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Boundedness of the Sublinear Operators on Anisotropic Herz-Slice Spaces

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Abstract. We will define the idea of anisotropic Herz-slice spaces and prove some properties of these spaces. As an application we obtain the bounds for sublinear operator on anisotropic Herz-slice spaces.

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Key Words and Phrases: Herz spaces, slice spaces, Herz-slice spaces, integral operators, atomic decomposition, boundedness

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1. Introduction

Let $p \in (0, \infty)$, the Lebesgue space is defined as

$$L^p(E) := \left\{ f \text{ is measurable: } I_p\left(\frac{f}{\gamma}\right) < \infty \text{ for some constant } \gamma > 0 \right\}$$

where

$$I_p(f) := \int_E |g(x)|^p \mathrm{d}x$$

and

$$\|f\|_{L^p(E)} := \inf\left\{ \gamma > 0 : I_p\left(\frac{f}{\gamma}\right) \leqslant 1 \right\}.$$

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Afterward, $L^p(E)$ is a Banach space, and $\|\cdot\|_{L^p(E)}$ is its norm.

Slice spaces were initially presented in [1] to study weak solutions of boundary value problems for time-independent elliptic systems in the upper half-plane. This concept was later expanded upon in [2], where generalized slice spaces were used to study the mapping properties of sublinear operators on tent spaces, the Hardy-Littlewood maximal operator M, and the Calderón-Zygmund operators. Let $u \in (0,\infty)$, $p \in [1,\infty)$ and $r \in (1,\infty)$, the slice space $(E_r^p)_u$ is defined as the set of all measurable functions g such that

$$\|f\|_{(E_r^p)_u} := \left\| \left(\frac{1}{|B(\cdot,u)|} \int\limits_{B(\cdot,u)} |f(y)|^r \, dy \right) \right\|_{L^p} < \infty.$$

More recently, great attention was paid to the study on the Herz spaces since there are several remarkable works to push forward the study on the Herz spaces. Conversely, Herz spaces, a class of function spaces, have been pivotal in real analysis due to their norms, which explicitly incorporate both local and global information of functions. For more results generalized versions of Herz space like boundedness of sublinear operators, fractional integrals, the commutator of singular integrals with BMO functions, and the commutator of fractional integrals with BMO functions see [3–25].

Let $\alpha \in \mathbb{R}$, $q \in (0,\infty]$, and $0 < p \leq \infty$, then the homogeneous version of Herz spaces $\dot{K}_p^{\alpha,q}$ are defined by

$$\dot{K}_p^{\alpha,q} = \left\{ g \in L_{\mathrm{loc}}^p(\mathbb{R}^n \setminus \{0\}) : \|g\|_{\dot{K}_p^{\alpha,q}} < \infty \right\},$$

where

$$\|g\|_{\dot{K}_p^{\alpha,q}} = \left(\sum_{\ell=-\infty}^{\infty} 2^{\ell\alpha q} \|g\chi_\ell\|_{L^p}^q \right)^{\frac{1}{q}}.$$

Firstly, we define the idea of anisotropic Herz-slice spaces by using anisotropic Herz spaces and slice spaces. We will establish the atomic decomposition in these spaces. Then, we obtain boundedness for sublinear in these spaces.

2. Preliminaries

Now we give some notations.

Let $l \in \mathbb{Z}$, define $B_l := \left\{ z \in \mathbb{R}^n : |z| \leqslant 2^l \right\}, R_l := B_l \backslash B_{l-1}$, and $\chi_l := \chi_{R_l}$. A $n \times n$ real matrix O is called dilation or expansive matrix if $|\gamma| > 1$, where γ is the eigenvalue. Let $\gamma_1, \cdots, \gamma_n$ are eigenvalues of O such that $1 < |\gamma_1| \leq \cdots \leq |\gamma_n|$ and γ_-, γ_+ are two numbers such that $1 < \gamma_- < |\gamma_1| \leq |\gamma_n| < \gamma_+$. Let $|\cdot|$ is the Euclidean norm, P is nondegenerate matrix of order $n \times n$ and $r > 1$, then the set $\Delta \subset \mathbb{R}^n$ is called ellipsoid if $\Delta = \{z \in \mathbb{R}^n : |Pz| < 1\}$. consider an ellipsoid Δ and $r > 1$, dilation O, $\Delta \subset r\Delta \subset A\Delta$ and $|\Delta| = 1$, $|\Delta|$ is the Lebesgue measure of Δ. If $B_\ell = O^\ell \Delta$ for $\ell \in \mathbb{Z}$, then we get $B_\ell \subset rB_\ell \subset B_{\ell+1}$, and $|B_\ell| = b^\ell$ such that $b = \prod_{i=1}^n |\gamma_i| = |\mathrm{det} O| > 1$. Assume that w is the smallest integer such that $2B_0 \subset O^w B_0 = B_w$. A quaisi-norm with expansive matrix O is a measurable mapping $\tau_O : \mathbb{R}^n \to [0, \infty)$ such that

$$\tau_O(y) > 0 \quad \textit{for } y \neq 0,$$

$$\tau_O(Ay) = |\det A| \tau(y) \quad \textit{for } y \in \mathbb{R}^n,$$

$$\tau_O(x + y) \leq C\left(\tau_O(x) + \tau_O(y)\right) \quad \textit{for } x, y \in \mathbb{R}^n,$$

where $C \geq 1$. The step homogeneous quasi-norm τ on $\mathbb{R}^n$ defined by dilation O is given as

$$\left(\tau(x) = b^j \quad \textit{for } x \in B_{j+1} \setminus B_j\right) \quad \textit{and} \quad \left(0 \quad \textit{for } x = 0\right).$$

If $x, y \in \mathbb{R}^n$, then we have

$$\tau(x + y) \leq b^w\left(\tau(x) + \tau(y)\right). \tag{2.1}$$

Definition 1. If $\alpha \in \mathbb{R}$, $u \in (0, \infty)$, $p, q \in [1, \infty)$ and $r \in (1, \infty)$, then the homogeneous version of anisotropic Herz-slice spaces $\left(\dot{K}E^{\alpha,p}_{q,r}\right)_u (O; \mathbb{R}^n)$ are defined by

$$\left(\dot{K}E^{\alpha,p}_{q,r}\right)_u (O; \mathbb{R}^n) = \left\{ g \in (E^p_r)_u : \|g\|_{\left(\dot{K}E^{\alpha,p}_{q,r}\right)_u (O; \mathbb{R}^n)} < \infty \right\},$$

where

$$\|g\|_{\left(\dot{K}E^{\alpha,p}_{q,r}\right)_u (O; \mathbb{R}^n)} = \left(\sum_{\ell=-\infty}^{\infty} b^{\ell \alpha q} \|g \chi_\ell\|^q_{(E^p_r)_u} \right)^{\frac{1}{q}}.$$

Now we state the Hölder's inequality for slice space.

Lemma 2. *[26] Let $1 \leq p \leq \infty$, $0 < u < \infty$ and $1 < r < \infty$,*

$$\|fg\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{(E^p_r)_u} \|g\|_{\left(E^{p'}_{r'}\right)_u}$$

where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{r} + \frac{1}{r'} = 1$.

Lemma 3. *[27] Let $u \in (0, \infty), p, r \in (1, \infty)$, if $x_0 \in \mathbb{R}^n$, and $1 < r_0 < \infty$, then the characteristic function on $B(x_0, r_0)$ fulfills*

$$\left\|\chi_{B(x_0, r_0)}\right\|_{(E^p_r)_u} \leq C r_0^{n/p}.$$

Remark 4. [27] Let $u \in (0, \infty), p.r \in (1, \infty), \ell \in \mathbb{Z}$, then the characteristic function on R_ℓ fulfills

$$\|\chi_{R_\ell}\|_{(E^p_r)_u} \leq \|\chi_{B_\ell}\|_{\,(E^p_r)_u)} \leq C b^{\ell n/p}.$$

3. Atomic Decomposition of anisotropic Herz-slice spaces

Definition 5. Let $\alpha \in \mathbb{R}$, and $p, q, r, u \in (0, \infty]$. Then

(i) A measurable function $a(y)$ is called central (α, p, r, u)-block if supp $a \subset B_l$ and $\|a\|_{(E_r^p)_u} \leq b^{-l\alpha}$.

(ii) A measurable function $a(y)$ is called central (α, p, r, u)-block if supp $a \subset B_l$.

Theorem 3.1. Let $\alpha \in \mathbb{R}$, and $p, q, r, u \in (0, \infty]$. Let b_ℓ be a central (α, p, r, u)-block with support d in B_ℓ and $\sum_{\ell=-\infty}^{\infty} |\gamma_\ell|^q < \infty$. Then the following two statements are equivalent:
(i) $g \in \left(\dot{K}E_{q,r}^{\alpha,p}\right)_u (O; \mathbb{R}^n)$.
(ii) g are given as

$$g(y) = \sum_{\ell=-\infty}^{\infty} \gamma_\ell b_\ell(y). \tag{2.1}$$

Proof.
We first prove (i) implies (ii). For every $g \in \left(\dot{K}E_{q,r}^{\alpha,p}\right)_u (O; \mathbb{R}^n)$, write

$$\begin{aligned} g(y) &= \sum_{\ell=-\infty}^{\infty} g(y)\chi_\ell(y) \\ &= \sum_{\ell=-\infty}^{\infty} b^{\ell\alpha} \|g\chi_\ell\|_{(E_r^p)_u} \frac{g(y)\chi_\ell(y)}{b^{\ell\alpha} \|\chi_\ell\|_{(E_r^p)_u}} \\ &= \sum_{\ell=-\infty}^{\infty} \gamma_\ell b_\ell(x), \end{aligned}$$

where $\gamma_\ell = b^{\ell\alpha} \|g\chi_\ell\|_{(E_r^p)_u}$ and $b_\ell(x) = \frac{g(y)\chi_\ell(y)}{b^{\ell\alpha}\|\chi_\ell\|_{(E_r^p)_u}}$.

It is easy to note that supp $b_\ell \subset B_\ell$ and $\|b_\ell\|_{(E_r^p)_u} = |B_\ell|^{-\alpha/n}$. So every b_ℓ is a central (α, p, r, u)-block with support contained in B_ℓ

$$\sum_{\ell=-\infty}^{\infty} |\gamma_\ell|^q = \sum_{\ell=-\infty}^{\infty} b^{\ell\alpha} \|g\chi_\ell\|_{(E_r^p)_u}^q = \|g\|^q_{\left(\dot{K}E_{q,r}^{\alpha,p}\right)_u (O;\mathbb{R}^n)} < \infty.$$

Now we prove (ii) implies (i). Let $g(y) = \sum_{\ell=-\infty}^{\infty} \gamma_\ell b_\ell(y)$, we get

$$\|g\chi_j\|_{(E_r^p)_u} \leq \sum_{\ell=j}^{\infty} |\gamma_\ell| \|b_\ell\|_{(E_r^p)_u}. \tag{3.1}$$

If $0 < q \leq 1$. From (3.1) it follows that

$$\|g\|^q_{\left(\dot{K}E_{q,r}^{\alpha,p}\right)_u (O;\mathbb{R}^n)} = \sum_{\ell=-\infty}^{\infty} b^{\ell\alpha} \|g\chi_\ell\|^q_{(E_r^p)_u}$$

$$
\begin{aligned}
&\leq \sum_{\ell=-\infty}^{\infty} b^{\ell\alpha q} \left\| g\chi_\ell \right\|^q_{\left(E_r^p\right)_u} \\
&\leq \sum_{\ell=-\infty}^{\infty} b^{\ell\alpha q} \left(\sum_{j=\ell}^{\infty} |\gamma_j|^q \left\| b_j \right\|^q_{\left(E_r^p\right)_u} \right).
\end{aligned}
$$

Let $I = \sum\limits_{\ell=-\infty}^{\infty} b^{\ell\alpha q} \left(\sum_{j=\ell}^{\infty} |\gamma_j|^q \left\| b_j \right\|^q_{\left(E_r^p\right)_u} \right)$. By using the fact $0 < \alpha < \infty$, we get

$$
\begin{aligned}
I &= \sum_{\ell=-\infty}^{\infty} b^{\ell\alpha q} \left(\sum_{j=\ell}^{\infty} |\gamma_j|^q \left\| b_j \right\|^q_{\left(E_r^p\right)_u} \right) \\
&\lesssim \sum_{\ell=-\infty}^{\infty} b^{\ell\alpha q} \sum_{j=\ell}^{\infty} |\gamma_j|^q \, b^{-j\alpha q} \\
&\lesssim \sum_{\ell=-\infty}^{\infty} \sum_{j=\ell}^{\infty} |\gamma_j|^q \, b^{(\ell-j)\alpha q} \\
&\lesssim \sum_{j=-\infty}^{\infty} \sum_{\ell=-\infty}^{j} |\gamma_j|^q \, b^{(\ell-j)\alpha q} \\
&\lesssim \sum_{j=-\infty}^{\infty} |\gamma_j|^q .
\end{aligned}
$$

If $1 < q < \infty$, we have
For I, if $0 < \alpha < \infty$, then (3.1) and Hölder's inequality yields

$$
\begin{aligned}
I &\lesssim \sum_{\ell=-\infty}^{\infty} b^{\ell\alpha q} \left(\sum_{j=\ell}^{\infty} |\gamma_j| \left\| b_j \right\|_{\left(E_r^p\right)_u} \right) \\
&\lesssim \sum_{\ell=-\infty}^{\infty} \left(\sum_{j=\ell}^{\infty} |\gamma_j| \, b^{(\ell-j)\alpha} \right)^q \\
&\lesssim \sum_{\ell=-\infty}^{\infty} \left(\sum_{j=\ell}^{\infty} |\gamma_j|^q \, b^{(\ell-j)\alpha q/2} \right) \left(\sum_{j=\ell}^{\infty} b^{(\ell-j)\alpha q'/2} \right)^{q/q'} \\
&\lesssim \sum_{\ell=-\infty}^{\infty} \left(\sum_{j=\ell}^{\infty} |\gamma_j|^q \, b^{(\ell-j)\alpha q/2} \right) \\
&\lesssim \sum_{j=-\infty}^{\infty} \sum_{\ell=-\infty}^{j} |\gamma_j|^q \, b^{(\ell-j)\alpha q/2}
\end{aligned}
$$

$$\lesssim \sum_{j=-\infty}^{\infty} |\gamma_j|^q .$$

Thus the proof of the Theorem is completed. Non-homogeneous version of the Theorem is obtained similarly.

Remark 6. By using the Theorem, note that if $g \in \left(\dot{K}E^{\alpha,p}_{q,r}\right)_u(O;\mathbb{R}^n)$ and $g(y) = \sum_{\ell=-\infty}^{\infty} \gamma_\ell b_\ell(x)$ is a central (α,p,r,u)-block decomposition, then

$$\|g\|_{\left(\dot{K}E^{\alpha,p}_{q,r}\right)_u(O;\mathbb{R}^n)} \approx \left(\sum_{\ell=-\infty}^{\infty} |\gamma_\ell|^q\right)^{1/q}.$$

4. Boundedness of sublinear operators on anisotropic Herz-slice spaces

As applications of the decomposition theorem, we will prove the boundedness of sublinear operators on anisotropic Herz-slice spaces.

Theorem 4.1. Let $\alpha \in \mathbb{R}$, with $0 < \alpha < 1/p'$ and $p,q,r,u \in (0,\infty]$. Let the sublinear T fulfills the size condition

$$|Tg(y)| \lesssim \int_{\mathbb{R}^n} \frac{|g(y)|}{\tau(x-y)} dy, \quad x \notin \operatorname{supp} g, \tag{4.1}$$

for any $g \in (E^p_r)_u$ with a compact support and T is bounded on $(E^p_r)_u$. Then T is bounded in $\left(\dot{K}E^{\alpha,p}_{q,r}\right)_u(O;\mathbb{R}^n)$.

Proof. Let $g \in \left(\dot{K}E^{\alpha,p}_{q,r}\right)_u(O;\mathbb{R}^n)/$ such that $g(y) = \sum_{a_0=-\infty}^{\infty} \gamma_{a_0} b_{a_0}(x)$ where b_{a_0} is a central $(\alpha, p.r, u)$-block with support contained in B_{a_0}, By applying the decomposition theorem, we get

$$\|g\|_{\left(\dot{K}E^{\alpha,p}_{q,r}\right)_u(O;\mathbb{R}^n)} \approx \left(\sum_{a_0=-\infty}^{\infty} |\gamma_{a_0}|^q\right)^{1/q}.$$

Therefore, we get

$$\begin{aligned}
\|Tg\|^q_{\left(\dot{K}E^{\alpha,p}_{q,r}\right)_u(O;\mathbb{R}^n)} &= \sum_{\ell=-\infty}^{\infty} b^{\ell\alpha q} \|(Tg)\chi_\ell\|^q_{(E^p_r)_u} \\
&\lesssim \sum_{\ell=-\infty}^{\infty} b^{\ell\alpha q} \|(Tg)\chi_\ell\|^q_{(E^p_r)_u} \\
&\lesssim \sum_{\ell=-\infty}^{\infty} b^{\ell\alpha q} \left(\sum_{a_0=-\infty}^{\ell-w-1} |\gamma_{a_0}| \left\|(Tb_{a_0})\chi_\ell\right\|_{(E^p_r)_u}\right)^q
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\ell=-\infty}^{\infty} b^{\ell\alpha q}\left(\sum_{a_0=\ell-w}^{\infty}|\gamma_{a_0}|\left\|(Tb_{a_0})\,\chi_\ell\right\|_{(E_r^p)_u}\right)^q\\
&=I_1+I_2.
\end{aligned}
$$

To find the estimate of I_1. Let $x\in C_\ell$, $y\in B_{a_0}$ such that $a_0\le \ell-w-1$,, then (2.1) yields

$$
b^{-w}(1-1/b)\tau(x)=b^{-w}\tau(x)-b^{-w-1}\tau(x)\le b^{-w}\tau(x)-\tau(y)\le\tau(x-y).
$$

Hence (4.1) and Hölder's inequality yields

$$
\begin{aligned}
|Tb_{a_0}(x)| &\le C\tau(x)^{-1}\int_{B_{a_0}}|b_{a_0}(y)|\,dy\\
&\le Cb^{-\ell}\left\|b_{a_0}\right\|_{(E_r^p)_u}\left\|\chi_{B_{a_0}}\right\|_{\left(E_{r'}^{p'}\right)_u}.
\end{aligned}
$$

Applying Lemmas 2 and 4, we get

$$
\begin{aligned}
\left\|(Tb_{a_0})\,\chi_\ell\right\|_{(E_r^p)_u} &\lesssim b^{-\ell}\left\|b_{a_0}\right\|_{(E_r^p)_u}\left\|\chi_{B_{a_0}}\right\|_{\left(E_{r'}^{p'}\right)_u}\left\|\chi_{B_\ell}\right\|_{(E_r^p)_u}\\
&\lesssim b^{-\ell}\left\|b_{a_0}\right\|_{(E_r^p)_u}\left(|B_\ell|\left\|\chi_{B_\ell}\right\|^{-1}_{\left(E_{r'}^{p'}\right)_u}\right)\left\|\chi_{B_{a_0}}\right\|_{\left(E_{r'}^{p'}\right)_u}\\
&\lesssim \left\|b_{a_0}\right\|_{(E_r^p)_u}\frac{\left\|\chi_{B_{a_0}}\right\|_{\left(E_{r'}^{p'}\right)_u}}{\left\|\chi_{B_\ell}\right\|_{\left(E_{r'}^{p'}\right)_u}}\\
&\lesssim b^{(a_0-\ell)/p'}\left\|b_{a_0}\right\|_{(E_r^p)_u}.
\end{aligned}
$$

Therefore, for $0<q\le 1$, and $0<\alpha<1/p'$, we have

$$
\begin{aligned}
I_1 &= \sum_{\ell=-\infty}^{\infty} b^{\ell\alpha q}\left(\sum_{a_0=-\infty}^{\ell-w-1}|\gamma_{a_0}|\left\|(Tb_{a_0})\,\chi_\ell\right\|_{(E_r^p)_u}\right)^q\\
&\lesssim \sum_{\ell=-\infty}^{\infty} b^{\ell\alpha q}\left(\sum_{a_0=-\infty}^{\ell-w-1}|\gamma_{a_0}|^q\, b^{[(a_0-\ell)/p'-a_0\alpha]q}\right)\\
&\lesssim \sum_{a_0=-\infty}^{-w-2}|\gamma_{a_0}|^q\sum_{\ell=a_0+w+1}^{\infty} b^{(a_0-\ell)[1/p'-\alpha]q}\\
&\lesssim \sum_{a_0=-\infty}^{-w-2}|\gamma_{a_0}|^q
\end{aligned}
$$

$$\lesssim \|g\|^q_{\left(\dot{K}E^{\alpha,p}_{q,r}\right)_u(O;\mathbb{R}^n)}.$$

When $1<q<\infty$, $0<\alpha<1/p'$, and the Hölder inequality yields

$$\begin{aligned}
I_1 &\lesssim \sum_{\ell=-\infty}^{\infty} b^{\ell\alpha q}\left(\sum_{a_0=-\infty}^{\ell-w-1}|\gamma_{a_0}|\, b^{(a_0-\ell)/p'-a_0\alpha}\right)^q \\
&\lesssim \sum_{\ell=-\infty}^{\infty}\left(\sum_{a_0=-\infty}^{\ell-w-1}|\gamma_{a_0}|^q\, b^{(a_0-\ell)[1/p'-\alpha]q/2}\right)\left(\sum_{a_0=-\infty}^{\ell-w-1} b^{(a_0-\ell)[1/p'-\alpha]q'/2}\right)^{q/q'} \\
&\lesssim \sum_{\ell=-\infty}^{\infty}\left(\sum_{a_0=-\infty}^{\ell-w-1}|\gamma_{a_0}|^q\, b^{(a_0-\ell)[1/p'-\alpha]q/2}\right) \\
&\lesssim \sum_{a_0=-\infty}^{-w-2}|\gamma_{a_0}|^q \sum_{\ell=a_0+w+1}^{-1} b^{(a_0-\ell)[1/p'-\alpha]q/2} \\
&\lesssim \sum_{a_0=-\infty}^{-w-2}|\gamma_{a_0}|^q \\
&\lesssim \|g\|^q_{\left(\dot{K}E^{\alpha,p}_{q,r}\right)_u(O;\mathbb{R}^n)}.
\end{aligned}$$

Next we find the estimate of I_2. If $0<q\leq 1$, by $(E^p_r)_u$ boundedness of T, we get

$$\begin{aligned}
I_2 &= \sum_{\ell=-\infty}^{\infty} b^{\ell\alpha q}\left(\sum_{a_0=\ell-w}^{\infty}|\gamma_{a_0}|\,\|(Tb_{a_0})\,\chi_\ell\|_{(E^p_r)_u}\right)^q \\
&\lesssim \sum_{\ell=-\infty}^{\infty} b^{\ell\alpha q}\left(\sum_{a_0=\ell-w}^{\infty}|\gamma_{a_0}|^q\,\|\, b_{a_0}|^q_{(E^p_r)_u}\right) \\
&\lesssim \sum_{\ell=-\infty}^{\infty} b^{\ell\alpha q}\left(\sum_{a_0=\ell-w}^{\infty}|\gamma_{a_0}|^q\, b^{-a_0\alpha q}\right) \\
&\lesssim \sum_{\ell=-\infty}^{\infty}\sum_{a_0=\ell-w}^{\infty}|\gamma_{a_0}|^q\, b^{(\ell-a_0)\alpha q} \\
&\lesssim \sum_{a_0=-\infty}^{\infty}|\gamma_{a_0}|^q\sum_{\ell=-\infty}^{a_0+w} b^{(\ell-a_0)\alpha q} \\
&\lesssim \sum_{a_0=-\infty}^{\infty}|\gamma_{a_0}|^q \\
&\lesssim \|g\|^q_{\left(\dot{K}E^{\alpha,p}_{q,r}\right)_u(O;\mathbb{R}^n)}.
\end{aligned}$$

If $1 < q < \infty$, by using $\left(E_r^p\right)_u$ boundedness of T and again Hölder inequality to obtain

$$\begin{aligned}
I_2 &\lesssim \sum_{\ell=-\infty}^{\infty} b^{\ell\alpha q}\left(\sum_{a_0=\ell-w}^{\infty}|\gamma_{a_0}|\,\|b_{a_0}\|_{\left(E_r^p\right)_u}\right)^q \\
&\lesssim \sum_{\ell=-\infty}^{\infty}\left(\sum_{a_0=\ell-w}^{\infty}|\gamma_{a_0}|\,b^{(\ell-a_0)\alpha}\right)^q \\
&\lesssim \sum_{\ell=-\infty}^{\infty}\left(\sum_{a_0=\ell-w}^{\infty}|\gamma_{a_0}|^q\,b^{(\ell-a_0)\alpha q/2}\right)\left(\sum_{a_0=\ell-w}^{\infty}b^{(\ell-a_0)\alpha(q)'/2}\right)^{q/q'} \\
&\lesssim \sum_{a_0=-\infty}^{\infty}|\gamma_{a_0}|^q\sum_{\ell=-\infty}^{a_0+w}b^{(\ell-a_0)\alpha q/2} \\
&\lesssim \sum_{a_0=-\infty}^{\infty}|\gamma_{a_0}|^q \\
&\lesssim \|g\|^q_{\left(\dot{K}E_{q,r}^{\alpha,p}\right)_u(O;\mathbb{R}^n)}.
\end{aligned}$$

Combining these estimates we get

$$\|Tg\|_{\left(\dot{K}E_{q,r}^{\alpha,p}\right)_u(O;\mathbb{R}^n)} \lesssim \|g\|_{\left(\dot{K}E_{q,r}^{\alpha,p}\right)_u(O;\mathbb{R}^n)}.$$

Thus, the proof of the Theorem 4.1 is completed.

5. Ethics declarations

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Ethics approval and consent to participate

This manuscript has not and will not be submitted to more than one journal for simultaneous consideration. The submitted work is original and will not be published elsewhere.

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