mathcal{A}, \mathcal{C}) = 1 - 0.10883 = 0.89117$$

Based on the calculation, region $\mathcal{C}$ has a higher degree of similarity to region $\mathcal{A}$ than $\mathcal{B}$. Thus, we conclude that region $\mathcal{C}$ has more similar characteristics of malnutrition with region $\mathcal{A}$.

4. Conclusions

In this paper, we propose a generalized similarity measure for collections of intuitionistic fuzzy sets considering the differences in their universal sets. Unlike an existing similarity measure, which typically assumes that the collections of intuitionistic fuzzy sets share an identical universal set, the proposed similarity measure overcomes this limitation by considering the different of the universal set of collections of intuitionistic fuzzy sets. The proposed similarity measure, developed using a distance-based approach by assuming complete knowledge of the membership degree, and nonmembership degree. It offers a more flexible and comprehensive method for comparing such collections. Furthermore, we introduce formal definitions of inferiority and equivalence relations to enhance the structural understanding of such collections. These relations also support the theoretical foundation of the proposed measure, ensuring that it satisfies the standard axioms of similarity measures. Through illustrative example as pattern recognition, we show that the proposed measure can effectively support the decision-making process. This approach opens further opportunities for application in real-world problems such as medical diagnosis, pattern recognition, and multi-criteria decision analysis.

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