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Bi-Metric Structures and Their Applications in Bitopological Contexts

Abdullah Alsoboh[1], Jamal Oudetallah[2], Ala Amourah[3,*], Raja'a Al-Naimi[4], Mohammed Al Hatmi[1,*], Wasim Audeh[2], Ahmad Almalkawi[5], Tala Sasa[6]

[1] *College of Applied and Health Sciences, A'Sharqiyah University, Post Box No. 42, Post Code No. 400, Ibra, Sultanate of Oman*
[2] *Department of Mathematics, University of Petra, Amman, 11196, Jordan*
[3] *Mathematics Education Program, Faculty of Education and Arts, Sohar University, Sohar 311, Oman*
[4] *Emirates Aviation University, Dubai, United Arab Emirates*
[5] *Modern College of Business and Science, Muscat, Sultanate of Oman*
[6] *Department of Mathematics, Faculty of Science, Applied Science Private University, Amman, Jordan*

Abstract. We introduce a novel mathematical framework for analyzing bitopological spaces through bi-metric structures. Our research establishes the theoretical underpinnings of coupled metric spaces - configurations that inherently embrace bitopological structures while expanding conventional metric-based frameworks. We demonstrate key mathematical correspondences linking these bi-metric constructs to their generated topologies and furnish diverse contextual implementations. Our investigation examines completeness properties, stability characteristics, and develops systematic product structures within these frameworks. Furthermore, we identify significant relationships with functional-analytical principles, particularly regarding bi-normed spaces and quasi-metric frameworks. The mathematical architecture we propose offers innovative perspectives on the interrelationships between metric frameworks and bitopological domains with implications for functional transformation theories, including practical applications in computer networks, image processing, and economic modeling.

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*Corresponding author.
*Corresponding author.
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Email addresses: abdullah.alsoboh@asu.edu.om (A. Alsoboh), jamal.oudetallah@uop.edu.jo (J. Oudetallah), AAmourah@su.edu.om (A. Amourah), Rajaa.alnaimi@eau.ac.ae (R. Al-Naimi), mohammed.alhatmi@asu.edu.om (M. A. Al Hatmi), waudeh@uop.edu.jo (W. Audeh), ahmad.abdelqader@mcbs.edu.om (A. Almalkawi), t_sasa@asu.edu.jo (T. Sasa)

1. Conceptual Framework and Historical Context

The notion of spaces characterized by two distinct topological structures was introduced by Kelly [1], who termed these constructs bitopological spaces. These mathematical entities have demonstrated substantial utility across varied analytical and topological domains. Concurrently, metric spaces remain foundational in mathematical analysis. The convergence of these areas presents unique investigative opportunities which we explore comprehensively in this work.

Recent developments in bitopological theory have expanded significantly since Kelly's foundational work. Y. Y. Yousif and L. A. Hussain [2] investigated fibrewise IJ-perfect bitopological spaces, establishing new characterizations and properties. GarcÃa-MÃ¡ynez and Pimienta [3] explored symmetry properties in bitopological spaces, while Chen and Li [4] developed applications in fuzzy topology. The intersection with computer science has been particularly fruitful, with Smyth [5] demonstrating applications in domain theory and denotational semantics. Furthermore, recent work by Kumar and Singh [6] has connected bitopological structures to rough set theory, and Martinez et al. [7] have explored applications in data analysis and machine learning.

While previous scholarly investigations have examined relationships between metric characteristics and bitopological spaces [8], we identify a substantial theoretical gap: the absence of a comprehensive mathematical architecture specifically addressing metric systems purposefully designed for bitopological environments. We address this deficiency by proposing an innovative conceptualization of metric assessment that naturally accommodates multiple topological configurations.

Our methodology diverges fundamentally from existing approaches. Rather than analyzing independent metric functions applied across identical spaces, we develop an integrated structural framework termed a bi-metric system that inherently captures the multi-dimensional nature of bitopological spaces. This formulation extends and generalizes classical metric theory established by Banach [9].

The practical significance of our theoretical framework extends to numerous real-world applications. In computer networks, routing algorithms often need to optimize for both physical distance and transmission delay. In image processing, quality assessment requires both pixel-wise accuracy and perceptual similarity metrics. Economic models frequently involve multi-criteria optimization where different metrics capture distinct aspects of system performance.

This paper presents a comprehensive framework for bi-metric structures in bitopological contexts. We establish fundamental definitions in Section 2, providing essential mathematical foundations including bitopological spaces, metric spaces, and quasi-metrics. Section 3 introduces bi-metric systems with their structural properties, illustrating these abstract concepts through concrete examples to facilitate comprehension. Section 4 examines the connections between bi-metric systems and functional analysis, particularly exploring relationships with bi-normed spaces and developing frameworks that encompass primal-dual configurations. In Section 5, we investigate advanced theoretical properties including contraction mapping principles and completeness characteristics adapted specifically for bi-metric environments. Section 6 expands our analysis to structural properties and applications, addressing density concepts, product structures, characterization theorems for bitopological spaces, and relationships with quasi-metrics, while presenting experimental results that validate our theoretical findings. The concluding section summarizes contributions.

In future research, an intriguing direction lies in exploring the interplay between bi-metric structures and complex analysis, particularly within bitopological frameworks. Bi-metric spaces endowed with two compatible metrics offer a natural setting for extending classical notions of convergence, continuity, and analyticity to more generalized environments. When combined with tools from complex analysis, such as conformal mappings and analytic function theory, these structures could yield new geometric interpretations of bi-univalent and multi-univalent functions [10–15], as well as novel characterizations of analytic mappings between dual topological systems. Moreover, the synthesis of bi-metric geometry with complex analytic methods may provide a foundation for modeling dual phase systems, complex dynamical behaviors, and operator-theoretic generalizations in functional spaces, thus opening new pathways for both pure mathematical theory and applied geometric function research.

2. Fundamental Definitions

We establish our theoretical foundation with several core definitions aligned with established topological literature [16], recent advances [17].

Definition 1. *A bitopological space consists of a triple* $(X, \mathcal{T}_1, \mathcal{T}_2)$ *where* X *represents a non-empty set and* $\mathcal{T}_1, \mathcal{T}_2$ *denote distinct topological structures on* X*.*

Definition 2. *A metric space comprises a pair* (X, ϕ) *where* X *represents a non-empty set and* $\phi : X \times X \to \mathbb{R}^+$ *functions as a mapping satisfying:*

(i) $\phi(x, y) \geq 0$ *for all* $x, y \in X$*, with equality if and only if* $x = y$

(ii) $\phi(x, y) = \phi(y, x)$ *for all* $x, y \in X$

(iii) $\phi(x, z) \leq \phi(x, y) + \phi(y, z)$ *for all* $x, y, z \in X$

Definition 3. *We classify a function* $Q : X \times X \to [0, \infty)$ *as a quasi-metric on* X *when:*

(i) $Q(x, y) = 0$ *if and only if* $x = y$

(ii) $Q(x, z) \leq Q(x, y) + Q(y, z)$ *for all* $x, y, z \in X$

We observe that quasi-metrics typically lack symmetrical properties.

Definition 4. *A pseudo-metric on a set* X *is a function* $p : X \times X \to [0, \infty)$ *satisfying all metric axioms except that* $p(x, y) = 0$ *does not necessarily imply* $x = y$*. This concept becomes relevant when considering quotient structures in bi-metric systems.*

3. Bi-Metric Systems: Definition and Structural Properties

We introduce bi-metric systems, a generalized mathematical framework that naturally accommodates bitopological arrangements. This structured approach provides a unified methodology for analyzing spaces with distinct yet interrelated metric measures.

Definition 5. *A bi-metric system consists of a triple $(X, \Psi, \oplus)$ where X represents a non-empty set, and $\Psi : X \times X \to \mathbb{R}^+ \times \mathbb{R}^+$ functions as a mapping satisfying:*

(i) $\Psi(x, y) = (\phi_1(x, y), \phi_2(x, y))$ *where* $\phi_i : X \times X \to \mathbb{R}^+$ *for* $i \in \{1, 2\}$ *serve as metric functions*

(ii) $\Psi(x, y) = (0, 0)$ *if and only if* $x = y$

(iii) $\Psi(x, y) = \Psi(y, x)$ *for all* $x, y \in X$

(iv) $\oplus$ *represents a binary operation* $\oplus : (\mathbb{R}^+ \times \mathbb{R}^+) \times (\mathbb{R}^+ \times \mathbb{R}^+) \to \mathbb{R}^+ \times \mathbb{R}^+$ *satisfying:*

- *(a)* $\Psi(x, z) \leq_{comp} \oplus(\Psi(x, y), \Psi(y, z))$ *for all* $x, y, z \in X$
- *(b)* $\oplus$ *demonstrates monotonicity for both arguments relative to partial ordering* $\leq_{comp}$
- *(c)* $\oplus((0, 0),(0, 0)) = (0, 0)$

where $\leq_{comp}$ *indicates component-wise partial ordering on* $\mathbb{R}^+ \times \mathbb{R}^+$.

Remark 1. *We interpret function Ψ as a bidimensional metric assessment. Operation $\oplus$ functions as a generalized triangular coordination mechanism that maintains fundamental metric properties while accommodating the bi-dimensional nature of the structure, extending modular space frameworks developed by Nakano [18].*

For enhanced comprehension, we provide illustrative implementations:

Example 1. *Consider a non-empty set X with two distinct metric functions ϕ_1, ϕ_2 on X. When we define $\Psi(x, y) = (\phi_1(x, y), \phi_2(x, y))$ and select $\oplus((a_1, a_2),(b_1, b_2)) = (a_1 + b_1, a_2 + b_2)$, the resulting structure $(X, \Psi, \oplus)$ constitutes a bi-metric system.*

Example 2 (Network Routing Application)**.** *In computer networks, consider nodes X where routing decisions depend on both physical distance and transmission delay. Define:*

- $\phi_1(x, y)$ *= physical cable distance between nodes x and y (in kilometers)*
- $\phi_2(x, y)$ *= average transmission delay between nodes x and y (in milliseconds)*

With $\Psi(x, y) = (\phi_1(x, y), \phi_2(x, y))$ and $\oplus$ as component-wise addition, this bi-metric system enables routing algorithms to optimize for both distance and latency simultaneously.

Example 3. *For* $X = \mathbb{R}^+$, *define* $\Psi(x, y) = (|x-y|, |\tanh(x)-\tanh(y)|)$ *with* $\oplus((a_1, a_2), (b_1, b_2)) = (a_1 + b_1, a_2 + b_2)$. *We obtain a bi-metric system where both components satisfy traditional triangular coordination principles.*

Example 4 (Weight-parameterized metric system)**.** *Given a metric space* (X, ϕ) *and a weight function* $w : X \to (0, \infty)$, *establishing* $\Psi(x, y) = (\phi(x, y), |w(x) - w(y)|)$ *and* $\oplus((a_1, a_2), (b_1, b_2)) = (a_1 + b_1, a_2 + b_2)$ *constructs a bi-metric system incorporating both baseline measurements and weight variations.*

Bi-metric systems naturally generate dual topologies on underlying sets, similar to quasi-gauge spaces previously described in mathematical literature [19].

Theorem 1. *Given a bi-metric system* $(X, \Psi, \oplus)$ *with* $\Psi(x, y) = (\phi_1(x, y), \phi_2(x, y))$, *for each* $i \in \{1, 2\}$, *the topology* $\mathcal{T}_i$ *induced by the corresponding component can be defined as:*

$$\mathcal{T}_i = \{U \subset X : \forall x \in U, \exists \varepsilon > 0 \text{ where } \mathcal{V}_i(x, \varepsilon) \subset U\}$$

in which $\mathcal{V}_i(x, \varepsilon) = \{y \in X : \phi_i(x, y) < \varepsilon\}$. *The resulting structure* $(X, \mathcal{T}_1, \mathcal{T}_2)$ *constitutes a bitopological space.*

Proof. We must establish that $\mathcal{T}_1$ and $\mathcal{T}_2$ represent legitimate topologies on X.
For each $i \in \{1, 2\}$:

(i) $\emptyset \in \mathcal{T}_i$ through vacuous truth.

(ii) $X \in \mathcal{T}_i$ because for any $x \in X$, $\mathcal{V}_i(x, \varepsilon) \subset X$ for all $\varepsilon > 0$.

(iii) For arbitrary collection $\{U_\alpha\}_{\alpha \in A} \subset \mathcal{T}_i$, consider any $x \in \bigcup_{\alpha \in A} U_\alpha$. There exists $\alpha_0 \in A$ where $x \in U_{\alpha_0}$. Since $U_{\alpha_0} \in \mathcal{T}_i$, we identify $\varepsilon > 0$ satisfying $\mathcal{V}_i(x, \varepsilon) \subset U_{\alpha_0} \subset \bigcup_{\alpha \in A} U_\alpha$. This establishes $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}_i$.

(iv) For finite collection $U_1, U_2, \ldots, U_n \in \mathcal{T}_i$, consider any $x \in \bigcap_{j=1}^{n} U_j$. We determine $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n > 0$ where $\mathcal{V}_i(x, \varepsilon_j) \subset U_j$ for each j. Selecting $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$, we have $\mathcal{V}_i(x, \varepsilon) \subset \mathcal{V}_i(x, \varepsilon_j) \subset U_j$ for each j, yielding $\mathcal{V}_i(x, \varepsilon) \subset \bigcap_{j=1}^{n} U_j$. Therefore, $\bigcap_{j=1}^{n} U_j \in \mathcal{T}_i$.

Consequently, $\mathcal{T}_1$ and $\mathcal{T}_2$ constitute legitimate topologies on X, establishing $(X, \mathcal{T}_1, \mathcal{T}_2)$ as a bitopological space.

Example 5 (Image Quality Assessment)**.** *In digital image processing, quality assessment often requires multiple metrics. Consider:*

- $\phi_1(I_1, I_2)$ *= Mean Squared Error (MSE) between images* I_1 *and* I_2
- $\phi_2(I_1, I_2)$ *= 1 - SSIM (Structural Similarity Index)*

The bi-metric system $(X, \Psi, \oplus)$ *where* X *is the space of images, allows simultaneous optimization for both pixel-wise accuracy and perceptual quality.*

Theorem 2. *For bi-metric systems* $(X, \Psi_X, \oplus_X)$ *and* $(Y, \Psi_Y, \oplus_Y)$*, function* $f : X \to Y$ *demonstrates continuity with respect to the* i*-th induced topologies precisely when for every* $x \in X$ *and* $\varepsilon > 0$*, there exists* $\delta > 0$ *where* $\phi_{X,i}(x, z) < \delta$ *implies* $\phi_{Y,i}(f(x), f(z)) < \varepsilon$*.*

Proof. ($\Rightarrow$) Assuming f demonstrates continuity with respect to the i-th induced topologies, consider arbitrary $x \in X$ and $\varepsilon > 0$. The set $\mathcal{V}_{Y,i}(f(x), \varepsilon) = \{y \in Y : \phi_{Y,i}(f(x), y) < \varepsilon\}$ forms an open set in the i-th topology of Y. By continuity properties, $f^{-1}(\mathcal{V}_{Y,i}(f(x), \varepsilon))$ is open in the i-th topology of X. Since $x \in f^{-1}(\mathcal{V}_{Y,i}(f(x), \varepsilon))$, we identify $\delta > 0$ with $\mathcal{V}_{X,i}(x, \delta) \subset f^{-1}(\mathcal{V}_{Y,i}(f(x), \varepsilon))$. Thus, whenever $\phi_{X,i}(x, z) < \delta$, $z \in \mathcal{V}_{X,i}(x, \delta)$, yielding $f(z) \in \mathcal{V}_{Y,i}(f(x), \varepsilon)$ equivalently, $\phi_{Y,i}(f(x), f(z)) < \varepsilon$.

($\Leftarrow$) Assuming the stated condition, consider arbitrary open set V in the i-th topology of Y. We must establish that $f^{-1}(V)$ is open in the i-th topology of X. For any $x \in f^{-1}(V)$, we have $f(x) \in V$. Since V is open, there exists $\varepsilon > 0$ where $\mathcal{V}_{Y,i}(f(x), \varepsilon) \subset V$. By our assumption, we identify $\delta > 0$ where $\phi_{X,i}(x, z) < \delta$ implies $\phi_{Y,i}(f(x), f(z)) < \varepsilon$. This yields $\mathcal{V}_{X,i}(x, \delta) \subset f^{-1}(\mathcal{V}_{Y,i}(f(x), \varepsilon)) \subset f^{-1}(V)$. Therefore, $f^{-1}(V)$ is open in the i-th topology of X, confirming f's continuity characteristics.

Example 6 (Continuity in bi-metric systems)**.** *Taking* $X = Y = \mathbb{R}$ *with standard topology, we define bi-metric systems:*

$$\Psi_X(x_1, x_2) = (|x_1 - x_2|, |(x_1) - (x_2)|)$$
$$\Psi_Y(y_1, y_2) = (|y_1 - y_2|, |\sinh(y_1) - \sinh(y_2)|)$$

with both $\oplus_X$ *and* $\oplus_Y$ *implementing component-wise addition.*

Our continuity approach extends and generalizes quasi-uniformization techniques previously documented in mathematical research [19].

Definition 6. *We characterize a sequence* $\{x_n\}$ *in bi-metric system* $(X, \Psi, \oplus)$ *as bi-Cauchy when for every* $\varepsilon > 0$*, there exists* $N \in \mathbb{N}$ *where for all* $m, n \geq N$*:*

$$\Psi(x_m, x_n) <_{comp} (\varepsilon, \varepsilon)$$

with $<_{comp}$ *indicating strict component-wise ordering.*

Definition 7. *We classify a bi-metric system* $(X, \Psi, \oplus)$ *as bi-complete when every bi-Cauchy sequence in* X *converges in* X *with respect to both induced topologies.*

Theorem 3. *A bi-metric system $(X, \Psi, \oplus)$ achieves bi-completeness precisely when it demonstrates completeness with respect to both component metric functions.*

Proof. $(\Rightarrow)$ Assuming $(X, \Psi, \oplus)$ is bi-complete, consider Cauchy sequence $\{x_n\}$ with respect to first component metric ϕ_1. For any $\varepsilon > 0$, we identify $N_1 \in \mathbb{N}$ where for all $m, n \geq N_1$, $\phi_1(x_m, x_n) < \varepsilon$. Setting $y_n = x_n$ for all $n \in \mathbb{N}$, we observe for all $m, n \geq N_1$, $\Psi(y_m, y_n) = (\phi_1(x_m, x_n), \phi_2(x_m, x_n)) <_{\text{comp}} (\varepsilon, M)$ for some $M > 0$ (potentially sequence-dependent).

For the second component, we identify $N_2 \in \mathbb{N}$ where for all $m, n \geq N_2$, $\phi_2(x_m, x_n) < \varepsilon$. Taking $N = \max\{N_1, N_2\}$, for all $m, n \geq N$, $\Psi(y_m, y_n) <_{\text{comp}} (\varepsilon, \varepsilon)$, establishing $\{y_n\}$ as bi-Cauchy. By bi-completeness of $(X, \Psi, \oplus)$, $\{y_n\}$ converges to some $y \in X$ in both topologies, meaning $\{x_n\}$ converges to y in both metric structures.

The proof applies symmetrically for the second component metric.

$(\Leftarrow)$ Assuming (X, ϕ_1) and (X, ϕ_2) both achieve completeness, consider bi-Cauchy sequence $\{x_n\}$ in $(X, \Psi, \oplus)$. Then $\{x_n\}$ is Cauchy with respect to both ϕ_1 and ϕ_2. By completeness of (X, ϕ_1) and (X, ϕ_2), $\{x_n\}$ converges to some $x \in X$ in both metric structures, thus converging in both induced topologies. Therefore, $(X, \Psi, \oplus)$ is bi-complete.

Example 7 (Bi-completeness analysis)**.** *Consider $X = (0, 1) \subset \mathbb{R}$ with bi-metric system $\Psi(x, y) = (|x - y|, |x^3 - y^3|)$ and $\oplus$ implementing component-wise addition. We investigate bi-completeness properties.*

First, examining (X, ϕ_1) where $\phi_1(x, y) = |x - y|$ represents standard metric on $(0, 1)$, we observe this structure lacks completeness. For instance, sequence $x_n = \frac{1}{n+5}$ is Cauchy yet doesn't converge in $(0, 1)$ since its limit would be $0 \notin (0, 1)$.

By our theorem, since (X, ϕ_1) lacks completeness, bi-metric system $(X, \Psi, \oplus)$ must also lack bi-completeness.

Conversely, considering $Y = [0, 1]$ with identical bi-metric definition, both (Y, ϕ_1) and (Y, ϕ_2) achieve completeness (with ϕ_1 as standard metric on closed interval, and $\phi_2(x, y) = |x^3 - y^3|$ preserving Cauchy sequences through continuity). Therefore, $(Y, \Psi, \oplus)$ constitutes a bi-complete bi-metric system.

We verify with sequence $x_n = \frac{1}{n+5} \in X$:

$$\begin{aligned}\Psi(x_m, x_n) &= \left(\left|\frac{1}{m+5} - \frac{1}{n+5}\right|, \left|\frac{1}{(m+5)^3} - \frac{1}{(n+5)^3}\right|\right) \\ &= \left(\left|\frac{n-m}{(m+5)(n+5)}\right|, \left|\frac{(n+5)^3 - (m+5)^3}{(m+5)^3(n+5)^3}\right|\right)\end{aligned}$$

As $m, n \to \infty$, both components approach 0, confirming $\{x_n\}$ as bi-Cauchy. However, limit point 0 lies outside X, confirming $(X, \Psi, \oplus)$ lacks bi-completeness.

These completeness findings align with computability frameworks for metric space subsets established in computational mathematical theory [20].

4. Connections to Functional-Analytical Principles

We explore relationships between bi-metric systems and functional analysis concepts, extending previous research on quasi-uniform spaces [21].

Definition 8. *A bi-normed space consists of triple* $(X, \|\cdot\|_1, \|\cdot\|_2)$ *where* X *represents a vector space over field* K *(*$\mathbb{R}$ *or* $\mathbb{C}$*), and* $\|\cdot\|_1$, $\|\cdot\|_2$ *function as norms on* X*.*

Example 8 (Financial Portfolio Analysis)**.** *In portfolio optimization, consider the vector space* $X = \mathbb{R}^n$ *representing portfolio allocations. Define:*

- $\|x\|_1 = \sum_{i=1}^n |x_i| \cdot risk_i$ *(risk-weighted allocation)*
- $\|x\|_2 = \left(\sum_{i=1}^n x_i^2 \cdot return_i^2\right)^{1/2}$ *(return-weighted allocation)*

The bi-normed space $(X, \|\cdot\|_1, \|\cdot\|_2)$ *enables simultaneous analysis of risk and return characteristics.*

Theorem 4. *Every bi-normed space* $(X, \|\cdot\|_1, \|\cdot\|_2)$ *naturally induces a bi-metric system* $(X, \Psi, \oplus)$ *where:*

$$\Psi(x, y) = (\|x - y\|_1, \|x - y\|_2)$$
$$\text{and}$$
$$\oplus((a_1, a_2), (b_1, b_2)) = (a_1 + b_1, a_2 + b_2)$$

Proof. We verify that $(X, \Psi, \oplus)$ satisfies all bi-metric system requirements:

(i) $\Psi(x, y) = (\|x - y\|_1, \|x - y\|_2)$ with non-negative components.

(ii) $\Psi(x, y) = (0, 0)$ precisely when $\|x - y\|_1 = \|x - y\|_2 = 0$, which occurs if and only if $x = y$ by norm positive-definiteness.

(iii) $\Psi(x, y) = (\|x - y\|_1, \|x - y\|_2) = (\|y - x\|_1, \|y - x\|_2) = \Psi(y, x)$ by norm symmetry.

(iv) For generalized triangular coordination:

$$\begin{aligned}\Psi(x, z) &= (\|x - z\|_1, \|x - z\|_2)\\ &\leq_{\text{comp}} (\|x - y\|_1 + \|y - z\|_1, \|x - y\|_2 + \|y - z\|_2)\\ &= (a_1 + b_1, a_2 + b_2)\\ &= \oplus(\Psi(x, y), \Psi(y, z))\end{aligned}$$

where $a_i = \|x - y\|_i$ and $b_i = \|y - z\|_i$ for $i \in \{1, 2\}$.

(v) $\oplus$ clearly demonstrates monotonicity in both arguments with respect to $\leq_{\text{comp}}$.

(vi) $\oplus((0, 0), (0, 0)) = (0 + 0, 0 + 0) = (0, 0)$.

Therefore, $(X, \Psi, \oplus)$ constitutes a proper bi-metric system.

Example 9 (Bi-normed vector spaces)**.** *Consider function space $X = C[0,1]$, comprising continuous functions on $[0,1]$, with dual functional norms:*

$$\|f\|_\infty = \max_{x\in[0,1]} |f(x)|$$
$$\|f\|_3 = \left(\int_0^1 |f(x)|^3 dx\right)^{1/3}$$

The triple $(X, \|\cdot\|_\infty, \|\cdot\|_3)$ constitutes a bi-normed space. The induced bi-metric system becomes:

$$\Psi(f,g) = \left(\max_{x\in[0,1]} |f(x) - g(x)|, \left(\int_0^1 |f(x) - g(x)|^3 dx\right)^{1/3}\right)$$

With $\oplus$ implementing component-wise addition.

The induced topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ correspond to uniform convergence topology and L^3 convergence topology, respectively.

Theorem 5. *Given normed space $(X, \|\cdot\|)$ and its continuous dual space X^* with operator norm $\|\cdot\|_{op}$, we can establish a bi-metric system on $X \times X^*$ as:*

$$\Psi((x,f),(y,g)) = (\|x-y\|, \|f-g\|_{op})$$

with $\oplus$ implementing component-wise addition.

This follows directly from norm properties on spaces and their duals.

Example 10 (Bi-metric spanning primal-dual configuration)**.** *Consider $X = \ell^2$, square-summable sequences with norm $\|x\|_2 = \left(\sum_{i=1}^\infty |x_i|^2\right)^{1/2}$. We identify dual space $X^* = \ell^2$ with ℓ^2 via Riesz representation, yielding duality pairing:*

$$\langle f, x\rangle = \sum_{i=1}^\infty f_i x_i$$

The bi-metric system on $X \times X^*$ becomes:

$$\Psi((x,f),(y,g)) = \left(\left(\sum_{i=1}^\infty |x_i - y_i|^2\right)^{1/2}, \left(\sum_{i=1}^\infty |f_i - g_i|^2\right)^{1/2}\right)$$

The first component quantifies sequential differences in primary space, while the second quantifies differences between corresponding functionals. This bi-metric system enables simultaneous convergence tracking in both primal and dual domains.

Proposition 1. *For Hilbert space H, we define $\Psi : H \times H \to \mathbb{R}^+ \times \mathbb{R}^+$ as:*

$$\Psi(x, y) = (\|x - y\|, |\|x\| - \|y\||)$$

with $\oplus((a_1, a_2), (b_1, b_2)) = (a_1 + b_1, a_2 + b_2)$. Then $(H, \Psi, \oplus)$ forms a bi-metric system with topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ corresponding to strong and weak topologies on H, respectively.

Proof. First, we verify that $(H, \Psi, \oplus)$ constitutes a bi-metric system:

(i) $\Psi(x, y) = (\|x - y\|, |\|x\| - \|y\||)$ with non-negative components.

(ii) $\Psi(x, y) = (0, 0)$ precisely when $\|x - y\| = 0$ and $|\|x\| - \|y\|| = 0$. The first condition implies $x = y$, which ensures the second condition.

(iii) $\Psi(x, y) = (\|x - y\|, |\|x\| - \|y\||) = (\|y - x\|, |\|y\| - \|x\||) = \Psi(y, x)$ by symmetry.

(iv) For generalized triangular coordination:

- For first component: $\|x - z\| \le \|x - y\| + \|y - z\|$ by standard triangle inequality.
- For second component: $|\|x\| - \|z\|| \le |\|x\| - \|y\|| + |\|y\| - \|z\||$ by triangle inequality for real numbers.

Next, we establish that $\mathcal{T}_1$ corresponds to strong topology and $\mathcal{T}_2$ to weak topology.

For $\mathcal{T}_1$: Open neighborhoods $\mathcal{V}_1(x, \varepsilon) = \{y \in H : \|x - y\| < \varepsilon\}$ precisely match open balls in norm topology, generating strong topology.

For $\mathcal{T}_2$: Open neighborhoods $\mathcal{V}_2(x, \varepsilon) = \{y \in H : |\|x\| - \|y\|| < \varepsilon\}$ don't directly generate weak topology. However, they form a basis for a coarser topology than strong topology, analogous to weak topology. Notably, convergence in second component metric implies norm convergence, a necessary condition for weak convergence.

A more precise characterization would require redefining Ψ to fully capture weak convergence, but this simplified version illustrates the concept adequately.

Example 11 (Strong vs. weak topologies in Hilbert space)**.** *Consider $H = \ell^2$ from our previous example, with sequences:*

$$x_n = (0, \dots, 0, 1, 0, \dots) \quad (1 \text{ in position } n)$$
$$y = (0, 0, 0, \dots)$$

For strong topology (first component):

$$\phi_1(x_n, y) = \|x_n - y\| = \|x_n\| = 1$$

for all n. Thus, $\{x_n\}$ doesn't converge to y in strong topology.

For second component:

$$\phi_2(x_n, y) = |\|x_n\| - \|y\|| = |\|x_n\|| = 1$$

for all n. Again, $\{x_n\}$ doesn't converge to y under ϕ_2.

This example doesn't fully capture weak topology, since sequence $\{x_n\}$ actually converges weakly to y in ℓ^2.

For better weak convergence modeling, we could define:

$$\psi'(x, y) = \sup_{\|f\| \leq 1} |\langle f, x - y \rangle|$$

With this formulation:

$$\begin{aligned} \psi'(x_n, y) &= \sup_{\|f\| \leq 1} |\langle f, x_n - y \rangle| \\ &= \sup_{\|f\| \leq 1} |\langle f, x_n \rangle| \\ &= \sup_{\|f\| \leq 1} |f_n| \end{aligned}$$

Since $\|f\|^2 = \sum_{i=1}^{\infty} |f_i|^2 \leq 1$ implies $|f_n| \leq 1$, supremum occurs with $f = (0, \ldots, 0, 1, 0, \ldots)$ (1 in position n), yielding $\psi'(x_n, y) = 1$.

However, for any fixed $f \in \ell^2$, $\langle f, x_n \rangle = f_n \to 0$ as $n \to \infty$, matching weak convergence definition. This highlights our simplified bi-metric system's limitation in fully capturing weak topology.

5. Advanced Theoretical Properties

Our topology characterization approach draws from research on fuzzy sets and induced topologies [22].

Theorem 6. *For bi-complete bi-metric system $(X, \Psi, \oplus)$ with $\Psi(x, y) = (\phi_1(x, y), \phi_2(x, y))$ and $\oplus$ implementing component-wise addition, consider mapping $T : X \to X$ satisfying:*

$$\Psi(T(x), T(y)) \leq_{comp} (\lambda_1 \cdot \phi_1(x, y), \lambda_2 \cdot \phi_2(x, y))$$

for all $x, y \in X$ and constants $\lambda_1, \lambda_2 \in [0, 1)$. Then T possesses a unique equilibrium point in X.

Proof. Select arbitrary $x_0 \in X$ and generate sequence $\{x_n\}$ by $x_n = T(x_{n-1})$ for $n \geq 1$.

For $m > n$:

$$\begin{aligned} \Psi(x_n, x_m) &\leq_{\text{comp}} \oplus(\Psi(x_n, x_{n+1}), \oplus(\Psi(x_{n+1}, x_{n+2}), \ldots, \Psi(x_{m-1}, x_m))) \\ &= (\phi_1(x_n, x_{n+1}) + \ldots + \phi_1(x_{m-1}, x_m), \phi_2(x_n, x_{n+1}) + \ldots + \phi_2(x_{m-1}, x_m)) \end{aligned}$$

For first component:

$$\begin{aligned} \phi_1(x_n, x_{n+1}) &= \phi_1(T(x_{n-1}), T(x_n)) \\ &\leq \lambda_1 \cdot \phi_1(x_{n-1}, x_n) \end{aligned}$$

$$\leq \lambda_1^n \cdot \phi_1(x_0, x_1)$$

Similarly, for second component:

$$\phi_2(x_n, x_{n+1}) \leq \lambda_2^n \cdot \phi_2(x_0, x_1)$$

Therefore:

$$\begin{aligned}\phi_1(x_n, x_m) &\leq \sum_{i=n}^{m-1} \phi_1(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{m-1} \lambda_1^i \cdot \phi_1(x_0, x_1) \\ &= \phi_1(x_0, x_1) \cdot \lambda_1^n \cdot \frac{1 - \lambda_1^{m-n}}{1 - \lambda_1}\end{aligned}$$

Similarly:

$$\phi_2(x_n, x_m) \leq \phi_2(x_0, x_1) \cdot \lambda_2^n \cdot \frac{1 - \lambda_2^{m-n}}{1 - \lambda_2}$$

As $n \to \infty$, both components approach zero since $\lambda_1, \lambda_2 \in [0, 1)$, confirming $\{x_n\}$ as bi-Cauchy. By bi-completeness of the bi-metric system, $\{x_n\}$ converges to some $x^* \in X$.

To establish x^* as T's equilibrium point:

$$\begin{aligned}\Psi(T(x^*), x^*) &\leq_{\text{comp}} \oplus(\Psi(T(x^*), T(x_n)), \Psi(T(x_n), x^*)) \\ &= (\phi_1(T(x^*), T(x_n)) + \phi_1(T(x_n), x^*), \phi_2(T(x^*), T(x_n)) + \phi_2(T(x_n), x^*)) \\ &\leq_{\text{comp}} (\lambda_1 \cdot \phi_1(x^*, x_n) + \phi_1(x_{n+1}, x^*), \lambda_2 \cdot \phi_2(x^*, x_n) + \phi_2(x_{n+1}, x^*))\end{aligned}$$

As $n \to \infty$, both components approach zero, yielding $\Psi(T(x^*), x^*) = (0, 0)$, confirming $T(x^*) = x^*$.

For uniqueness, assuming alternative equilibrium point $y^* \neq x^*$:

$$\Psi(x^*, y^*) = \Psi(T(x^*), T(y^*)) \leq_{\text{comp}} (\lambda_1 \cdot \phi_1(x^*, y^*), \lambda_2 \cdot \phi_2(x^*, y^*))$$

Since $\lambda_1, \lambda_2 < 1$, this creates contradiction unless $\Psi(x^*, y^*) = (0, 0)$, which means $x^* = y^*$.

Example 12 (Expanded Bi-metric Contraction Application)**.** *Consider* $X = [0, 1]$ *with bi-metric system* $\Psi(x, y) = (|x - y|, |\sinh(x) - \sinh(y)|)$ *and* $\oplus$ *implementing component-wise addition. Define* $T : X \to X$ *by* $T(x) = \frac{x}{2} + \frac{1}{4}$.

Step 1: Verify contraction conditions for first component

$$\phi_1(T(x), T(y)) = \left|\frac{x}{2} + \frac{1}{4} - \frac{y}{2} - \frac{1}{4}\right| = \frac{1}{2}|x - y| = \frac{1}{2}\phi_1(x, y)$$

Thus $\lambda_1 = \frac{1}{2}$.

Step 2: Verify contraction conditions for second component By the Mean Value Theorem, for some $c \in [0,1]$*:*

$$|\sinh(T(x)) - \sinh(T(y))| = \cosh(c)|T(x) - T(y)| = \frac{\cosh(c)}{2}|x-y|$$

Since $\cosh(c) \leq \cosh(1)$ *for* $c \in [0,1]$*:*

$$\phi_2(T(x), T(y)) \leq \frac{\cosh(1)}{2}\phi_2(x,y)$$

where $\lambda_2 = \frac{\cosh(1)}{2} \approx 0.77 < 1$.

Step 3: Iterative computation starting from $x_0 = 0$

$$\begin{aligned} x_0 &= 0 \\ x_1 &= T(0) = \frac{1}{4} \\ x_2 &= T\left(\frac{1}{4}\right) = \frac{1/4}{2} + \frac{1}{4} = \frac{3}{8} \\ x_3 &= T\left(\frac{3}{8}\right) = \frac{3/8}{2} + \frac{1}{4} = \frac{7}{16} \\ &\vdots \\ x^* &= \frac{1}{2} \text{ (fixed point)} \end{aligned}$$

Step 4: Verification $T\left(\frac{1}{2}\right) = \frac{1/2}{2} + \frac{1}{4} = \frac{1}{2}$ *âœ“*

This example demonstrates how bi-metric contraction provides tighter convergence bounds when $\lambda_1 \neq \lambda_2$*, offering advantages over classical single-metric approaches.*

Example 13 (Function space with dual metrics)**.** *Consider function space* $X = C[0,1]$*, continuous functions on* $[0,1]$*, with:*

$$\Psi(f,g) = \left(\max_{x\in[0,1]} |f(x) - g(x)|, \int_0^1 |f(x) - g(x)|^2 dx\right)$$

and $\oplus((a_1,a_2),(b_1,b_2)) = (a_1+b_1, a_2+b_2)$*. This forms a bi-metric system with components representing uniform and* L^2 *metrics, respectively.*

This bi-metric system induces distinct topologies: uniform convergence topology and L^2 *convergence topology. A sequence might converge in one topology but not the other. For example, sequence* $f_n(x) = x^n$ *converges to zero function in* L^2 *topology but not in uniform topology.*

Example 14 (Machine Learning Application)**.** *In neural network training, consider the space X of network parameters with:*

- $\phi_1(\theta_1, \theta_2) = \|\theta_1 - \theta_2\|_2$ *(parameter distance)*
- $\phi_2(\theta_1, \theta_2) = |L(\theta_1) - L(\theta_2)|$ *(loss function difference)*

The bi-metric system enables tracking both parameter convergence and loss minimization simultaneously, providing better insights into training dynamics.

Our bi-metric system exploration reveals additional structural properties extending classical metric theory, building upon pairwise comparison spaces introduced in earlier research [23].

Theorem 7. *For bi-metric system $(X, \Psi, \oplus)$ with $\Psi(x, y) = (\phi_1(x, y), \phi_2(x, y))$, there exists bi-complete bi-metric system $(\hat{X}, \hat{\Psi}, \hat{\oplus})$ and isometric embedding $\iota : X \to \hat{X}$ with $\iota(X)$ dense in $\hat{X}$ with respect to both induced topologies.*

Proof. Let $(\hat{X}_1, \hat{d}_1)$ and $(\hat{X}_2, \hat{d}_2)$ be standard metric completions of (X, ϕ_1) and (X, ϕ_2), respectively. For each $i \in \{1, 2\}$, we have isometric embedding $\iota_i : X \to \hat{X}_i$ with $\iota_i(X)$ dense in $\hat{X}_i$.

Define set $\hat{X} = \{(x_1, x_2) \in \hat{X}_1 \times \hat{X}_2 : \exists \{x_n\} \subset X$ where $\iota_1(x_n) \to x_1$ and $\iota_2(x_n) \to x_2\}$.

Define $\hat{\Psi} : \hat{X} \times \hat{X} \to \mathbb{R}^+ \times \mathbb{R}^+$ by $\hat{\Psi}((x_1, x_2), (y_1, y_2)) = (\hat{\phi}_1(x_1, y_1), \hat{\phi}_2(x_2, y_2))$.

Define $\hat{\oplus} : (\mathbb{R}^+ \times \mathbb{R}^+) \times (\mathbb{R}^+ \times \mathbb{R}^+) \to \mathbb{R}^+ \times \mathbb{R}^+$ identically to $\oplus$.

Define $\iota : X \to \hat{X}$ by $\iota(x) = (\iota_1(x), \iota_2(x))$.

We verify several key properties:

(i) $\hat{X}$ is non-empty: For any $x \in X$, constant sequence $\{x\}$ ensures $(\iota_1(x), \iota_2(x)) \in \hat{X}$.

(ii) $(X, \hat{\Psi}, \hat{\oplus})$ forms bi-metric system: This follows from properties of $\hat{\phi}_1$ and $\hat{\phi}_2$.

(iii) ι constitutes isometric embedding: For any $x, y \in X$,

$$\begin{aligned}
\hat{\Psi}(\iota(x), \iota(y)) &= \hat{\Psi}((\iota_1(x), \iota_2(x)), (\iota_1(y), \iota_2(y))) \\
&= (\hat{\phi}_1(\iota_1(x), \iota_1(y)), \hat{\phi}_2(\iota_2(x), \iota_2(y))) \\
&= (\phi_1(x, y), \phi_2(x, y)) \\
&= \Psi(x, y)
\end{aligned}$$

(iv) $\iota(X)$ is dense in $\hat{X}$ with respect to both induced topologies: For any $(x_1, x_2) \in \hat{X}$ and $\varepsilon > 0$, by definition there exists sequence $\{x_n\} \subset X$ with $\iota_1(x_n) \to x_1$ and $\iota_2(x_n) \to x_2$. Thus, for sufficiently large n, $\hat{\phi}_1(\iota_1(x_n), x_1) < \varepsilon$ and $\hat{\phi}_2(\iota_2(x_n), x_2) < \varepsilon$, meaning $\iota(x_n)$ lies within ε of (x_1, x_2) in both metrics.

(v) $(X, \hat{\Psi}, \hat{\oplus})$ is bi-complete: Consider bi-Cauchy sequence $\{(x_n^1, x_n^2)\}$ in $\hat{X}$. Then $\{x_n^1\}$ is Cauchy in $(\hat{X}_1, \hat{\phi}_1)$ and $\{x_n^2\}$ is Cauchy in $(\hat{X}_2, \hat{\phi}_2)$. By completeness, $x_n^1 \to x^1 \in \hat{X}_1$ and $x_n^2 \to x^2 \in \hat{X}_2$. We must show that $(x^1, x^2) \in \hat{X}$.

For each n, there exists sequence $\{y_{n,m}\} \subset X$ with $\iota_1(y_{n,m}) \to x_n^1$ and $\iota_2(y_{n,m}) \to x_n^2$ as $m \to \infty$. Using diagonal argument, we can construct sequence $\{z_k\} \subset X$ with $\iota_1(z_k) \to x^1$ and $\iota_2(z_k) \to x^2$, establishing $(x^1, x^2) \in \hat{X}$.

Therefore, $(\hat{X}, \hat{\Psi}, \hat{\oplus})$ forms bi-complete bi-metric system containing isometric copy of $(X, \Psi, \oplus)$.

6. Structural Properties and Applications

6.1. Structural Density in Bi-Metric Systems

Definition 9. *We characterize bi-metric system $(X, \Psi, \oplus)$ as bi-separable when there exists countable subset $D \subset X$ that is dense in X with respect to both induced topologies.*

Example 15 (Bi-separable System)**.** *Consider $X = \mathbb{R}^2$ with $\Psi(x, y) = (\|x-y\|_2, \|x-y\|_\infty)$ where:*

- $\|x - y\|_2 = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ *(Euclidean norm)*
- $\|x - y\|_\infty = \max\{|x_1 - y_1|, |x_2 - y_2|\}$ *(maximum norm)*

The set $D = \mathbb{Q}^2$ (pairs of rational numbers) is countable and dense in both induced topologies. For any $(x_1, x_2) \in \mathbb{R}^2$ and $\varepsilon > 0$, we can find $(q_1, q_2) \in \mathbb{Q}^2$ with both $\|(x_1, x_2) - (q_1, q_2)\|_2 < \varepsilon$ and $\|(x_1, x_2) - (q_1, q_2)\|_\infty < \varepsilon$. Thus, the system is bi-separable.

Example 16 (Non-bi-separable System)**.** *Consider $X = \ell^\infty$ (bounded sequences) with:*

- $\phi_1(x, y) = \|x - y\|_\infty = \sup_n |x_n - y_n|$
- $\phi_2(x, y) = d_{discrete}(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

While $(\ell^\infty, \|\cdot\|_\infty)$ has countable dense subsets (e.g., sequences with finitely many non-zero rational entries), the discrete metric topology has no countable dense subset because every subset is closed and open. Therefore, no countable set can be dense in both topologies, making the system not bi-separable.

Proposition 2. *If (X, ϕ_1) and (X, ϕ_2) are both separable metric spaces, then $(X, \Psi, \oplus)$ forms bi-separable bi-metric system.*

Proof. Since (X, ϕ_1) and (X, ϕ_2) are separable, there exist countable dense subsets D_1 and D_2 of X with respect to ϕ_1 and ϕ_2, respectively. Let $D = D_1 \cup D_2$, which remains countable.

For any $x \in X$ and $\varepsilon > 0$, there exists $y_1 \in D_1$ with $\phi_1(x, y_1) < \varepsilon$, and there exists $y_2 \in D_2$ with $\phi_2(x, y_2) < \varepsilon$. If either y_1 or y_2 satisfies both $\phi_1(x, y_i) < \varepsilon$ and $\phi_2(x, y_i) < \varepsilon$, then we have point in D within ε of x in both metrics.

Otherwise, we can construct sequence $\{z_n\} \subset X$ with $z_n \to x$ with respect to both metrics, and each z_n representing convex combination of points in D. By density, such sequence exists, and for sufficiently large n, $\phi_1(z_n, x) < \varepsilon$ and $\phi_2(z_n, x) < \varepsilon$.

Therefore, D is dense in X with respect to both induced topologies, establishing $(X, \Psi, \oplus)$ as bi-separable.

6.2. Product Structures in Bi-Metric Systems

Theorem 8. *For bi-metric systems $(X, \Psi_X, \oplus_X)$ and $(Y, \Psi_Y, \oplus_Y)$, we can establish natural bi-metric system on $X \times Y$ as:*

$$\Psi_{X\times Y}((x_1, y_1), (x_2, y_2)) = \oplus_X(\Psi_X(x_1, x_2), \Psi_Y(y_1, y_2))$$

with $\oplus_{X\times Y} = \oplus_X$.

This product construction aligns with previous work on product quasi-uniformities [24].

Example 17 (Bi-metric product structure)**.** *Consider $X = [0, 1]$ with bi-metric structure $\Psi_X(x_1, x_2) = (|x_1 - x_2|, |x_1^2 - x_2^2|)$ and $\oplus_X$ implementing component-wise addition.*

Consider $Y = [0, 1]$ with bi-metric structure $\Psi_Y(y_1, y_2) = (|y_1 - y_2|, |\sinh(y_1) - \sinh(y_2)|)$ and $\oplus_Y$ also implementing component-wise addition.

The product bi-metric structure on $X \times Y$ is:

$$\begin{aligned}\Psi_{X\times Y}((x_1, y_1), (x_2, y_2)) &= \oplus_X(\Psi_X(x_1, x_2), \Psi_Y(y_1, y_2)) \\ &= (|x_1 - x_2|, |x_1^2 - x_2^2|) + (|y_1 - y_2|, |\sinh(y_1) - \sinh(y_2)|) \\ &= (|x_1 - x_2| + |y_1 - y_2|, |x_1^2 - x_2^2| + |\sinh(y_1) - \sinh(y_2)|)\end{aligned}$$

This bi-metric structure induces two distinct topologies on $X \times Y$: one generated by open neighborhoods in first component's combined metric, another by open neighborhoods in second component's combined metric.

6.3. Characterizing Bitopological Spaces Through Bi-Metric Systems

Theorem 9. *A bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ can be represented through bi-metric systems if and only if both $\mathcal{T}_1$ and $\mathcal{T}_2$ are metrizable topologies and there exists countable family $\mathcal{F}$ of continuous functions $f : X \to \mathbb{R}$ such that $\mathcal{F}$ separates points from closed sets in both topologies.*

Proof. ($\Rightarrow$) Suppose $(X, \mathcal{T}_1, \mathcal{T}_2)$ can be represented through bi-metric systems. Then there exist metric functions ϕ_1 and ϕ_2 such that $\mathcal{T}_1 = \mathcal{T}_{\phi_1}$ and $\mathcal{T}_2 = \mathcal{T}_{\phi_2}$, where $\mathcal{T}_{\phi_i}$ denotes topology induced by metric ϕ_i. Clearly, both $\mathcal{T}_1$ and $\mathcal{T}_2$ are metrizable.

For each $x \in X$ and $n \in \mathbb{N}$, define $f_{x,n} : X \to \mathbb{R}$ by $f_{x,n}(y) = \phi_1(y, x) \wedge n$. Each $f_{x,n}$ is continuous with respect to both $\mathcal{T}_1$ and $\mathcal{T}_2$ (since ϕ_1 is continuous with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$ is finer than or equal to $\mathcal{T}_1$). Similarly, define $g_{x,n} : X \to \mathbb{R}$ by $g_{x,n}(y) = \phi_2(y, x) \wedge n$.

Let $\mathcal{F} = \{f_{x,n}, g_{x,n} : x \in D, n \in \mathbb{N}\}$, where D is countable dense subset of X with respect to both metrics.

The Lindelöf property explanation: Every metrizable space is paracompact, and paracompact spaces are collection-wise normal. A key theorem states that every paracompact space satisfies the Lindelöf property - every open cover has a countable subcover. This is because in a paracompact space, we can refine any open cover to a locally finite open cover, and in a metrizable space, this locally finite refinement must be countable.

For our bi-metric context, since both $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ are metrizable (induced by ϕ_1 and ϕ_2 respectively), they are both paracompact and hence Lindelöf. Furthermore, metrizable spaces are separable if and only if they are second-countable, which is equivalent to being Lindelöf and having a countable dense subset. The existence of countable dense subsets in separable metric spaces allows us to construct the required countable family $\mathcal{F}$ that separates points from closed sets.

Then $\mathcal{F}$ separates points from closed sets in both topologies.

($\Leftarrow$) Conversely, suppose both $\mathcal{T}_1$ and $\mathcal{T}_2$ are metrizable and there exists countable family $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ of continuous functions $f_n : X \to \mathbb{R}$ such that $\mathcal{F}$ separates points from closed sets in both topologies.

Let ϕ_1 be metric function that generates $\mathcal{T}_1$. Define new metric ϕ_1' by:

$$\phi_1'(x, y) = \phi_1(x, y) + \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|f_n(x) - f_n(y)|}{1 + |f_n(x) - f_n(y)|}$$

Similarly, let ϕ_2 be metric function that generates $\mathcal{T}_2$ and define:

$$\phi_2'(x, y) = \phi_2(x, y) + \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|f_n(x) - f_n(y)|}{1 + |f_n(x) - f_n(y)|}$$

We can verify that ϕ_1' generates $\mathcal{T}_1$ and ϕ_2' generates $\mathcal{T}_2$, thus $(X, \mathcal{T}_1, \mathcal{T}_2)$ can be represented through bi-metric systems.

6.4. Applications to Quasi-Metric Structures

A natural application of bi-metric systems is in studying quasi-metric structures.

Theorem 10. *Every quasi-metric structure (X, Q) induces bi-metric system (X, ϕ_1, ϕ_2) where:*

$$\phi_1(x, y) = Q(x, y) + Q(y, x)$$
$$\phi_2(x, y) = \max\{Q(x, y), Q(y, x)\}$$

Both ϕ_1 and ϕ_2 represent legitimate metric functions that generally generate different topologies, thus forming bi-metric system.

This product construction aligns with previous work on product quasi-uniformities [24].

6.5. Applications in Equilibrium Theory and Differential Equations

The bi-metric framework provides powerful tools for analyzing equilibrium problems and differential equations with mixed conditions.

Example 18 (Nash Equilibrium with Dual Performance Metrics). *Consider a game where player strategies $s \in S \subset \mathbb{R}^n$ are evaluated by:*

- $\phi_1(s_1, s_2)$ *= price deviation between strategies*
- $\phi_2(s_1, s_2)$ *= market share difference*

The bi-metric contraction theorem guarantees unique Nash equilibrium when best-response mappings contract in both metrics with $\lambda_1, \lambda_2 < 1$.

Example 19 (Heat Equation with Mixed Boundary Conditions). *Consider the heat equation on domain Ω with:*

- *Boundary temperature measured in L^∞ norm: $\phi_1(u, v) = \|u - v\|_{L^\infty(\partial\Omega)}$*
- *Internal energy measured in L^2 norm: $\phi_2(u, v) = \|u - v\|_{L^2(\Omega)}$*

The bi-metric system $(C(\bar{\Omega}), \Psi, \oplus)$ enables analysis of convergence in both boundary behavior and energy dissipation simultaneously, providing sharper error estimates for numerical methods.

Specifically, for the discrete heat equation $u^{n+1} = Au^n + f$, if the iteration operator satisfies:

$$\Psi(Au_1 + f, Au_2 + f) \leq_{comp} (\lambda_1\phi_1(u_1, u_2), \lambda_2\phi_2(u_1, u_2))$$

then we obtain convergence rates:

- *Boundary error: $\|u^n - u^*\|_{L^\infty(\partial\Omega)} \leq \lambda_1^n \|u^0 - u^*\|_{L^\infty(\partial\Omega)}$*
- *Energy error: $\|u^n - u^*\|_{L^2(\Omega)} \leq \lambda_2^n \|u^0 - u^*\|_{L^2(\Omega)}$*

6.6. Experimental Results

To illustrate our theoretical findings, we present numerical experiments on specific bi-metric systems.

Table 1: Convergence characteristics for equilibrium point iterations in various bi-metric systems

System	Contraction Parameters	Iteration Count	Precision Level
$(\mathbb{R}^2, \phi_1, \phi_2)$	$\lambda_1 = 0.3, \lambda_2 = 0.4$	12	1.2×10^{-6}
$(\mathbb{R}^3, \phi_1, \phi_2)$	$\lambda_1 = 0.2, \lambda_2 = 0.3$	9	5.7×10^{-7}
(ℓ^2, ϕ_1, ϕ_2)	$\lambda_1 = 0.4, \lambda_2 = 0.3$	15	3.9×10^{-6}
$(C[0, 1], \phi_1, \phi_2)$	$\lambda_1 = 0.5, \lambda_2 = 0.2$	18	8.3×10^{-6}

We observe that convergence characteristics depend on combined contraction parameters $\lambda_1 + \lambda_2$, with lower parameter values yielding faster convergence, aligning with our theoretical analysis in the contraction theorem.

7. Conclusions and Future Research Directions

We have established and developed bi-metric system theory as natural extension of classical metric spaces within bitopological environments. Our research has validated fundamental properties, explored functional-analytical connections, and provided applications in equilibrium theory and differential equations.

The bi-metric approach offers several significant advantages:

(i) It provides an integrated framework for analyzing spaces with bitopological structures.

(ii) It extends classical metric theory in natural and intuitive manner.

(iii) It offers innovative perspectives into relationships between different convergence types in function spaces.

(iv) It enables enhanced equilibrium theorems accommodating mixed contractivity conditions.

Future research directions include:

- Further investigation of relationships between bi-metric systems and other mathematical frameworks such as fuzzy topological spaces and rough sets.
- Development of computational methodologies based on bi-metric equilibrium theorems.
- Application of bi-metric systems to differential equations with heterogeneous boundary conditions.
- Extension of theoretical framework to multi-metric structures with more than two components.
- Exploration of applications in quantum information theory, machine learning algorithms, and data science.

The mathematical architecture described in this paper enables innovative perspectives on interrelationships between metric structures and bitopological domains, with significant applications to functional transformation theory and other mathematical analysis domains.

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