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Weak Filters of Sheffer Stroke Hilbert Algebras Based on the Intuitionistic Fuzzy Set

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Abstract. Using the concept of intuitionistic fuzzy points, the weak filter in Sheffer stroke Hilbert algebras is addressed. The notion of intuitionistic fuzzy weak filters in Sheffer stroke Hilbert algebras is introduced, and their properties are investigated. Conditions under which the intuitionistic fuzzy set becomes an intuitionistic fuzzy weak filter are examined. Characterizations of intuitionistic fuzzy weak filters are considered, and conditions under which the intuitionistic fuzzy set becomes an intuitionistic fuzzy weak filter are discussed. The $(0, 1)$-set for the intuitionistic fuzzy set is established, and the phases in which it can be a weak filter are explored. Conditions for an intuitionistic level set and an intuitionistic q-set to be weak filters are provided.

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Key Words and Phrases: Weak filter, intuitionistic fuzzy point, intuitionistic level set, intuitionistic q-set, $(0, 1)$-set, intuitionistic fuzzy weak filter

1. Introduction

The Sheffer operation (or, Sheffer stroke) is a logical operation in Boolean algebra that produces a false result only when both of its inputs are true. It is also known as the NAND operation, and is often symbolized as "|" or sometimes as "↑". The Sheffer stroke has been applied to several algebraic structures, for example, Boolean algebra, BCK-algebra, MV-algebra, BL-algebra, and ortholattices, etc., and it is also being dealt with in the fuzzy environment (see [1–11]). In 2021, Oner et al. [5] applied the Sheffer stroke to Hilbert algebras. They introduced Sheffer stroke Hilbert algebra and investigated several properties. In [4], Oner et al. introduced the notion of deductive system and filter of Sheffer stroke Hilbert algebras, and dealt with their fuzzification. Oner et al. [5]

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also introduced the concept of ideal and examined its properties in Sheffer stroke Hilbert algebras. The intuitionistic fuzzy set, which is a generalization of fuzzy sets, is introduced by K. Athanasov in 1986, and it is a useful tool for better modeling uncertainties and ambiguity. The fuzzy set only considers the degree of membership of the element, while the intuitionistic fuzzy set deals with membership and non-membership simultaneously, along with the degree of hesitation (or uncertainty) about the element. An intuitionistic fuzzy point (see [12]) is an extension of the classical concept of a point in set theory, which is adapted to the framework of the intuitionistic fuzzy set and plays an important role in intuitionistic fuzzy sets. Jun et al. [13] introduced the concept of weak filters that have weakened the filter conditions in the Sheffer stroke Hilbert algebra and investigated several properties. They presented how to make weak filters using ideals, and examined the shape of the weak filter in the Cartesian product of Sheffer stroke Hilbert algebras.

The purpose of this paper is to study weak filters in Sheffer stroke Hilbert algebras using the concept of intuitionistic fuzzy points. We introduce the notion of intuitionistic fuzzy weak filters in Sheffer stroke Hilbert algebras, and investigates their properties. We examine the conditions under which the intuitionistic fuzzy set becomes an intuitionistic fuzzy weak filter. We discuss the characterization of intuitionistic fuzzy weak filters, and consider the conditions under which the intuitionistic fuzzy set becomes an intuitionistic fuzzy weak filter. We build a $(0, 1)$-set for the intuitionistic fuzzy set, and discuss the phases in which it can be a weak filter. It provides conditions for an intuitionistic level set and an intuitionistic q-set to be weak filters.

2. Preliminaries

Definition 1 ([14])**.** *Let $\mathcal{B} := (B, |)$ be a groupoid. Then the operation "$|$" is said to be Sheffer stroke or Sheffer operation if it satisfies:*

(s1) $(\forall \mathfrak{a}, \mathfrak{b} \in B)\ (\mathfrak{a}|\mathfrak{b} = \mathfrak{b}|\mathfrak{a})$,

(s2) $(\forall \mathfrak{a}, \mathfrak{b} \in B)\ ((\mathfrak{a}|\mathfrak{a})|(\mathfrak{a}|\mathfrak{b}) = \mathfrak{a})$,

(s3) $(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in B)\ (\mathfrak{a}|((\mathfrak{b}|\mathfrak{c})|(\mathfrak{b}|\mathfrak{c})) = ((\mathfrak{a}|\mathfrak{b})|(\mathfrak{a}|\mathfrak{b}))|\mathfrak{c})$,

(s4) $(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in B)\ ((\mathfrak{a}|((\mathfrak{a}|\mathfrak{a})|(\mathfrak{b}|\mathfrak{b})))|(\mathfrak{a}|((\mathfrak{a}|\mathfrak{a})|(\mathfrak{b}|\mathfrak{b}))) = \mathfrak{a})$.

Let $\mathbb{X} := (X, |)$ be a groupoid. For every element $\mathfrak{a} \in X$, consider the following mapping:

$$\eth_{\mathfrak{a}} : X \to X,\ \mathfrak{b} \mapsto \mathfrak{a}|(\mathfrak{b}|\mathfrak{b}).$$

Definition 2 ([5])**.** *A Sheffer stroke Hilbert algebra is a groupoid $\mathbb{X} := (X, |)$ with a Sheffer stroke "$|$" that satisfies:*

(sH1) $(\mathfrak{a}|(\eth_{\mathfrak{b}}(\mathfrak{c})|\eth_{\mathfrak{b}}(\mathfrak{c})))|((\eth_{\mathfrak{a}}(\mathfrak{b})|(\eth_{\mathfrak{a}}(\mathfrak{c})|\eth_{\mathfrak{a}}(\mathfrak{c})))|(\eth_{\mathfrak{a}}(\mathfrak{b})|(\eth_{\mathfrak{a}}(\mathfrak{c})|\eth_{\mathfrak{a}}(\mathfrak{c})))) = \eth_{\mathfrak{a}}(\mathfrak{a})$,

(sH2) $\eth_{\mathfrak{a}}(\mathfrak{b}) = \eth_{\mathfrak{b}}(\mathfrak{a}) = \eth_{\mathfrak{a}}(\mathfrak{a}) \ \Rightarrow\ \mathfrak{a} = \mathfrak{b}$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X$.

Recall that every Sheffer stroke Hilbert algebra $\mathbb{X} := (X, |)$ satisfies $\eth_{\mathfrak{a}}(\mathfrak{a}) = \eth_{\mathfrak{b}}(\mathfrak{b})$ for all $\mathfrak{a}, \mathfrak{b} \in X$. It means that $\mathbb{X} := (X, |)$ has an algebraic constant which is denoted by "1" (see [5]).

Let $\mathbb{X} := (X, |)$ be a Sheffer stroke Hilbert algebra. Then the order relation "$\preceq$" on X is defined as follows:

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{a} \preceq \mathfrak{b} \;\Leftrightarrow\; \eth_{\mathfrak{a}}(\mathfrak{b}) = 1). \tag{1}$$

We observe that the relation " $\preceq$ " is a partial order in a Sheffer stroke Hilbert algebra $\mathbb{X} := (X, |)$ (see [5]).

Proposition 1 ([5]). *Every Sheffer stroke Hilbert algebra* $\mathbb{X} := (X, |)$ *satisfies:*

$$\eth_{\mathfrak{a}}(\mathfrak{a}) = 1,\ \eth_{\mathfrak{a}}(1) = 1,\ \eth_{1}(\mathfrak{a}) = \mathfrak{a}, \tag{2}$$
$$\mathfrak{a} \preceq \eth_{\mathfrak{b}}(\mathfrak{a}), \tag{3}$$
$$\eth_{\mathfrak{a}}(\mathfrak{b})|(\mathfrak{b}|\mathfrak{b}) = \eth_{\mathfrak{b}}(\mathfrak{a})|(\mathfrak{a}|\mathfrak{a}), \tag{4}$$
$$(\eth_{\mathfrak{a}}(\mathfrak{b})|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}) = \eth_{\mathfrak{a}}(\mathfrak{b}), \tag{5}$$
$$\mathfrak{a}|(\eth_{\mathfrak{b}}(\mathfrak{c})|\eth_{\mathfrak{b}}(\mathfrak{c})) = \mathfrak{b}|(\eth_{\mathfrak{a}}(\mathfrak{c})|\eth_{\mathfrak{a}}(\mathfrak{c})), \tag{6}$$
$$\mathfrak{a} \preceq \mathfrak{b} \;\Rightarrow\; \eth_{\mathfrak{c}}(\mathfrak{a}) \preceq \eth_{\mathfrak{c}}(\mathfrak{b}),\ \eth_{\mathfrak{b}}(\mathfrak{c}) \preceq \eth_{\mathfrak{a}}(\mathfrak{c}), \tag{7}$$
$$\mathfrak{a}|(\eth_{\mathfrak{b}}(\mathfrak{c})|\eth_{\mathfrak{b}}(\mathfrak{c})) = (\eth_{\mathfrak{a}}(\mathfrak{b})|(\eth_{\mathfrak{a}}(\mathfrak{c})|\eth_{\mathfrak{a}}(\mathfrak{c})), \tag{8}$$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X$.

By (2), we know that the algebraic constant 1 is the greatest element in $\mathbb{X} := (X, |)$ with respect to the order $\preceq$.

Proposition 2. *Let* $\mathbb{X} := (X, |)$ *be a Sheffer stroke Hilbert algebra with the smallest element* 0. *Then*

$$0|0 = 1,\ \ 1|1 = 0, \tag{9}$$
$$\eth_{1}(0) = 0,\ \ \eth_{0}(0) = 1. \tag{10}$$

Definition 3 ([4]). *Let* $\mathbb{X} := (X, |)$ *be a Sheffer stroke Hilbert algebra. A subset F of X is called a filter of* $\mathbb{X} := (X, |)$ *if it satisfies:*

$$1 \in F, \tag{11}$$
$$(\forall \mathfrak{a}, \mathfrak{b} \in X)(\mathfrak{b} \in F \;\Rightarrow\; \eth_{\mathfrak{a}}(\mathfrak{b}) \in F), \tag{12}$$
$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)(\mathfrak{b}, \mathfrak{c} \in F \;\Rightarrow\; (\mathfrak{a}|(\mathfrak{b}|\mathfrak{c}))|(\mathfrak{b}|\mathfrak{c}) \in F). \tag{13}$$

Let $\mathbb{X} := (X, |)$ be a Sheffer stroke Hilbert algebra. If a subset F of X satisfies (11) and (12), we say that F is a *weak filter* of $\mathbb{X} := (X, |)$ (see [13]).

Let X be a set. An *intuitionistic fuzzy set* $\mathcal{B}^*$ in X (see [15]) is an object having the form

$$\mathcal{B}^* := \{\langle \mathfrak{a}, f_B(\mathfrak{a}), g_B(\mathfrak{a})\rangle \mid f_B(\mathfrak{a}) + g_B(\mathfrak{a}) \leq 1,\ \mathfrak{a} \in X\},$$

which is simply denoted by $\mathcal{B}^* := (X; f_B, g_B)$ where f_B and g_B are fuzzy sets in X, The intuitionistic fuzzy set $\mathcal{B}^* := (X; f_B, g_B)$ in X can be represented as follows:

$$\mathcal{B}^* := (X; f_B, g_B) : X \to [0,1] \times [0,1], \mathfrak{a} \mapsto (f_B(\mathfrak{a}), g_B(\mathfrak{a}))$$

such that $f_B(\mathfrak{a}) + g_B(\mathfrak{a}) \leq 1$.

An intuitionistic fuzzy set $\mathcal{B}^* := (X; f_B, g_B)$ in a set X of the form

$$\mathcal{B}^* := (X; f_B, g_B) : X \to [0,1] \times [0,1], \mathfrak{b} \mapsto \begin{cases} (s,t) \in (0,1] \times [0,1) & \text{if } \mathfrak{b} = \mathfrak{a}, \\ (0,1) & \text{if } \mathfrak{b} \neq \mathfrak{a}, \end{cases}$$

is said to be an *intuitionistic fuzzy point* with support $\mathfrak{a}$ and value (s,t) such that $s+t \leq 1$, and is denoted by $\mathfrak{a}_{(s,t)}$.

Given an intuitionistic fuzzy set $\mathcal{B}^* := (X; f_B, g_B)$ and intuitionistic fuzzy point $\mathfrak{a}_{(s,t)}$ in X, we say

$$\mathfrak{a}_{(s,t)} \in \mathcal{B}^* \text{ if } f_B(\mathfrak{a}) \geq s \text{ and } g_B(\mathfrak{a}) \leq t. \tag{14}$$

$$\mathfrak{a}_{(s,t)}\, q\, \mathcal{B}^* \text{ if } f_B(\mathfrak{a}) + s > 1 \text{ and } g_B(\mathfrak{a}) + t < 1. \tag{15}$$

$$\mathfrak{a}_{(s,t)} \in\!\vee q\, \mathcal{B}^* \text{ if } \mathfrak{a}_{(s,t)} \in \mathcal{B}^* \text{ or } \mathfrak{a}_{(s,t)}\, q\, \mathcal{B}^*. \tag{16}$$

Given $(s,t) \in (0,1] \times [0,1)$ and an intuitionistic fuzzy set $\mathcal{B}^* := (X; f_B, g_B)$ in X, consider the following sets:

$$\begin{aligned}
&(f_B, s)_\in := \{\mathfrak{a} \in X \mid f_B(\mathfrak{a}) \geq s\}, \\
&(g_B, t)_\in := \{\mathfrak{a} \in X \mid g_B(\mathfrak{a}) \leq t\}, \\
&(f_B, s)_q := \{\mathfrak{a} \in X \mid f_B(\mathfrak{a}) + s > 1\}, \\
&(g_B, t)_q := \{\mathfrak{a} \in X \mid g_B(\mathfrak{a}) + t < 1\}. \\
&(f_B, s)_{\in\vee q} := \{\mathfrak{a} \in X \mid f_B(\mathfrak{a}) \geq s \text{ or } f_B(\mathfrak{a}) + s > 1\}. \\
&(g_B, t)_{\in\vee q} := \{\mathfrak{a} \in X \mid g_B(\mathfrak{a}) \leq t \text{ or } g_B(\mathfrak{a}) + t < 1\}.
\end{aligned}$$

Also, we consider the sets below.

$$\begin{aligned}
&(\mathcal{B}^*, (s,t))_\in := (f_B, s)_\in \cap (g_B, t)_\in, \\
&(\mathcal{B}^*, (s,t))_q := (f_B, s)_q \cap (g_B, t)_q, \\
&(\mathcal{B}^*, (s,t))_{\in\vee q} := (f_B, s)_{\in\vee q} \cap (g_B, t)_{\in\vee q},
\end{aligned}$$

which are called the *intuitionistic level set*, *intuitionistic q-set* and *intuitionistic $\in\!\vee q$-set* of $\mathcal{B}^* := (X; f_B, g_B)$, respectively.

Definition 4 ([16]). *An intuitionistic fuzzy set $\mathcal{B}^* := (X; f_B, g_B)$ in a Sheffer stroke Hilbert algebra $\mathbb{X} := (X, |)$ is called an intuitionistic fuzzy filter of $\mathbb{X} := (X, |)$ if it satisfies:*

$$(\forall x \in X)(f_B(1) \geq f_B(x),\ g_B(1) \leq g_B(x)), \tag{17}$$

$$(\forall x, y \in X)(f_B(\eth_x(y)) \geq f_B(y),\ g_B(\eth_x(y)) \leq g_B(y)), \tag{18}$$

$$(\forall x, y, z \in X) \left(\begin{array}{l} f_B((x|(y|z))|(y|z)) \geq \min\{f_B(y), f_B(z)\} \\ g_B((x|(y|z))|(y|z)) \leq \max\{g_B(y), g_B(z)\} \end{array} \right). \tag{19}$$

3. Intuitionistic fuzzy weak filters

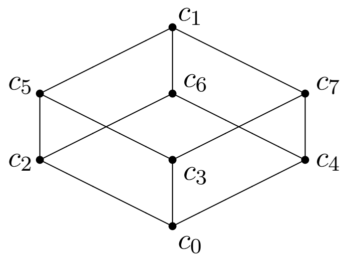
In what follows, $\mathbb{X} := (X, |)$ stands for a Sheffer stroke Hilbert algebra, unless otherwise stated.

Definition 5. *An intuitionistic fuzzy set $\mathcal{B}^* := (X; f_B, g_B)$ in X is called an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$ if it satisfies:*

$$(\forall x \in X)(\forall (s,t) \in (0,1] \times [0,1)) \left(x_{(s,t)} \in \mathcal{B}^* \ \Rightarrow\ 1_{(s,t)} \in \mathcal{B}^* \right), \tag{20}$$

$$(\forall x, y \in X)(\forall (s,t) \in (0,1] \times [0,1)) \left(y_{(s,t)} \in \mathcal{B}^* \ \Rightarrow\ \eth_x(y)_{(s,t)} \in \mathcal{B}^* \right). \tag{21}$$

Example 1. *Let $X = \{c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7\}$ be a set with the following Hasse diagram:*



Define a Sheffer stroke "|" on X by Table 1
Then $\mathbb{X} := (X, |)$ is a Sheffer stroke Hilbert algebra where the algebraic constant is c_1 (see [5]).

Let $\mathcal{B}^* := (X; f_B, g_B)$ be an intuitionistic fuzzy set in X where

$$f_B : X \to [0,1],\ x \mapsto \begin{cases} 0.66 & \text{if } x = c_1, \\ 0.52 & \text{if } x \in \{c_5, c_6\}, \\ 0.44 & \text{if } x = c_7, \\ 0.35 & \text{otherwise.} \end{cases}$$

and

$$g_B : X \to [0,1],\ x \mapsto \begin{cases} 0.27 & \text{if } x = c_1, \\ 0.32 & \text{if } x \in \{c_6, c_7\}, \\ 0.34 & \text{otherwise.} \end{cases}$$

It is routine to verify that $\mathcal{B}^* := (X; f_B, g_B)$ is an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$.

Table 1: Cayley table for the Sheffer stroke "|"

$\mid$	c_0	c_2	c_3	c_4	c_5	c_6	c_7	c_1
c_0	c_1	c_1	c_1	c_1	c_1	c_1	c_1	c_1
c_2	c_1	c_7	c_1	c_1	c_7	c_7	c_1	c_7
c_3	c_1	c_1	c_6	c_1	c_6	c_1	c_6	c_6
c_4	c_1	c_1	c_1	c_5	c_1	c_5	c_5	c_5
c_5	c_1	c_7	c_6	c_1	c_4	c_7	c_6	c_4
c_6	c_1	c_7	c_1	c_5	c_7	c_3	c_5	c_3
c_7	c_1	c_1	c_6	c_5	c_6	c_5	c_2	c_2
c_1	c_1	c_7	c_6	c_5	c_4	c_3	c_2	c_0

It is obvious that every intuitionistic fuzzy filter is an intuitionistic fuzzy weak filter, but the converse may not be true. In fact, the intuitionistic fuzzy weak filter $\mathcal{B}^* := (X; f_B, g_B)$ in Example 1 is not an intuitionistic fuzzy filter of $\mathbb{X} := (X, |)$ because of

$$g_B((c_4|(c_6|c_7))|(c_6|c_7)) = g_B(c_4) = 0.34 \nleq 0.32 = \max\{g_B(c_6), g_B(c_7)\}.$$

Theorem 1. *An intuitionistic fuzzy set $\mathcal{B}^* := (X; f_B, g_B)$ in X is an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$ if and only if it satisfies:*

$$(\forall x \in X)(f_B(1) \geq f_B(x),\ g_B(1) \leq g_B(x)), \tag{22}$$

$$(\forall x, y \in X)(f_B(\eth_x(y)) \geq f_B(y),\ g_B(\eth_x(y)) \leq g_B(y)). \tag{23}$$

Proof. Assume that $\mathcal{B}^* := (X; f_B, g_B)$ in X is an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$. Since $x_{(s,t)} \in \mathcal{B}^*$ for $s = f_B(x)$ and $t = g_B(x)$, we have $1_{(s,t)} \in \mathcal{B}^*$ by (20). Hence $f_B(1) \geq s = f_B(x)$ and $g_B(1) \leq t = g_B(x)$, i.e., (22) is valid. Since $y_{(f_B(y),g_B(y))} \in \mathcal{B}^*$ for all $y \in X$, it follows from (21) that $\eth_x(y)_{(f_B(y),g_B(y))} \in \mathcal{B}^*$ for all $x \in X$. Hence $f_B(\eth_x(y)) \geq f_B(y)$ and $g_B(\eth_x(y)) \leq g_B(y)$, i.e., (23) is valid.

Conversely, let $\mathcal{B}^* := (X; f_B, g_B)$ be an intuitionistic fuzzy set in X that satisfies (22) and (23). Let $x \in X$ and $(s,t) \in (0,1] \times [0,1)$ be such that $x_{(s,t)} \in \mathcal{B}^*$. Then $f_B(x) \geq s$ and $g_B(x) \leq t$. If $1_{(s,t)} \overline{\in} \mathcal{B}^*$, then $f_B(1) < s \leq f_B(x)$ or $g_B(1) > t \geq g_B(x)$ which is a contradiction. Thus $1_{(s,t)} \in \mathcal{B}^*$. Let $x, y \in X$ and $(s,t) \in (0,1] \times [0,1)$ be such that $y_{(s,t)} \in \mathcal{B}^*$. Then $f_B(y) \geq s$ and $g_B(y) \leq t$. If $\eth_x(y)_{(s,t)} \overline{\in} \mathcal{B}^*$, then $f_B(\eth_x(y)) < s \leq f_B(y)$ or $g_B(\eth_x(y)) > t \geq g_B(y)$ which is a contradiction. Thus $\eth_x(y)_{(s,t)} \in \mathcal{B}^*$. Therefore $\mathcal{B}^* := (X; f_B, g_B)$ is an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$.

Theorem 2. *An intuitionistic fuzzy set $\mathcal{B}^* := (X; f_B, g_B)$ in X is an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$ if and only if the nonempty sets $(f_B, s)_\in$ and $(g_B, t)_\in$ are weak filters of $\mathbb{X} := (X, |)$ for all $(s,t) \in (0,1] \times [0,1)$.*

Proof. Assume that $\mathcal{B}^* := (X; f_B, g_B)$ is an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$ and let $(s,t) \in (0,1] \times [0,1)$ be such that $(f_B, s)_\in \neq \emptyset \neq (g_B, t)_\in$, say $x \in (f_B, s)_\in$ and

$\mathfrak{a} \in (g_B, t)_\in$. Then $f_B(1) \geq f_B(x) \geq s$ and $g_B(1) \leq g_B(\mathfrak{a}) \leq t$ by (22). Hence $1 \in (f_B, s)_\in$ and $1 \in (g_B, t)_\in$. Let $y \in (f_B, s)_\in$ and $\mathfrak{b} \in (g_B, t)_\in$. Then $f_B(\eth_x(y)) \geq f_B(y) \geq s$ and $\eth_{\mathfrak{a}}(\mathfrak{b}) \leq g_B(\mathfrak{b}) \leq t$ for all $x, \mathfrak{a} \in X$ by (23). It follows that $\eth_x(y) \in (f_B, s)_\in$ and $\eth_{\mathfrak{a}}(\mathfrak{b}) \in (g_B, t)_\in$. Therefore $(f_B, s)_\in$ and $(g_B, t)_\in$ are weak filters of $\mathbb{X} := (X, |)$.

Conversely, suppose that the nonempty sets $(f_B, s)_\in$ and $(g_B, t)_\in$ are weak filters of $\mathbb{X} := (X, |)$ for all $(s, t) \in (0, 1] \times [0, 1)$. If (22) is not valid, then $f_B(1) < f_B(\mathfrak{a})$ or $g_B(1) > g_B(\mathfrak{b})$ for some $\mathfrak{a}, \mathfrak{b} \in X$. It follows that $1 \notin (f_B, s)_\in$ or $1 \notin (g_B, t)_\in$ where $s := f_B(\mathfrak{a})$ and $t := g_B(\mathfrak{b})$. This is a contradiction, and so (22) is valid. Suppose that (23) is not valid. Then $f_B(\eth_x(y)) < f_B(y)$ or $g_B(\eth_{\mathfrak{a}}(\mathfrak{b})) > g_B(\mathfrak{b})$. If we take $s := f_B(y)$ and $t := g_B(\mathfrak{b})$, then $\eth_x(y) \notin (f_B, s)_\in$ or $\eth_{\mathfrak{a}}(\mathfrak{b}) \notin (g_B, t)_\in$, a contradiction. Thus (23) is valid. Consequently, $\mathcal{B}^* := (X; f_B, g_B)$ is an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$ by Theorem 1.

Corollary 1. *If $\mathcal{B}^* := (X; f_B, g_B)$ is an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$, then its nonempty intuitionistic level set $(\mathcal{B}^*, (s, t))_\in$ is a weak filter of $\mathbb{X} := (X, |)$ for all $(s, t) \in (0, 1] \times [0, 1)$.*

We examine the conditions under which the intuitionistic fuzzy set becomes an intuitionistic fuzzy weak filter.

Theorem 3. *If an intuitionistic fuzzy set $\mathcal{B}^* := (X; f_B, g_B)$ in X satisfies* (20) *and*

$$x_{(s_1,t_1)} \in \mathcal{B}^*,\ \eth_x(y)_{(s_2,t_2)} \in \mathcal{B}^* \ \Rightarrow\ y_{(\min\{s_1,s_2\},\max\{t_1,t_2\})} \in \mathcal{B}^* \tag{24}$$

for all $x, y \in X$ and $(s_1, t_1), (s_2, t_2) \in (0, 1] \times [0, 1)$, then $\mathcal{B}^* := (X; f_B, g_B)$ is an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$.

Proof. Let $\mathcal{B}^* := (X; f_B, g_B)$ be an intuitionistic fuzzy set in X that satisfies (20) and (24). We first show that the condition (24) is equivalent to the following facts.

$$(\forall x, y \in X) \left(\begin{array}{l} f_B(y) \geq \min\{f_B(x), f_B(\eth_x(y))\} \\ g_B(y) \leq \max\{g_B(x), g_B(\eth_x(y))\} \end{array} \right). \tag{25}$$

Suppose that $\mathcal{B}^* := (X; f_B, g_B)$ satisfies (24) and let $x, y \in X$. If we take $(s_1, t_1) = (f_B(x), g_B(x))$ and $(s_2, t_2) = (f_B(\eth_x(y)), g_B(\eth_x(y)))$, then $x_{(s_1,t_1)} \in \mathcal{B}^*$ and $\eth_x(y)_{(s_2,t_2)} \in \mathcal{B}^*$. It follows from (24) that $y_{(\min\{s_1,s_2\},\max\{t_1,t_2\})} \in \mathcal{B}^*$. Hence $f_B(y) \geq \min\{s_1, s_2\} = \min\{f_B(x), f_B(\eth_x(y))\}$ and

$$g_B(y) \leq \max\{t_1, t_2\} = \max\{g_B(x), g_B(\eth_x(y))\}.$$

Now, assume that (25) is valid and let $x_{(s_1,t_1)} \in \mathcal{B}^*$ and $\eth_x(y)_{(s_2,t_2)} \in \mathcal{B}^*$ for all $(s_1, t_1), (s_2, t_2) \in (0, 1] \times [0, 1)$. Then $f_B(x) \geq s_1$, $g_B(x) \leq t_1$, $f_B(\eth_x(y)) \geq s_2$, and $g_B(\eth_x(y)) \leq t_2$. Using (25), we have

$$f_B(y) \geq \min\{f_B(x), f_B(\eth_x(y))\} \geq \min\{s_1, s_2\}$$

and $g_B(y) \leq \max\{g_B(x), g_B(\eth_x(y))\} \leq \max\{t_1, t_2\}$. Hence $y_{(\min\{s_1,s_2\},\max\{t_1,t_2\})} \in \mathcal{B}^*$, and therefore (24) is valid. The combination of (1) and (3) induces $y|(\eth_x(y)|\eth_x(y)) = 1$ for all $x, y \in X$. It follows from (22) and (25) that

$$\begin{aligned} f_B(\eth_x(y)) &\geq \min\{f_B(y), f_B(y|(\eth_x(y)|\eth_x(y)))\} \\ &= \min\{f_B(y), f_B(1)\} = f_B(y) \end{aligned}$$

and

$$\begin{aligned} g_B(\eth_x(y)) &\leq \max\{g_B(y), g_B(y|(\eth_x(y)|\eth_x(y)))\} \\ &= \max\{g_B(y), g_B(1)\} = g_B(y) \end{aligned}$$

for all $x, y \in X$. Therefore $\mathcal{B}^* := (X; f_B, g_B)$ is an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$ by Theorem 1.

In the following theorem, we use weak filters to form intuitionistic fuzzy weak filters.

Theorem 4. *For every nonempty subset F of X, consider an intuitionistic fuzzy set $\mathcal{B}^*_F := (X; f^F_B, g^F_B)$ in X which is given by*

$$\mathcal{B}^*_F := (X; f^F_B, g^F_B) : X \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (s_1, t_1) & \text{if } x \in F, \\ (s_2, t_2) & \text{otherwise} \end{cases}$$

where $(s_1, t_1), (s_2, t_2) \in (0,1] \times [0,1)$ with $s_1 > s_2$ and $t_1 < t_2$. Then $\mathcal{B}^*_F := (X; f^F_B, g^F_B)$ is an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$ if and only if F is a weak filter of $\mathbb{X} := (X, |)$.

Proof. Let $(s_1, t_1), (s_2, t_2) \in (0,1] \times [0,1)$ be such that $s_1 > s_2$ and $t_1 < t_2$. Suppose that $\mathcal{B}^*_F := (X; f^F_B, g^F_B)$ is an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$. Since $f_B(1) = s_1$ and $g_B(1) = t_1$ by (22), we have $1 \in F$. Let $x \in X$ and $y \in F$. Using (23), we have $f_B(\eth_x(y)) \geq f_B(y) = s_1$ and $g_B(\eth_x(y)) \leq g_B(y) = t_1$. Thus $f_B(\eth_x(y)) = s_1$ and $g_B(\eth_x(y)) = t_1$ which shows that $\eth_x(y) \in F$. Hence F is a weak filter of $\mathbb{X} := (X, |)$.

Conversely, assume that F is a weak filter of $\mathbb{X} := (X, |)$. Then $1 \in F$, and so $f_B(1) = s_1 \geq f_B(x)$ and $g_B(1) = t_1 \leq g_B(x)$ for all $x \in X$. Let $x, y \in X$. If $y \notin F$, then $f_B(y) = s_2 \leq f_B(\eth_x(y))$ and $g_B(y) = t_2 \geq g_B(\eth_x(y))$. If $y \in F$, then $\eth_x(y) \in F$, and thus $f_B(\eth_x(y)) = s_1 = f_B(y)$ and $g_B(\eth_x(y)) = t_1 = g_B(y)$. It follows from Theorem 1 that $\mathcal{B}^*_F := (X; f^F_B, g^F_B)$ is an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$.

The example below illustrates Theorem 4.

Example 2. *Consider the Sheffer stroke Hilbert algebra $\mathbb{X} := (X, |)$ in Example 1. We can observe that $F := \{c_1, c_5, c_6, c_7\}$ is a weak filter of $\mathbb{X} := (X, |)$. Hence the intuitionistic fuzzy set $\mathcal{B}^*_F := (X; f^F_B, g^F_B)$ given by*

$$\mathcal{B}^*_F := (X; f^F_B, g^F_B) : X \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (\frac{0.7}{n}, \frac{0.5}{2n}) & \text{if } x \in F, \\ (\frac{0.7}{2n}, \frac{0.5}{n}) & \text{otherwise,} \end{cases}$$

where n is a natural number, is an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$.

Corollary 2. *For every* $\mathfrak{a} \in X$ *and* $(s_1, t_1), (s_2, t_3) \in (0, 1] \times [0, 1)$ *with* $s_1 > s_2$ *and* $t_1 < t_2$, *consider an intuitionistic fuzzy set* $\mathcal{B}^*_{\vec{\mathfrak{a}}} := (X; f_B^{\vec{\mathfrak{a}}}, g_B^{\vec{\mathfrak{a}}})$ *in* X *which is defined by*

$$\mathcal{B}^*_{\vec{\mathfrak{a}}} := (X; f_B^{\vec{\mathfrak{a}}}, g_B^{\vec{\mathfrak{a}}}) : X \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (s_1, t_1) & \text{if } x \in \vec{\mathfrak{a}}, \\ (s_2, t_2) & \text{otherwise} \end{cases}$$

is an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$ *where*

$$\vec{\mathfrak{a}} := \{x \in X \mid \eth_{\mathfrak{a}}(x) = 1\}.$$

Proof. Since $\vec{\mathfrak{a}}$ is a weak filter of $\mathbb{X} := (X, |)$ for all $\mathfrak{a} \in X$ (see [13]), it follows from Theorem 4 that $\mathcal{B}^*_{\vec{\mathfrak{a}}} := (X; f_B^{\vec{\mathfrak{a}}}, g_B^{\vec{\mathfrak{a}}})$ is an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$ for all $\mathfrak{a} \in X$.

Corollary 3. *For every* $\mathfrak{b} \in X$ *and* $(s_1, t_1), (s_2, t_3) \in (0, 1] \times [0, 1)$ *with* $s_1 > s_2$ *and* $t_1 < t_2$, *consider an intuitionistic fuzzy set* $\mathcal{B}^*_{X^{\mathfrak{b}}} := (X; f_B^{X^{\mathfrak{b}}}, g_B^{X^{\mathfrak{b}}})$ *in* X *which is defined by*

$$\mathcal{B}^*_{X^{\mathfrak{b}}} := (X; f_B^{X^{\mathfrak{b}}}, g_B^{X^{\mathfrak{b}}}) : X \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (s_1, t_1) & \text{if } x \in X^{\mathfrak{b}}, \\ (s_2, t_2) & \text{otherwise} \end{cases}$$

is an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$ *where*

$$X^{\mathfrak{b}} := \{x \in X \mid \eth_{\mathfrak{b}}(x) = x\}.$$

Proof. Note that $X^{\mathfrak{b}}$ is a weak filter of $\mathbb{X} := (X, |)$ for all $\mathfrak{b} \in X$ (see [13]). Hence $\mathcal{B}^*_{X^{\mathfrak{b}}} := (X; f_B^{X^{\mathfrak{b}}}, g_B^{X^{\mathfrak{b}}})$ is an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$ for all $\mathfrak{b} \in X$ by Theorem 4.

Corollary 4. *Let* F *be a subset of* X. *For every* $\mathfrak{b} \in X$ *and* $(s_1, t_1), (s_2, t_2) \in (0, 1] \times [0, 1)$ *with* $s_1 > s_2$ *and* $t_1 < t_2$, *let* $\mathcal{B}^*_{F_{\mathfrak{b}}} := (X; f_B^{F_{\mathfrak{b}}}, g_B^{F_{\mathfrak{b}}})$ *be an intuitionistic fuzzy set in* X *which is defined by*

$$\mathcal{B}^*_{F_{\mathfrak{b}}} := (X; f_B^{F_{\mathfrak{b}}}, g_B^{F_{\mathfrak{b}}}) : X \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (s_1, t_1) & \text{if } x \in F_{\mathfrak{b}}, \\ (s_2, t_2) & \text{otherwise} \end{cases}$$

where $F_{\mathfrak{b}} := \{z \in X \mid \eth_{\mathfrak{b}}(x) = z, x \in F\}$. *If* F *is a weak filter of* $\mathbb{X} := (X, |)$, *then* $\mathcal{B}^*_{F_{\mathfrak{b}}} := (X; f_B^{F_{\mathfrak{b}}}, g_B^{F_{\mathfrak{b}}})$ *is an intuitionistic fuzzy weak filter of* $\mathbb{X} := (X, |)$.

Proof. If F is a weak filter of $\mathbb{X} := (X, |)$, then $F_{\mathfrak{b}}$ is a weak filter of $\mathbb{X} := (X, |)$ (see [13]). Thus $\mathcal{B}^*_{F_{\mathfrak{b}}} := (X; f_B^{F_{\mathfrak{b}}}, g_B^{F_{\mathfrak{b}}})$ is an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$ by Theorem 4.

Corollary 5. *Let* G *be a subset of* X. *For every* $(s_1, t_1), (s_2, t_2) \in (0, 1] \times [0, 1)$ *with* $s_1 > s_2$ *and* $t_1 < t_2$, *let* $\mathcal{B}^*_G := (X; f_B^G, g_B^G)$ *in* X *which is given by*

$$\mathcal{B}^*_G := (X; f_B^G, g_B^G) : X \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (s_1, t_1) & \text{if } x \in G^*, \\ (s_2, t_2) & \text{otherwise} \end{cases}$$

where $G^* := \{x \in X \mid (\forall y \in G)(\eth_x(y) = 1 \ \Rightarrow\ y = 1)\}$. *If* G *is a weak filter of* $\mathbb{X} := (X, |)$, *then* $\mathcal{B}^*_G := (X; f_B^G, g_B^G)$ *is an intuitionistic fuzzy weak filter of* $\mathbb{X} := (X, |)$.

Proof. If G is a weak filter of $\mathbb{X} := (X, |)$, then G^* is a weak filter of $\mathbb{X} := (X, |)$ (see [13]). Hence $\mathcal{B}^*_G := (X; f^G_B, g^G_B)$ is an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$ by Theorem 4.

Theorem 5. *Let* $\{F_i \mid i \in \Gamma \subseteq (0, 1]\}$ *and* $\{G_i \mid i \in \Lambda \subseteq [0, 1)\}$ *be collections of weak filters of* $\mathbb{X} := (X, |)$ *that satisfy* $\bigcup_{i\in\Gamma} F_i = X = \bigcup_{i\in\Lambda} G_i$, *and*

$$F_j \subset F_i \;\Leftrightarrow\; i < j \;\Leftrightarrow\; G_j \subset G_i$$

for all $i, j \in \Gamma \cup \Lambda$. *If we define an intuitionistic fuzzy set* $\mathcal{B}^* := (X; f_B, g_B)$ *in* X *by*

$$f_B(x) = \sup\{i \in \Gamma \mid x \in F_i\} \text{ and } g_B(x) = \inf\{i \in \Lambda \mid x \in G_i\}$$

for all $x \in X$, *then it is an intuitionistic fuzzy weak filter of* $\mathbb{X} := (X, |)$.

Proof. It is sufficient to show that $(f_B, s)_\in$ and $(g_B, t)_\in$ are weak filters of $\mathbb{X} := (X, |)$ for all $(s, t) \in (0, f(1)] \times [g_B(1), 1)$ according to Theorem 2. If $s = \sup\{s_\mathfrak{a} \in \Gamma \mid s_\mathfrak{a} < s\}$, then

$$x \in (f_B, s)_\in \;\Leftrightarrow\; (\forall s_\mathfrak{a} < s)(x \in F_{s_\mathfrak{a}}) \;\Leftrightarrow\; x \in \bigcap_{s_\mathfrak{a}<s} F_{s_\mathfrak{a}}.$$

Hence $(f_B, s)_\in = \bigcap_{s_\mathfrak{a}<s} F_{s_\mathfrak{a}}$ is a weak filter of $\mathbb{X} := (X, |)$. Suppose that $s \neq \sup\{s_\mathfrak{a} \in \Gamma \mid s_\mathfrak{a} < s\}$. If $x \in \bigcup_{s_\mathfrak{a}\geq s} F_{s_\mathfrak{a}}$, then $x \in F_{s_\mathfrak{a}}$ for some $s_\mathfrak{a} \geq s$, and so $f_B(x) = \sup\{s \in \Gamma \mid x \in F_s\} \geq s_\mathfrak{a} \geq s$, i.e., $x \in (f_B, s)_\in$. This shows that $\bigcup_{s_\mathfrak{a}\geq s} F_{s_\mathfrak{a}} \subseteq (f_B, s)_\in$. If $x \notin \bigcup_{s_\mathfrak{a}\geq s} F_{s_\mathfrak{a}}$, then $x \notin F_{s_\mathfrak{a}}$ for all $s_\mathfrak{a} \geq s$. Since $s \neq \sup\{s_\mathfrak{a} \in \Gamma \mid s_\mathfrak{a} < s\}$, there exists $\varepsilon_\mathfrak{a} > 0$ such that $(s - \varepsilon_\mathfrak{a}, s) \cap \Gamma = \emptyset$. Hence $x \notin F_{s_\mathfrak{a}}$ for all $s_\mathfrak{a} > s - \varepsilon_\mathfrak{a}$, and so if $x \in F_{s_\mathfrak{a}}$ then $s_\mathfrak{a} \leq s - \varepsilon_\mathfrak{a}$. Thus $f_B(x) \leq s - \varepsilon_\mathfrak{a} < s$, that is, $x \notin (f_B, s)_\in$. Therefore $(f_B, s)_\in = \bigcup_{s_\mathfrak{a}\geq s} F_{s_\mathfrak{a}}$, and it is a weak filter of $\mathbb{X} := (X, |)$. In order to show that $(g_B, t)_\in$ is a weak filter of $\mathbb{X} := (X, |)$, we need to consider the following two cases:

$$t \neq \inf\{t_\mathfrak{b} \in \Lambda \mid t_\mathfrak{b} > t\} \text{ and } t = \inf\{t_\mathfrak{b} \in \Lambda \mid t_\mathfrak{b} > t\}.$$

If the first case is valid, then $(t, t + \varepsilon_\mathfrak{b})$ and Λ are disjoint for some $\varepsilon_\mathfrak{b} > 0$. We will verify that $(g_B, t)_\in = \bigcup_{t_\mathfrak{b}\leq t} G_{t_\mathfrak{b}}$. If $y \in \bigcup_{t_\mathfrak{b}\leq t} G_{t_\mathfrak{b}}$, then $y \in G_{t_\mathfrak{b}}$ for some $t_\mathfrak{b} \leq t$. It follows that $g_B(y) = \inf\{t \in \Lambda \mid y \in G_t\} \leq t_\mathfrak{b} \leq t$, i.e., $y \in (g_B, t)_\in$. If $y \notin \bigcup_{t_\mathfrak{b}\leq t} G_{t_\mathfrak{b}}$, then $y \notin G_{t_\mathfrak{b}}$ for all $t_\mathfrak{b} \leq t < t + \varepsilon_\mathfrak{b}$. This shows that if $y \in G_{t_\mathfrak{b}}$, then $t_\mathfrak{b} \geq t + \varepsilon_\mathfrak{b} > t$, i.e., $y \notin (g_B, t)_\in$. Hence $(g_B, t)_\in = \bigcup_{t_\mathfrak{b}\leq t} G_{t_\mathfrak{b}}$ which is a weak filter of $\mathbb{X} := (X, |)$. For the second case, we have

$$y \in (g_B, t)_\in \;\Leftrightarrow\; (\forall t_\mathfrak{b} > t)(y \in G_{t_\mathfrak{b}}) \;\Leftrightarrow\; y \in \bigcap_{t_\mathfrak{b}>t} G_{t_\mathfrak{b}}.$$

Hence $(g_B, t)_\in = \bigcap_{t_\mathfrak{b} > t} G_{t_\mathfrak{b}}$ is a weak filter of $\mathbb{X} := (X, |)$. This completes the proof.

Given an intuitionistic fuzzy set $\mathcal{B}^* := (X; f_B, g_B)$ in X, consider the following set:

$$X_{(0,1)} := \{x \in X \mid f_B(x) \neq 0,\, g_B(x) \neq 1\}$$

which is called the $(0,1)$-*set* of $\mathcal{B}^* := (X; f_B, g_B)$.

We explore the conditions under which the nonempty $(0,1)$-set is a weak filter.

Theorem 6. *If $\mathcal{B}^* := (X; f_B, g_B)$ is an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$, then its nonempty $(0,1)$-set is a weak filter of $\mathbb{X} := (X, |)$.*

Proof. Let $\mathcal{B}^* := (X; f_B, g_B)$ be an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$. Suppose $X_{(0,1)} \neq \emptyset$, say $x \in X_{(0,1)}$. Then $f_B(1) \geq f_B(x) \neq 0$ and $g_B(1) \leq g_B(x) \neq 1$ by (22). Thus $1 \in X_{(0,1)}$. Let $x \in X$ and $y \in X_{(0,1)}$. Then $f_B(\eth_x(y)) \geq f_B(y) \neq 0$ and $g_B(\eth_x(y)) \leq g_B(y) \neq 1$ by (23). Hence $\eth_x(y) \in X_{(0,1)}$, and therefore $X_{(0,1)}$ is a weak filter of $\mathbb{X} := (X, |)$.

In the following example, we can observe that the converse of Theorem 6 is not true in general.

Example 3. *Let $\mathbb{X} := (X, |)$ be the Sheffer stroke Hilbert algebra described in Example 1 and let $\mathcal{B}^* := (X; f_B, g_B)$ be an intuitionistic fuzzy set in X given as follows:*

$$\mathcal{B}^* := (X; f_B, g_B) : X \to [0,1] \times [0,1],$$
$$\mathfrak{b} \mapsto \begin{cases} \left(\frac{0.45}{n}, \frac{0.87}{2n}\right) & \text{if } \mathfrak{b} = c_1, \\ \left(\frac{0.69}{n}, \frac{0.56}{2n}\right) & \text{if } \mathfrak{b} \in \{c_5, c_6\}, \\ (0.00, 1.00) & \textit{otherwise} \end{cases}$$

where n is a natural number. Then $X_{(0,1)} = \{c_1, c_5, c_6\}$ which is a weak filter of $\mathbb{X} := (X, |)$. We can observe that $f_B(c_1) = \frac{0.45}{n} < \frac{0.69}{n} = f_B(c_5)$ and/or $g_B(c_1) = \frac{0.87}{2n} > \frac{0.56}{2n} = g_B(c_6)$, that is, (22) is not valid. Hence $\mathcal{B}^* := (X; f_B, g_B)$ is not an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$.

Theorem 7. *If an intuitionistic fuzzy set $\mathcal{B}^* := (X; f_B, g_B)$ in X satisfies:*

$$(\forall x \in X)(\forall (s,t) \in (0,1] \times [0,1)) \left(x_{(s,t)} \in \mathcal{B}^* \;\Rightarrow\; 1_{(s,t)}\, q\, \mathcal{B}^*\right), \tag{26}$$

$$(\forall x, y \in X)(\forall (s,t) \in (0,1] \times [0,1)) \left(y_{(s,t)} \in \mathcal{B}^* \;\Rightarrow\; \eth_x(y)_{(s,t)}\, q\, \mathcal{B}^*\right), \tag{27}$$

then its nonempty $(0,1)$-set is a weak filter of $\mathbb{X} := (X, |)$.

Proof. Let $\mathcal{B}^* := (X; f_B, g_B)$ be an intuitionistic fuzzy set in X that satisfies (26) and (27). Suppose that $X_{(0,1)} \neq \emptyset$ and say $x \in X_{(0,1)}$. Then $f_B(x) \neq 0$ and $g_B(x) \neq 1$. Since $x_{(f_B(x), g_B(x))} \in \mathcal{B}^*$, we have $1_{(f_B(x), g_B(x))} \in \mathcal{B}^*$ by (26). Hence $f_B(1) \geq f_B(x) \neq 0$ and $g_B(1) \leq g_B(x) \neq 1$, and so $1 \in X_{(0,1)}$. Let $x \in X$ and $y \in X_{(0,1)}$. Then $f_B(y) \neq 0$, $g_B(y) \neq 1$ and $y_{(f_B(y), g_B(y))} \in \mathcal{B}^*$. It follows from (27) that $\eth_x(y)_{(f_B(y), g_B(y))} \in \mathcal{B}^*$. Thus $f_B(\eth_x(y)) \geq f_B(y) \neq 0$ and $g_B(\eth_x(y)) \leq g_B(y) \neq 1$ which imply that $\eth_x(y) \in X_{(0,1)}$. Therefore $X_{(0,1)}$ is a weak filter of $\mathbb{X} := (X, |)$.

Theorem 8. *If an intuitionistic fuzzy set* $\mathcal{B}^* := (X; f_B, g_B)$ *in* X *satisfies:*

$$(\forall x \in X)(\forall (s,t) \in (0,1] \times [0,1)) \left(x_{(s,t)} \, q \, \mathcal{B}^* \;\Rightarrow\; 1_{(s,t)} \in \mathcal{B}^* \right), \tag{28}$$

$$(\forall x, y \in X)(\forall (s,t) \in (0,1] \times [0,1)) \left(y_{(s,t)} \, q \, \mathcal{B}^* \;\Rightarrow\; \eth_x(y)_{(s,t)} \in \mathcal{B}^* \right), \tag{29}$$

then its nonempty $(0,1)$*-set is a weak filter of* $\mathbb{X} := (X, |)$.

Proof. Let $\mathcal{B}^* := (X; f_B, g_B)$ be an intuitionistic fuzzy set in X that satisfies (28) and (29). Suppose that $X_{(0,1)} \neq \emptyset$ and say $x \in X_{(0,1)}$. Then $f_B(x) \neq 0$ and $g_B(x) \neq 1$. Thus $f_B(x) + 1 > 1$ and $g_B(x) + 0 < 1$, i.e., $x_{(1,0)} \, q \, \mathcal{B}^*$. It follows from (28) that $1_{(1,0)} \in \mathcal{B}^*$, that is, $f_B(1) \geq 1$ and $g_B(1) \leq 0$. Thus $1 \in X_{(0,1)}$. If $x \in X$ and $y \in X_{(0,1)}$, then $f_B(y) \neq 0$ and $g_B(y) \neq 1$, and so $f_B(y) + 1 > 1$ and $g_B(y) + 0 < 1$, i.e., $y_{(1,0)} \, q \, \mathcal{B}^*$. Using (29) induces $\eth_x(y)_{(1,0)} \in \mathcal{B}^*$. Hence $f_B(\eth_x(y)) \geq 1$ and $g_B(\eth_x(y)) \leq 0$ which shows that $\eth_x(y) \in X_{(0,1)}$. Therefore $X_{(0,1)}$ is a weak filter of $\mathbb{X} := (X, |)$.

Theorem 9. *If an intuitionistic fuzzy set* $\mathcal{B}^* := (X; f_B, g_B)$ *in* X *satisfies:*

$$(\forall x \in X)(\forall (s,t) \in (0,1] \times [0,1)) \left(x_{(s,t)} \, q \, \mathcal{B}^* \;\Rightarrow\; 1_{(s,t)} \, q \, \mathcal{B}^* \right), \tag{30}$$

$$(\forall x, y \in X)(\forall (s,t) \in (0,1] \times [0,1)) \left(y_{(s,t)} \, q \, \mathcal{B}^* \;\Rightarrow\; \eth_x(y)_{(s,t)} \, q \, \mathcal{B}^* \right), \tag{31}$$

then its nonempty $(0,1)$*-set is a weak filter of* $\mathbb{X} := (X, |)$.

Proof. Let $\mathcal{B}^* := (X; f_B, g_B)$ be an intuitionistic fuzzy set in X that satisfies (30) and (31). Suppose that $X_{(0,1)} \neq \emptyset$ and say $x \in X_{(0,1)}$. Then $f_B(x) \neq 0$ and $g_B(x) \neq 1$. Thus $f_B(x) + 1 > 1$ and $g_B(x) + 0 < 1$, i.e., $x_{(1,0)} \, q \, \mathcal{B}^*$. Using (30) induces $1_{(1,0)} \, q \, \mathcal{B}^*$, that is, $f_B(1) + 1 > 1$ and $g_B(1) + 0 < 1$. Thus $f_B(1) \neq 0$ and $g_B(1) \neq 1$, i.e., $1 \in X_{(0,1)}$. If $x \in X$ and $y \in X_{(0,1)}$, then $f_B(y) \neq 0$ and $g_B(y) \neq 1$, and so $f_B(y) + 1 > 1$ and $g_B(y) + 0 < 1$, i.e., $y_{(1,0)} \, q \, \mathcal{B}^*$. It follows from (31) that $\eth_x(y)_{(1,0)} \, q \, \mathcal{B}^*$. Hence $f_B(\eth_x(y)) + 1 > 1$ and $g_B(\eth_x(y)) + 0 < 1$, and so $\eth_x(y) \in X_{(0,1)}$. Therefore $X_{(0,1)}$ is a weak filter of $\mathbb{X} := (X, |)$.

We provide conditions for the intuitionistic level set and intuitionistic q-set to be weak filters.

Theorem 10. *If an intuitionistic fuzzy set* $\mathcal{B}^* := (X; f_B, g_B)$ *in* X *satisfies:*

$$f_B(x) \leq \max\{f_B(1), 0.5\}, \; g_B(x) \geq \min\{g_B(1), 0.5\}, \tag{32}$$

$$f_B(y) \leq \max\{f_B(\eth_x(y)), 0.5\}, \; g_B(y) \geq \min\{g_B(\eth_x(y)), 0.5\} \tag{33}$$

for all $x, y \in X$*, then its nonempty intuitionistic level set* $(\mathcal{B}^*, (s,t))_\in$ *is a weak filter of* $\mathbb{X} := (X, |)$ *for all* $(s,t) \in (0.5, 1] \times [0, 0.5)$.

Proof. Let $(s,t) \in (0.5, 1] \times [0, 0.5)$ be such that $(\mathcal{B}^*, (s,t))_\in \neq \emptyset$, say $x \in (\mathcal{B}^*, (s,t))_\in$. Then $x \in (f_B, s)_\in \cap (g_B, t)_\in$, which implies from (32) that

$$\max\{f_B(1), 0.5\} \geq f_B(x) \geq s > 0.5$$

and $\min\{g_B(1), 0.5\} \leq g_B(x) \leq t < 0.5$. Hence $f_B(1) \geq s$ and $g_B(1) \leq t$, and so $1 \in (f_B, s)_\in \cap (g_B, t)_\in = (\mathcal{B}^*, (s,t))_\in$. If $x \in X$ and $y \in (\mathcal{B}^*, (s,t))_\in$, then $f_B(y) \geq s$ and $g_B(y) \leq t$. It follows from (33) that

$$0.5 < s \leq f_B(y) \leq \max\{f_B(\eth_x(y)), 0.5\}$$

and $0.5 > t \geq g_B(y) \geq \min\{g_B(\eth_x(y)), 0.5\}$. Hence $f_B(\eth_x(y)) \geq s$ and $g_B(\eth_x(y)) \leq t$, which imply that $\eth_x(y) \in (f_B, s)_\in \cap (g_B, t)_\in = (\mathcal{B}^*, (s,t))_\in$. Therefore $(\mathcal{B}^*, (s,t))_\in$ is a weak filter of $\mathbb{X} := (X, |)$.

Theorem 11. *If $\mathcal{B}^* := (X; f_B, g_B)$ is an intuitionistic fuzzy weak filter of $\mathbb{X} := (X, |)$, then its nonempty intuitionistic q-set $(\mathcal{B}^*, (s,t))_q$ is a weak filter of $\mathbb{X} := (X, |)$ for all $(s,t) \in (0,1] \times [0,1)$.*

Proof. Let $(s,t) \in (0,1] \times [0,1)$ be such that $(\mathcal{B}^*, (s,t))_q \neq \emptyset$. Since $f_B(1) \geq f_B(x)$ and $g_B(1) \leq g_B(x)$ for $x \in (\mathcal{B}^*, (s,t))_q$, we have $f_B(1) \geq f_B(x) > 1 - s$ and $g_B(1) \leq g_B(x) < 1 - t$. Hence $1 \in (f_B, s)_q \cap (g_B, t)_q = (\mathcal{B}^*, (s,t))_q$. If $y \in (\mathcal{B}^*, (s,t))_q$, then $f_B(\eth_x(y)) \geq f_B(y) > 1 - s$ and $g_B(\eth_x(y)) \leq g_B(y) < 1 - t$ for all $x \in X$ by (23). Thus $\eth_x(y) \in (f_B, s)_q \cap (g_B, t)_q = (\mathcal{B}^*, (s,t))_q$. Therefore $(\mathcal{B}^*, (s,t))_q$ is a weak filter of $\mathbb{X} := (X, |)$.

Proposition 3. *Let $\mathcal{B}^* := (X; f_B, g_B)$ be an intuitionistic fuzzy set in X. If its intuitionistic q-set $(\mathcal{B}^*, (s,t))_q$ is a weak filter of $\mathbb{X} := (X, |)$ for all $(s,t) \in (0, 0.5] \times [0.5, 1)$, then $1 \in (\mathcal{B}^*, (s,t))_\in$ and*

$$y \in (\mathcal{B}^*, (s,t))_q \;\Rightarrow\; \eth_x(y) \in (\mathcal{B}^*, (s,t))_\in$$

for all $x, y \in X$ and $(s,t) \in (0, 0.5] \times [0.5, 1)$.

Proof. Let $x, y \in X$ and $(s,t) \in (0, 0.5] \times [0.5, 1)$. Assume that $(\mathcal{B}^*, (s,t))_q$ is a weak filter of $\mathbb{X} := (X, |)$. Then $1 \in (\mathcal{B}^*, (s,t))_q$ and so $f_B(1) > 1 - s \geq s$ and $g_B(1) < 1 - t \leq t$. Hence $1 \in (f_B, s)_\in \cap (g_B, t)_\in = (\mathcal{B}^*, (s,t))_\in$. If $y \in (\mathcal{B}^*, (s,t))_q$, then $\eth_x(y) \in (\mathcal{B}^*, (s,t))_q$. It follows that $f_B(\eth_x(y)) > 1 - s \geq s$ and $g_B(\eth_x(y)) < 1 - t \leq t$. Thus $\eth_x(y) \in (f_B, s)_\in \cap (g_B, t)_\in = (\mathcal{B}^*, (s,t))_\in$.

Proposition 4. *Given an intuitionistic fuzzy set $\mathcal{B}^* := (X; f_B, g_B)$ in X, if its intuitionistic q-set $(\mathcal{B}^*, (s,t))_q$ is a weak filter of $\mathbb{X} := (X, |)$ for all $(s,t) \in (0.5, 1] \times [0, 0.5)$, then the following is valid.*

$$y \in (\mathcal{B}^*, (s,t))_\in \;\Rightarrow\; \eth_x(y) \in (\mathcal{B}^*, (s,t))_q$$

for all $x, y \in X$ and $(s,t) \in (0.5, 1] \times [0, 0.5)$.

Proof. Let $x, y \in X$ and $(s,t) \in (0.5, 1] \times [0, 0.5)$. If $y \in (\mathcal{B}^*, (s,t))_\in$, then $f_B(y) \geq s > 1 - s$ and $g_B(y) \leq t < 1 - t$. Thus $y \in (f_B, s)_q \cap (g_B, t)_q = (\mathcal{B}^*, (s,t))_q$, and so $\eth_x(y) \in (\mathcal{B}^*, (s,t))_q$.

Theorem 12. *If an intuitionistic fuzzy set* $\mathcal{B}^* := (X; f_B, g_B)$ *in* X *satisfies:*

$$x_{(s,t)}\, q\, \mathcal{B}^* \;\Rightarrow\; 1_{(s,t)} \in\!\vee q\, \mathcal{B}^*, \tag{34}$$

$$y_{(s,t)}\, q\, \mathcal{B}^* \;\Rightarrow\; \eth_x(y)_{(s,t)} \in\!\vee q\, \mathcal{B}^*, \tag{35}$$

for all $x, y \in X$ *and* $(s,t) \in (0.5, 1] \times [0, 0.5)$*, then its nonempty intuitionistic q-set* $(\mathcal{B}^*, (s,t))_q$ *is a weak filter of* $\mathbb{X} := (X, |)$ *for all* $(s,t) \in (0.5, 1] \times [0, 0.5)$

Proof. Let $(s,t) \in (0.5, 1] \times [0, 0.5)$ be such that $(\mathcal{B}^*, (s,t))_q \neq \emptyset$, say $x \in (\mathcal{B}^*, (s,t))_q$. Then $x_{(s,t)}\, q\, \mathcal{B}^*$, and so $1_{(s,t)} \in\!\vee q\, \mathcal{B}^*$ by (34), i.e., $1_{(s,t)} \in \mathcal{B}^*$ or $1_{(s,t)}\, q\, \mathcal{B}^*$. If $1_{(s,t)}\, q\, \mathcal{B}^*$, then $1 \in (\mathcal{B}^*, (s,t))_q$. If $1_{(s,t)} \in \mathcal{B}^*$, then $f_B(1) \geq s > 1 - s$ and $g_B(1) \leq t < 1 - t$. Hence $1 \in (f_B, s)_q \cap (g_B, t)_q = (\mathcal{B}^*, (s,t))_q$. Let $y \in (\mathcal{B}^*, (s,t))_q$. Then $y_{(s,t)}\, q\, \mathcal{B}^*$, and so $\eth_x(y)_{(s,t)} \in\!\vee q\, \mathcal{B}^*$ by (35), that is, $\eth_x(y)_{(s,t)} \in \mathcal{B}^*$ or $\eth_x(y)_{(s,t)}\, q\, \mathcal{B}^*$. If $\eth_x(y)_{(s,t)}\, q\, \mathcal{B}^*$, then $\eth_x(y) \in (\mathcal{B}^*, (s,t))_q$. If $\eth_x(y)_{(s,t)} \in \mathcal{B}^*$, then $f_B(\eth_x(y)) \geq s > 1 - s$ and $g_B(\eth_x(y)) \leq t < 1 - t$. Hence $\eth_x(y) \in (f_B, s)_q \cap (g_B, t)_q = (\mathcal{B}^*, (s,t))_q$. Consequently, $(\mathcal{B}^*, (s,t))_q$ is a weak filter of $\mathbb{X} := (X, |)$.

Theorem 13. *Given a weak filter* F *of* $\mathbb{X} := (X, |)$*, if an intuitionistic fuzzy set* $\mathcal{B}^* := (X; f_B, g_B)$ *in* X *satisfies* $\mathcal{B}^*(x) = (0, 1)$*, i.e.,* $f_B(x) = 0$ *and* $g_B(x) = 1$*, for* $x \in X \setminus F$ *and* $x \in (\mathcal{B}^*, (0.5, 0.5))_\in$ *for* $x \in F$*, then its nonempty intuitionistic q-set* $(\mathcal{B}^*, (s,t))_q$ *is a weak filter of* $\mathbb{X} := (X, |)$ *for all* $(s,t) \in (0.5, 1] \times [0, 0.5)$*.*

Proof. Let $(s,t) \in (0.5, 1] \times [0, 0.5)$ be such that $(\mathcal{B}^*, (s,t))_q \neq \emptyset$, say $y \in (\mathcal{B}^*, (s,t))_q$. Then $y_{(s,t)}\, q\, \mathcal{B}^*$, and so $f_B(y) + s > 1$ and $g_B(y) + t < 1$. If $y \in X \setminus F$, then $1 < f_B(y) + s = 0 + s = s$ and $1 > g_B(y) + t = 1 + s$ which is a contradiction. Hence $y \in F$, and thus $y \in (\mathcal{B}^*, (0.5, 0.5))_\in$, that is, $g_B(y) \leq 0.5 \leq f_B(y)$. Since $1 \in F$, we have $1 \in (\mathcal{B}^*, (0.5, 0.5))_\in$, that is, $f_B(1) \geq 0.5$ and $g_B(1) \leq 0.5$. If $1_{(s,t)} \overline{\in} \mathcal{B}^*$, then $f_B(1) < s$ or $g_B(1) > t$. At this time, the following three cases should be considered.

(i) $f_B(1) < s$ and $g_B(1) > t$.

(ii) $f_B(1) < s$ and $g_B(1) \leq t$.

(iii) $f_B(1) \geq s$ and $g_B(1) > t$.

The first case induces $f_B(1) + s > 2f_B(1) \geq 1$ and $g_B(1) + t < 2g_B(1) \leq 1$. For the case (ii), we get $f_B(1) + s > 2f_B(1) \geq 1$ and $g_B(1) + t < 2t \leq 1$. The third case implies that $f_B(1) + s > 2s \geq 1$ and $g_B(1) + t < 2g_B(1) \leq 1$. This shows that $1_{(s,t)}\, q\, \mathcal{B}^*$ and consequently $1_{(s,t)} \in\!\vee q\, \mathcal{B}^*$. Since $y \in F$, we have $\eth_x(y) \in F$ for all $x \in X$ because F is a weak filter of $\mathbb{X} := (X, |)$. Thus $\eth_x(y) \in (\mathcal{B}^*, (0.5, 0.5))_\in$, that is, $f_B(\eth_x(y)) \geq 0.5$ and $g_B(\eth_x(y)) \leq 0.5$. If $\eth_x(y)_{(s,t)} \overline{\in} \mathcal{B}^*$, then $f_B(\eth_x(y)) < s$ or $g_B(\eth_x(y)) > t$. At this time, the following three cases should be considered.

(iv) $f_B(\eth_x(y)) < s$ and $g_B(\eth_x(y)) > t$.

(v) $f_B(\eth_x(y)) < s$ and $g_B(\eth_x(y)) \leq t$.

(vi) $f_B(\eth_x(y)) \geq s$ and $g_B(\eth_x(y)) > t$.

The case (iv) induces $f_B(\eth_x(y))+s > 2f_B(\eth_x(y)) \geq 1$ and $g_B(\eth_x(y))+t < 2g_B(\eth_x(y)) \leq 1$. For the case (v), we hace $f_B(\eth_x(y))+s > 2f_B(\eth_x(y)) \geq 1$ and $g_B(\eth_x(y))+t < 2t \leq 1$. The case (vi) implies that $f_B(\eth_x(y)) + s > 2s \geq 1$ and $g_B(\eth_x(y)) + t < 2g_B(\eth_x(y)) \leq 1$. This shows that $\eth_x(y)_{(s,t)}\, q\, \mathcal{B}^*$ and consequently $\eth_x(y)_{(s,t)} \in\!\vee q\, \mathcal{B}^*$. It follows from Theorem 12 that $(\mathcal{B}^*, (s,t))_q$ is a weak filter of $\mathbb{X} := (X, |)$.

4. Conclusion

This work aims to advance the theoretical framework of Schaeffer stroke Hilbert algebras using intuitionistic fuzzy points to develop weak filters. We have introduced the concept of intuitionistic fuzzy weak filters in Sheffer stroke Hilbert algebras, and have investigated several properties. We have explored the conditions under which the intuitionistic fuzzy set becomes an intuitionistic fuzzy weak filter. We have discussed the characterization of intuitionistic fuzzy weak filters. Forming the $(0;1)$-set and q-set for the intuitionistic fuzzy set, we have discussed the phases in which it can be a weak filter. Based on the ideas and results of this paper, We will study the several types of substructures using intuitionistic fuzzy points in the study of Sheffer stroke theory for various forms of logical algebras.

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References

[1] I. Chajad. Sheffer operation in ortholattices. *Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica*, 44(1):19–23, 2005.

[2] T. Katican. Branchesm and obstinate sbe-filters of sheffer stroke be-algebras. *Bulletin of the International Mathematical Virtual Institute*, 12(1):41–50, 2022.

[3] V. Kozarkiewicz and A. Grabowski. Axiomatization of boolean algebras based on sheffer stroke. *Formalized Mathematics*, 12(3):355–361, 2004.

[4] T. Oner, T. Katican, and A. Borumand Saeid. Fuzzy filters of sheffer stroke hilbert algebras. *Journal of Intelligent & Fuzzy Systems*, 40(1):759–772, 2021.

[5] T. Oner, T. Katican, and A. Borumand Saeid. Relation between sheffer stroke and hilbert algebras. *Categories and General Algebraic Structures with Applications*, 14(1):245–268, 2021.

[6] T. Oner, T. Katican, and A. Borumand Saeid. Bl-algebras defined by an operator. *Honam Mathematical Journal*, 44(2):18–31, 2022.

[7] T. Oner, T. Katican, and A. Borumand Saeid. Class of sheffer stroke bck-algebras. *Analele Ştiinţifice ale Universităţii "Ovidius" Constanţa*, 30(1):247–269, 2022.

[8] T. Oner, T. Katican, A. Borumand Saeid, and M. Terziler. Filters of strong sheffer stroke non-associative mv-algebras. *Analele Ştiinţifice ale Universităţii "Ovidius" Constanţa*, 29(1):143–164, 2021.

[9] T. Oner, N. Rajesh, A. Iampan, and A. Borumand Saeid. Soft subalgebras and ideals of sheffer stroke hilbert algebras based on n-structures. *European Journal of Pure and Applied Mathematics*, 18(2):6018, 2025.

[10] N. Rajesh, T. Oner, A. Iampan, and A. Rezaei. Investigating length and mean-fuzzy subalgebras in sheffer stroke hilbert algebras. *European Journal of Pure and Applied Mathematics*, 18(2):5914, 2025.

[11] N. Rajesh, T. Oner, A. Iampan, and I. Senturk. On length and mean fuzzy ideals of sheffer stroke hilbert algebras. *European Journal of Pure and Applied Mathematics*, 18(1):5779, 2025.

[12] P. M. Pu and Y. M. Liu. Fuzzy topology i, neighborhood structure of a fuzzy point and moore-smith convergence. *Journal of Mathematical Analysis and Applications*, 76:571–599, 1980.

[13] Y. B. Jun and T. Oner. Weak filters and multipliers in sheffer stroke hilbert algebras. *Palestine Journal of Mathematics*, 14(2):749–761, 2025.

[14] H. M. Sheffer. A set of five independent postulates for boolean algebras. *Transactions of the American Mathematical Society*, 14(4):481–488, 1913.

[15] K. T. Atanassov. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20(1):87–96, 1986.

[16] A. Borumand Saeid, T. Oner, and Y. B. Jun. Intuitionistic fuzzy filters in sheffer stroke hilbert algebras. *Journal of Mathematical Extension*, 18(10):1–20, 2024.