EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6522
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Stability Analysis and Numerical Simulation of Fractional-Order Models for Wolbachia Transmission in Aedes aegypti Mosquitoes with Seasonal Effects

Mohammad Yar[1,2,*], Shumaila Javeed[1], Tanveer Abbas Khan[1]

[1] *Department of Mathematics, COMSATS University Islamabad, Park Road, Chak Shahzad Islamabad, 45550, Pakistan*
[2] *Department of Mathematics, Kabul Polytechnic University, Kabul 1001, Afghanistan*

Abstract. Mosquito-borne diseases have historically impacted people and remain a global problem. Wolbachia represents an innovative vector management strategy capable of diminishing mosquito populations and reducing the threat of mosquito-borne diseases. In order to provide optimal and efficient control, Wolbachia should be released at each step of the mosquito life cycle. Employing fractional calculus enables the capture of memory effects and hereditary characteristics of this process. In this study, we develop and analyze a fractional-order mathematical model to investigate Wolbachia transmission dynamics, which accounts for imperfect maternal transmission and infection loss. The model considers Wolbachia-infected and uninfected subpopulations of Aedes aegypti mosquitoes, assuming equal numbers of adult males and females. We establish the positivity and boundedness of solutions for non-negative initial conditions. The invasive reproduction number $R_{0w|\bar{w}}$ has been found to determine whether the Wolbachia infection spreads. We consider two fractional-order models: one that ignores the impacts of seasons on the mosquito populations and another that incorporates these effects. Furthermore, the stability of the model is analyzed using Lyapunov functions and Ulam-Hyers stability theories. The models are numerically solved using the Adams-Bashforth-Moulton technique, demonstrating the effects of model parameters and fractional-order values. Based on the results, we find that the fractional order $\alpha = 0.5$ is optimal, and the corresponding conditions could be applied in real-world experiments to increase the population of Wolbachia-infected mosquitoes. These results highlight the importance of fractional-order modeling.

2020 Mathematics Subject Classifications: 34A08, 92D30, 34D20

Key Words and Phrases: Wolbachia, Aedes aegypti, Fractional calculus, Lyapunov function, Ulam-Hyers stability, Seasonal effects, Adams-Bashforth-Moulton method

---

1. Introduction

Mosquito-borne diseases are predominantly transmitted by female mosquitoes while feeding on the blood of living organisms, including humans, animals, and birds. A female mosquito infected with a parasite, virus, or bacterium can transmit these pathogens to humans [1]. Dengue fever is widespread globally, with approximately 3.9 billion individuals at risk and 390 million new cases annually [2]. Various strategies exist for managing Aedes aegypti

---

[*] Corresponding author.

DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6522

Email addresses: `myar@kpu.edu.af` (M. Yar),
shumaila$_{j}aveed@comsats.edu.pk$ $(S.Javeed)$, `tanveerabbas947@gmail.com` $(T.A.Khan)$

mosquitoes, including insecticide spraying, sterile insect techniques, incompatible insect techniques, combined approaches, and genetic engineering [3, 4].

Integrated vector management is a promising control strategy currently under investigation [5]. Substituting the Aedes aegypti mosquito population with alternative vector agents that cannot transmit viruses, rather than only preventing human-vector interactions, has shown a decrease in dengue distribution, and this method seems effective in managing other mosquito-borne diseases like Zika, malaria, West Nile virus, and Chikungunya [6–8]. Wolbachia is a genus of bacteria that reduces mosquitoes' capacity for transmitting viruses [9]. Wolbachia-infected (WI) mosquitoes have less ability to transmit diseases compared to Wolbachia-uninfected (WU) mosquitoes. Moreover, Wolbachia infection frequently causes cytoplasmic incompatibility, resulting in premature embryonic mortality when infected males mate with uninfected females. When a Wolbachia-infected mosquito bites a virus-infected human, it may become infectious, but it is unable to transmit the virus to an uninfected human. The Wolbachia method would significantly diminish the mosquito population density and facilitate the elimination of mosquito-borne diseases [10, 11].

On the other hand, imperfect maternal transmission can also prevent WI mosquitoes from effectively controlling disease transmission [12–14]. To implement the method extensively, comprehensive details about introducing Wolbachia into wild mosquitoes are required. The application of mathematical modeling in the decision-making process has facilitated the use of numerous widely recognized control policies. It plays an important role in assessing the impacts of various factors on the dynamics of infectious diseases. Several mathematical models have been established on the introduction of Wolbachia in mosquitoes, each illustrating the conditions that facilitate the rapid expansion of WI mosquitoes [15–21].

Additionally, fractional calculus (FC) is an emerging field of mathematical analysis that is developing rapidly. Numerous scientists have established several models of real processes using FC. A realistic representation of a physical phenomenon requires not just the current time but also the history of the preceding time, which FC can provide [22, 23]. In other words, fractional derivative definitions naturally allow for the incorporation of memory and hereditary characteristics in the modeling of diverse materials and processes [24]. Several models of infectious diseases based on FC have already been discussed in the literature [25–28].

Recently, researchers have developed mathematical models to investigate the key factors that influence Wolbachia's effectiveness in managing viral infections. Wan and Xu [29] created a mathematical model using optimal control theory to understand how Wolbachia infections and dengue spread work together, aiming to find the best ways to release young mosquitoes that die quickly. They found that increasing larval mortality prevented Wolbachia establishment and dengue treatments efficiency. Dianavinnarasi et al. [30] formulated a fractional-order Wolbachia invasion model, utilising impulsive control approaches to maintain infection in Aedes aegypti mosquitoes. The results showed that quick releases of Wolbachia can lower the number of local mosquitoes while keeping the infected ones alive. A fractional-order dengue transmission model by Vijayalakshmi and Ariyanatchi [31] looked at how both Wolbachia-infected and uninfected mosquitoes behave. They found that Wolbachia reduces dengue transmission and improves mosquito control. Dianavinnarasi et al. [32] used a fractional-order mathematical model to study how different Wolbachia strains spread in Aedes aegypti mosquitoes, with the goal of finding the best strain for long-term use. They demonstrated that Wolbachia is an efficacious approach for managing mosquito-borne illnesses. Ufuktepe [33] created a system that looks at how Wolbachia spreads in mosquito groups over time, considering how the presence of fewer wild insects can affect this spread. They examined the rivalry between released mosquitoes and indigenous mosquitoes. The investigation revealed that Wolbachia-infected mosquitoes can surpass native populations under certain conditions, resulting in possible mosquito control.

Additionally, it is expected that climate change will alter the distribution and seasonal behaviors of mosquitoes, significantly affecting the persistence and seasonality of vector-borne diseases [34]. This is expected to become increasingly important in the coming years as environmental conditions are anticipated to become more variable.

The novelty of this study is in the use of a fractional-order mathematical model to investigate the dynamics of Wolbachia invasion and the Effects of changes in the seasons on the development and spread of Wolbachia within the Aedes aegypti mosquito population. The model is constructed using the Caputo fractional derivative, which provides benefits including the ability to capture memory-dependent effects and enhance model accuracy. We incorporate the consequences of incomplete maternal transmission and the absence of Wolbachia infection. A mosquito population comprising two types, WU and WI, is analyzed. The model is evaluated for positivity and boundedness to ensure biological feasibility. The invasive reproduction number is determined to evaluate the efficacy of Wolbachia-infected mosquitoes in dengue control. The stability of the equilibrium points is examined using Lyapunov's direct method. Numerical simulations, performed using the fractional Adams–Bashforth–Moulton technique, illustrate the influence of model parameters and fractional-order values on the spread and control of Wolbachia infection. Furthermore, we investigate the impact of seasonal fluctuations on the introduction and transmission of Wolbachia within the Aedes aegypti mosquito population, emphasizing their effects across various life stages of the mosquito. The paper is structured as follows: An introductory section sets the context. Section 2 introduces the model formulation, fundamental definitions, and essential properties of the fractional operator in the Caputo sense. Section 3 discusses the positivity and boundedness of solutions. In Section 4, we examine the model by identifying the disease-free and endemic equilibrium points and derive the expression for the control reproduction number. Section 5 focuses on the stability analysis of equilibrium points and provides some graphical representations of the model's dynamics. Section 6 presents the Ulam-Hyers stability of the model, and Section 7 examines the seasonal effect of the proposed model. Finally, Section 8 concludes the study with a brief discussion.

2. Description of model framework

We develop an innovative fractional-order mathematical model that represents the life stages of Aedes aegypti mosquitoes. The Aedes aegypti mosquito population is categorized into two subpopulations: WI (w) and WU ($\bar{w}$). Assuming an equal distribution of adult male and female mosquitoes, we designate adult female mosquitoes free of Wolbachia as $F_{\bar{w}}$ and adult WI female mosquitoes as F_w [18, 35, 36]. The fractional mathematical model for Wolbachia invasion, considering imperfect maternal transmission and infection loss, is as follows:

$$\begin{cases} {}^{\boldsymbol{C}}\boldsymbol{\mathcal{D}}^{\boldsymbol{\alpha}} Q_{\bar{w}} = \left[\dfrac{\phi_{\bar{w}} F_{\bar{w}}^2 + \rho_1 \phi_w F_w^2 + \rho_2 \phi_w F_w F_{\bar{w}}}{F_{\bar{w}} + F_w}\right]\left(1 - \dfrac{Q}{K}\right) - (\mu_a + \psi) Q_{\bar{w}}, \\ {}^{\boldsymbol{C}}\boldsymbol{\mathcal{D}}^{\boldsymbol{\alpha}} Q_w = \left[\dfrac{(1-\rho_1)\phi_w F_w^2 + (1-\rho_2)\phi_w F_w F_{\bar{w}}}{F_{\bar{w}} + F_w}\right]\left(1 - \dfrac{Q}{K}\right) - (\mu_a + \psi) Q_w, \\ {}^{\boldsymbol{C}}\boldsymbol{\mathcal{D}}^{\boldsymbol{\alpha}} F_{\bar{w}} = \dfrac{\psi}{2} Q_{\bar{w}} + \sigma F_w - \mu_{\bar{w}} F_{\bar{w}}, \\ {}^{\boldsymbol{C}}\boldsymbol{\mathcal{D}}^{\boldsymbol{\alpha}} F_w = \dfrac{\psi}{2} Q_w - \sigma F_w - \mu_w F_w. \end{cases} \tag{1}$$

Here, $Q_{\bar{w}}(0) = 200000$, $Q_w(0) = 500000$, $F_{\bar{w}}(0) = 900000$, and $F_w(0) = 600000$ are the initial conditions. In the above equation, ${}^{\boldsymbol{C}}\boldsymbol{\mathcal{D}}^{\boldsymbol{\alpha}}$ denotes the Caputo fractional derivative where α is the order of derivative, $0 < \alpha \leq 1$. K stands for the aquatic phase's carrying capacity, $\phi_{\bar{w}}$ denotes the egg-laying rate for Wolbachia-free mosquitoes and, ϕ_w is the egg-laying rate for mosquitoes with Wolbachia infection. The ρ_1 is the fraction of eggs that are Wolbachia free when adult female and male mosquitoes with Wolbachia infection mate, ρ_2 is the fraction of eggs that are Wolbachia free when adult Wolbachia free males and females with Wolbachia infection mate, σ denotes loss of Wolbachia infection, ψ is the maturation rate, μ_a represents the aquatic death rate, $\mu_{\bar{w}}$ is the death rate of Wolbachia-free mosquitoes, and μ_w denotes the death rate of mosquitoes with Wolbachia infection. The following definitions and properties of fractional operators will be useful throughout our work.

Definition 1. *The fractional integral for a function* $f : (t_0, \infty) \to \mathbf{R}$ *is defined by*

$$\boldsymbol{I_t^{\alpha}} f(t) := \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-x)^{\alpha-1} f(x) dx,$$

where α *is the order of integral with* $0 < \alpha \leqslant 1$, $n \in N$ *and* $\Gamma(.)$ *denotes the gamma function.*

Definition 2. *The Caputo fractional derivative of order* α *with* $n - 1 < \alpha \leqslant n$ *is defined as:*

$${}^{\boldsymbol{C}}\boldsymbol{D_t^{\alpha}} f(t) := \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(x)}{(t-x)^{\alpha-n+1}} dx.$$

Theorem 1. *Let* $Re(\alpha) > 0, n = [Re(v)] + 1$. *Then*

$$\boldsymbol{I_t^{\alpha}} \boldsymbol{D_t^{\alpha}} f(t) = f(x) - \sum_{k=1}^{n} \frac{(D_t^k f)}{k!} x^k. \tag{2}$$

Lemma 1. *[37] Let* $x(t)$ *be a positive real, continuous and differentiable function. Then for any* $t \geq t_0$ *and* $0 < \alpha \leqslant 1$, *and* $x^* > 0$, *we have*

$${}^{\boldsymbol{C}}\boldsymbol{D_t^{\alpha}} \left[x(t) - x^* - x^* \ln\left(\frac{x(t)}{x^*}\right)\right] \leq \left(1 - \frac{x*}{x(t)}\right) \boldsymbol{D_t^{\alpha}} x(t). \tag{3}$$

We select the Caputo derivative; however, various fractional derivatives, like the Riemann-Liouville derivative and the Caputo-Fabrizio derivative, are also relevant. A key advantage of adopting the Caputo fractional derivative is its ability to facilitate the solution of problems with standard initial and boundary conditions. Real-world problems seem to be simple to physically understand using the Caputo derivative [19].

Description	Parameter	Estimate[Range]	Unit	References
Aquatic stage's carrying capacity	K	$10^6[10^4, 10^8]$	Aquatic mosquito	[38]
Egg laying rate for WU mosquitoes	$\phi_{\bar{w}}$	13[12-18]	Eggs per day	[39, 40]
Egg laying rate for WI mosquitoes	ϕ_w	11[8-12]	Eggs per day	[12, 39]
Fraction of eggs which is WU when adult female and male mosquitoes with WI mate	ρ_1	0.05 [0-0.11]	Dimensionless	[12]
Fraction of eggs which is WU when adult WU male and WI female mosquitoes mate	ρ_2	0.05[0-0.1]	Dimensionless	[12]
Per capita reduction of Wolbachia infection	σ	0.04[0-0.1]	Per day	[38]
Fraction of eggs that develop as females	a	0.5[0.34-0.6]	Dimensionless	[36, 41]
Per capita maturation rate	ψ	0.11 [0.1-0.12]	Per day	[12, 39]
Per capita aquatic death rate	μ_a	0.02	Per day	[18]
Per-capita mortality rate of WU mosquitoes	$\mu_{\bar{w}}$	0.061[0.02-0.09]	Per day	[12, 39]
Per-capita mortality rate of WI mosquitoes	μ_w	0.068[0.03-0.14]	Per day	[12, 42]

Table 1: Overview of model parameters and their corresponding descriptions

3. Positivity and boundedness of solutions

Establishing the positivity and boundedness of all solutions of the model will be significant because the problem is a biological model, which means that all of the solutions of the system (1) should remain non-negative and bounded $\forall t \geq 0$. First, from a biological perspective, the initial data $Q_{\bar{w}}(0)$, $Q_w(0)$, $F_{\bar{w}}(0)$, and $F_w(0)$ must be larger than or equal to zero. For the positivity, consider system (1) in the form:

$$
\begin{aligned}
&{}^{\mathcal{C}}\mathcal{D}^{\alpha} Q_{\bar{w}}|_{Q_{\bar{w}}=0} = \left[\frac{\phi_{\bar{w}} F_{\bar{w}}^2 + \rho_2 \phi_w F_w F_{\bar{w}}}{F_{\bar{w}} + F_w}\right]\left(1 - \frac{Q}{K}\right) \geqslant 0,\\
&{}^{\mathcal{C}}\mathcal{D}^{\alpha} Q_w|_{Q_w=0} = \left[\frac{\phi_w F_w^2 + (1-\rho_2)\phi_w F_w F_{\bar{w}}}{F_{\bar{w}} + F_w}\right]\left(1 - \frac{Q}{K}\right) \geqslant 0,\\
&{}^{\mathcal{C}}\mathcal{D}^{\alpha} F_{\bar{w}}|_{F_{\bar{w}}=0} = \frac{\psi}{2} Q_{\bar{w}} \geqslant 0,\\
&\text{and}\\
&{}^{\mathcal{C}}\mathcal{D}^{\alpha} F_w|_{F_w=0} = \frac{\psi}{2} Q_w \geqslant 0.
\end{aligned}
$$

According to Lemmas 5 and 6 in [43], all the solutions of the model remain non-negative. Next, we show the boundedness of the solutions. Following [44], we analyze the entire population and define the total population as:

$$
M(t) = Q_{\bar{w}}(t) + Q_w(t) + F_{\bar{w}}(t) + F_w(t).
$$

By adding all equations of the system (1), we will have

$$
{}^{\mathcal{C}}\mathcal{D}^{\alpha} M(t) = {}^{\mathcal{C}}\mathcal{D}^{\alpha} Q_{\bar{w}}(t) + {}^{\mathcal{C}}\mathcal{D}^{\alpha} Q_w(t) + {}^{\mathcal{C}}\mathcal{D}^{\alpha} F_{\bar{w}}(t) + {}^{\mathcal{C}}\mathcal{D}^{\alpha} F_w(t)
$$

$$
\begin{aligned}
=&\left[\frac{\phi_{\bar{w}}F_{\bar{w}}^2+\phi_w F_w F_{\bar{w}}+\phi_w F_w^2}{F_{\bar{w}}+F_w}\right]\left(1-\frac{Q}{K}\right)-\mu_a(Q_{\bar{w}}+Q_w)\\
&-\frac{\psi}{2}(Q_{\bar{w}}+Q_w)-\mu_{\bar{w}}F_{\bar{w}}-\mu_w F_w. \qquad (4)
\end{aligned}
$$

Since $Q_{\bar{w}} < K$, $Q_w < K$, it follows from model(1), $F_{\bar{w}} \leqslant \frac{\psi K}{2\mu_1}$ and $F_w \leqslant \frac{\psi K}{2\mu_1}$, where $\mu_1 = min(\mu_{\bar{w}}, \mu_w, \mu_a)$. Thus, Eq. (4) becomes

$$
{}^{\mathcal{C}}\mathcal{D}^{\alpha} M(t) \leqslant \Lambda_3 - \mu_1 M(t),
$$

where, $\Lambda_3 = \frac{\psi K(\phi_{\bar{w}}+2\phi_w)}{4\mu_1}$. So we deduce

$$
M(t) \leqslant M(0)E_\alpha(-\mu_1 \cdot t^\alpha) + \frac{\Lambda_3}{\mu_1}(1 - E_\alpha(-\mu_1 \cdot t^\alpha)).
$$

Where E_α is the Mittag-Leffler function of parameter α. Since $0 < E_\alpha(-\mu_1 \cdot t^\alpha) \leqslant 1$ and $1 - E_\alpha(-\mu_1 \cdot t^\alpha) \leqslant 1$. We obtain $M(t) \leqslant M(0) + \frac{\Lambda_3}{\mu_1}$. Therefore, the solutions of model (1) are uniformly bounded.

4. Equilibrium points

The equilibrium points of the non-integer order system (1) are obtained by solving the corresponding nonlinear algebraic system under the conditions ${}^{\mathcal{C}}\mathcal{D}^{\alpha} Q_{\bar{w}} = 0$, ${}^{\mathcal{C}}\mathcal{D}^{\alpha} Q_w = 0$, ${}^{\mathcal{C}}\mathcal{D}^{\alpha} F_{\bar{w}} = 0$ and ${}^{\mathcal{C}}\mathcal{D}^{\alpha} F_w = 0$. The model (1), with $\rho_1 = 0$ and $\sigma = 0$, yields four equilibrium points: $E_1 = (0,0,0,0)$ - demonstrating the absence of mosquitoes; $E_2 = (Q_{\bar{w}}^*, 0, F_{\bar{w}}^*, 0)$ - it illustrates the dominance of WU mosquitoes; $E_3 = (0, Q_w^*, 0, F_w^*)$ - demonstrating the existence of mosquitoes with Wolbachia infection and $E_4 = (Q_{\bar{w}}^*, Q_w^*, F_{\bar{w}}^*, F_w^*)$ - showing coexistence of both WU and WI mosquitoes. Determining the nature of stability points is important for controlling arboviral infections transmitted by Aedes aegypti mosquitoes.

4.1. No mosquitoes

The equilibrium point E_1 is trivial but uninteresting since it is biologically unrealistic. However, by looking at a particular example in which there is no contact between WI and WU mosquitoes can offer additional insight into the dynamics of this steady-state solution. We derived

$$
R_{0\bar{w}} = \frac{\phi_{\bar{w}}\psi}{2\mu_{\bar{w}}(\mu_\alpha + \psi)}, \qquad (5)
$$

and

$$
R_{0w} = \frac{\phi_w \psi}{2\mu_w(\mu_a + \psi)}, \qquad (6)
$$

which represent threshold values determining whether each group will survive or go extinct in the absence of contact. In the absence of contact between infected and uninfected mosquitoes, the thresholds in Eqs. (5) and (6) are obtained from the stability criteria of the appropriate Jacobian matrix under conditions of no contact among the two groups of mosquitoes. In other words, each group is autonomous from the others. Equivalent methods for the dynamics explicitly including the male mosquito states were presented in [18]. Thus, for the model (1), both populations are extinguished when $R_{0w} < 1$ and $R_{0\bar{w}} < 1$, as the reproductive parameters are insufficient to maintain the populations. Furthermore, given that the solutions remain non-negative for non-negative initial conditions, the solutions tend to the no-mosquito equilibrium point. However, and except for the biological implications of using insecticides, applying insecticides and destroying breeding sites have been an effective method in reducing mosquito populations.

4.2. Wolbachia uninfected mosquitoes only

The equilibrium point of WU mosquitoes is specified as $E_2 = (Q^*_{\bar{w}}, 0, F^*_{\bar{w}}, 0)$ where

$$Q^*_{\bar{w}} = K\left(1 - \frac{1}{R_{0\bar{w}}}\right), \tag{7}$$

and

$$F^*_{\bar{w}} = \frac{\psi K}{2\mu_{\bar{w}}}\left(1 - \frac{1}{R_{0\bar{w}}}\right). \tag{8}$$

Hence, for this equilibrium point, it is necessary that $R_{0\bar{w}} > 1$; otherwise, WU mosquitoes will not persist. We now establish the invasive reproduction number $R_{0w|\bar{w}}$ using the next-generation matrix method. WI populations can be divided into two categories: one characterized by the emergence rate of new WI mosquitoes ($\mathcal{F}$) and the other transition rates, including mortality and development into adult mosquitoes infected with Wolbachia ($\mathcal{V}$) [12].

$$\mathcal{F} = \begin{pmatrix} \left(\frac{\phi_w F_w^2 + (1-\rho_2)\phi_w F_w F_{\bar{w}}}{F_{\bar{w}} + F_w}\right)\left(1 - \frac{Q}{K}\right) \\ 0 \end{pmatrix},$$

$$\mathcal{V} = \begin{pmatrix} (\mu_a + \psi) Q_w \\ \frac{-\psi Q_w}{2} + \mu_w F_w \end{pmatrix}.$$

Furthermore, we introduce the matrices

$$F_{ij} = \frac{\partial \mathcal{F}_i}{\partial x_j}|_{E_2}, \tag{9}$$

and

$$V_{ij} = \frac{\partial \mathcal{V}_i}{\partial x_j}|_{E_2}, \tag{10}$$

where x_j show the infected population Q_w and F_w. Thus,

$$F = \begin{pmatrix} 0 & \frac{\phi_w (K - Q^*_{\bar{w}})(1-\rho_2)}{K} \\ 0 & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} \mu_a + \psi & 0 \\ \frac{-\psi}{2} & \mu_w \end{pmatrix},$$

the next generation matrix is

$$FV^{-1} = \begin{pmatrix} \frac{\psi\phi_w(K-Q^*_{\bar{w}})(1-\rho_2)}{2(\mu_a+\psi)\mu_w K} & \frac{\phi_w(K-Q^*_{\bar{w}})(1-\rho_2)}{\mu_w K} \\ 0 & 0 \end{pmatrix}.$$

Hence, the invasive reproduction number is

$$R_{0w|\bar{w}} = \lambda(FV^{-1}) = \frac{\phi_w \mu_{\bar{w}}(1-\rho_2)}{\mu_w K} = \frac{R_{0w}(1-\rho_2)}{R_{0\bar{w}}}. \tag{11}$$

Here, $\lambda(FV^{-1})$ denotes the spectral radius of FV^{-1}. The number $R_{0w|\bar{w}}$ represents the expected number of infected offspring produced by a single WI mosquito introduced into a population of WU mosquitoes at equilibrium. Biologically, this metric quantifies the potential for Wolbachia to invade and establish in a WU dominated population, under the assumption that infected mosquitoes do not lose their infection. The factor $(1-\rho_2)$ in this equation reflects the effect of imperfect maternal transmission: some offspring from WI females mating with WU males are uninfected, which diminishes the chances of successful invasion. This factor thus demonstrates how the ratio of aquatic-stage mosquitoes that are WI affects the likelihood of WI mosquitoes displacing WU mosquitoes when WU males mate with WI females. If $R_{0w|\bar{w}} < 1$, the infection is unable to establish itself, even with cytoplasmic incompatibility present. Therefore, cytoplasmic incompatibility by itself is not enough for successful invasion; the reliability of maternal transmission is also critical.

4.3. Wolbachia infected mosquitoes-only

In this instance, the equilibrium point is $E_3 = (0, Q^*_w, 0, F^*_w)$ where,

$$Q^*_w = K\left(1 - \frac{1}{R_{0w}}\right), \tag{12}$$

and

$$F^*_w = \frac{\psi K}{2\mu_w}\left(1 - \frac{1}{R_{0w}}\right). \tag{13}$$

4.4. Coexistence of Wolbachia infected and Wolbachia uninfected mosquitoes

The presence of both WI and WU mosquitoes within the Aedes aegypti population is a significant situation. In this case, it is desirable for the majority of the mosquito population to be Wolbachia-infected, as this can reduce the transmission of arboviral diseases. The coexistence equilibrium point of system (1) is defined as

$$E_4 = (d_1 F^*_{\bar{w}}, d_2 F^*_w, d_3 F^*_w, F^*_w), \tag{14}$$

where $F^*_w = \frac{K\psi}{2(\mu_{\bar{w}} d_3 + \mu_w)}\left[\frac{R_{0w}(1+(1-\rho_2)d_3)-(1+d_3)}{R_{0w}(1+(1-\rho_2)d_3)}\right]$, $d_1 = \frac{2\mu_{\bar{w}}}{\psi}$, $d_2 = \frac{2\mu_w}{\psi}$ and, $d_3 = \left[\frac{R_{0w|\bar{w}}(\mu_{\bar{w}}-\rho_2\mu_w)}{\mu_{\bar{w}}(1-\rho_2)(1-R_{0w|\bar{w}})}\right]$.

5. Global stability of equilibrium points

Lyapunov's direct method, also known as the second Lyapunov method, gives us an effective way to analyze the global behavior of a system without explicitly solving it. Classical Lyapunov functions in mathematical biology are linear combinations of linear, common quadratic, and Volterra-type functions [27, 37, 45]. We now examine the global asymptotic stability of the WU equilibrium point E_2, the WI mosquito equilibrium point E_3, and the combined WU and WI equilibrium point E_4 of system (1). The equilibrium point for adult female WI mosquitoes derived from Eq. (11) can be written as

$$F_w^* = \frac{\psi K}{2\mu_w}\left(1 - \frac{1-\rho_2}{R_{0w|\bar{w}} R_{0\bar{w}}}\right). \tag{15}$$

The above expression clearly shows that when $R_{0w|\bar{w}} < 1$, the WI mosquitoes-only equilibrium may exist. The conventional infectious diseases modeling work indicates a backward bifurcation in the presence of endemic equilibria for $R_{0w|\bar{w}} < 1$ [46]. This equilibrium is unstable when $R_{0w|\bar{w}} < \frac{1-\rho_2}{R_{0\bar{w}}}$ ($\Longrightarrow R_{0w} < 1$) and locally asymptotically stable even when $R_{0w|\bar{w}} < 1$, since $R_{0\bar{w}} > 1$ and $\mu_{\bar{w}} > \rho_2\mu_w$. Whenever $R_{0w|\bar{w}} < 1$, both equilibrium points E_2 and E_3 are locally asymptotically stable since $R_{0\bar{w}} > 1$ for E_2, $R_{0w} > 1$ and $\mu_{\bar{w}} > \rho_2\mu_w$ for E_3.

Theorem 2. *The equilibrium point* $E_2 = (Q^*_{\bar{w}}, 0, F^*_{\bar{w}}, 0)$ *is globally asymptotically stable whenever* $R_{0w|\bar{w}} > 1$ *and* $R_{0\bar{w}} > 1$.

Proof. Let us consider the following Lyapunov function:

$$\begin{aligned} V_1(Q_{\bar{w}}(t), F_{\bar{w}}(t)) =& \frac{\psi\phi_{\bar{w}}}{2\mu_{\bar{w}}(\mu_a+\psi)} \int_{Q^*_{\bar{w}}}^{Q_{\bar{w}}} \left(1 - \frac{Q^*_{\bar{w}}}{x}\right) dx + \frac{1}{\mu_{\bar{w}}} \int_{F^*_{\bar{w}}}^{F_{\bar{w}}} \left(1 - \frac{F^*_{\bar{w}}}{x}\right) dx \\ =& \frac{\psi\phi_{\bar{w}}}{2\mu_{\bar{w}}(\mu_a+\psi)} \left[Q_{\bar{w}} - Q^*_{\bar{w}} - Q^*_{\bar{w}} \ln\left(\frac{Q^*_{\bar{w}}}{Q_{\bar{w}}}\right)\right] + \frac{1}{\mu_{\bar{w}}} \left[F_{\bar{w}} - F^*_{\bar{w}} - F^*_{\bar{w}} \ln\left(\frac{F_{\bar{w}}}{F_{\bar{w}}}\right)\right]. \end{aligned} \tag{16}$$

The V_1 is a well-defined, continuous, and positive definite function. Utilizing the property of fractional derivatives as defined in the lemma 1, we compute the derivative of Eq. (16) with respect to time along the solution of system (1) and we demonstrate that the corresponding fractional derivative is negative definite, using Lemma 3.1 in [45], we have

$${}^C D^\alpha V_1(Q_{\bar{w}}(t), F_{\bar{w}}(t)) \leq \frac{\psi\phi_{\bar{w}}}{2\mu_{\bar{w}}(\mu_a+\psi)} \left(1 - \frac{Q^*_w}{Q_{\bar{w}}}\right) {}^C D^\alpha Q_{\bar{w}}(t) + \frac{1}{\mu_{\bar{w}}} \left(1 - \frac{F^*_{\bar{w}}}{F_{\bar{w}}}\right) {}^C D^\alpha F_{\bar{w}}(t). \tag{17}$$

Substituting the expression for the model (1) we have,

$$\begin{aligned} \frac{\psi\phi_{\bar{w}}}{2\mu_{\bar{w}}(\mu_a+\psi)} \left(1 - \frac{Q^*_{\bar{w}}}{Q_{\bar{w}}}\right) {}^C D^\alpha Q_{\bar{w}}(t) =& \frac{\psi\phi_{\bar{w}}}{2\mu_{\bar{w}}(\mu_a+\psi)} \left(1 - \frac{Q^*_{\bar{w}}}{Q_{\bar{w}}}\right) \\ & \left[\Lambda_1 \left(1 - \frac{Q}{K}\right) - (\mu_a+\psi) Q_{\bar{w}}\right] \\ =& \frac{\psi\phi_{\bar{w}}\Lambda_1}{2\mu_{\bar{w}}(\mu_a+\psi)} \left(1 - \frac{Q^*_{\bar{w}}}{Q_{\bar{w}}}\right) \left(1 - \frac{Q}{K}\right) \\ & - \frac{\psi Q_{\bar{w}}}{2\mu_{\bar{w}}} + \frac{\psi Q^*_{\bar{w}}}{2\mu_{\bar{w}}}, \end{aligned} \tag{18}$$

where $\Lambda_1 = \left(\frac{\phi_{\bar{w}} F_{\bar{w}}^2 + \rho_2 \phi_w F_w F_{\bar{w}}}{F_{\bar{w}} + F_w}\right)$ and,

$$\begin{aligned}\frac{1}{\mu_{\bar{w}}}\left(1-\frac{F_{\bar{w}}^*}{F_{\bar{w}}}\right){}^C D^\alpha F_{\bar{w}}(t) =& \frac{1}{\mu_{\bar{w}}}\left(1-\frac{F_{\bar{w}}^*}{F_{\bar{w}}}\right)\left(\frac{\psi}{2}Q_{\bar{w}} - \mu_{\bar{w}} F_{\bar{w}}\right)\\ =& \frac{\psi Q_{\bar{w}}}{2\mu_{\bar{w}}} - \frac{\psi Q_{\bar{w}} F_{\bar{w}}^*}{2\mu_{\bar{w}} F_{\bar{w}}} - F_{\bar{w}} + F_{\bar{w}}^*.\end{aligned} \tag{19}$$

Adding Eqs.(18) and (19) yields

$$\begin{aligned}{}^C D^\alpha V_1(Q_{\bar{w}}(t), F_{\bar{w}}(t)) \leq& \frac{\psi \phi_{\bar{w}} \Lambda_1}{2\mu_{\bar{w}}(\mu_a + \psi)}\left(1-\frac{Q_{\bar{w}}^*}{Q_{\bar{w}}}\right)\left(1-\frac{Q}{K}\right)\\ &+ \frac{\psi Q_{\bar{w}}}{2\mu_{\bar{w}}} - \frac{\psi Q_{\bar{w}} F_{\bar{w}}^*}{2\mu_{\bar{w}} F_{\bar{w}}} - F_{\bar{w}} + F_{\bar{w}}^*.\end{aligned} \tag{20}$$

Using the relations at the steady state and some algebraic manipulations, we get

$$\begin{aligned}{}^C D^\alpha V_1(Q_{\bar{w}}(t), F_{\bar{w}}(t)) \leq& \frac{\psi \phi_{\bar{w}} \Lambda_1}{2\mu_{\bar{w}}(\mu_a + \psi)}\left(1-\frac{Q_{\bar{w}}^*}{Q_{\bar{w}}}\right)\left(1-\frac{Q}{K}\right)\\ &+ F_{\bar{w}}^*\left(2 - \frac{Q_{\bar{w}} F_{\bar{w}}^*}{Q_{\bar{w}}^* F_{\bar{w}}} - \frac{Q_{\bar{w}}^* F_{\bar{w}}}{Q_{\bar{w}} F_{\bar{w}}^*}\right) - F_{\bar{w}}\left(1 - \frac{Q_{\bar{w}}^*}{Q_{\bar{w}}}\right).\end{aligned} \tag{21}$$

Thus,

$$\begin{aligned}{}^C D^\alpha V_1(Q_{\bar{w}}(t), F_{\bar{w}}(t)) \leq& F_{\bar{w}}\left(1-\frac{Q_{\bar{w}}^*}{Q_{\bar{w}}}\right)\left[\frac{R_{o\bar{w}}(F_{\bar{w}}\phi_{\bar{w}} + \rho_2 \phi_w F_w)}{F_{\bar{w}} + F_w}\left(1-\frac{Q}{K}\right) - 1\right]\\ &+ F_{\bar{w}}^*\left(2 - \frac{Q_{\bar{w}} F_{\bar{w}}^*}{Q_{\bar{w}}^* F_{\bar{w}}} - \frac{Q_{\bar{w}}^* F_{\bar{w}}}{Q_{\bar{w}} F_{\bar{w}}^*}\right).\end{aligned} \tag{22}$$

Eq. (22) can be reformulated as

$$\begin{aligned}{}^C D^\alpha V_1(Q_{\bar{w}}(t), F_{\bar{w}}(t)) \leq& F_{\bar{w}}^*\left(\frac{F_{\bar{w}}^*}{F_{\bar{w}}} + \frac{Q_{\bar{w}} F_{\bar{w}}^*}{Q_{\bar{w}}^* F_{\bar{w}}} - 2\right)\left[\frac{R_{o\bar{w}}(F_{\bar{w}}\phi_{\bar{w}} + \rho_2 \phi_w F_w)}{F_{\bar{w}} + F_w}\left(1-\frac{Q}{K}\right) - 1\right]\\ &+ F_{\bar{w}}^*\left(2 - \frac{Q_{\bar{w}} F_{\bar{w}}^*}{Q_{\bar{w}}^* F_{\bar{w}}} - \frac{Q_{\bar{w}}^* F_{\bar{w}}}{Q_{\bar{w}} F_{\bar{w}}^*}\right)\left[\frac{R_{o\bar{w}}(F_{\bar{w}}\phi_{\bar{w}} + \rho_2 \phi_w F_w)}{F_{\bar{w}} + F_w}\left(1-\frac{Q}{K}\right)\right].\end{aligned} \tag{23}$$

The second component on the right side of Eq. (23) is less than or equal to zero due to

$$\left(2 - \frac{Q_{\bar{w}} F_{\bar{w}}^*}{Q_{\bar{w}}^* F_{\bar{w}}} - \frac{Q_{\bar{w}}^* F_{\bar{w}}}{Q_{\bar{w}} F_{\bar{w}}^*}\right) \leq 0.$$

Additionally, $Q_{\bar{w}}^* \leq Q_{\bar{w}} \leq K$ results in the first term either less than or equal to zero. When $0 \leq Q_{\bar{w}} \leq Q_{\bar{w}}^*$, Eq. (23) is less than zero because $\left(1 - \frac{Q_{\bar{w}}^*}{Q_{\bar{w}}}\right) < 0$. Hence, ${}^C D^\alpha V_1(Q_{\bar{w}}(t), F_{\bar{w}}(t)) \leq 0$. So, the function $V_1(Q_{\bar{w}}(t), F_{\bar{w}}(t))$ is negative definite for $0 < \alpha < 1$. Then the equilibrium state E_2 is globally asymptotically stable.

Theorem 3. *The equilibrium point* $E_3 = (0, Q_w^*, 0, F_w^*)$ *is globally asymptotically stable whenever* $R_{0w|\bar{w}} > 1$, $R_{0w} > 1$ *and* $\mu_{\bar{w}} > \rho_2 \mu_w$.

Proof. When $R_{0w|\bar{w}} > 1$ then $R_{0w} > \frac{R_{0\bar{w}}}{1-\rho_2}$ and this implies $R_{0w} > R_{0\bar{w}}$. We consider the following Volterra-type Lyapunov function as

$$V_2(Q_w(t), F_w(t)) = \frac{\psi}{2\mu_w(\mu_a+\psi)} \int_{Q_w^*}^{Q_w} \left(1 - \frac{Q_w^*}{x}\right) dx + \frac{1}{\mu_w} \int_{F_w^*}^{F_w} \left(1 - \frac{F_w^*}{x}\right) dx. \tag{24}$$

By calculating the α-order derivative of $V_2(Q_w(t), F_w(t))$, one has,

$${}^{C}D^{\alpha} V_2(Q_w(t), F_w(t)) \leq \frac{\psi}{2\mu_w(\mu_a+\psi)} \left(1 - \frac{Q_w^*}{Q_w}\right) {}^{C}D^{\alpha} Q_w(t) + \frac{1}{\mu_w} \left(1 - \frac{F_w^*}{F_w}\right) {}^{C}D^{\alpha} F_w(t). \tag{25}$$

Substituting the expression for the model (1) we have,

$$\begin{aligned} \frac{\psi}{2\mu_w(\mu_a+\psi)} \left(1 - \frac{Q_w^*}{Q_w}\right) {}^{C}D^{\alpha} Q_w(t) =& \frac{\psi}{2\mu_w(\mu_a+\psi)} \left(1 - \frac{Q_w^*}{Q_w}\right) \\ & \left[\Lambda_2 \left(1 - \frac{Q}{K}\right) - (\mu_a+\psi) Q_w\right], \end{aligned} \tag{26}$$

where $\Lambda_2 = \left[\frac{(1-\rho_1)\phi_w F_w^2 + (1-\rho_2)\phi_w F_w F_{\bar{w}}}{F_{\bar{w}} + F_w}\right]$ and

$$\frac{1}{\mu_w} \left(1 - \frac{F_w^*}{F_w}\right) {}^{C}D^{\alpha} F_w(t) = \frac{1}{\mu_w} \left(1 - \frac{F_w^*}{F_w}\right) \left(\frac{\psi}{2} Q_w - \mu_w F_w\right). \tag{27}$$

Form Eq. (26),

$$\begin{aligned} \frac{\psi}{2\mu_w(\mu_a+\psi)} \left(1 - \frac{Q_w^*}{Q_w}\right) {}^{C}D^{\alpha} Q_w(t) =& \frac{\psi \Lambda_2}{2\mu_w(\mu_a+\psi)} \left(1 - \frac{Q_w^*}{Q_w}\right) \left(1 - \frac{Q}{K}\right) \\ & - \frac{\psi Q_w}{2\mu_w} + \frac{\psi Q_w^*}{2\mu_w}, \end{aligned} \tag{28}$$

and from Eq. (27),

$$\frac{1}{\mu_w} \left(1 - \frac{F_w^*}{F_w}\right) {}^{C}D^{\alpha} F_w(t) = \frac{\psi Q_w}{2\mu_w} - \frac{\psi Q_w F_w^*}{2\mu_w F_w} - F_w + F_w^*. \tag{29}$$

By adding Eqs. (28) and (29), and after rearrangements, we obtain

$$\begin{aligned} {}^{C}D^{\alpha} V_2(Q_w(t), F_w(t)) \leq & F_w \left(1 - \frac{Q_w^*}{Q_w}\right) \left[\frac{R_{0w}(F_w + (1-\rho_2)F_{\bar{w}})}{F_w + F_{\bar{w}}} \left(1 - \frac{Q}{K}\right) - 1\right] \\ & + F_w^* \left(2 - \frac{Q_w F_w^*}{Q_w^* F_w} - \frac{Q_w^* F_w}{Q_w F_w^*}\right). \end{aligned} \tag{30}$$

Eq. (30), which can then be simplified to

$$\begin{aligned} {}^{C}D^{\alpha} V_2(Q_w(t), F_w(t)) \leq & F_w^* \left(\frac{F_w^*}{F_w} + \frac{Q_w F_w^*}{Q_w^* F_w} - 2\right) \left[\frac{R_{0w}(F_w + (1-\rho_2)F_{\bar{w}})}{F_w + F_{\bar{w}}} \left(1 - \frac{Q}{K}\right) - 1\right] \\ & + F_w^* \left(2 - \frac{Q_w F_w^*}{Q_w^* F_w} - \frac{Q_w^* F_w}{Q_w F_w^*}\right) \left[\frac{R_{0w}(F_w + (1-\rho_2)F_{\bar{w}})}{F_w + F_{\bar{w}}} \left(1 - \frac{Q}{K}\right)\right]. \end{aligned} \tag{31}$$

Thus, ${}^{C}D^{\alpha} V_2(Q_w(t), F_w(t)) \leq 0$. So, the function $V_2(Q_w(t), F_w(t))$ is negative definite for $0 < \alpha < 1$. Then the equilibrium state E_3 is globally asymptotically stable.

Theorem 4. *The equilibrium point* $E_4 = (d_1F^*_{\bar{w}}, d_2F^*_w, d_3F^*_w, F^*_w)$ *of the non-integer order system (1) is globally asymptotically stable if* $R_{0w|\bar{w}} > 1$, $R_{0w} > 1$ *and* $R_{0\bar{w}} > 1$.

Proof. The following positive definite Lyapunov function is used to analyze the global asymptotic stability of the equilibrium point E_4:

$$V_3(t) = \mathcal{T}_1 Q(Q_{\bar{w}}(t)) + \mathcal{T}_2 Q(Q_w(t)) + \mathcal{T}_3 F(F_{\bar{w}}(t)) + \mathcal{T}_4 F(F_w(t)), \tag{32}$$

where,

$$\begin{aligned}
&Q(Q_w(t)) = Q_w - Q^*_w - Q^*_w \ln\left(\frac{Q^*_w}{Q_w}\right), Q(Q_{\bar{w}}(t)) = Q_{\bar{w}} - Q^*_{\bar{w}} - Q^*_{\bar{w}} \ln\left(\frac{Q^*_{\bar{w}}}{Q_{\bar{w}}}\right),\\
&F(F_{\bar{w}}(t)) = F_{\bar{w}} - F^*_{\bar{w}} - F^*_{\bar{w}} \ln\left(\frac{F^*_{\bar{w}}}{F_{\bar{w}}}\right), F(F_w(t)) = F_w - F^*_w - F^*_w \ln\left(\frac{F^*_w}{F_w}\right),\\
&\mathcal{T}_1 = \frac{\psi\phi_{\bar{w}}}{2\mu_{\bar{w}}(\mu_a+\psi)}, \mathcal{T}_2 = \frac{\psi}{2\mu_w(\mu_a+\psi)}, \mathcal{T}_3 = \frac{1}{\mu_{\bar{w}}} \quad \textit{and} \quad \mathcal{T}_4 = \frac{1}{\mu_w}.
\end{aligned}$$

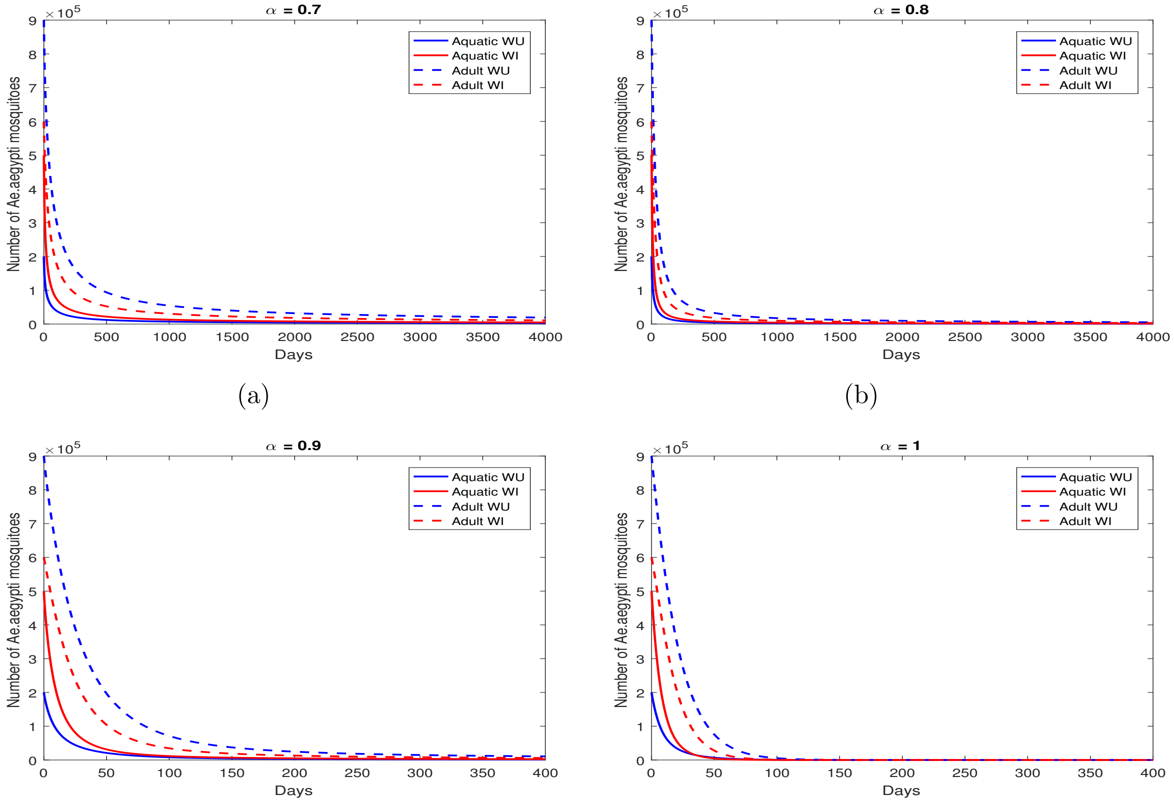
Differentiating $V_3(t)$ with respect to time and by applying Lemma 1, one has

$$\begin{aligned}
{}^C D^\alpha V_3(t) \;\leq\; & \frac{\psi\phi_{\bar{w}}}{2\mu_{\bar{w}}(\mu_a+\psi)}\left(1-\frac{Q^*_w}{Q_{\bar{w}}}\right){}^C D^\alpha Q_{\bar{w}}(t) + \frac{\psi}{2\mu_w(\mu_a+\psi)}\left(1-\frac{Q^*_w}{Q_w}\right){}^C D^\alpha Q_w(t)\\
& + \frac{1}{\mu_{\bar{w}}}\left(1-\frac{F^*_{\bar{w}}}{F_{\bar{w}}}\right){}^C D^\alpha F_{\bar{w}}(t) + \frac{1}{\mu_w}\left(1-\frac{F^*_w}{F_w}\right){}^C D^\alpha F_w(t).
\end{aligned} \tag{33}$$

Substituting the expressions for the derivatives into Eq. (33), and subsequently rearranging and manipulating the terms, gives

$$\begin{aligned}
{}^C D^\alpha V_3(t) \;\leq\; & F^*_{\bar{w}}\left(\frac{F^*_{\bar{w}}}{F_{\bar{w}}} + \frac{Q_{\bar{w}}F^*_{\bar{w}}}{Q^*_{\bar{w}}F_{\bar{w}}} - 2\right)\left[\frac{R_{o\bar{w}}(F_{\bar{w}}\phi_{\bar{w}} + \rho_2\phi_w F_w)}{F_{\bar{w}} + F_w}\left(1-\frac{Q}{K}\right) - 1\right]\\
& + F^*_{\bar{w}}\left(2 - \frac{Q_{\bar{w}}F^*_{\bar{w}}}{Q^*_{\bar{w}}F_{\bar{w}}} - \frac{Q^*_{\bar{w}}F_{\bar{w}}}{Q_{\bar{w}}F^*_{\bar{w}}}\right)\left[\frac{R_{o\bar{w}}(F_{\bar{w}}\phi_{\bar{w}} + \rho_2\phi_w F_w)}{F_{\bar{w}} + F_w}\left(1-\frac{Q}{K}\right)\right]\\
& + F^*_w\left(\frac{F^*_w}{F_w} + \frac{Q_wF^*_w}{Q^*_wF_w} - 2\right)\left[\frac{R_{0w}(F_w + (1-\rho_2)F_{\bar{w}})}{F_w + F_{\bar{w}}}\left(1-\frac{Q}{K}\right) - 1\right]\\
& + F^*_w\left(2 - \frac{Q_wF^*_w}{Q^*_wF_w} - \frac{Q^*_wF_w}{Q_wF^*_{\bar{w}}}\right)\left[\frac{R_{0w}(F_w + (1-\rho_2)F_{\bar{w}})}{F_w + F_{\bar{w}}}\left(1-\frac{Q}{K}\right)\right].
\end{aligned} \tag{34}$$

Hence, we conclude by saying that since $V_3(t)$ is negative definite for $0 < \alpha < 1$, then the equilibrium state E_4 is globally asymptotically stable.

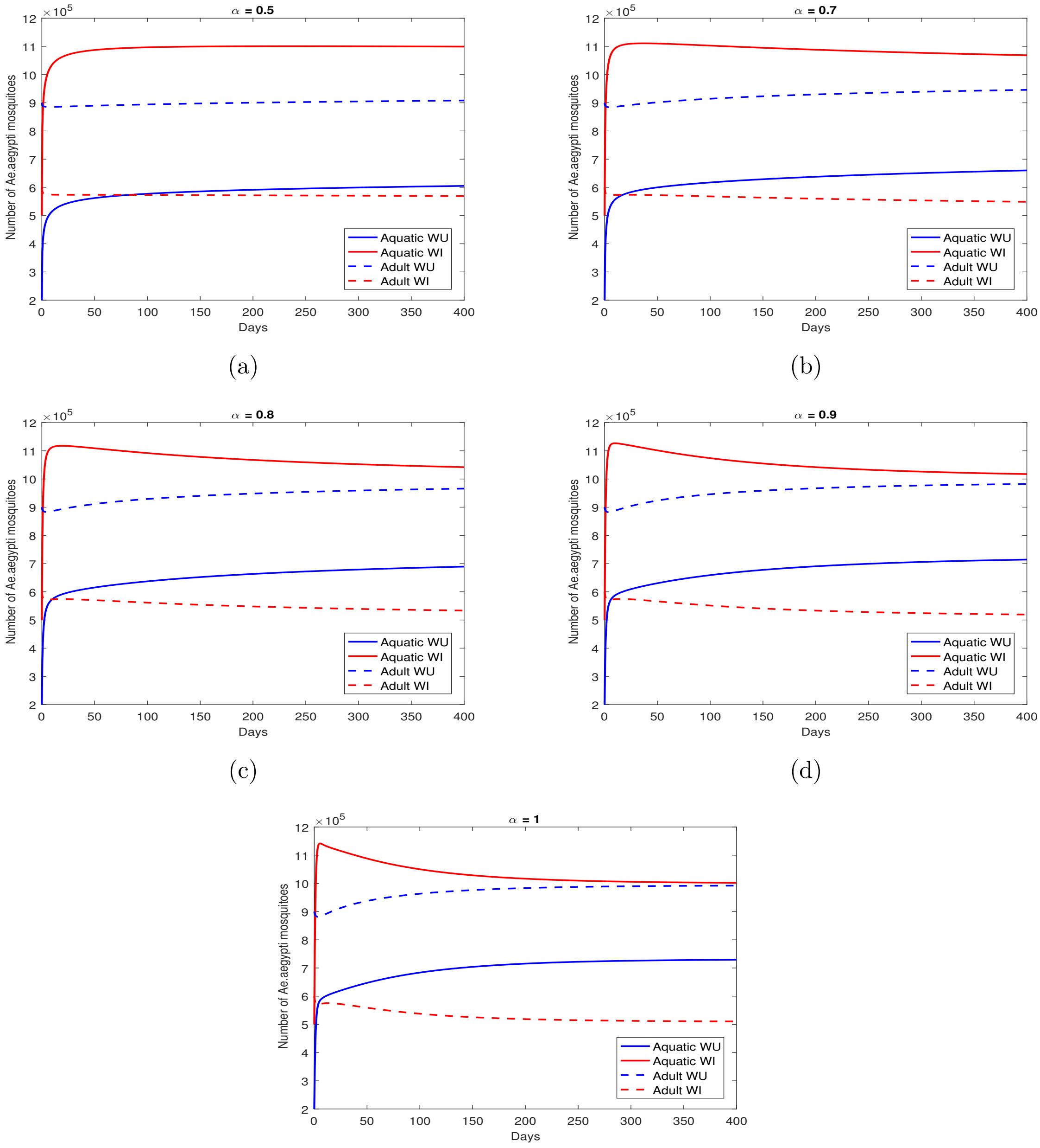


(a) (b)

(c) (d)

Figure 1: This figure illustrates the numerical simulation of aquatic-stage WI and WU mosquitoes, as well as adult WI and WU mosquitoes, across various fractional order values of α. The plots illustrate the impact of the memory effect, governed by the fractional order, on the temporal dynamics of these populations as α approaches 1. We set $\phi_w = 0.01$, $\phi_{\bar{w}} = 0.01$, $K = 2 \times 10^6$, $Q_{\bar{w}}(0) = 2 \times 10^5$, $Q_w(0) = 5 \times 10^5$, $F_{\bar{w}}(0) = 9 \times 10^5$, and $F_w(0) = 6 \times 10^5$. Other parameters used for these graphs simulations are provided in Table 1.

Hence, for $R_{0w} < 1$ and $R_{0\bar{w}} < 1$ both populations die out (see Figure 1) because the populations can not be sustained in reproductive terms. Furthermore, the solutions converge to the non-mosquito equilibrium point, as they remain non-negative with non-negative initial conditions. All populations take much time to die out as the order of α decreases (see Figure 1).



(a)

(b)

(c)

(d)

(e)

Figure 2: This figure presents the numerical simulation of the model under different values of the fractional order α. In these graphs we set $\phi_w = 2$, $\phi_{\bar{w}} = 1$, $K = 2 \times 10^6$, $Q_{\bar{w}}(0) = 2 \times 10^5$, $Q_w(0) = 5 \times 10^5$, $F_{\bar{w}}(0) = 9 \times 10^5$, and $F_w(0) = 600000$. Table 1 contains a list of additional parameters that are used for the simulations of these graphs.

The $\alpha = 0.5$ is the most suitable value for this model. The number of WI mosquitoes rises as the value of α falls from 1, while the number of WU mosquitoes decreases slowly. The WI mosquito population continues to increase, but at a very slow rate when the value of α falls below 0.5. At this point, however, the WU mosquito population continues to decrease, as shown in Figure 2. WU mosquito populations started increasing when $\alpha < 0.5$.

6. Ulam-Hyers stability analysis

The stability theory is a fundamental aspect of the qualitative analysis of differential equations. The concept of Ulam stability was introduced by Ulam [47]. in 1940. He proposed to investigate the degree of approximation between the approximate solution and the exact solution of the equations. In 1941, Hyers [48] addressed Ulam's notion and established the concept of Ulam-Hyers stability for equations. It addresses short-term perturbations and analyzes how minor alterations in the equation and initial conditions affect the solution inside a local neighborhood. Furthermore, given that stability is necessary for an approximate solution, we focus on the Ulam-Hyers stability for the model (1) by using the technique of nonlinear functional analysis. We will first present some lemmas essential for stability. We now reformulate the model (1) in the following manner.

$$\begin{cases} {}^{\mathcal{C}}\boldsymbol{\mathcal{D}}^{\boldsymbol{\alpha}} Q_{\bar{w}} = \boldsymbol{\Theta}_1(t, Q_{\bar{w}}, Q_w, F_{\bar{w}}, F_w), \\ {}^{\mathcal{C}}\boldsymbol{\mathcal{D}}^{\boldsymbol{\alpha}} Q_w = \boldsymbol{\Theta}_2(t, Q_{\bar{w}}, Q_w, F_{\bar{w}}, F_w), \\ {}^{\mathcal{C}}\boldsymbol{\mathcal{D}}^{\boldsymbol{\alpha}} F_{\bar{w}} = \boldsymbol{\Theta}_3(t, Q_{\bar{w}}, Q_w, F_{\bar{w}}, F_w), \\ {}^{\mathcal{C}}\boldsymbol{\mathcal{D}}^{\boldsymbol{\alpha}} F_w = \boldsymbol{\Theta}_4(t, Q_{\bar{w}}, Q_w, F_{\bar{w}}, F_w). \end{cases} \tag{35}$$

Where

$$\begin{cases} \boldsymbol{\Theta}_1(t, Q_{\bar{w}}, Q_w, F_{\bar{w}}, F_w) = \left[\dfrac{\phi_{\bar{w}} F_{\bar{w}}^2 + \rho_1 \phi_w F_w^2 + \rho_2 \phi_w F_w F_{\bar{w}}}{F_{\bar{w}} + F_w}\right]\left(1 - \dfrac{Q}{K}\right) - (\mu_a + \psi) Q_{\bar{w}}, \\ \boldsymbol{\Theta}_2(t, Q_{\bar{w}}, Q_w, F_{\bar{w}}, F_w) = \left[\dfrac{(1-\rho_1)\phi_w F_w^2 + (1-\rho_2)\phi_w F_w F_{\bar{w}}}{F_{\bar{w}} + F_w}\right]\left(1 - \dfrac{Q}{K}\right) - (\mu_a + \psi) Q_w, \\ \boldsymbol{\Theta}_3(t, Q_{\bar{w}}, Q_w, F_{\bar{w}}, F_w) = \dfrac{\psi}{2} Q_{\bar{w}} + \sigma F_w - \mu_{\bar{w}} F_{\bar{w}}, \\ \boldsymbol{\Theta}_4(t, Q_{\bar{w}}, Q_w, F_{\bar{w}}, F_w) = \dfrac{\psi}{2} Q_w - \sigma F_w - \mu_w F_w. \end{cases} \tag{36}$$

Thus, the proposed model (1) takes the form

$$\begin{cases} D_t^{\alpha} \boldsymbol{\nu}(t) = R(t, \boldsymbol{\nu}(t)); t \in [0, b], 0 < \alpha \leqslant 1, \\ \boldsymbol{\nu}(0) = \boldsymbol{\nu}_0. \end{cases} \tag{37}$$

In view of this the problem (37) is given by,

$$\begin{aligned} \boldsymbol{\nu}(t) &= \boldsymbol{\nu}_0 + \boldsymbol{I}_{\boldsymbol{0}}^{\boldsymbol{\alpha}} R(t, \boldsymbol{\nu}(t)) \\ &= \boldsymbol{\nu}_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-k)^{\alpha-1} R(k, \boldsymbol{\nu}(k)) dk. \end{aligned} \tag{38}$$

Let $\varepsilon > 0$ and consider the inequality given below

$$|D_t^{\alpha} \overline{\boldsymbol{\nu}}(t) - R(t, \overline{\boldsymbol{\nu}}(t))| \leq \varepsilon, \quad t \in [0, b], \tag{39}$$

where $\varepsilon = max(\varepsilon_j)^T, j = 1, 2, 3, 4$.

Definition 3. *The proposed model (37) which is equivalent to system (1) is Ulam-Hyers stable if there exists* $C_R > 0$*, such that for every* $\varepsilon > 0$ *and a solution* $\overline{\boldsymbol{\nu}}(t) \in \boldsymbol{\mathcal{B}}$ *satisfies Eq. (39), there exists a unique solution* $\boldsymbol{\nu}(t) \in \boldsymbol{\mathcal{B}}$ *of system (1), with*

$$|\overline{\boldsymbol{\nu}}(t) - \boldsymbol{\nu}(t)| \leq X_R\varepsilon, \quad t \in J, \tag{40}$$

where $X_R = max(C_{Rj})^T$*.*

Remark 1. *A function* $\overline{\boldsymbol{\nu}}(t) \in \boldsymbol{\mathcal{B}}$ *satisfies inequality (39) if and only if there exists a function* $h \in \boldsymbol{\mathcal{B}}$ *with properties below:*

(i) $|h(t)| \leq \varepsilon, \quad t \in J.$

(ii) $D_t^\alpha \overline{\boldsymbol{\nu}}(t) = R(t, \overline{\boldsymbol{\nu}}(t)) + h(t), \quad t \in J.$

Lemma 2. *Assume that* $\overline{\boldsymbol{\nu}}(t) \in \boldsymbol{\mathcal{B}}$ *satisfies inequality (39), then* $\overline{\boldsymbol{\nu}}(t)$ *satisfies the integral inequality described by*

$$|\overline{\boldsymbol{\nu}}(t) - \overline{\boldsymbol{\nu}}_0 - \frac{1}{\Gamma(\alpha)}\int_0^t (t-k)^{\alpha-1} R(k, \overline{\boldsymbol{\nu}}(k))dk| \leq \Omega\varepsilon. \tag{41}$$

Proof. Thanks to (ii) of Remark 1,

$$D_t^\alpha \overline{\boldsymbol{\nu}}(t) = R(t, \overline{\boldsymbol{\nu}}(t)) + h(t),$$

and Eq. (38), gives

$$\begin{aligned} \overline{\boldsymbol{\nu}}(t) =& \overline{\boldsymbol{\nu}}_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-k)^{\alpha-1} R(k, \overline{\boldsymbol{\nu}}(k))dk \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-k)^{\alpha-1} h(k)dk. \end{aligned} \tag{42}$$

Using (i) of Remark 1, we get

$$|\overline{\boldsymbol{\nu}}(t) - \overline{\boldsymbol{\nu}}_0 - \frac{1}{\Gamma(\alpha)}\int_0^t (t-k)^{\alpha-1} R(k, \overline{\boldsymbol{\nu}}(k))dk| \leqslant \frac{1}{\Gamma(\alpha)}\int_0^t (t-k)^{\alpha-1} |h(k)|dk \leqslant \Omega\varepsilon, \tag{43}$$

where, $\Omega = \frac{b^\alpha}{\Gamma(\alpha+1)}$, $t \in [a, b]$.

Theorem 5. *Suppose that* $R : J \times \mathbf{R^4} \to \mathbf{R}$ *is continuous for every* $\boldsymbol{\nu}(t) \in \boldsymbol{\mathcal{B}}$ *then the system (37) which is the equivalence of system (1) is Ulam-Hyers stable.*

Proof. Suppose that $\overline{\boldsymbol{\nu}}(t) \in \boldsymbol{\mathcal{B}}$ holds (39) and $\overline{\boldsymbol{\nu}}(t)$ is a unique solution of system (37). Therefore, $\forall\varepsilon > 0$, $t \in J$ and Lemma 2, it is obtained.

$$\begin{aligned} |\overline{\boldsymbol{\nu}}(t) - \boldsymbol{\nu}(t)| &\leq \max_{t\in J} |\overline{\boldsymbol{\nu}}(t) - \boldsymbol{\nu}_0 - \frac{1}{\Gamma(\alpha)}\int_0^t (t-k)^{\alpha-1} R(k, \boldsymbol{\nu}(k))dk| \\ &\leq \max_{t\in J} |\overline{\boldsymbol{\nu}}(t) - \overline{\boldsymbol{\nu}}_0 - \frac{1}{\Gamma(\alpha)}\int_0^t (t-k)^{\alpha-1} R(k, \overline{\boldsymbol{\nu}}(k))dk| \end{aligned}$$

$$
\begin{aligned}
&+\max_{t\in J}\frac{1}{\Gamma(\alpha)}\int_0^t (t-k)^{\alpha-1}|R(k,\overline{\boldsymbol{\nu}}(k)-R(k,\boldsymbol{\nu}(k))|dk \\
&\le |\boldsymbol{\nu}(t)-\overline{\boldsymbol{\nu}}_0-\frac{1}{\Gamma(\alpha)}\int_0^t (t-k)^{\alpha-1}R(k,\overline{\boldsymbol{\nu}}(k)dk| \\
&+\left[\frac{L_R}{\Gamma(\alpha)}\right]\int_0^t (t-k)^{\alpha-1}|\overline{\boldsymbol{\nu}}(k)-\boldsymbol{\nu}(k)|dk \\
&\le \Omega\varepsilon+\Omega L_R|\overline{\boldsymbol{\nu}}(k)-\boldsymbol{\nu}(k)|.
\end{aligned}
$$

So,

$$||\overline{\boldsymbol{\nu}}(t)-\boldsymbol{\nu}(t)||\le X_R\varepsilon.$$

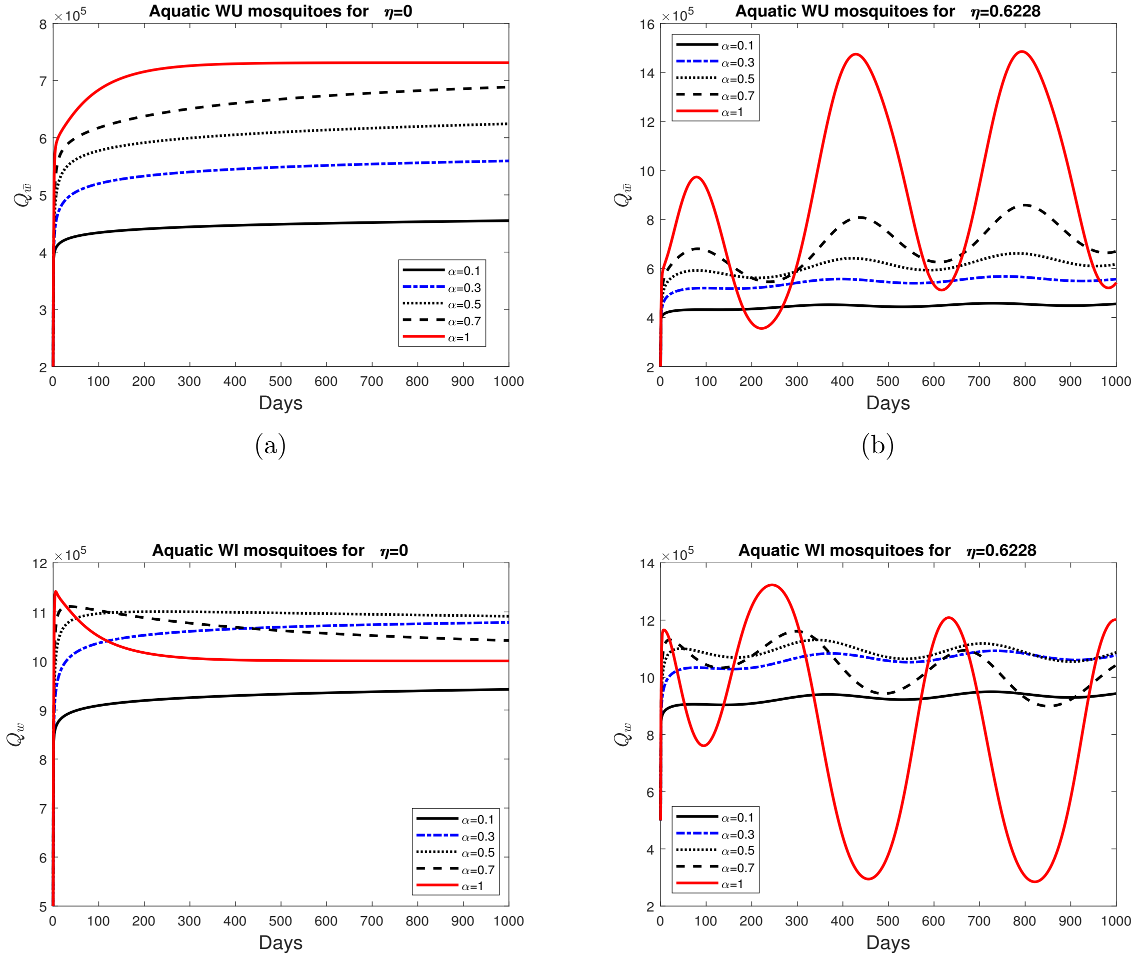
From Definition 3, the problem (1) has the Ulam-Hyers stability on $t\in J$.

7. Seasonal Effect

The environmental conditions, including temperature, humidity, and rainfall, significantly affect the mortality of adult WI mosquitoes [34], leading to a sinusoidally forced death rate.

$$
\begin{aligned}
{}^{\mathcal{C}}\boldsymbol{\mathcal{D}}^{\boldsymbol{\alpha}}Q_{\bar{w}} &= \left[\frac{\phi_{\bar{w}}F_{\bar{w}}^2+\rho_1\phi_w F_w^2+\rho_2\phi_w F_w F_{\bar{w}}}{F_{\bar{w}}+F_w}\right]\left(1-\frac{Q}{K}\right)-(\mu_a+\psi)Q_{\bar{w}}, && (44)\\
{}^{\mathcal{C}}\boldsymbol{\mathcal{D}}^{\boldsymbol{\alpha}}Q_{w} &= \left[\frac{(1-\rho_1)\phi_w F_w^2+(1-\rho_2)\phi_w F_w F_{\bar{w}}}{F_{\bar{w}}+F_w}\right]\left(1-\frac{Q}{K}\right)-(\mu_a+\psi)Q_{w}, && (45)\\
{}^{\mathcal{C}}\boldsymbol{\mathcal{D}}^{\boldsymbol{\alpha}}F_{\bar{w}} &= \frac{\psi}{2}Q_{\bar{w}}+\sigma F_w-\left[\mu_{\bar{w}}\left(1-\eta\cos\left(\frac{2\pi(t+\omega)}{365}\right)\right)\right]F_{\bar{w}}, && (46)\\
{}^{\mathcal{C}}\boldsymbol{\mathcal{D}}^{\boldsymbol{\alpha}}F_{w} &= \frac{\psi}{2}Q_{w}-\sigma F_w-\left[\mu_{w}\left(1-\eta\cos\left(\frac{2\pi(t+\omega)}{365}\right)\right)\right]F_{w}, && (47)
\end{aligned}
$$

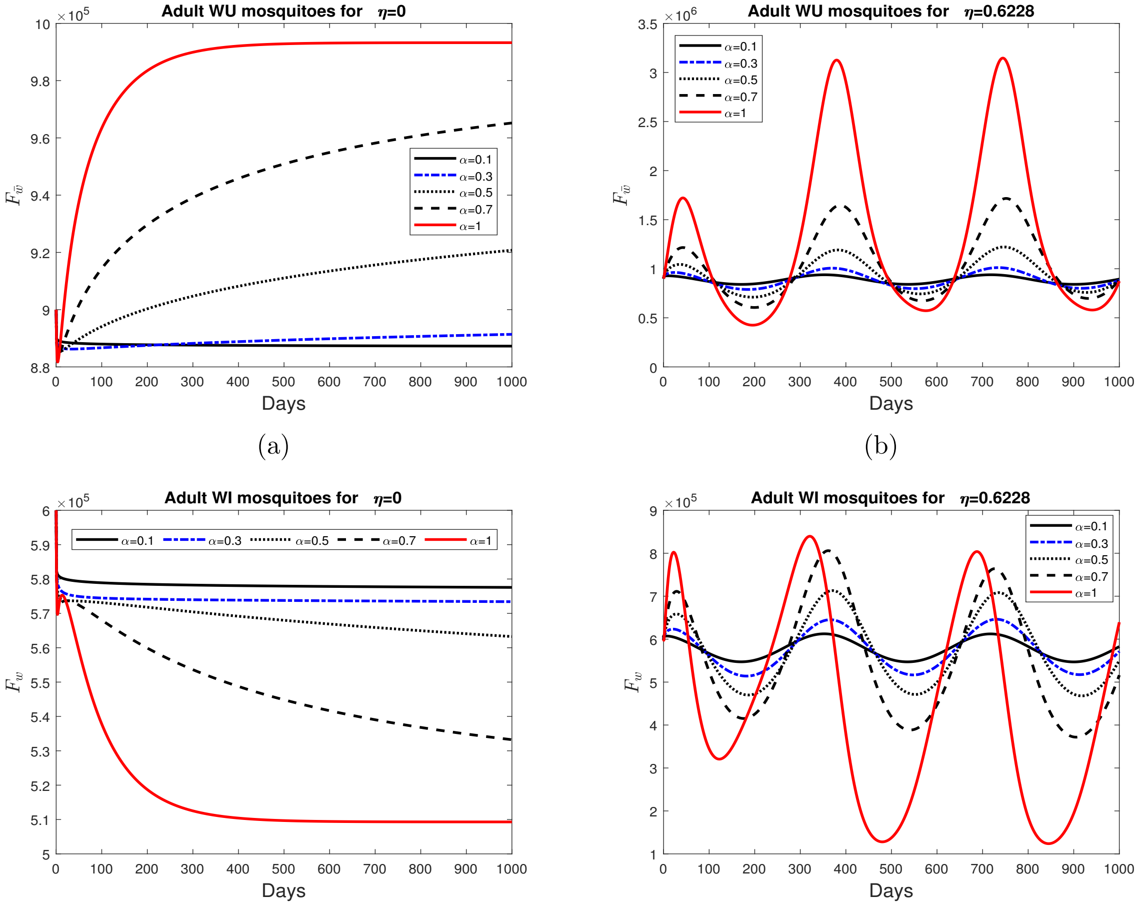
Here, η represents the level of seasonal influence on the adult mortality rate, μ_w and $\mu_{\bar{w}}$ denote the average mortality rates of adult WI and WU mosquitoes, respectively; t denotes time, and ω indicates phase shift, which adjusts the seasonal phase in the cosine function. Because the mosquito population is explicitly modeled, it is not necessary to introduce external seasonal forcing terms. Since the mosquito population size is highly sensitive to the death rate, this parameter was selected for seasonal forcing. As a result, the adult mosquito population fluctuates seasonally as needed. This leads to seasonal changes in the aquatic population which are appropriate since the mating function depends on population size. Seasonal effects are shown in Figures 3 and 4.



(a) (b)

(c) (d)

Figure 3: Comparison between WI and WU mosquitoes with and without seasonal effects at different values of α.



(a) (b) (c) (d)

Figure 4: Comparison between adult WI and WU mosquitoes with and without seasonal effect at different values of α.

Figures 3 and 4 show the time based dynamics of aquatic-stage and adult-stage mosquitoes, both WI and WU, influenced by seasonally varying parameters. In this context, η denotes the impact of seasonal forcing applied on the adult mortality rate μ_w, whereas α indicates the fractional-order derivative integrating memory effects inside the system. As η varies, seasonal variations in mortality cause periodic changes in mosquito population levels, simulating authentic environmental dynamics. Furthermore, when α decreases, the oscillation amplitude is reduced, signifying a slower system reaction and enhanced memory retention. This combined influence of seasonal forcing and fractional-order dynamics suggests that the model (44)–(47) effectively captures the recurrent and long-term behaviour of mosquito populations under realistic ecological conditions.

8. Conclusions

This study developed and analyzed fractional-order mathematical models to investigate the introduction of Wolbachia into the Aedes aegypti mosquito population, including imperfect maternal transmission and infection loss. The positivity and boundedness of the solutions were demonstrated, validating the biological feasibility of the system. The invasive reproduction number $R_{0w|\bar{w}}$ was derived to determine the conditions for Wolbachia persistence. If $R_{0w|\bar{w}} < 1$, the Wolbachia infection is not likely to spread and the WI population will eventually die out after introduction into the wild mosquito population. If $R_{0w|\bar{w}} > 1$, then the WI mosquito population persists after being introduced into the wild population. The equilibrium points were derived, and global stability was analyzed using Volterra-type Lyapunov functions. In addition, the proposed model was shown to be Ulam-Hyers stable, indicating that the approximate solution remains valid under small perturbations. The numerical simulations, conducted using the Adams-Bashforth-Moulton approach, further aid in understanding the dynamics across different fractional orders. The results indicate that as the fractional order α decreases, the WI mosquito population increases, while the WU mosquito population decreases. Additionally, the seasonal term was included to study the seasonal effects on mosquito populations and observed fluctuations in the curves for different orders of α. The study suggests that $\alpha = 0.5$ yields optimal outcomes for increasing the WI mosquito population, and the conditions used for $\alpha = 0.5$ can be applied in real-life experiments to increase the population of WI mosquitoes. This study offers important insights that may assist in the examination of other mosquito-borne diseases, including Yellow fever, malaria, Zika virus, and West Nile virus.

References

[1] R. V. Gibbons. Dengue: an escalating problem. *BMJ*, 324(7353):1563–1566, June 2002.

[2] M. B. Khan et al. Dengue overview: An updated systematic review. *Journal of Infection and Public Health*, 16(10):1625–1642, August 2023.

[3] L. Alphey et al. Sterile-insect methods for control of mosquito-borne diseases: An analysis. *Vector-Borne and Zoonotic Diseases*, 10(3):295–311, September 2009.

[4] J. Bouyer and T. Lefrançois. Boosting the sterile insect technique to control mosquitoes. *Trends in Parasitology*, 30(6):271–273, April 2014.

[5] World Health Organization (WHO). *Global strategy for dengue prevention and control 2012-2020.* Geneva, 2012.

[6] E.-E. Ooi, K.-T. Goh, and D. J. Gubler. Dengue prevention and 35 years of vector control in singapore. *Emerging Infectious Diseases*, 12(6):887–893, June 2006.

[7] A. A. Hoffmann et al. Successful establishment of wolbachia in aedes populations to suppress dengue transmission. *Nature*, 476(7361):454–457, August 2011.

[8] H. L. C. Dutra et al. Wolbachia blocks currently circulating zika virus isolates in brazilian aedes aegypti mosquitoes. *Cell Host and Microbe*, 19(6):771–774, May 2016.

[9] L. A. Moreira et al. A wolbachia symbiont in aedes aegypti limits infection with dengue, chikungunya, and plasmodium. *Cell*, 139(7):1268–1278, December 2009.

[10] A. P. Turley, L. A. Moreira, S. L. O'Neill, and E. A. McGraw. Wolbachia infection reduces blood-feeding success in the dengue fever mosquito, aedes aegypti. *PLoS Neglected*

Tropical Diseases, 3(9):e516, September 2009.

[11] T. H. Ant, C. S. Herd, V. Geoghegan, A. A. Hoffmann, and S. P. Sinkins. The wolbachia strain wau provides highly efficient virus transmission blocking in aedes aegypti. *PLoS Pathogens*, 14(1):e1006815, January 2018.

[12] T. Walker et al. The wmel wolbachia strain blocks dengue and invades caged aedes aegypti populations. *Nature*, 476(7361):450–453, August 2011.

[13] P. A. Ross, S. A. Ritchie, J. K. Axford, and A. A. Hoffmann. Loss of cytoplasmic incompatibility in wolbachia-infected aedes aegypti under field conditions. *PLoS Neglected Tropical Diseases*, 13(4):e0007357, April 2019.

[14] M. Turelli. Cytoplasmic incompatibility in populations with overlapping generations. *Evolution*, 64(1):232–241, August 2009.

[15] D. E. Campo-Duarte, O. Vasilieva, D. Cardona-Salgado, and M. Svinin. Optimal control approach for establishing wmelpop wolbachia infection among wild aedes aegypti populations. *Journal of Mathematical Biology*, 76(7):1907–1950, February 2018.

[16] P. Saha, G. C. Sikdar, J. K. Ghosh, and U. Ghosh. Disease dynamics and optimal control strategies of a two-serotypes dengue model with co-infection. *Mathematics and Computers in Simulation*, 209:16–43, February 2023.

[17] M. Rafikov, M. E. M. Meza, D. P. F. Correa, and A. P. Wyse. Controlling aedes aegypti populations by limited wolbachia-based strategies in a seasonal environment. *Mathematical Methods in the Applied Sciences*, 42(17):5736–5745, March 2019.

[18] L. Xue, C. A. Manore, P. Thongsripong, and J. M. Hyman. Two-sex mosquito model for the persistence of wolbachia. *Journal of Biological Dynamics*, 11(sup1):216–237, September 2016.

[19] S. Javeed, D. Baleanu, A. Waheed, M. S. Khan, and H. Affan. Analysis of homotopy perturbation method for solving fractional order differential equations. *Mathematics*, 7(1):40, January 2019.

[20] J. Dianavinnarasi, R. Raja, J. Alzabut, J. Cao, M. Niezabitowski, and O. Bagdasar. Application of caputo–fabrizio operator to suppress the aedes aegypti mosquitoes via wolbachia: An lmi approach. *Mathematics and Computers in Simulation*, 201:462–485, February 2021.

[21] A. I. Adekunle, M. T. Meehan, and E. S. McBryde. Mathematical analysis of a wolbachia invasive model with imperfect maternal transmission and loss of wolbachia infection. *Infectious Disease Modelling*, 4:265–285, January 2019.

[22] K. S. Miller and B. Ross. *An introduction to the fractional calculus and fractional differential equations.* Wiley, 1993.

[23] I. Podlubny. *Fractional differential equations.* Academic Press, San Diego, CA, USA, 1999.

[24] A. A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and applications of fractional differential equations.* Elsevier Science Limited, 2006.

[25] M. A. Taneco-Hernández and C. Vargas-De-León. Stability and lyapunov functions for systems with atangana–baleanu caputo derivative: An hiv/aids epidemic model. *Chaos Solitons and Fractals*, 132:109586, January 2020.

[26] H. Khan, N. Kamran, S. Maqsood, D. K. Almutairi, and J. Alzabut. A logistic growth epidemiological seir model with computational and qualitative results. *European Journal of Pure and Applied Mathematics*, 18(2):5944, May 2025.

[27] A. Boukhouima, K. Hattaf, E. M. Lotfi, M. Mahrouf, D. F. M. Torres, and N. Yousfi. Lyapunov functions for fractional-order systems in biology: Methods and applications.

Chaos Solitons and Fractals, 140:110224, August 2020.

[28] A. S. Rashed, M. M. Mahdy, S. M. Mabrouk, and R. Saleh. Fractional order mathematical model for predicting and controlling dengue fever spread based on awareness dynamics. *Computation*, 13(5):122, May 2025.

[29] H. Wan and J. Xu. How does the enhanced mortality of wolbachia-infected immature mosquitoes affect dengue transmission? *International Journal of Biomathematics*, April 2024.

[30] J. Dianavinnarasi, R. Raja, J. Alzabut, M. Niezabitowski, and O. Bagdasar. Controlling wolbachia transmission and invasion dynamics among aedes aegypti population via impulsive control strategy. *Symmetry*, 13(3):434, March 2021.

[31] G. M. Vijayalakshmi and M. Ariyanatchi. Fractional order modelling of wolbachia-carrying mosquito population dynamics for dengue control. *Modeling Earth Systems and Environment*, 11(4), May 2025.

[32] D. Joseph, R. Ramachandran, J. Alzabut, S. A. Jose, and H. Khan. A fractional-order density-dependent mathematical model to find the better strain of wolbachia. *Symmetry*, 15(4):845, April 2023.

[33] U. Ufuktepe. Discrete wolbachia diffusion in mosquito populations with allee effects. *European Journal of Pure and Applied Mathematics*, 15(4):1613–1622, October 2022.

[34] H. M. Yang, M. L. G. Macoris, K. C. Galvani, M. T. M. Andrighetti, and D. M. V. Wanderley. Assessing the effects of temperature on the population of aedes aegypti, the vector of dengue. *Epidemiology and Infection*, 137(8):1188–1202, February 2009.

[35] B. Zheng, M. Tang, J. Yu, and J. Qiu. Wolbachia spreading dynamics in mosquitoes with imperfect maternal transmission. *Journal of Mathematical Biology*, 76(1–2):235–263, June 2017.

[36] J. Arrivillaga and R. Barrera. Food as a limiting factor for aedes aegypti in water-storage containers. *PubMed*, 29(1):11–20, June 2004.

[37] A. Boukhouima, H. Zine, E. M. Lotfi, M. Mahrouf, D. F. M. Torres, and N. Yousfi. Lyapunov functions and stability analysis of fractional-order systems. In *Elsevier eBooks*, pages 125–136. 2022.

[38] S. T. Ogunlade, A. I. Adekunle, M. T. Meehan, D. P. Rojas, and E. S. McBryde. Modeling the potential of wau-wolbachia strain invasion in mosquitoes to control aedes-borne arboviral infections. *Scientific Reports*, 10(1), October 2020.

[39] A. A. Hoffmann et al. Stability of the wmel wolbachia infection following invasion into aedes aegypti populations. *PLoS Neglected Tropical Diseases*, 8(9), September 2014.

[40] C. J. McMeniman and S. L. O'Neill. A virulent wolbachia infection decreases the viability of the dengue vector aedes aegypti during periods of embryonic quiescence. *PLoS Neglected Tropical Diseases*, 4(7):e748, July 2010.

[41] L. P. Lounibos and R. L. Escher. Sex ratios of mosquitoes from long-term censuses of florida tree holes. *Journal of the American Mosquito Control Association*, 24(1):11–15, March 2008.

[42] C. J. McMeniman et al. Stable introduction of a life-shortening wolbachia infection into the mosquito aedes aegypti. *Science*, 323(5910):141–144, January 2009.

[43] A. Boukhouima, K. Hattaf, and N. Yousfi. Dynamics of a fractional order hiv infection model with specific functional response and cure rate. *International Journal of Differential Equations*, 2017:1–8, January 2017.

[44] H.-L. Li, L. Zhang, C. Hu, Y.-L. Jiang, and Z. Teng. Dynamical analysis of a fractional-order predator-prey model incorporating a prey refuge. *Journal of Applied Mathematics*

and Computing, 54(1–2):435–449, May 2016.

[45] C. Vargas-De-León. Volterra-type lyapunov functions for fractional-order epidemic systems. *Communications in Nonlinear Science and Numerical Simulation*, 24(1–3):75–85, January 2015.

[46] J. Dushoff, W. Huang, and C. Castillo-Chavez. Backwards bifurcations and catastrophe in simple models of fatal diseases. *Journal of Mathematical Biology*, 36(3):227–248, February 1998.

[47] S. M. Ulam. *A Collection of Mathematical Problems.* Interscience Publishers, New York, 1960.

[48] D. H. Hyers. On the stability of the linear functional equation. *Proceedings of the National Academy of Sciences*, 27(4):222–224, April 1941.