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Equivalence and Stability of Compactness in Operator Spaces over the Non-Commutative Torus

Mortada S. Ali[1,*], Abd Elmotaleb A. M. A.[2], Ibtisam M. O. Mohammed[1]

[1] *Department of Mathematics, College of Science, Al-Baha University, P.O. Box 1988, KSA.*
[2] *Department of Mathematics, College of Science and Humanity, Prince Sattam bin Abdulaziz University, Sulail, Al-Kharj 11942, KSA.*

Abstract. We investigate compactness in operator spaces over the non-commutative torus A_θ, applying the structure of non-commutative C^*-algebras as well as compact operators acting on Hilbert A_θ-modules, and provide a characterization of compactness in the framework of operator spaces. Key results include the equivalence between classical and complete compactness, and the stability of compactness under tensor products. Applications and examples of compact operators in operator spaces over the non-commutative torus A_θ are presented. We also discuss limitations and propose future research directions to extend these results to more general settings.

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Key Words and Phrases: Compactness, Operator Spaces, Non-Commutative Torus, Hilbert Modules, Quantum Metric Spaces, Tensor Products

1. Introduction

Compactness plays a foundational role in the theory of operators on Banach and Hilbert spaces, as highlighted in [1, 2]. Such an operator is called *compact* if it sends bounded subsets to relatively compact subsets, and it has significant properties in operator theory, as well as connections to compactness in topological spaces, spectral theory, and even functional calculus. While the classical theory of compact operators focuses on spaces of functions and matrices, the study of non-commutative operator algebras presents an intriguing new context for this theory.

The non-commutative torus represents one of the simplest and most studied examples of a non-commutative C^*-algebra[3, 4]. It plays a central role in various branches of mathematics and theoretical physics, ranging from quantum mechanics and topological

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Email addresses: mortada@bu.edu.sa (M. S. Ali), aa.alameen@psau.edu.sa (A. E. A. M. A. Elamin), a.mahjoub@bu.edu.sa (I. M. O. Mohammed)

dynamics to non-commutative geometric spaces. The non-commutative torus A_θ is defined as the algebra generated by two elements, U and V, with the relation

$$UV = e^{2\pi i\theta} VU,$$

where θ is a positive angle argument. For instance, this algebra possesses a rich representation theory and serves as a non-commutative generalization of the classical two-torus. In other words, the non-commutative torus has a rich history, related to studies in spectral theory and representation theory of C^*-algebras[5].

In this paper, we study the compactness properties of operators on operator spaces related to A_θ. Operator spaces are subspaces of operator algebras equipped with a matrix norm structure that satisfies the *Ruan axioms*[6–8]. Operator spaces provide a rigorous framework for analyzing operator behavior, particularly in the context of completely bounded maps. These spaces have demonstrated their fundamental role as a powerful tool across various disciplines, including quantum computing, harmonic analysis, and operator algebras[9, 10].

Compactness for operator spaces is of special importance in the non-commutative realm, as it combines classical notions of compactness with the well-developed theory of completely bounded maps. This provides a synthesis of functional analysis, operator algebras, and the structure of the non-commutative torus for compact operators on operator spaces over A_θ[11–13] . The notion of compactness in operator spaces, although distinct from its classical analogue in Banach spaces, serves as a conceptual bridge between these two domains.

Earlier studies and research works have introduced notions of compactness in Hilbert C^*-modules, the behavior of compact operators, their uses in geometry, as well as in quantum and non-commutative contexts[14–17]. However, previous studies have not systematically explored the relationship between classical compactness and complete compactness within the context of operator spaces over the non-commutative torus.

This study bridges these two notions and demonstrates the stability of compactness under multi-fold tensor products, which is an important feature describing the structure of non-commutative operator spaces [18–20]. In addition, we also describe theoretical applications that illustrate the effect of compact operators in quantum metric spaces, thus broadening the applied scope of our results in comparison to previous work. Hence, this study makes a significant contribution to the deeper understanding of non-commutative operator space theory and the rising opportunities for applications in modern mathematics and physics.

Our motivation arises from fundamental questions in non-commutative geometry and quantum theory [16, 21], which are further explored in the applications section. We further establish connections to quantum theory, in which compactness plays a key role in determining the behavior of quantum channels [22, 23]. Our results build on what other researchers have done before and give us new ways to think about compact operators in non-commutative spaces.

2. Comparison with Related Work

The concept of compactness in operator algebras and in Hilbert C^*-modules has been investigated in both classical and non-commutative settings. Earlier work (e.g., Lance[15]) developed a theory for compact operators on Hilbert C^*-modules. Ruan[8] and Pisier[10] generalized these concepts to operator spaces with completely boundedness and matrix norm structures.

Moreover, recent developments have introduced refined and innovative tools that build upon these classical foundations. Junge and Sherman [7] provide an overview of operator spaces in noncommutative ℓ_p spaces, which builds on this more analytic foundation. Hiai and Ueda[12] have explored novel aspects of non-commutative operator theory, particularly concerning bounded and compact maps, bringing new insights into the structure and behavior of such mappings.

In particular, a modern approach to compact quantum metric spaces was proposed by Latrémolière[23], unifying compactness and geometry via quantum Gromov–Hausdorff convergence. We contribute to this framework by relating operator-theoretic compactness specifically through adjointable maps and complete compactness, while also establishing the stability of these properties under the tensor product operation.

Similarly, Caspers and Skalski[22] studied the behavior of compactness in quantum information channels, and how this concept can be leveraged to maximize the transfer of information. Our paper builds on this foundation by providing a functional-analytic perspective on compact operators in operator spaces over the non-commutative torus explicitly highlighting how compact operators act as information-preserving maps.

Besnard and Latrémolière[3] focused on convergence issues in quantum metric geometry, whereas our results concentrate on functional structures and equivalence theorems for classical and non-classical compactness in operator spaces. Therefore, this paper serves both as a complement and an extension to the modern literature, offering a unified treatment of compactness for non-commutative tori with broad theoretical and applied implications.

3. Preliminaries

In this section, we review key concepts related to compact operators, Hilbert A_θ-Modules, Compactness, and Complete Compactness.

Definition 1. ***Compact Operators*** [19]

In Hilbert C^*-modules, an operator is said to be *compact* if it lies in the norm closure of operators of the form

$$\xi \mapsto \eta\langle\zeta, \xi\rangle,$$

where η, ζ are fixed elements of the module, and $\langle\cdot,\cdot\rangle$ denotes the C^*-valued inner product.

Definition 2. ***Hilbert*** A_θ***-Modules*** [3, 13]

A *Hilbert A_θ-module* is a right A_θ-module equipped with an A_θ-valued inner product

$$\langle \cdot, \cdot \rangle : E \times E \to A_\theta,$$

which is positive-definite and complete with respect to the norm induced by this inner product.

Definition 3. ***Compactness*** [15]

An operator $T : E \to E$ on a Hilbert A_θ-module E is said to be *compact* if it can be represented as the norm limit of finite-rank operators:

$$T_k(\xi) = \sum_{i=1}^{n_k} \eta_i^{(k)} \cdot \langle \zeta_i^{(k)}, \xi \rangle,$$

where $\eta_i^{(k)}, \zeta_i^{(k)} \in E$, with order $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \|T - T_k\| = 0.$$

Definition 4. ***Complete Compactness*** [18, 20, 22]

Let $T : E \to E$ be an adjointable operator on a Hilbert A_θ-module E. The operator T is said to be *completely compact* if its matrix amplifications

$$T_n := I_n \otimes T : M_n(E) \to M_n(E)$$

are compact for all $n \in \mathbb{N}$.

Definition 5. ***Stability under Tensor Products***[19, 20]
Let T, S be compact adjointable operators on Hilbert A_θ-modules E, F respectively. Then the tensor product operator

$$T \otimes S : E \otimes F \to E \otimes F$$

is also compact.

Definition 6. ***completely bounded*** [10]
Let $T : E \to F$ be a compact linear-transformation between operator spaces E and F. The operator T is said to be *completely bounded* if the sequence of amplifications

$$T_n : M_n(E) \to M_n(F)$$

is uniformly bounded, that is,

$$\|T\|_{\text{cb}} := \sup_{n\in\mathbb{N}} \|T_n\| < \infty,$$

where T_n denotes the matrix amplification of T acting on $M_n(E)$.

4. Main Results and Applications

4.1. Main Results

In this subsection, we will prove our main theorems on compactness in operator spaces over A_θ.

Theorem 1. *An operator T is compact if and only if it is completely compact in the operator space sense, provided that T is an adjointable operator.*

Proof. Let $T \in \mathcal{L}(E)$ be adjointable. If T is compact, then by definition, there exists a sequence of finite-rank operators $\{T_k\}$ such that

$$\lim_{k\to\infty} \|T - T_k\| = 0,$$

where each T_k is of the form

$$T_k(\xi) = \sum_{i=1}^{n_k} \eta_i^{(k)} \cdot \langle \zeta_i^{(k)}, \xi \rangle, \quad \text{with } \eta_i^{(k)}, \zeta_i^{(k)} \in E.$$

For each $n \in \mathbb{N}$, consider the matrix amplification

$$T_n := I_n \otimes T : M_n(E) \to M_n(E).$$

Since matrix norms in operator spaces are defined via Ruan's axioms, the compactness of T implies that the sequence $T_{n,k} := I_n \otimes T_k$ converges uniformly to T_n. Therefore, T_n is compact for all n, and thus T is completely compact.

Conversely, if T is completely compact, then for every n, T_n is compact. In particular, $T_1 = T$ is compact. Hence, the two notions coincide under adjointability.

Corollary 1. *Let $T : E \to E$ be an adjointable operator on a Hilbert A_θ-module E. If T^* is compact, then T is compact.*

Proof. In a Hilbert A_θ-module, the polar decomposition $T = U|T|$ exists, where U is a partial isometry and adjointable.Theorem 1 establishes that compactness and complete compactness coincide for adjointable operators.

Since $|T| = (T^*T)^{1/2}$ is obtained via continuous functional calculus applied to the compact operator T^*, it follows that $|T|$ is compact. Then $T = U|T|$ is the product of an adjointable operator and a compact operator, and hence is compact.

Proposition 1. *A compact operator T is completely bounded if $\|T\|_{\text{cb}} = \|T\|$. The converse does not necessarily hold.*

Proof. Suppose that T is compact and that $\|T\|_{\text{cb}} = \|T\|$.

Since T is a bounded linear operator between operator spaces, and $\|T\|_{\text{cb}} < \infty$, it follows directly from the definition that T is completely bounded. Therefore, the condition $\|T\|_{\text{cb}} = \|T\|$ implies $\|T\|_{\text{cb}} < \infty$, which confirms that T is completely bounded.

However, the converse does not necessarily hold. That is, there exist compact operators T that are completely bounded but satisfy $\|T\|_{\text{cb}} > \|T\|$.

Theorem 2. *Let $T : E \to E$ and $S : F \to F$ be compact adjointable operators on Hilbert A_θ-modules E and F, respectively. Then the tensor product operator*

$$T \otimes S : E \otimes_{A_\theta} F \to E \otimes_{A_\theta} F$$

is also compact.

Moreover, the compactness property is preserved under finite internal direct sums of compact adjointable operators. That is, if T_1, T_2 are compact on E_1, E_2, then $T_1 \oplus T_2$ is compact on $E_1 \oplus E_2$.

Proof. Since T and S are compact, there exist sequences of finite-rank operators $\{T^{(k)}\}$ and $\{S^{(\ell)}\}$ such that

$$\lim_{k\to\infty} \|T - T^{(k)}\| = 0, \quad \lim_{\ell\to\infty} \|S - S^{(\ell)}\| = 0.$$

Then the tensor product

$$T \otimes S = \lim_{k,\ell\to\infty} T^{(k)} \otimes S^{(\ell)}$$

is a norm-limit of finite-rank operators (since the tensor of two finite-rank operators is again finite-rank), and hence is compact.

For the direct sum part: let T_1 and T_2 be compact on E_1 and E_2, respectively. Then their direct sum

$$T_1 \oplus T_2 : E_1 \oplus E_2 \to E_1 \oplus E_2$$

is defined by $(T_1 \oplus T_2)(x_1, x_2) = (T_1x_1, T_2x_2)$, and since both components are compact, so is the operator. This follows from the fact that the operator norm and compactness are stable under finite direct sums.

Corollary 2. *Let $T_1, \dots, T_m$ be compact adjointable operators acting on Hilbert A_θ-modules $E_1, \dots, E_m$, respectively. Then the (internal) m-fold tensor product*

$$T_1 \widehat{\otimes} \cdots \widehat{\otimes} T_m$$

is compact on the tensor-product module

$$E_1 \widehat{\otimes} \cdots \widehat{\otimes} E_m.$$

Proof. Theorem 2 gives the result for $m = 2$.

We proceed by induction. Suppose the result holds for $m = k$. Then for $m = k + 1$, consider the operator

$$(T_1 \widehat{\otimes} \cdots \widehat{\otimes} T_k) \widehat{\otimes} T_{k+1}.$$

By the induction hypothesis, the k-fold tensor product is compact, and since T_{k+1} is compact,Theorem 2 implies that their tensor product is compact.

Thus, by induction, the m-fold tensor product is compact.

4.2. Applications and Examples

In this section, we illustrate applications and examples of compact operators in operator spaces over the non-commutative torus A_θ.

Example 1 (Finite-Rank Operator). *Let* $T : A_\theta^n \to A_\theta^n$ *be defined by*

$$T(\xi) = \eta\langle\zeta, \xi\rangle,$$

for fixed $\eta, \zeta \in A_\theta^n$*. Then* T *is a finite-rank operator, since its range is contained in the span of* η*. Hence,* T *is compact.*

Example 2 (Toeplitz-Type Operators). *Suppose* $\{e_n\}_{n=1}^{\infty}$ *is an orthonormal basis of a Hilbert* A_θ*-module, and define an operator* T *by*

$$T(e_n) = \lambda_n e_n,$$

where $\lambda_n \to 0$ *as* $n \to \infty$*. Then* T *is the norm limit of finite-rank diagonal operators and is therefore compact.*

*This structure is similar to classical **Toeplitz** and **Hankel** operators in Hilbert spaces, where sequences* $\{\lambda_n\}$ *represent symbol decay. A detailed comparison with Hankel operators (see [6, 9]) may yield further insights into non-self-adjoint analogues in* A_θ*-modules.*

Example 3 (Numerical Example – Truncated Matrix Representation). *Let* A_θ *be approximated by finite matrices (via rational* $\theta \approx p/q$*). Consider* $A_\theta \cong M_q(\mathbb{C})$ *for* $q = 3$*. Let*

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix},$$

which has eigenvalues tending to zero. Then T *is compact as a limit of finite-rank diagonal matrices. This illustrates compactness numerically in quantum tori approximated by finite-dimensional matrix algebras.*

Example 4 (Quantum Information Channels). *Let* $\Phi : A_\theta \to A_\theta$ *be a quantum channel defined by*

$$\Phi(x) = \sum_{i=1}^{k} V_i x V_i^*,$$

where $\sum_i V_i^* V_i = I$ *and* $V_i \in A_\theta$*. If* $\Phi = \Psi + \text{noise}$ *with* Ψ *completely compact, then* Φ *has effectively finite-dimensional range, preserving key properties in quantum information.*

Such structure is useful for error correction, compression, and modeling decoherence. Compactness ensures containment of quantum evolution within finite metric spaces, as noted in [22].

Application 1 (Quantum Metric Spaces).

In Rieffel's framework of compact quantum metric spaces, define the Lipschitz seminorm:

$$L(x) = \sup \{\|[D, x]\| : D = D^*, \ D \text{ unbounded}\},$$

and consider the unit ball $\{x \in A_\theta : L(x) \leq 1, \ \|x\| \leq 1\}$. Compact operators help define the metric on the state space via approximations of the identity operator.

This concept is critical for analyzing convergence in non-commutative Gromov–Hausdorff spaces (see [23]).

Application 2 (Non-Commutative Harmonic Analysis).

Let $T(f) = u * f * v^*$, where u, v are unitaries in A_θ. If u has rapidly decaying Fourier coefficients (i.e., in the smooth subalgebra of A_θ), then T acts as a **low-pass filter**, and is compact.

Such compact operators localize frequency energy in non-commutative spaces, enabling signal representation with geometric and spectral coherence. This parallels classical harmonic filters and contributes to a developing theory of non-commutative signal analysis.

5. Conclusion

We have characterized compact operators in operator spaces over the non-commutative torus, including studying their action on tensor products and their relation to complete boundedness. This structure has many important applications in quantum geometry and harmonic analysis, and these results are anticipated to inspire further developments in the realm of quantum groups. A key direction for future work involves the generalization of these results to more general operator spaces, in particular those of non-adjointable or unbounded operators. Furthermore, the theoretical framework could be further strengthened by incorporating numerical simulations or case studies from quantum computation and non-commutative signal analysis.

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