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Finite Elements Method for Solving Elliptic and Hyperbolic Partial Differential Equations

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Abstract. In various disciplines of engineering such as hydrodynamics, heat conduction, geomechanics, civil engineering, nuclear, and even biomedical, the finite element method (FEM) is known to be an effective and sophisticated technique to tackle complex computational problems. The main idea of FEM is to break down a complicated space or domain into a number of small, countable, and finite elements, or "finite elements" by approximation methods whose behavioral outputs could be predicted using simpler equations. In this work, we will analyze the comparison between the Finite Difference Method (FDM) and FEM in order to see the effectiveness of each method. Within the scope of this research, several types of partial differential equations (PDEs) will be solved with FEM to estimate a solution that is as accurate as possible to the exact answer. Computations will be completed with the MATLAB software, and a discussion will follow.

2020 Mathematics Subject Classifications: 76S05, 76D10, 80A32, 35Q79

Key Words and Phrases: Boundary value problem, Finite difference method, Galerkin Finite element method, Poisson's equation, Wave equation

1. Introduction

Physicists and engineers apply Poisson's equation to solve problems in areas including electrostatics, magnetostatics, quantum mechanics, and fluid dynamics.Poisson's equation explains the relationship between the electric potential and the charge distribution in electrostatics. It links the current distribution and the magnetic potential in magnetostatics. This equation is vital for scientists and engineers since it aids in comprehending how various physical systems behave [1–4].

The wave equation describes the propagation of sound, light, water waves, and other waves through various media. That makes the equation useful in applied mathematics, physics, and engineering. This second-order PDE, also known as the wave equation, is crucial for analyzing the dynamic nature of physical phenomena. The wave equation is vital when studying dynamic systems[5, 6].

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Real-world problems come with differential equations that are complicated and often have no explicit solution formulas. When they do exist, they involve integrals that have to be calculated through numerical quadrature methods. Numerical analysis has become extremely important in solving differential equations—under specific conditions of initial or boundary values—making this approach effective for overcoming numerous challenges. Partial differential equations (PDEs) can be solved numerically using the Finite Volume Method (FVM), FDM, and FEM. FDM creates approximations of solutions to differential equations via local Taylor series expansion. When dealing with increasingly complex geometries in higher dimensions, the topologically square network of lines may present certain challenges in the discretization of PDEs in FDM. These restrictions made integral-based formulations more desirable, which allowed for the creation of the two less sophisticated FVM and FDM techniques [1, 5, 7].

The primary application of FEM is in the solution of differential equations with boundary conditions. The solution is calculated by partitioning the domain into finite number of regions, or elements, and applying basis functions to them. Hydrodynamic and structural analysis, as well as multidisciplinary problems are some of the many fields where FEM is implemented today. The reason for this tremendous use is due to the method's precision in modeling and numerically solving intricate problems. By enabling the simulation of natural phenomena and the analysis of engineering designs, both scientific and engineering evaluation are possible. This permits the prediction of system performance under design conditions and natural phenomena, which aids in the development of safer, cost-effective systems. The risk of disasters can also be reduced with the aid of FEM. The quality of life is highly influenced by the technology we have today; this is gradually advanced by a growing understanding of nature through FEM's versatile nature [8] .

Galerkin finite element method (GFEM) is formulated as a modification of the basic process of FEM, construction by Galerkin's method. It is simpler than other methods because GFEM uses a systematic approach with piecewise trial functions for the approximation of partial differential equations over finite elements [6, 9, 10].

This study is an extension of the work done by S. S. Almutairi, A. M. Saeed [11] to further investigate and scrutinize alternative approximate methods for solving various types of boundary value problems.

This study explores the applications of GFEM and compares it with FDM to determine which approach is more effective. The paper is structured into five sections. Section 2 provides a brief overview of the FDM. Section 3 presents the GFEM formulations for solving PDEs of both elliptic and hyperbolic types. Section 4 discusses numerical experiments and compares the two methods. Finally, the paper concludes with a summary of findings and final remarks.

2. Brief Description of Finite difference Method

FDM is a numerical method that approximates the solution of differential equations by means of discretization. Breaking continuous functions into discrete values simplifies

calculations and enhances various types of quantitative analysis. This transformation makes it easier to work with data, optimize processes, and apply computational methods effectively. It is quite popular in various fields of science and technologie that involve differential equations, like physics, engineering, or even finance. There are three different kinds of finite difference approximations: forward distinctions, backward distinctions, and center distinctions. By evaluating the function at two future or present values, the forward difference approach calculates the derivative. Both the values at the current location and the previous one are of interest to backward differences. Center is more accurate than the others, as it takes the average of forward and backward difference, making it a more accurate form of approximation. The Forward Difference Method is one of the numerical techniques developed over many decades [12, 13]

In two dimensions, the Poisson's equation can be expressed as [14, 15].

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(x, y), \tag{1}$$

with the boundary constraint $u = f(x, y)$ along the boundary C.
In this case, we additionally assumed that the mesh points are consistent across the x and y dimensions. This assumption makes it possible to reduce the (1) central difference approximation to

$$u_{i,j} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - h^2 g_{i,j}),$$

where $g_{i,j} = g(x_i, y_i)$.

Let $i, j = 0, 1, 2, 3, 4$ and $u = 0$ along the boundary C. For $j = 0, 1, 2, 3, 4$, then $u_{i,0} = 0, u_{i,4} = 0$, and $u_{0,j} = 0, u_{4,j} = 0$. When $i = 0, 1, 2, 3, 4$. In Figure 1, the boundary values (filled circles) are displayed.

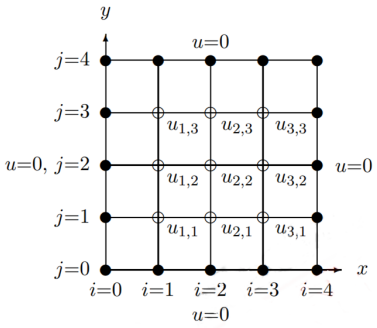

Figure 1: FDM mesh points.

An example of this would be $i, j = 0, 1, 2, 3$. As a result, equation (1) transforms into a system of nine equations with nine unknowns. These equations can be represented in matrix form for a more structured and efficient notation.

$$\begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} u_{1,1} \\ u_{1,2} \\ u_{1,3} \\ u_{2,1} \\ u_{2,2} \\ u_{2,3} \\ u_{3,1} \\ u_{3,2} \\ u_{3,3} \end{pmatrix} = \begin{pmatrix} -h^2 g_{1,1} \\ -h^2 g_{1,2} \\ -h^2 g_{1,3} \\ -h^2 g_{2,1} \\ -h^2 g_{2,2} \\ -h^2 g_{2,3} \\ -h^2 g_{3,1} \\ -h^2 g_{3,2} \\ -h^2 g_{3,3} \end{pmatrix}$$

Consequently, a system of N equations was created by (1). Here, n represents the number of subintervals along the x and y axes. The coefficient matrix is both positive definite and symmetric, with many elements being zero, making it sparse. Due to this sparsity, the system of equations is best solved using an iterative approach rather than a direct method.

We consider a hyperbolic PDE, the numerical solution to the wave equation. The differential equation provides the wave equation [16] :

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < l, \quad t > 0, \tag{2}$$

based on the terms

$$\begin{aligned} &u(0,t) = u(l,t) = 0, \quad \text{for} \quad t > 0, \\ &u(x,0) = f(x), \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x), \quad \text{for} \quad 0 \le x \le l, \end{aligned}$$

where α is a constant that depends on the problem's physical circumstances.
To define the x-axis grid points using $h = l/m$, choose an integer $m > 0$. Choose a time-step size $k > 0$ as well. The definition of the mesh points (x_i, t_j) is

$$x_i = ih, \quad \text{and} \quad t_j = jk,$$

for each $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots$.
The wave equation changes to (x_i, t_j) at any inner mesh point.

$$\frac{\partial^2 u}{\partial t^2}(x_i, t_j) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x_i, t_j) = 0. \tag{3}$$

The difference technique is generated using the centered-difference quotient for the second partial derivatives given by

$$\frac{\partial^2 u}{\partial t^2}(x_i, t_j) = \frac{u(x_i, t_{j+1}) - 2u(x_i, t_j) + u(x_i, t_{j-1})}{k^2} - \frac{k^2}{12} \frac{\partial^4 u}{\partial t^4}(x_i, \mu_j),$$

where $\mu_j \in (t_{j-1}, t_{j+1})$, and

$$\frac{\partial^2 u}{\partial x^2}(x_i,t_j)=\frac{u(x_{i+1},t_j)-2u(x_i,t_j)+u(x_{i-1},t_j)}{h^2}-\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i,t_j),$$

where $\xi_i\in(x_{i-1},x_{i+1})$. Substituting these into (3) gives

$$\frac{u(x_i,t_{j+1})-2u(x_i,t_j)+u(x_i,t_{j-1})}{k^2}-\alpha^2\frac{u(x_{i+1},t_j)-2u(x_i,t_j)+u(x_{i-1},t_j)}{h^2}$$
$$=\frac{1}{12}\left[k^2\frac{\partial^4 u}{\partial t^4}(x_i,\mu_j)-\alpha^2h^2\frac{\partial^4 u}{\partial x^4}(\xi_i,t_j)\right].$$

Ignoring the incorrect phrase

$$\tau_{i,j}=\frac{1}{12}\left[k^2\frac{\partial^4 u}{\partial t^4}(x_i,\mu_j)-\alpha^2h^2\frac{\partial^4 u}{\partial x^4}(\xi_i,t_j)\right], \tag{4}$$

causes the distinction, formula

$$\frac{w_{i,j+1}-2w_{i,j}+w_{i,j-1}}{k^2}-\alpha^2\frac{w_{i+1,j}-2w_{i,j}+w_{i-1,j}}{h^2}=0.$$

Define $\lambda=\alpha k/h$. The difference equation can therefore be written as

$$w_{i,j+1}-2w_{i,j}+w_{ij-1}-\lambda^2w_{i+1,j}+2\lambda^2w_{i,j}-\lambda^2w_{i-1,j}=0.$$

Then calculate the most advanced time-step estimate, w_{ij+1}, to obtain

$$w_{i,j+1}=2\left(1-\lambda^2\right)w_{i,j}+\lambda^2\left(w_{i+1,j}+w_{i-1,j}\right)-w_{i,j-1}. \tag{5}$$

For any $i=1,2,\ldots,m-1$ and $j=1,2,\ldots$, this equation is true. The boundary conditions give

$$w_{0,j}=w_{m,j}=0,\quad \text{for each } j=1,2,3,\ldots, \tag{6}$$

and the initial condition implies that

$$w_{i,0}=f(x_i),\quad \text{for each } i=1,2,\ldots,m-1.$$

This set of equations can be written in matrix form to give

$$\begin{pmatrix} w_{1,j+1}\\ w_{2,j+1}\\ \vdots\\ w_{m-1,j+1}\end{pmatrix}=\begin{pmatrix} 2(1-\lambda^2) & \lambda^2 & 0\cdots & 0\\ \lambda^2 & 2(1-\lambda^2) & \lambda^2\cdots & 0\\ 0 & \vdots & \cdots & \cdots\\ \vdots & \vdots & \ddots & \vdots\\ 0 & \vdots & \lambda^2 & 2(1-\lambda^2)\end{pmatrix}=\begin{pmatrix} w_{1,j}\\ w_{2,j}\\ \vdots\\ w_{m-1,j}\end{pmatrix}-\begin{pmatrix} w_{1,j-1}\\ w_{2,j-1}\\ \vdots\\ w_{m-1,j-1}\end{pmatrix} \tag{7}$$

According to equations (4) and (7), values from the j th and $(j-1)$ st time steps are needed for the $(j+1)$ st time step. Figure 2 is referred to. Because (6) gives values for $j=0$, but the starting-velocity condition has to give values for $j=1$, which are needed in (4) to compute $w_{i,2}$, this leads to a minor initial problem.

$$\frac{\partial u}{\partial t}(x,0)=g(x), \quad 0\leq x\leq l.$$

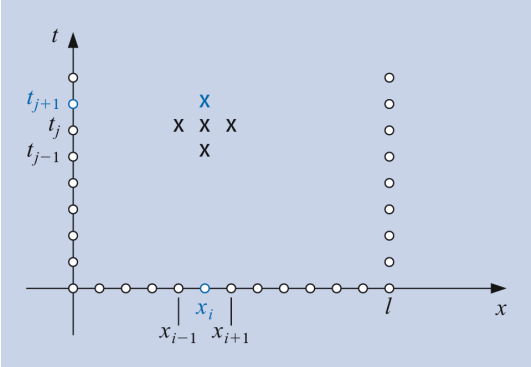

Figure 2: Discretization of mesh points

One method is to use a forward-difference approximation to replace $\partial u/\partial t$.

$$\frac{\partial u}{\partial t}\left(x_i,0\right)=\frac{u\left(x_i,t_1\right)-u\left(x_i,0\right)}{k}-\frac{k}{2}\frac{\partial^2 u}{\partial t^2}\left(x_i,\tilde{\mu}_i\right),$$

for some $\tilde{\mu}_i$ in $(0,t_1)$.The equation's $u\left(x_i,t_1\right)$ can be solved to yield

$$\begin{aligned}u\left(x_i,t_1\right)&=u\left(x_i,0\right)+k\frac{\partial u}{\partial t}\left(x_i,0\right)+\frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}\left(x_i,\tilde{\mu}_i\right)\\&=u\left(x_i,0\right)+kg\left(x_i\right)+\frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}\left(x_i,\tilde{\mu}_i\right).\end{aligned}$$

When the truncation phrase is removed, the approximate

$$w_{i,1}=w_{i,0}+kg\left(x_i\right), \quad \text{for each } i=1,\ldots,m-1.$$

The truncation error in our approximation is only $O(k)$, whereas the truncation error in (5) is $O\left(k^2\right)$. A better approximation to $u\left(x_i,0\right)$ is obtained by expanding in t, $u\left(x_i,t_1\right)$ is a second Maclaurin polynomial. Then,

$$u\left(x_i,t_1\right)=u\left(x_i,0\right)+k\frac{\partial u}{\partial t}\left(x_i,0\right)+\frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}\left(x_i,0\right)+\frac{k^3}{6}\frac{\partial^3 u}{\partial t^3}\left(x_i,\hat{\mu}_i\right),$$

for some $\hat{\mu}_i$ in $(0,t_1)$. If f'' exists, then

$$\frac{\partial^2 u}{\partial t^2}\left(x_i,0\right)=\alpha^2\frac{\partial^2 u}{\partial x^2}\left(x_i,0\right)=\alpha^2\frac{d^2 f}{dx^2}\left(x_i\right)=\alpha^2 f''\left(x_i\right),$$

and

$$u(x_i, t_1) = u(x_i, 0) + kg(x_i) + \frac{\alpha^2 k^2}{2} f''(x_i) + \frac{k^3}{6} \frac{\partial^3 u}{\partial t^3}(x_i, \hat{\mu}_i).$$

An estimate with error $O\left(k^3\right)$ is therefore produced:

$$w_{i1} = w_{i0} + kg(x_i) + \frac{\alpha^2 k^2}{2} f''(x_i).$$

The difference equation can be used if $f \in C^4[0,1]$ but $f''(x_i)$ is not readily available.

$$f''(x_0) = \frac{1}{h^2}\left[f(x_0 - h) - 2f(x_0) + f(x_0 + h)\right] - \frac{h^2}{12} f^{(4)}(\xi), \tag{8}$$

to write

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} - \frac{h^2}{12} f^{(4)}\left(\tilde{\xi}_i\right),$$

for some $\tilde{\xi}_i$ in (x_{i-1}, x_{i+1}). This implies that

$$u(x_i, t_1) = u(x_i, 0) + kg(x_i) + \frac{k^2\alpha^2}{2h^2}\left[f(x_{i+1}) - 2f(x_i) + f(x_{i-1})\right] + O\left(k^3 + h^2k^2\right).$$

Because $\lambda = k\alpha/h$, we can write this as

$$\begin{aligned} u(x_i, t_1) &= u(x_i, 0) + kg(x_i) + \frac{\lambda^2}{2}\left[f(x_{i+1}) - 2f(x_i) + f(x_{i-1})\right] + O\left(k^3 + h^2k^2\right) \\ &= \left(1 - \lambda^2\right) f(x_i) + \frac{\lambda^2}{2} f(x_{i+1}) + \frac{\lambda^2}{2} f(x_{i-1}) + kg(x_i) + O\left(k^3 + h^2k^2\right). \end{aligned}$$

Consequently, the difference equation,

$$w_{i,1} = \left(1 - \lambda^2\right) f(x_i) + \frac{\lambda^2}{2} f(x_{i+1}) + \frac{\lambda^2}{2} f(x_{i-1}) + kg(x_i).$$

For all $i = 1, 2, \ldots, m-1$, $w_{i,1}$ may be found using this method. We apply the system in (7) to determine subsequent approximates.

3. Formulation of the Finite element Method

GFEM solves differential equations numerically by dividing the domain into smaller finite elements. Within each of these elements, the function is approximated using piecewise trial functions, allowing for an efficient and flexible solution approach [17, 18]. The following steps make up the GFEM for solving a differential equation [17, 18]:

(i) The differential equation is multiplied by a weight function $\omega(x)$ they make up the integral for the entire domain;

(ii) Integrate by parts if required to lower the highest order term's order;

(iii) Choose the interpolation order—whether linear, quadratic, or higher—and define the corresponding shape functions, N_i, for $i = 1...m$. Using the trial function $p = \widetilde{p}(x) = \sum_{i=1}^{m} N_i(x)p_i$, the function is approximated based on these selected shape functions;

(iv) Calculate all integrals within each element, either through exact analytical methods or numerical approximation, to establish a system of equations for the unknown values $p_{i's}$;

(v) Solve the system of equations to determine the unknown values $p_{i's}$, ensuring accuracy in the numerical solution process.

Let's examine the model problem: Poisson's equation, subject to homogeneous Dirichlet boundary conditions [19].

$$-\Delta u = f \quad in \quad \Omega, \tag{9}$$
$$u = 0 \ on \ \partial\Omega.$$

In this case, it is clear that $a(\cdot,\cdot)$ is both symmetric and bilinear, satisfying the property $a(u,u) = |u|^2_{1,\Omega} := ||u||^2$. Furthermore, when $a(u,u) = 0$, it implies that $\nabla u = 0$, meaning that u must be a constant function throughout the domain. The value of this constant should be zero as $u|_\Gamma = 0$. Consequently, the Riesz representation theorem indicates that (9) has a unique solution since $a(\cdot,\cdot)$ forms an inner product on V. The Galerkin methods are a class of techniques that are used to estimate the solution to (9). Let $V_h \subset V$ be a finite dimensional subspace. Limit the variational formulation to the subspace V_h, meaning that the goal is to find $u_h \in V$ such that it satisfies the given conditions within this restricted space.

$$a(u,v) = (f,v), \ for \ \ all \ v \in V, \tag{10}$$

where

$$a(u,v) = \int_\Omega \nabla u \nabla v \ \ dx, \quad (f,v) = \int_\Omega fv \ \ dx \ \ for \ \ all \ \ f \in L^2(\Omega). \tag{11}$$

It is evident that in this scenario, $a(\cdot,\cdot)$ is symmetric and bilinear, and $a(u,u) = |u|^2_{1,\Omega} := ||u||^2$ Moreover, $a(u,u) = 0$ suggests that $\nabla u = 0$, and as a result, u is constant. Since $u|_\Gamma = 0$, this constant's value should be 0. Consequently, the Riesz representation theorem indicates that (9) has a unique solution since $a(\cdot,\cdot)$ forms an inner product on V. The Galerkin methods are a class of techniques that are used to estimate the solution to (9). Let $V_h \subset V$ be a subspace of finite dimensions. Find $u_h \in V$ s.t. by restricting the variational form in the subspace V_h.

$$a(u_h, v_h) = (f, v_h), \quad for \ \ all \ \ v_h \in V. \tag{12}$$

$V_h = span\{\varphi_1, \varphi_2, ..., \varphi_N\}$ allows us to Every function $v \in V_h$ has a unique representation: To calculate $v = \sum_{i=1}^{N} v_i\varphi_i$ As a result, one can define an isomorphism. $V_h \cong \mathbb{R}^N$ by

$$v = \sum_{i=1}^{N} v_i\varphi_i \longleftrightarrow v = (v_1, ..., v_N)^T.$$

We then refer to v as the coordinate vector of v with regard to the basis $\{\varphi\}_{i=1}^{N}$. We present the stiffness matrix after introducing the elasticity language.

$$A = (a_{ij})_{N\ X\ N} \quad with \quad a_{ij} = a(\varphi_i, \varphi_j),$$

and the load vector $f = \{\langle f, \varphi\rangle\}_{k=1}^{N} \in \mathbb{R}^N$. The following linear algebraic system can therefore be used to formulate the variational (12) on V_h.

$$Au = f.$$

The $a(\cdot,\cdot)$ -inner product of two functions, $vu \in V_h$ is realized via the matrix product by definition.

$$a(u_h, v_h) = a(\sum_i u_i\varphi_i, \sum v_j\varphi_j) = \sum_{i,j} a(\varphi_i, \varphi_j)u_iv_j = v^tAu.$$

Therefore, for any vector $u \in \mathbb{R}^N$, $u^TAu = a(u,u) \geq 0$ and equals 0 if and only if u is zero. Namely, A is an SPD matrix, and thus the solution $u = A^{-1}f$ exists and is unique. Once we get the coefficient vector u, we can use a linear combination of basis functions to generate u_h. A well-known and frequently applied example of Galerkin methods is the FEM, which builds a finite-dimensional subspace V_h from domain triangulations τ_h. The domain's decomposition into a limited number of pieces is the source of the name. Piecewise polynomials are commonly used to define a finite dimensional space.

The scalar wave equation that models the propagation of acoustic waves in a bounded domain Ω^3 with boundary Γ is examined : [20–22].

$$\begin{aligned} \frac{1}{c(\mathrm{x})^2}\frac{\partial^2 u}{\partial t^2} - \triangle u = 0, \quad & \text{in } \Omega \times (0,T), \\ u(\cdot,0) = 0, \quad \frac{\partial u}{\partial t}(\cdot,0) = f, \quad & \text{in } \Omega, \\ \partial_n u|_{\Gamma_1} = 0, \ & \text{on } \Gamma \times (0,T). \end{aligned} \tag{13}$$

Where t is the time variable, T is a final time, and f is a load vector. $u(\mathrm{x}, t)$ is the pressure, and $c(\mathrm{x})$ is the wave speed based on $\mathrm{x} = (x_1, x_2, x_3) \in \Omega$.
We now use continuous piecewise linear functions in space and time to build a FEM for (13). The standard method for discretizing $\Omega \times (0,T)$ is to use $K_h = \{K\}$ to partition the domain Ω into tetrahedra $K(h = h(\mathrm{x})$, where $J_k = \{J\}$ is a partition of the time interval $(0,T)$ into time intervals $J = (t_{k-1}, t_k)\,.k = 1, \ldots, N$, of uniform length $\tau = t_k - t_{k-1}$.

The finite element approach for (13) is formulated by introducing the finite element space W_h^u, which is defined as follows :

$$\begin{aligned} W^u &:= \left\{u \in H^1(\Omega \times J) : u(\cdot, 0) = 0, \partial_n u|_\Gamma = 0\right\}, \\ W_h^u &:= \left\{u \in W^u : u|_{K\times J} \in P_1(K) \times P_1(J), \forall K \in K_h, \forall J \in J_k\right\}. \end{aligned} \tag{14}$$

The set of piecewise linear functions on K and J, respectively, is denoted by $P_1(K)$ and $P_1(J)$.

We use CG(1) in space and time to construct a discrete scheme, and we look for a discrete solution in the space W_h^u for u spanned by the functions.

$$u(x,t) = \sum_{l=0}^{N}\sum_{i=1}^{M} u_i^l \varphi_i(x)\psi_l(t). \tag{15}$$

In space and time, the typical continuous piecewise linear functions $\varphi_i(x)$ and $\psi_l(t)$ are represented, respectively.
The system of linear equations that results from substituting this into (13) is as follows:

$$M\left(\mathbf{u}^{k+1} - 2\mathbf{u}^k + \mathbf{u}^{k-1}\right) = \tau^2 F^k - \tau^2 A\left(\frac{1}{6}\mathbf{u}^{k-1} + \frac{2}{3}\mathbf{u}^k + \frac{1}{6}\mathbf{u}^{k+1}\right), \quad k = 1, \ldots, N-1, \tag{16}$$

with initial conditions :

$$u(0) = \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0.$$

Here, F^k is the load vector at time level t_k, where$k = 1, 2, 3 \ldots$, $mathbfu$ is the unknown discrete field values of u, M is the mass matrix in space, A is the stiffness matrix, and τ is the time step.

For each element e, the explicit formulas for the entries in system (16) can be provided as follows :

$$\begin{aligned} M_{i,j}^e &= \left(\frac{1}{c^2}\varphi_i, \varphi_j\right)_e, \\ K_{i,j}^e &= \left(\nabla\varphi_i, \nabla\varphi_j\right)_e, \\ F_j^e &= \left(f, \varphi_j\right)_e. \end{aligned}$$

Where the $L_2(e)$ scalar product is indicated by $(.,.)_e$. The contribution of element e to the global assembled matrix in space M is represented by the matrix M_e, while the contribution of element e to the global assembled matrix K is represented by the matrix K^e.

W_h^u is a vector space with finite dimensions, $\dim W_h^u = M \times N$, with a basis consisting of the standard continuous piecewise linear functions $\{\varphi_i\}_{i=1}^M$ in space and hat functions $\{\psi_l\}_{l=1}^N$ in time, where:

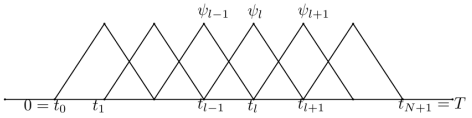

Figure 3: Discretization of $\psi(t)$

$$\psi_l(t) = \begin{cases} 0, & \text{if } t \notin [t_{l-1}, t_{l+1}], \\ \frac{t-t_{l-1}}{t_i-t_{l-1}}, & \text{if } t \in [t_{l-1}, t_l], \\ \frac{t_{l+1}-t}{t_{l+1}-t_l}, & \text{if } t \in [t_l, t_{l+1}]. \end{cases} \tag{17}$$

The discrete system of equations Using basis of functions $\{\varphi_i\}_{i=1}^{M}$ in space and $\{\psi_l\}_{l=1}^{N}$ in time we have: $u_h = \sum_{l=0}^{N}\sum_{i=1}^{M} u_{hi}^{l}\varphi_i(x)\psi_l(t)$. Substituting into (13) we get:

$$\begin{aligned} &-\sum_{l=0}^{N}\sum_{i=1}^{M} u_{h_i^l}^{l}\int_\Omega \frac{1}{c^2}\varphi_i(x)\int_{t_{l-1}}^{t_{l+1}} \frac{\partial\psi_l(t)}{\partial t}\frac{\partial v(x,t)}{\partial t}dxdt \\ &+\sum_{l=0}^{N}\sum_{i=1}^{M} u_{h_i}^{l}\int_\Omega\int_{t_{l-1}}^{t_{l+1}} \nabla\varphi_i(x)\nabla v(x,t)\psi_l(t)dxdt \\ &=\int_\Omega\int_0^T f(x)v(x,t)dxdt \quad \forall v \in W_h^u. \end{aligned}$$

We take $v(x,t) = \varphi_j(x)\psi_m(t)$ and get:

$$\begin{aligned} &-\sum_{l=0}^{N}\sum_{i=1}^{M} u_{hi}^{l}\int_\Omega \frac{1}{c^2}\varphi_i(x)\varphi_j(x)\int_{t_{l-1}}^{t_{l+1}} \frac{\partial\psi_l(t)}{\partial t}\frac{\partial\psi_m(t)}{\partial t}dxdt \\ &+\sum_{l=0}^{N}\sum_{i=1}^{M} u_{hi}^{l}\int_\Omega \nabla\varphi_i(x)\nabla\varphi_j(x)\int_{t_{l-1}}^{t_{l+1}} \psi_l(t)\psi_m(t)dxdt \\ &=\int_\Omega\int_0^T f(x)v(x,t)dxdt \quad \forall v \in W_h^u. \end{aligned} \tag{18}$$

The vector of unknown coefficients is represented by $U = u_{hi}^{l}$, $M = (m_{ij})$, and $A = (a_{ij})$, are stiffness and mass matrices. Accordingly, $M \times M$ in space, with coefficients

$$\begin{aligned} m_{ij} &= \int_\Omega \varphi_i(x)\varphi_j(x)dx, \\ a_{ij} &= \int_\Omega \nabla\varphi_i(x)\nabla\varphi_j(x)dx. \end{aligned}$$

$K = (k_{lm}), P = (p_{lm})$ are matrices of mass and stiffness. $N \times N$ with coefficients in time

$$k_{lm} = \int_{t_{l-1}}^{t_{l+1}} \frac{\partial \psi_l(t)}{\partial t} \frac{\partial \psi_m(t)}{\partial t} dt,$$
$$k_{lm} = \int_{t_{l-1}}^{t_{l+1}} \psi_l(t)\psi_m(t) dt,$$

as well as the load vector $b = (b_{jm})$ with coefficients

$$b_{jm} = \int_{\Omega} \int_{t_{l-1}}^{t_{l+1}} f(x)\varphi_j(x)\psi_m(t) dx dt.$$

$K = (k_{lm})$ and $P = (p_{lm})$ are first calculated. As you can see, $k_{lm} = 0, p_{lm} = 0$ unless when $l = m-1, l = m, l = m+1$. Using the test function specification (17), we calculate the first diagonal elements k_{ll} :

$$\begin{aligned} k_{ll} &= \int_{t_{l-1}}^{t_{l+1}} \psi_l'(t)\psi_l'(t) dt \\ &= \int_{t_{l-1}}^{t_l} \left(\frac{1}{\tau}\right)^2 dt + \int_{t_l}^{t_{l+1}} \left(\frac{-1}{\tau}\right)^2 dt = \frac{2}{\tau}, \\ p_{ll} &= \int_{t_{l-1}}^{t_{l+1}} \psi_l(t)\psi_l(t) dt \\ &= \int_{t_{l-1}}^{t_l} \left(\frac{t - t_{l-1}}{\tau}\right)^2 dt + \int_{t_l}^{t_{l+1}} \left(\frac{t_{l+1} - t}{\tau}\right)^2 dt = \frac{2}{3}\tau. \end{aligned}$$

Similarly, the other elements can be written as

$$\begin{aligned} k_{l,l+1} &= \int_{t_{l-1}}^{t_{l+1}} \psi_l'(t)\psi_{l+1}'(t) dt \\ &= \int_{t_l}^{t_{l+1}} \frac{-1}{\tau}\frac{1}{\tau} dt \\ &= -\frac{1}{\tau}, \\ k_{l-1,l} &= \int_{t_{l-1}}^{t_{l+1}} \psi_{l-1}'(t)\psi_l'(t) dt \\ &= \int_{t_{l-1}}^{t_l} \frac{-1}{\tau}\frac{1}{\tau} dt \\ &= -\frac{1}{\tau}. \end{aligned}$$

$$
\begin{aligned}
p_{l,l+1} &= \int_{t_{l-1}}^{t_{l+1}} \psi_l(t)\psi_{l+1}(t)dt \\
&= \int_{t_l-1}^{t_l} \left(\frac{t-t_{l-1}}{\tau}\right)\cdot\left(\frac{t-t_l}{\tau}\right)dt \\
&= \frac{1}{6}\tau, \\
p_{l-1,l} &= \int_{t_{l-1}}^{t_{l+1}} \psi_{l-1}(t)\psi_l(t)dt \\
&= \int_{t_l}^{t_{l+1}} \left(\frac{t_{l+1}-t}{\tau}\right)\cdot\left(\frac{t_l-t}{\tau}\right)dt \\
&= \frac{1}{6}\tau.
\end{aligned} \tag{19}
$$

Verification: in (19), write (17) for ψ_{l+1} and ψ_{l-1}.

Using the same method, we calculate the coefficients of b to obtain:

$$
\begin{aligned}
b_{jm} &= \int_\Omega \int_{t_{m-1}}^{t_{m+1}} f(x)\varphi_j(x)\psi_m(t)dxdt \\
&= \int_\Omega f(x)\varphi_j(x) \int_{t_{m-1}}^{t_m} \frac{t-t_{m-1}}{\tau}dxdt \\
&+ \int_\Omega f(x)\varphi_j(x) \int_{t_m}^{t_{m+1}} \frac{t_{m+1}-t}{\tau}dxdt \\
&\approx \tau \int_\Omega f\left(x_j\right)\varphi_j(x)dx.
\end{aligned}
$$

The system of linear equations (16) is obtained by substituting estimated coefficients to (18):

$$
\begin{aligned}
M\left(\mathbf{u}^{k+1} - 2\mathbf{u}^k + \mathbf{u}^{k-1}\right) &= \tau^2 F^k \\
&- \tau^2 A\left(\frac{1}{6}\mathbf{u}^{k-1} + \frac{2}{3}\mathbf{u}^k + \frac{1}{6}\mathbf{u}^{k+1}\right), \quad k = 1,\ldots,N-1.
\end{aligned} \tag{20}
$$

To get an explicit approach, M is approximated using the lumped mass matrix M^L in space, the diagonal approximation obtained by taking the row sum of M, and mass lumping in time by replacing the terms $\frac{1}{6}\mathbf{u}^{k-1} + \frac{2}{3}\mathbf{u}^k + \frac{1}{6}\mathbf{u}^{k+1}$ by $\mathbf{u}^k$ and (20) is multiplied by $\left(M^L\right)^{-1}$ to produce an effective explicit formulation:

$$
\mathbf{u}^{k+1} = \tau^2 F^k + 2\mathbf{u}^k - \tau^2\left(M^L\right)^{-1} K\mathbf{u}^k - \mathbf{u}^{k-1}, \quad k = 1,\ldots,N-1. \tag{21}
$$

4. Numerical Experiments

Example 1. This experiment have been done by S. S. Almutairi, A. M. Saeed [11] but we extended it to additional points to ensure the accuracy of the method.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = xy. \tag{22}$$

Depending on the boundary conditions $u(x,0) = 0$, $u(x,2) = 0$, $u(0,y) = 0$, $u(2,y) = 0$. With the exact solution of the PDE is

$$u = \sum_{m\geq 1}\sum_{n\geq 1} \frac{-16}{\pi^2}\frac{(-1)^{n+m}}{nm\lambda_{nm}} \sin(\frac{n\pi x}{2}) \sin(\frac{m\pi y}{2}). \tag{23}$$

Table 1: A comparison of the approximation solutions obtained using FDM and GFEM, alongside the analytical solution of the given problem.

NODES	FDM	GFEM	Analytical Solution
1	0	0	0
2	−0.0684455218855219	−0.066074304273125	−0.0679026913
3	−0.119349193976091	−0.122668821548822	−0.1226832533
4	−0.11477288471148	−0.117143703703704	−0.1176222268
5	−0.119349193976091	−0.122668821548822	−0.1226832533
6	−0.146756094276094	−0.14343667851387	-0.1473700856
7	−0.216687033077673	−0.221429225589226	−0.222990496
8	-0.216574276094276	-0.213255170322128	-0.2194532299
9	−0.213255170322128	−0.216574276094276	−0.2194532299
10	-0.240081885521885	-0.237711277602432	-0.2512047163
11	-0.267567407407407	-0.262825492159407	-0.2707333253
12	-0.262825492159407	-0.267567407407407	-0.2707333253
13	-0.273941548821549	-0.270622555204864	-0.2809825828
14	-0.329065589225589	-0.324323898288515	-0.3354518272

Table 1 presents the approximate solutions obtained using GFEM and FDM, alongside the analytical solution of the proposed problem. The results indicate that GFEM performs better than FDM, as its approximations are closer to the analytical solution. This is because GFEM relies on a solid mathematical foundation, using weighted residuals and carefully chosen basis functions to minimize errors. It's especially effective for handling complex geometries and localized phenomena, making it ideal for simulations that demand precision. On the other hand, FDM is straightforward and computationally efficient, but its accuracy depends heavily on maintaining a structured mesh. This can lead to larger discretization errors, especially in cases where the problem involves irregular domains or varying material properties. Because of this limitation, FDM may not be the best choice when high precision is required. Simply put, GFEM stands out because of its adaptability

and accuracy, making it a better choice for solving differential equations in situations where precision matters most.

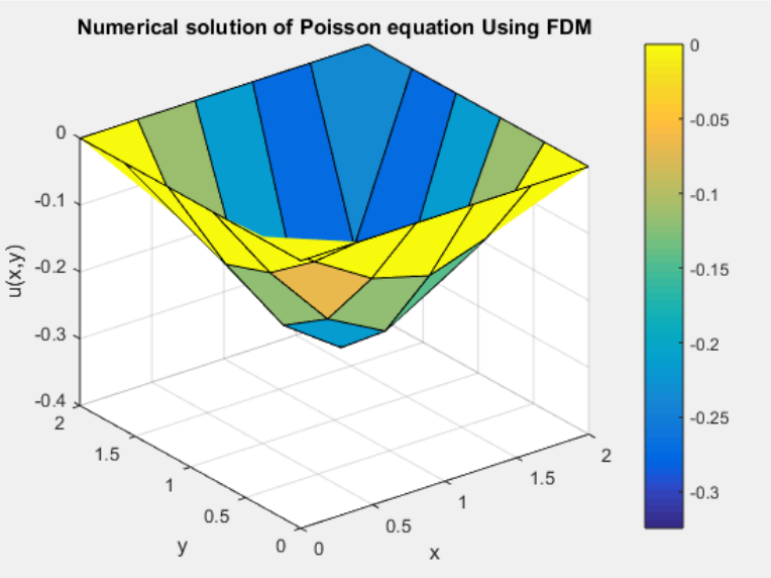

Figure 4: Graph representing the numerical solution of the Poisson equation computed using FDM.

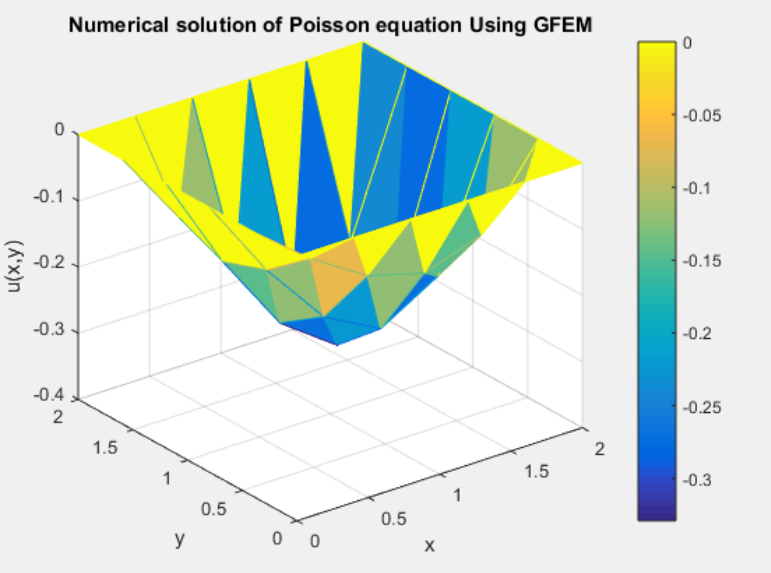

Figure 5: Graph representing the numerical solution of the Poisson equation computed using GFEM.

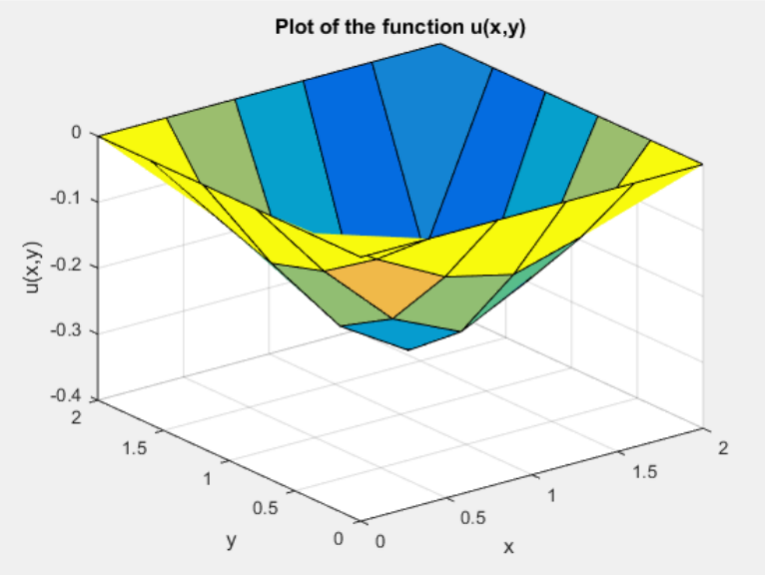

Figure 6: Graph representing the analytical solution.

Figures 4, 5 and 6 clearly show the numerical solution of Poisson's equation using FDM, FEM, and analytical solution.

Example 2. Consider wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),$$

with boundary conditions

$$u(0,y,t) = u(a,y,t) = 0, \quad 0 \le y \le b,\ t > 0,$$
$$u(x,0,t) = u(x,b,t) = 0, \quad 0 \le x \le a,\ t > 0.$$

And initial conditions

$$u(x,y,0) = f(x,y), \qquad (x,y) \in R,$$
$$u(x,y,0) = g(x,y), \qquad (x,y) \in R.$$

The general solution is

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\left(B_{mn}\cos\left(c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}t\right) + B^*_{mn}\sin\left(c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}t\right)\right). \tag{24}$$

Table 2: Comparison results of the approximation solution FDM and analytical solution.

X	Y	Analytical Solution	FDM
0.15789	0.15789	-0.091701	-0.095024
0.78947	1.1053	-3.117	-3.0759
0.94737	0.63158	-2.3209	-2.2537
1.1053	1.4211	-4.3357	-4.3143
1.5789	0.47368	-2.1181	-2.0426
1.7368	2.5263	-2.0561	-1.9891
2.0526	2.3684	-2.3209	-2.2537
2.3684	1.8947	-2.5588	-2.4973
2.8421	1.2632	-0.64947	-0.66045

The error of FDM are L_1 error: 0.031623, L_2 error: 0.041968 and L_∞ error: 0.083343.

Figure 7: Comparison of the analytical solution and the FDM approximation solution for wave equation

Table 3: Comparison results of the approximation solution GFEM and analytical solution.

X	Y	Analytical Solution	GFEM
0.15	1.95	-0.56165	-0.56281
0.225	0.075	-0.061833	-0.062814
0.9	2.85	-0.50533	-0.50546
1.2	2.7	-1.2571	-1.2247
1.575	0.075	-0.30975	-0.31918
1.95	2.85	-0.56165	-0.56281
2.025	0.825	-2.9906	-2.9721
2.325	1.425	-2.9808	-2.9531
2.775	1.125	-0.89784	-0.88465

The errors of FEM are L_1 error: 2.722e-05, L_2 error: 0.00099633 and L_∞ error:

0.056977.

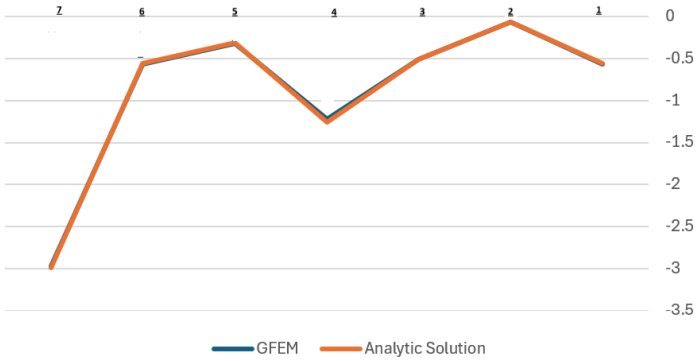

Figure 8: Comparison of the analytical solution and the GFEM approximation solution for wave equation

The GFEM exhibits lower error rates than the FDM, which suggests that the GFEM is more reliable for solving these kinds of equations, according to tables 2 and 3. In response, this study suggests using GFEM as a substitute for elliptic and hyperbolic type PDEs. Also figures 7 and 8 clearly demonstrate that the GFEM solution agrees more closely with the analytical solution than FDM.

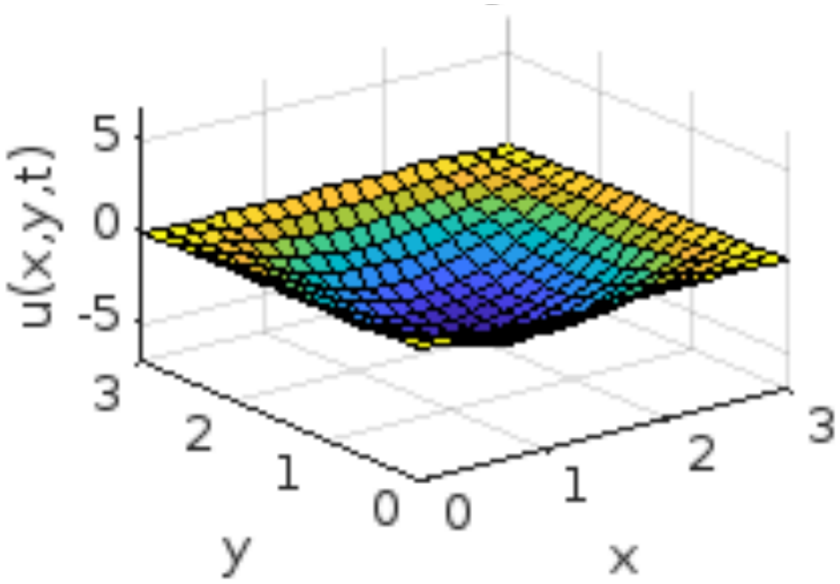

Figure 9: Graph of the numerical solution of wave equation Using FDM.

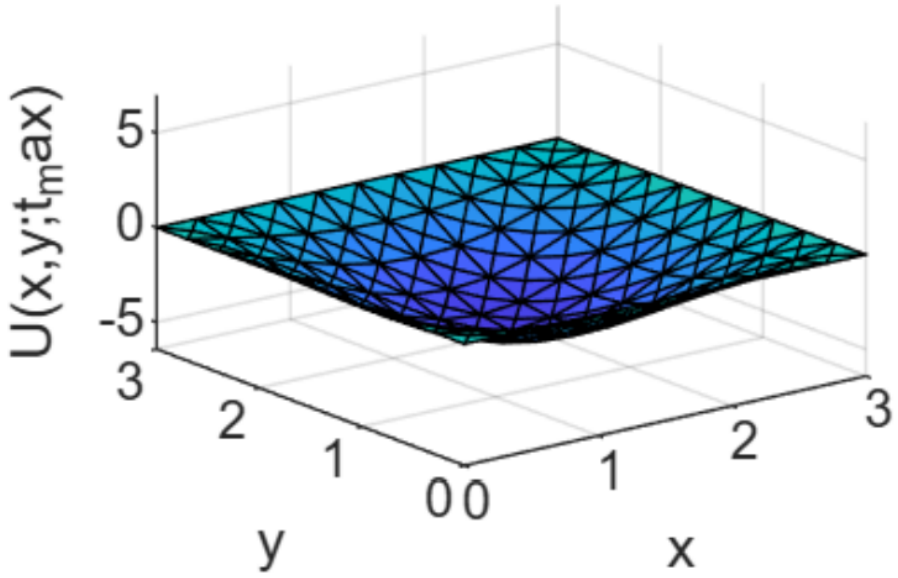

Figure 10: Graph of the numerical solution of wave equation Using GFEM.

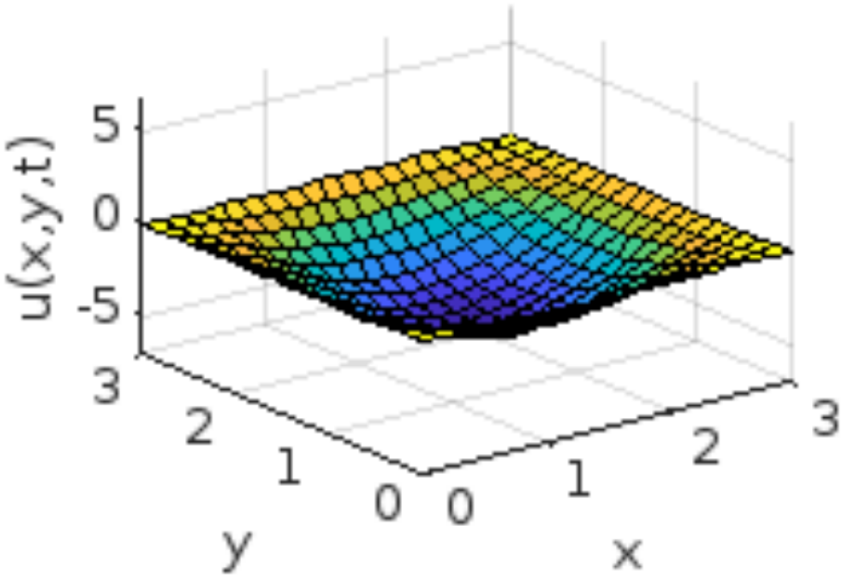

Figure 11: Graph of the analytical solution of wave equation.

Figures 9, 10, and 11 display the solutions of the wave equation obtained using the FDM, GFEM, and the analytical solution. The results show that the GFEM achieves higher accuracy compared to the FDM when applied to the wave equation. Based on these findings, we conclude that GFEM provides more precise and stable solutions, making it a more effective choice for solving elliptic and hyperbolic PDEs.

5. Conclusion

In this paper, Poisson's equation and the Wave equation have been solved using the GFEM. To assess its effectiveness relative to the FDM, several numerical experiments . The results consistently indicate that GFEM achieves superior accuracy, particularly in capturing fine wave structures and handling boundary conditions with greater stability. This improved performance is most evident in scenarios involving complex geometries and high-frequency wave propagation, where FDM tends to suffer from numerical dispersion and phase errors. Based on these findings, we conclude that GFEM presents a reliable and efficient alternative for solving both elliptic and hyperbolic partial differential equations. Furthermore, given its flexibility in mesh adaptation and compatibility with higher-order basis functions, GFEM holds strong potential for broader applications in engineering, fluid dynamics, material science, and other computational fields that require robust and precise numerical modeling.

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