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On Some Subcategories of Probabilistic Convergence Groups

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Dedicated to Prof. Dr. Abdul Moyeen Khan on the occasion of his 78th birthday

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Abstract. In this article, we focus on discussing some subcategories of the category of probabilistic convergence groups, **PConvGrp**. In so doing, we introduce a category of probabilistic neighborhood spaces, **PNeigh**, and a category of probabilistic topological neighborhood spaces, **PTopNeigh**. We identify probabilistic metric spaces as probabilistic neighborhood spaces. Introducing a category of probabilistic neighborhood groups, **PNeighGrp**, and a category of probabilistic topological neighborhood groups, **PTopNeighGrp**, we show that the category of probabilistic neighborhood groups, **PNeighGrp** - a topological category, and the category of probabilistic pretopological groups, **PPreTopGrp** are isomorphic, and find a categorical relationship of probabilistic neighborhood groups with probabilistic convergence groups. Furthermore, we introduce a category of probabilistic pre Cauchy groups, **PPreChyGrp** showing that the category of probabilistic Cauchy groups, **PChyGrp** - a category introduced earlier, is a full subcategory of the category **PPreChyGrp**. In this respect, we prove that in presence of probabilistic convergence group, the category of probabilistic Cauchy groups, **PChyGrp** and the category of strongly normal probabilistic limit groups, **SNPLimGrp** are isomorphic. Moreover, following a notion of probabilistic normed group - a notion studied earlier, we show that the category of probabilistic normed groups, **PNormedGrp** is isomorphic to the well-known category of probabilistic metric groups, **PMetGrp**; finally, we present probabilistic version of so-called invariance of norm theorem.

2020 Mathematics Subject Classifications: 54A20, 54E70, 54H11, 54B30

Key Words and Phrases: Probabilistic metric space, probabilistic Tardiff neighborhood system, probabilistic neighborhood space, probabilistic convergence space, probabilistic convergence groups, probabilistic pre-Cauchy space, probabilistic neighborhood group, probabilistic metric group, probabilistic normed group.

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1. Introduction

Since the inception of the notion of Menger probabilistic metric spaces, [27] (see also, [34]), enormous quantity of work surfaced over the years in a wide variety of areas, we include

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here a tiny list for the convenience of the reader, [3, 5–8, 14, 18, 22, 23, 29, 31, 33–39]. Following the category of probabilistic convergence spaces, **PConv**, [22, 32] - a supercategory of the category of topological spaces, **Top**(see also, [30]), in [3] the authors introduced the concept of probabilistic convergence groups; upon using Tardiff neighborhood system as introduced in [38], they showed that every probabilistic metric group is a probabilistic convergence group; further they discussed the unifmormizability and metrizability of probabilistic convergence groups. It goes without saying that the notion of probabilistic normed spaces play a vital role in functional analysis, [12, 20], and as such, quite a good number of papers appeared in recent years, see f.i., [29, 31] where the notion of probabilistic normed groups and their non categorical aspects are discussed. In [3], the authors of this article also introduced the notion of norm groups; in 1963, Šerstnev, [35] studied probabilistic norm on linear spaces. In almost all of these work except [3] where notion of Tardiff neighborhood system is considered, and in other cases, the notion of so-called strong topology, [34] are used.
Our motivations are to introduce a category of probabilistic neighborhood groups - a topological category, and a subcategory probabilistic topological groups. We identify probabilistic neighborhood groups with probabilistic metric groups, among others fundamental results, show that the category of probabilistic neighborhood groups is isomorphic with the category of probabilistic pretopological groups.This probabilistic metric group gives rise to a probabilistic convergence group, [3]. A detail study is made on quantale-valued Cauchy tower spaces and quantale-valued convergence tower in [23] presenting their various connection with quantale-valued metric spaces; moreover, in [23] a thorough investigation is made on completeness of quantale-valued Cauchy tower spaces. In this article, we extract some the results in the light of probabilistic Cauchy spaces, and more importantly, we discuss the category of probabilistic pre-Cauchy spaces, and probabilistic pre-Cauchy groups, and their relationship with probabilistic Cauchy groups. Following a notion of so-called strong normality as introduced in [9], we introduce a notion of strongly normal probabilistic limit group, showing that the category of probabilistic Cauchy groups is isomorphic to the category of strongly norm limit groups under the largest triangle function τ. Finally, following the notion of probabilistic norm group as introduced in [3], we show that there exists a one-to-one correspondence between probabilistic metrics on groups and probabilistic norm-groups leading to the fact that the category of probabilistic normed groups is isomorphic to the category of probabilistic metric groups.

2. Preliminaries

If $(A, \leq)$ is an ordered set, we denote by $\bigwedge_{j\in J} \alpha_j$ the infimum, while $\bigvee_{j\in J} \alpha_j$ denotes the supremum, if they exist, of the set $\{\alpha_j : j \in J\} \subseteq A$. In case of a two-point set $\{\alpha, \beta\}$ we write $\alpha \wedge \beta$ and $\alpha \vee \beta$, respectively. A function $\varphi : [0, \infty] \longrightarrow [0, 1]$, which is non-decreasing, left-continuous on $(0, \infty)$ and satisfies $\varphi(0) = 0$ and $\varphi(\infty) = 1$, is called a *distance distribution function* [34]. The set of all distance distribution functions is denoted by Δ^+. For example, for each $0 \leq a < \infty$ the functions

$$\epsilon_a(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq a \\ 1 & \text{if } a < x \leq \infty \end{cases} \qquad \text{and} \qquad \epsilon_\infty(x) = \begin{cases} 0 & \text{if } 0 \leq x < \infty \\ 1 & \text{if } x = \infty \end{cases}$$

belong to Δ^+. It is well-known that $(\Delta^+, \leq)$ is a complete lattice with minimal element ϵ_∞ and maximal element ϵ_0. Thus for any nonempty set J and any family $(\varphi_j)_{j\in J}$ of distribution functions in Δ^+, the function $\varphi = \bigvee_{j\in J} \varphi_j$ is also in Δ^+. The following result is mentioned in Schweizer and Sklar [34].

Lemma 1. *(i) If $\varphi, \psi \in \Delta^+$, then also $\varphi \wedge \psi \in \Delta^+$.*

(ii) If $\varphi_j \in \Delta^+$ for all $j \in J$, then also $\bigvee_{j\in J} \varphi_j \in \Delta^+$.

Here, $\varphi \wedge \psi$ denotes the point-wise minimum of φ and ψ in $(\Delta^+, \leq)$ and $\bigvee_{j\in J} \varphi_j$ denotes the pointwise supremum of the family $\{\varphi_j : j \in J\}$ in $(\Delta^+, \leq)$. On the set Δ^+ we consider the *modified Lévy metric* [37], which is defined below for the convenience of the reader.
Let $\varphi, \psi \in \Delta^+$ and $\epsilon > 0$. Consider the following properties

$$A(\varphi, \psi; \epsilon) \Longleftrightarrow \varphi(x - \epsilon) - \epsilon \leq \psi(x), \text{ if } x \in [0, \tfrac{1}{\epsilon});$$
$$\text{and } B(\varphi, \psi; \epsilon) \Longleftrightarrow \varphi(x + \epsilon) + \epsilon \geq \psi(x), \text{ if } x \in [0, \tfrac{1}{\epsilon}).$$

Then the *modified Lévy metric* d_L on $\Delta^+ \times \Delta^+$ is given by

$$d_L(\varphi, \psi) = \bigwedge\{\epsilon > 0\colon A(\varphi, \psi; \epsilon) \text{ and } B(\varphi, \psi; \epsilon) \text{ hold }\}.$$

Definition 1. *A binary operation $\tau : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$, which is commutative, associative, non-decreasing in each place satisfying the boundary condition $\tau(\varphi, \epsilon_0) = \varphi$ for all $\varphi \in \Delta^+$, is called a* triangle function *[34]. The largest triangle function is the point-wise minimum $\mu(\varphi, \psi) = \varphi \wedge \psi$. It is easy to prove that a triangle function that is idempotent, i.e., for which $\tau(\varphi, \varphi) = \varphi$ for all $\varphi \in \Delta^+$, must be the largest triangle function.*

For a good survey on triangle functions, see e.g. [33]. A triangle function is called *continuous* [34, 38] if it is a continuous function with respect to the topology and product topology induced by the modified Lévy metric. A triangle function is called *sup-continuous* [34, 38] if $\tau(\bigvee_{j\in J} \varphi_j, \psi) = \bigvee_{j\in J} \tau(\varphi_j, \psi)$ for all $\varphi_j, \psi \in \Delta^+$ $(j \in J)$. For further study on sup-continuity and its relation to continuity, we refer to [34].
For a set S, we denote $P(S)$ its power set. We denote the set of filters on the set S by $\mathbb{F}(S)$. We order this set by set inclusion and we denote for $p \in S$ the point filter by $[p] = \{F \subseteq S : p \in F\}$. If $\mathbb{F} \in \mathbb{F}(S)$ and $\mathbb{G} \in \mathbb{F}(T)$, then the filter on $S \times T$ generated by the sets of the form $\{F \times G : F \in \mathbb{F}, G \in \mathbb{G}\}$ is denoted by $\mathbb{F} \times \mathbb{G}$. If $(S, \cdot)$ is a group and $\mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$, then we define $\mathbb{F} \odot \mathbb{G}$ as a filter generated by the sets $F \cdot G = \{pq : p \in F, q \in G\}$, where $F \in \mathbb{F}$ and $G \in \mathbb{G}$. The filter $\mathbb{F}^{-1}$ is generated by the sets $F^{-1} = \{p^{-1} : p \in F\}$ for $F \in \mathbb{F}$.
For notions of category theory we refer to Adámek et. al. [1].

A *construct* is a category $\mathcal{C}$ whose objects are structured sets (S,ξ) and morphisms are suitable mappings between the underlying sets. A construct is called *topological* if it allows *initial constructions*, i.e., if for any source $(f_j\colon S \longrightarrow (S_j,\xi_j))_{j\in J}$, there is a unique structure ξ on S such that a mapping $g\colon (T,\eta) \longrightarrow (S,\xi)$ is a morphism if and only if for each $j \in J$ the composition $f_j \circ g\colon (T,\eta) \longrightarrow (S_j,\xi_j)$ is a morphism, where (T,η) is a structured set.
A topological construct is called *Cartesian closed* if for each pair of objects $(S,\xi),(T,\eta)$, there is a structure on the set $\mathcal{C}(S,T)$ of morphisms from S to T such that mapping $ev\colon \mathcal{C}(S,T)\times S \longrightarrow T$, defined for any $f \in \mathcal{C}(S,T)$ and $s \in S$ by $ev(f,s) = f(s)$ (called an *evaluation mapping*) is a morphism, and for each object (Z,ζ), and each morphism $f\colon S \times Z \longrightarrow T$ the mapping $\hat{f}\colon Z \longrightarrow \mathcal{C}(S,T)$ defined by $\hat{f}(z)(x) = f(x,z)$ is a morphism.

Lemma 2. *[2] Let S and T be groups, $\mathbb{F},\mathbb{G},\mathbb{H} \in \mathbb{F}(S)$ and $f\colon S \longrightarrow T$ a group-homomorphism, then we have*

(a) $\mathbb{F}\odot\mathbb{F}^{-1} \le [e]$ *and* $\mathbb{F}^{-1}\odot\mathbb{F} \le [e]$;

(b) $[s]\odot[s]^{-1} = [s]^{-1}\odot[s] = [e]$;

(c) $[s^{-1}] = [s]^{-1}$;

(d) $[s\cdot t] = [s]\cdot[t]$;

(e) $(\mathbb{F}\odot\mathbb{G})\odot\mathbb{H} = \mathbb{F}\odot(\mathbb{G}\odot\mathbb{H})$;

(f) $(\mathbb{F}^{-1})^{-1} = \mathbb{F}$;

(g) $(\mathbb{F}\odot\mathbb{G})^{-1} = \mathbb{G}^{-1}\odot\mathbb{F}^{-1}$;

(h) $[e]\odot\mathbb{F} = \mathbb{F}\odot[e] = \mathbb{F}$;

(i) $(\mathbb{F}\wedge\mathbb{G})^{-1} = \mathbb{F}^{-1}\wedge\mathbb{G}^{-1}$;

(j) $(\mathbb{F}\wedge\mathbb{G})\odot\mathbb{H} = (\mathbb{F}\odot\mathbb{H})\wedge(\mathbb{G}\odot\mathbb{H})$;

(k) $f(\mathbb{F}\odot\mathbb{G}) = f(\mathbb{F})\odot f(\mathbb{G})$;

(l) $f(\mathbb{F}^{-1}) = (f(\mathbb{F}))^{-1}$.

3. Probabilistic metric spaces, their associated Tardiff's neighborhood systems, probabilistic closure operator and probabilistic convergence structures

Definition 2. *[34]* A probabilistic metric space under a triangle function τ *is a pair* (S,F) *where* $F\colon S\times S \longrightarrow \Delta^+$ *such that for all* $p,q \in S$ *the following properties hold:*
(PM1) $F(p,q) = \epsilon_0 \Longleftrightarrow p = q$;

(PM2) $F(p,q)=F(q,p)$;
(PM3) $\tau(F(p,q),F(q,r))\leq F(p,r)$.
We use the index notation $F_{p,q}$ for $F(p,q)$.
A mapping $f\colon (S,F)\longrightarrow (S',F')$ between probabilistic metric spaces is called non-expansive *if $F_{p,q}\leq F'_{f(p),f(q)}$ for all $p,q\in S$.*
The category of all probabilistic metric spaces and non-expansive mappings is denoted by ***PMet***.

We recall Tardiff's neighborhood systems from [38] that are based on the so-called *profile functions* introduced in [18]. A profile function is in fact just an element $\varphi\in\Delta^+$, where $\varphi(x)$, $x>0$, is *interpreted as the maximum probability assigned to the event that the distance between p and q is less than x.* Given a $\varphi\in\Delta^+$, $\epsilon>0$ and $p\in S$, the (φ,ϵ)-*neighborhood of p* is defined by

$$N_p^{\varphi,\epsilon}=\{q\in S\,:\,F_{p,q}(x+\epsilon)+\epsilon\geq\varphi(x),\forall x\in[0,\frac{1}{\epsilon})\}.$$

The set $\{N_p^{\varphi,\epsilon}\,:\,\epsilon>0\}$ is a filter basis, and the filter generated by this basis is denoted by $\mathfrak{N}_p^{\varphi}$. Note that $\mathfrak{N}_p^{\varphi}$ satisfies:

(PMTN1) $\mathfrak{N}_p^{\varphi}\in\mathbb{F}(S)$;

(PMTN2) $\mathfrak{N}_p^{\varphi}\leq[p]$;

(PMTN3) $\varphi\leq\psi$ implies $\mathfrak{N}_p^{\varphi}\leq\mathfrak{N}_p^{\psi}$.

Definition 3. *[22] Let S be a set. A family of mappings $(c_\varphi\colon\mathbb{F}(S)\longrightarrow P(S))_{\varphi\in\Delta^+}$ which satisfies the axioms*
(PCS1) $p\in c_\varphi([p])$, $p\in S$, $\varphi\in\Delta^+$;
(PCS2) if $\mathbb{F}\leq\mathbb{G}$, then $c_\varphi(\mathbb{F})\subseteq c_\varphi(\mathbb{G})$, $\forall\mathbb{F},\mathbb{G}\in\mathbb{F}(S)$ and $\forall\varphi\in\Delta^+$;
(PCS3) if $\varphi\leq\psi$, then $c_\psi(\mathbb{F})\subseteq c_\varphi(\mathbb{F})$, $\forall\mathbb{F},\mathbb{G}\in\mathbb{F}(S)$ and $\forall\varphi,\psi\in\Delta^+$;
(PCS4) $p\in c_{\epsilon_\infty}(\mathbb{F})$ $\forall p\in S$, $\mathbb{F}\in\mathbb{F}(S)$,
is called a probabilistic convergence structure on S. *The pair $\left(S,\overline{c}=(c_\varphi)_{\varphi\in\Delta^+}\right)$ is called a (distance distribution function indexed)* probabilistic convergence space.
If $(S,\overline{c})$ satisfies further the axiom (PCS5)

$$c_\varphi(\mathbb{F})\cap c_\varphi(\mathbb{G})\subseteq c_\varphi\left(\mathbb{F}\wedge\mathbb{G}\right),\ \forall\varphi\in\Delta^+,\ \forall\mathbb{F},\mathbb{G}\in\mathbb{F}(S),$$

then we speak of a probabilistic limit space. *The statement: $p\in c_\varphi(\mathbb{F})$ is interpreted in [22] as,* the probability that the distance between p and the limit points of $\mathbb{F}$ is smaller than x is at least $\varphi(x)$. *A probabilistic convergence space $\left(S,\overline{c}=(c_\varphi)_{\varphi\in\Delta^+}\right)$ is called a* probabilistic pretopological space *if it satisfies the axiom (PPT):*
*for all $p\in S$, $\varphi\in\Delta^+$ and $\mathbb{F}\in\mathbb{F}(S)$, we have $p\in c_\varphi(\mathbb{F})$ if and only if $\mathbb{F}\geq\mathfrak{N}_p^{\varphi}$ whence $\mathfrak{N}_p^{\varphi}=\bigwedge_{p\in c_\varphi(\mathbb{F})}\mathbb{F}$, called the φ-*neighborhood filter of p.
It is pointed out in [22] that the probabilistic convergence space $\left(S,\overline{c^F}=(c_\varphi^F)_{\varphi\in\Delta^+}\right)$ associated with probabilistic metric space (S,F), is a probabilistic pretopological space.

A mapping $f\colon \left(S,\overline{c^S}\right) \longrightarrow \left(T,\overline{c^T}\right)$ *between probabilistic convergence spaces (resp. between probabilistic pretopological spaces) is called* continuous *if* $f(p) \in c^T_\varphi(f(\mathbb{F}))$ *whenever* $p \in c^S_\varphi(\mathbb{F})$ *for every* $p \in S$ *and for every* $\mathbb{F} \in \mathbb{F}(S)$*, and* $\varphi \in \Delta^+$*; for convergence notations, we will suppress* S *and* T *if no confusion arises.*
PConv *denotes the category of probabilistic convergence spaces as objects and continuous mappings as morphisms while* ***PPreTop*** *denotes the category of probabilistic pretopological spaces.*

The dual of the following notion called φ-*interior operator* can be found details in [22]. Here we just present the notion of φ-closure operator for a particular result that we included herein this text without further elaboration. We intend to discuss this issue in a future paper in conjunction with so-called probabilistic approximation spaces.

Definition 4. *[22, 38] For a* $\varphi \in \Delta^+$*, a* φ-closure operator *on a set* S *is a mapping* $\mathbb{C}^\varphi\colon P(S) \longrightarrow P(S)$ *between power set of* S *satisfying following conditions:*

(c1) $\mathbb{C}^\varphi(\emptyset) = \emptyset$ *for all* $\varphi \in \Delta^+$*;*

(c2) $A \subseteq \mathbb{C}^\varphi(A)$ *for all* $A \in P(S)$*, and for all* $\varphi \in \Delta^+$*;*

(c3) $A \subseteq B$ *implies* $\mathbb{C}^\varphi(A) \subseteq \mathbb{C}^\varphi(B)$ *for all* $A, B \in P(S)$ *and for all* $\varphi \in \Delta^+$*;*

(c4) for all $\varphi, \psi \in \Delta^+$ *with* $\varphi \leq \psi$ *implies* $\mathbb{C}^\psi(A) \subseteq \mathbb{C}^\varphi(A)$ *for all* $A \in P(S)$*;*

Then the pair $\left(S, (\mathbb{C}^\varphi)_{\varphi\in\Delta^+}\right)$ *is called a* probabilistic closure space*. It is called* probabilistic topological space under a triangle function τ *if it further satisfies axiom (c5)*

$$\mathbb{C}^\varphi\left(\mathbb{C}^\psi(A)\right) \subseteq \mathbb{C}^{\tau(\varphi,\psi)} \text{ for all } A \neq \emptyset \text{ and for all } (\varphi,\psi) \in \Delta^+ \times \Delta^+.$$

Note that the axiom (c3) is equivalent to the axiom (c3)′: $\mathbb{C}^\varphi(A) \cup \mathbb{C}^\varphi(B) \subseteq \mathbb{C}^\varphi(A \cup B)$*. If* $(S, \mathbb{C})$ *is a probabilistic closure space and* $A \subseteq S$*, then the set* $\mathbb{C}^\varphi(A)$ *is called the* φ-closure of A *in* $(S, \mathbb{C})$*.*

Remark 1. *In any probabilistic convergence space* $\left(S, \overline{c} = (c_\varphi)_{\varphi\in\Delta^+}\right)$*, one can define* φ-closure *in the following way:* $\mathbb{C}_\varphi(A) = \{p \in S\colon\ \exists\ \mathbb{F} \in \mathbb{F}(S)\ \ni p \in c_\varphi(\mathbb{F}) \Rightarrow A \in \mathbb{F}\}$*, for any subset* $A \subseteq S$*.*

Remark 2. *The category* ***PConv*** *is a Cartesian closed topological category [22]. In particular, the initial probabilistic convergence structure for a given source* $\Big(f_i\colon S \longrightarrow \left(S_i, (c^i_\varphi)_{\varphi\in\Delta^+}\right)\Big)_{i\in J}$ *is given by*

$$p \in c_\varphi(\mathbb{F}) \Longleftrightarrow f_i(p) \in c^i_\varphi\left(f_i(\mathbb{F})\right),\ \forall i \in J,$$

where $p \in S$*,* $\mathbb{F} \in \mathbb{F}(S)$ *and* $\varphi \in \Delta^+$*.*

4. Probabilistic neighborhood spaces and probabilistic convergence spaces

Definition 5. *A family of filters* $\mathfrak{N} = (\mathfrak{N}_p^{\varphi})_{(p,\varphi)\in S\times\Delta^+}$*, where for each* $\varphi \in \Delta^+$*,* $\mathfrak{N}^{\varphi}\colon S \longrightarrow P(P(S)), p \longmapsto \mathfrak{N}_p^{\varphi}$ *is said to be a* probabilistic neighborhood system *on* S *if*

(PNS1) $\forall\ p \in S, \forall \varphi \in \Delta^+, \mathfrak{N}_p^{\varphi} \leq [p]$*;*

(PNS2) $\forall\ \varphi, \psi \in \Delta^+$ *with* $\varphi \leq \psi$*, we have* $\mathfrak{N}_p^{\varphi} \leq \mathfrak{N}_p^{\psi}$*;*

(PNS3) $\mathfrak{N}_p^{\epsilon_\infty} = [S], \forall p \in S$.

Then the pair $\left(S, \mathfrak{N} = (\mathfrak{N}_p^{\varphi})_{(p,\varphi)\in S\times\Delta^+}\right)$ *is called a* probabilistic neighborhood space*. We call* $\mathfrak{N}_p^{\varphi}$ *a* φ*-neighborhood filter in* p *while the elements of* $\mathfrak{N}_p^{\varphi}$ *are called a* φ*-* neighborhood of p.
A mapping $f\colon (S, \mathfrak{N}) \longrightarrow (T, \mathfrak{M})$ *between probabilistic neighborhood spaces* $(S, \mathfrak{N})$ *and* $(T, \mathfrak{M})$ *is called* continuous *if* $\mathfrak{M}_{f(p)}^{\varphi} \leq f\left(\mathfrak{N}_p^{\varphi}\right)$.
The category of probabilistic neighborhood spaces as objects and morphisms are all continuous mappings as morphisms is denoted by ***PNeigh****.*

Remark 3. *A probabilistic neighborhood space* $\left(S, \mathfrak{N} = (\mathfrak{N}_p^{\varphi})_{(p,\varphi)\in S\times\Delta^+}\right)$ *is said to be a* probabilistic topological neighborhood space or a probabilistic topological space under a continuous triangle function τ *if*

(PLCC) for all $\emptyset \neq M \subset \Delta^+$*,* $\bigvee_{\varphi\in M} \mathfrak{N}_p^{\varphi} = \mathfrak{N}_p^{\bigvee M}$*;*

(τ*-PK) for all* $p, q \in S$ *and for all* $\varphi, \psi \in \Delta^+$*:*

$$\mathfrak{N}_p^{\tau(\varphi,\psi)} \leq \kappa\left(\mathfrak{N}_p^{\psi}, (\mathfrak{N}_q^{\varphi})_{q\in S}\right).$$

The category of probabilistic topological neighborhood spaces under continuous function τ as objects and continuous mapping between them as morphisms is denoted by **PTop-Neigh**.

Lemma 3. *Let* $\left(S, \mathfrak{N} = (\mathfrak{N}_p^{\varphi})_{(p,\varphi)\in S\times\Delta}\right)$ *be a probabilistic neighborhood space. Define a probabilistic convergence structure* $\overline{c^{\mathfrak{N}}}$ *on* S *by :* $p \in c_{\varphi}^{\mathfrak{N}}(\mathbb{F}) \iff \mathbb{F} \geq \mathfrak{N}_p^{\varphi}$*, for any* $\mathbb{F} \in \mathbb{F}(S)$*, and* $\varphi \in \Delta^+$*. Then* $\left(S, \overline{c}^{\mathfrak{N}}\right)$ *is a probabilistic convergence space.*

Proof. (PC1) Since by (PNS1), for any $p \in S$ and $\varphi \in \Delta^+$, $\mathfrak{N}_p^{\varphi} \leq [p]$, immediately, we get $p \in c_{\varphi}^{\mathfrak{N}}([p])$. (PC2) Let $\mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$ with $\mathbb{F} \leq \mathbb{G}$, and $p \in c_{\varphi}^{\mathfrak{N}}(\mathbb{F})$. Then $\mathbb{F} \geq \mathfrak{N}_p^{\varphi}$. But then $\mathbb{G} \geq \mathfrak{N}_p^{\varphi}$ implying $p \in c_{\varphi}^{\mathfrak{N}}(\mathbb{G})$. (PC3) Let $\varphi, \psi \in \Delta^+$ with $\varphi \leq \psi$ and let $\mathbb{F} \in \mathbb{F}(S)$. If $p \in c_{\psi}^{\mathfrak{N}}(\mathbb{F})$, then $\mathbb{F} \geq \mathfrak{N}_p^{\psi} \geq \mathfrak{N}_p^{\varphi}$ implying $\mathbb{F} \geq \mathfrak{N}_p^{\varphi}$ which yields that $p \in c_{\varphi}^{\mathfrak{N}}(\mathbb{F})$. Finally, to check (PC4), let $\mathbb{F} \in \mathbb{F}(S)$ and $p \in S$. Then $\mathfrak{N}_p^{\epsilon_\infty} = [\{S\}] \leq \mathbb{F}$, i.e., $\mathfrak{N}_p^{\epsilon_\infty} \leq \mathbb{F}$ which in turn yields that $p \in c_{\epsilon_\infty}^{\mathfrak{N}}(\mathbb{F})$.

Remark 4. *Under the definition given in the preceding Lemma 3, essentially, this probabilistic convergence space is a pretopological space. In fact, in view of the Lemma 4.2 [22], if we take for any* $\varphi \in \Delta^+$ *and a family of filters* $(\mathbb{F}_j)_{j\in J}$, $p \in \bigcap_{j\in J} c_\varphi(\mathbb{F}_j)$, *then for all* $j \in J$, $p \in c_\varphi(\mathbb{F}_j)$ *which is equivalent to saying* $\mathbb{F}_j \geq \mathfrak{N}_p^\varphi$, *this is true for all* $j \in J$, *that is,* $\bigwedge_{j\in J} \mathbb{F}_j \geq \mathfrak{N}_p^\varphi$ *which is again equivalent to:* $p \in c_\varphi\left(\bigwedge_{j\in J} \mathbb{F}_j)\right)$. *Hence* $\bigcap_{j\in J} c_\varphi(\mathbb{F}_j) \subseteq c_\varphi\left(\bigwedge_{j\in J} \mathbb{F}_j)\right)$ *which is precisely the Lemma 4.2 [22].*

Lemma 4. *Let* $\left(S, \overline{c} = (c_\varphi)_{\varphi\in\Delta^+}\right)$ *be a probabilistic pretopological space. Then* $\left(S, \mathfrak{N}^{\overline{c},\varphi} = (\mathfrak{N}_p^{\overline{c},\varphi})_{(p,\varphi)\in S\times\Delta^+}\right)$ *is a probabilistic neighborhood space.*

Proof. For $\varphi \in \Delta^+$ and $p \in S$, we define *φ-neighborhood filter of p* by $\mathfrak{N}_p^{\overline{c},\varphi} = \bigwedge_{p\in c_\varphi(\mathbb{F})} \mathbb{F}$. Then the result follows from the definition.

Lemma 5. *Let* $(S, \mathfrak{N})$ *and* $(T, \mathfrak{M})$ *be probabilistic neighborhood spaces and* $f\colon S \longrightarrow T$ *be a mapping. Then the following are equivalent:*

(1) $\forall p \in S$, $\forall \varphi \in \Delta^+$, $\mathfrak{M}_{f(p)}^\varphi \leq f\left(\mathfrak{N}_p^\varphi\right)$;

(2) $\forall p \in S$, $\forall \varphi \in \Delta^+$, $\forall U \in \mathfrak{M}_{f(p)}^\varphi$ *there exists a* $V \in \mathfrak{N}_p^\varphi$ *such that* $f(V) \leq U$;

(3) $\forall p \in S$, $\forall \varphi \in \Delta^+$, $\forall \mathbb{F} \in \mathbb{F}(S)$, $p \in c_\varphi(\mathbb{F})$ *implies* $f(p) \in c_\varphi\left(f(\mathbb{F})\right)$.

Proof. (1) $\iff$ (2) follows from the fact that $f(V) \subseteq U \iff V \subseteq f^{-1}(U)$. We check the equivalency of (1) and (3). First, let (1) holds, and let $p \in c_\varphi(\mathbb{F})$, then $\mathbb{F} \geq \mathfrak{N}_p^\varphi$. But then $f(\mathbb{F}) \geq f\left(\mathfrak{N}_p^\varphi\right) \geq \mathfrak{M}_{f(p)}^\varphi$ yields $f(\mathbb{F}) \geq \mathfrak{M}_{f(p)}^\varphi$ which is equivalent to $f(p) \in c_\varphi\left(f(\mathbb{F})\right)$. Next, assume (3) holds. Then $f\left(\mathfrak{N}_p^\varphi\right) = f\left(\bigwedge_{p\in c_\varphi(\mathbb{F})} \mathbb{F}\right) = \bigwedge_{p\in c_\varphi(\mathbb{F})} f(\mathbb{F}) \geq \bigwedge_{f(p)\in c_\varphi(f(\mathbb{F}))} f(\mathbb{F}) = \mathfrak{M}_{f(p)}^\varphi$.

Lemma 6. *[22] Let* $(S, \overline{c})$ *and* $\left(T, \overline{c'}\right)$ *be probabilistic pretopological spaces and* $f\colon S \longrightarrow T$ *be a mapping. Then* f *is continuous if and only if for all* $p \in S$ *and for all* $\varphi \in \Delta^+$, *we have* $\mathfrak{N}_{f(p)}^{\overline{c'},\varphi} \leq f\left(\mathfrak{N}_p^{\overline{c},\varphi}\right)$.

Lemma 7. *Every probabilistic metric space* (S, F) *under triangle function* τ *gives rise to a probabilistic neighborhood space* $\left(S,\ \left({}^F\mathfrak{N}_p^\varphi\right)_{(p,\varphi)\in S\times\Delta^+}\right)$.

Proof. Given a $\varphi \in \Delta^+$, $\epsilon > 0$ and $p \in S$, the (φ, ϵ)-neighborhood of p defined by

$$N_p^{\varphi,\epsilon} = \{q \in S\colon F_{p,q}(x + \epsilon) + \epsilon \geq \varphi(x), \forall x \in [0, \tfrac{1}{\epsilon})\}$$

yields that $\mathfrak{N}_p^\varphi = [\{N_p^{\varphi,\epsilon}\colon \epsilon > 0\}]$ the filter satisfying all the conditions (PCS1)-(PCS3). We may call this probabilistic neighborhood space $(S, ({}^F\mathfrak{N}_p^\varphi)_{(p,\varphi)\in S\times\Delta})$ a *probabilistic metric-generated* probabilistic Tardiff-neighborhood space.

Hence in view of Lemma 6.4[22] in conjunction with preceding Lemma 4, and denoting **PMetNeigh** the category having probabilistic Tardiff-neighborhood spaces as objects and morphisms as described in Lemma 6.4[22], we get the following

Corollary 1.

$$\mathfrak{B}:\begin{cases} \textbf{\textit{PMetNeigh}} & \longrightarrow & \textbf{\textit{PNeigh}} \\ \left(S,\mathfrak{N}^F_\varphi\right) & \longmapsto & (S,\mathfrak{N}) \\ f & \longmapsto & f. \end{cases}$$

is a functor

Proposition 1. ***PNeigh*** *is a topological category.*

Proof. Let $\left(f_j\colon S\longrightarrow\left(S_j,\ \mathfrak{N}^{j,\varphi}_{f_j(p)}\right)\right)_j$ be a source. Then the initial probabilistic neighborhood system on S is given for $p\in S$ and $\varphi\in\Delta^+$ by $\mathfrak{N}^\varphi_p=\bigvee_{j\in J}f_j^{-1}\left(\mathfrak{N}^{j,\varphi}_{f_j(p)}\right)$. For the remaining proof see f.i. the Proposition 4.1 [24]).

Remark 5. *If $(S,\mathfrak{N})$ is a probabilistic neighborhood space and $A\subseteq S$, then the subspace $(A,\mathfrak{N}_A)$, the probabilistic neighborhood system on A is given for any $p\in A$ by ${}^A\mathfrak{N}^\varphi_p=\{A\cap V\colon V\in\mathfrak{N}^\varphi_p\}$. For the product probabilistic neighborhood spaces, we just consider $f_j=pr_j$ where $pr_j\colon\prod_{i\in J}S_i\longrightarrow S_j$ are the projections. In fact, if $(S_j,\mathfrak{N}_j)$ is a family of probabilistic neighborhood systems, then their product structure on $S=\prod_{j\in J}S_j$ is given by $\mathfrak{N}=\bigvee_{j\in J}pr_j^{-1}\left(\mathfrak{N}^{j,\varphi}_{pr_j(p)}\right)$, for $p\in S$, $\varphi\in\Delta$, where pr_j are the projections. If for instance $j=1,2$, then the product structure $(\mathfrak{N}\times\mathfrak{M})_{(p,q)}$ on $S_1\times S_2$ is given for any $(p,q)\in S_1\times S_2$ by its basis $\mathfrak{B}=\{pr_1^{-1}(U)\cap pr_2^{-1}(V)\colon U\in\mathfrak{N}^\varphi_p, V\in\mathfrak{M}^\varphi_q\}$.*

5. Probabilistic neighborhood groups and probabilistic convergence groups

Definition 6. *Let $(S,\cdot)$ be a group, and $(S,\mathfrak{N})$ a probabilistic neighborhood space. Then the triple $(S,\cdot,\mathfrak{N})$ is called a* probabilistic neighborhood group under triangle function τ *if and only if the following conditions are fulfilled:*

(PNGM) For all $p,q\in S$ and for all $\varphi,\psi\in\Delta$, $\mathfrak{N}^{\tau(\varphi,\psi)}_{pq}\leq\mathfrak{N}^\varphi_p\odot\mathfrak{N}^\psi_q$;

(PNGI) for all $p\in S$ and for all $\forall\varphi\in\Delta^+$, $\mathfrak{N}^\varphi_{p^{-1}}\leq(\mathfrak{N}^\varphi_p)^{-1}$.

Lemma 8. *Let $\left(S,\cdot,\mathfrak{N}=(\mathfrak{N}^\varphi_p)_{(p,\varphi)\in S\times\Delta^+}\right)$ be a probabilistic neighborhood group. Then the statement (PNGM)′ is equivalent to the statements (PNGM) and (PNGI) as given below:*

(PNGM)′ For all $p,q\in S$ and for all $\varphi,\psi\in\Delta$, $\mathfrak{N}^{\tau(\varphi,\psi)}_{p^{-1}q}\leq(\mathfrak{N}^\varphi_p)^{-1}\odot\mathfrak{N}^\psi_q$;

(PNGM) For all $p,q\in S$ and for all $\varphi,\psi\in\Delta$, $\mathfrak{N}^{\tau(\varphi,\psi)}_{pq}\leq\mathfrak{N}^\varphi_p\odot\mathfrak{N}^\psi_q$;

(PNGI) for all $p\in S$ and for all $\forall\varphi\in\Delta^+$, $\mathfrak{N}^\varphi_{p^{-1}}\leq(\mathfrak{N}^\varphi_p)^{-1}$.

Proof. (PNGM)′⇒ (PNGM)+(PNGI): First, we prove (PNGI). For let $p \in S$ and $\varphi \in \Delta^+$. Then upon using the boundary condition $\tau(\varphi, \epsilon_0) = \varphi$, we have $\mathfrak{N}^{\varphi}_{p^{-1}} = \mathfrak{N}^{\tau(\varphi,\epsilon_0)}_{p^{-1}e} \leq (\mathfrak{N}^{\varphi}_p)^{-1} \odot \mathfrak{N}^{\epsilon_0}_e \leq (\mathfrak{N}^{\varphi}_p)^{-1} \odot [e] = (\mathfrak{N}^{\varphi}_p)^{-1}$, i.e., $\mathfrak{N}^{\varphi}_{p^{-1}} \leq (\mathfrak{N}^{\varphi}_p)^{-1}$. Next, to prove (PNGM), let $p, q \in S$ and $\varphi, \psi \in \Delta^+$. Then using (PNGI) along with Lemma 2(f) and the fact that $pq = (p^{-1})^{-1}q$, we have from the assumption that $\mathfrak{N}^{\tau(\varphi,\psi)}_{pq} = \mathfrak{N}^{\tau(\varphi,\psi)}_{(p^{-1})^{-1}q} \leq \left(\mathfrak{N}^{\varphi}_{p^{-1}}\right)^{-1} \odot \mathfrak{N}^{\psi}_q = \mathfrak{N}^{\varphi}_p \odot \mathfrak{N}^{\psi}_q$, i.e., $\mathfrak{N}^{\tau(\varphi,\psi)}_{pq} \leq \mathfrak{N}^{\varphi}_p \odot \mathfrak{N}^{\psi}_q$.
Conversely, we prove (PNGM)+(PNGI)⇒ (PNGM)′: Let $p, q \in S$ and $\varphi, \psi \in \Delta^+$. Upon using (PNGM) and then using (PNGI), we have $\mathfrak{N}^{\tau(\varphi,\psi)}_{p^{-1}q} \leq \mathfrak{N}^{\varphi}_{p^{-1}} \odot \mathfrak{N}^{\psi}_q \leq (\mathfrak{N}^{\varphi}_p)^{-1} \odot \mathfrak{N}^{\psi}_q$. Hence $\mathfrak{N}^{\tau(\varphi,\psi)}_{p^{-1}q} \leq (\mathfrak{N}^{\varphi}_p)^{-1} \odot \mathfrak{N}^{\psi}_q$.

Due to Lemma 6.5[3] and Lemma 5, we have the following

Lemma 9. *If $\left(S, \cdot, \mathfrak{N}^F_{\varphi}\right)$ is a probabilistic metric topological Tardiff-neighborhood group under continuous and the largest triangle function τ, i.e., $\tau(\varphi, \varphi) = \varphi$, then it is a probabilistic topological neighborhood group.*

Proof. We only need check the continuity of the group operations. But that follows from Lemma 6.5[3] in conjunction with Lemma 8, i.e., for any $p, q \in S$ and $\psi, \psi \in \Delta^+$ that
$\mathfrak{N}^{\tau(\varphi,\psi)}_{p^{-1}q} \leq (\mathfrak{N}^{\varphi}_p)^{-1} \odot \mathfrak{N}^{\psi}_q$.

Lemma 10. *Let $\left(S, \mathfrak{N} = (\mathfrak{N}^{\varphi}_p)_{(p,\varphi)\in S\times\Delta^+}\right)$ be a probabilistic neighborhood space. If $\tau(\varphi, \varphi) = \varphi$ for all $\varphi \in \Delta^+$. Then $(S, \cdot, \mathfrak{N})$ is a probabilistic neighborhood group under a triangle function τ if and only if $m\colon (S \times S, \mathfrak{N} \times \mathfrak{N}) \longrightarrow (S, \mathfrak{N}), (p, q) \longmapsto pq$ and $\jmath\colon (S, \mathfrak{N}) \longrightarrow (S, \mathfrak{N}), p \longmapsto p^{-1}$ are continuous.*

Proof. Let $W^{\tau(\varphi,\psi)}_{pq} \in \mathfrak{N}^{\tau(\varphi,\psi)}_{pq}$. Since m is continuous, we choose $U^{\varphi}_p \in \mathfrak{N}^{\varphi}_p$ and $V^{\psi}_q \in \mathfrak{N}^{\psi}_q$ such that $U^{\varphi}_p \odot V^{\psi}_q \subseteq W^{\tau(\varphi,\psi)}_{pq}$. This implies $W^{\tau(\varphi,\psi}_{pq} \in \mathfrak{N}^{\tau(\varphi,\psi)}_{pq}$. Hence $\mathfrak{N}^{\tau(\varphi,\psi)}_{pq} \leq \mathfrak{N}^{\varphi}_p \odot \mathfrak{N}^{\psi}_q$.
Conversely, we need show the mapping m is continuous. For, let $W \in \mathfrak{N}^{\varphi}_{pq}$, i.e., $W \in \mathfrak{N}^{\tau(\varphi,\varphi)}_{pq}$, implying $W \in \mathfrak{N}^{\varphi}_p \odot \mathfrak{N}^{\psi}_q = m\left(\mathfrak{N}^{\varphi}_p \times \mathfrak{N}^{\varphi}_q\right)$. Consequently, there are $U \in \mathfrak{N}^{\varphi}_p$ and $V \in \mathfrak{N}^{\varphi}_q$ such that $m(U \times V) \subseteq W$.

Lemma 11. *Let $(S, \cdot, \mathfrak{N})$ be a probabilistic neighborhood group and $x \in S$. Then*

(1) The $\mathcal{L}_x\colon S \longrightarrow S$, $z \longmapsto zx$ the left and $\mathcal{R}_x\colon S \longrightarrow S$, $z \longmapsto xz$ the right translations, and the inversion mapping $\jmath\colon S \longrightarrow S, z \longmapsto z^{-1}$ are homeomorphisms;

(2) the inner automorphism operator $\mathcal{I}_x\colon S \longrightarrow S, z \longmapsto xzx^{-1}$ is an isomorphism;

(3) the mapping $\mathcal{T}_{(a,b)}\colon S \longrightarrow S, z \longmapsto azb$ is a homeomorphism;

(4) $N \in \mathfrak{N}^{\varphi}_e \iff N^{-1} \in \mathfrak{N}^{\varphi}_e$, or alternatively, $(\mathfrak{N}^{\varphi}_e)^{-1} = \mathfrak{N}^{\varphi}_e$, for all $\varphi \in \Delta^+$;

(5) $N \in \mathfrak{N}_x^{\varphi} \iff x^{-1} \odot N \in \mathfrak{N}_e^{\varphi} \iff N \odot x^{-1} \in \mathfrak{N}_e^{\varphi}$, *or alternatively,* $\mathfrak{N}_p^{\varphi} = [p^{-1}] \odot \mathfrak{N}_e^{\varphi} = \mathfrak{N}_e^{\varphi} \odot [p^{-1}]$, *for all* $p \in S$ *and for all* $\varphi \in \Delta$;

(6) $N \in \mathfrak{N}_e^{\varphi} \iff x \odot N \in \mathfrak{N}_x^{\varphi} \iff N \odot x \in \mathfrak{N}_x^{\varphi}$, *or alternatively,* $\mathfrak{N}_e^{\varphi} = [p] \odot \mathfrak{N}_p^{\varphi} = \mathfrak{N}_p^{\varphi} \odot [p]$, *for all* $p \in S$ *and for all* $\varphi \in \Delta$.

Proof. Follows almost similar way as in classical cases (see f.i. [13, 15]).

Definition 7. *[3] A triple* $\left(S, \cdot, \overline{c} = (c_{\varphi})_{\varphi \in \Delta^+}\right)$ *is called a* probabilistic convergence group *under the triangle function* τ *if for all* $\varphi, \psi \in \Delta^+$ *and for all* $\mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$

(PCG1) $(S, \cdot)$ *is a group;*

(PCG2) $(S, \overline{c})$ *is a probabilistic convergence space;*

(PCGM) $pq \in c_{\tau(\varphi,\psi)}(\mathbb{F} \odot \mathbb{G})$ *whenever* $p \in c_{\varphi}(\mathbb{F})$ *and* $q \in c_{\psi}(\mathbb{G})$;

(PCGI) $p^{-1} \in c_{\varphi}(\mathbb{F}^{-1})$ *whenever* $p \in c_{\varphi}(\mathbb{F})$.

Furthermore, if we consider in (PCG2), $(S, \overline{c})$ *a probabilistic limit space, then the triple* $\left(S, \cdot, \overline{c} = (c_{\varphi})_{\varphi \in \Delta^+}\right)$ *is called a* probabilistic limit group. *A probabilistic convergence group is a* probabilistic pretopological group *if it satisfies axiom (PLCC). The category of probabilistic convergence groups and continuous group homomorphisms is denoted by* ***PConvGrp*** *while* ***PLimGrp*** *denotes the category of probabilistic limit groups whence objects are probabilistic limit spaces and morphisms are continuous group homomorphisms. We denote by* ***PPreTop****, the category of pretopological groups.*

Lemma 12. *Let* $(S, \cdot, \overline{c})$ *be a probabilistic convergence group. Then the statement (PCGM)′ is equivalent to the statements (PCGM) and (PCGI) as given below:*

(PCGM)′ If for all $\mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$ *and for all* $\varphi, \psi \in \Delta^+$, $p \in c_{\varphi}(\mathbb{F})$ *and* $q \in c_{\psi}(\mathbb{G})$, *then* $p^{-1}q \in c_{\tau(\varphi,\psi)}\left(\mathbb{F}^{-1} \odot \mathbb{G}\right)$.

(PCGM) If for all $\mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$ *and for all* $\varphi, \psi \in \Delta^+$, $p \in c_{\varphi}(\mathbb{F})$ *and* $q \in c_{\psi}(\mathbb{G})$, *then* $pq \in c_{\tau(\varphi,\psi)}(\mathbb{F} \odot \mathbb{G})$;

(PCGI) for all $\mathbb{F} \in \mathbb{F}(S)$ *and for all* $\varphi \in \Delta^+$, $p \in c_{\varphi}(\mathbb{F})$, *then* $p^{-1} \in c_{\varphi}(\mathbb{F}^{-1})$.

Proof. Assume (PCGM)′ holds. First we show (PCGI): Let $\mathbb{F} \in \mathbb{F}(S)$, $\varphi \in \Delta^+$ and $p \in c_{\varphi}(\mathbb{F})$. Then by (PCMI), $e \in c_{\epsilon_0}([e])$ and hence by (PCGM)′, $ep^{-1} \in c_{\tau(\epsilon_0,\varphi)}([e] \odot \mathbb{F}^{-1})$ which implies $p^{-1} \in c_{\varphi}(\mathbb{F}^{-1})$ since $\tau(\epsilon_0, \varphi) = \tau(\varphi, \epsilon_0) = \varphi$. Next, to show (PCGM), let $\mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$ and $\varphi, \psi \in \Delta^+$. Furthermore, assume $p \in c_{\varphi}(\mathbb{F})$ and $q \in c_{\psi}(\mathbb{G})$. Since $p \in c_{\varphi}(\mathbb{F})$ and, by (PCGI), $p^{-1} \in c_{\psi}(\mathbb{F}^{-1})$, these together in conjunction with (PCGM)′ imply that $(p^{-1})^{-1}q \in c_{\tau(\varphi,\psi)}\left((\mathbb{F}^{-1})^{-1} \odot \mathbb{G}\right)$ meaning $pq = (p^{-1})^{-1}q \in c_{\tau(\varphi,\psi)}(\mathbb{F} \odot \mathbb{G})$, i.e., $pq \in c_{\tau(\varphi,\psi)}(\mathbb{F} \odot \mathbb{G})$.
Now assume that the statements (PCGM) and (PCGI) hold; we verify (PCGM)′. Let $p \in c_{\varphi}(\mathbb{F})$ and $q \in c_{\psi}(\mathbb{G})$. Since by (PCGI), $p^{-1} \in c_{\psi}(\mathbb{F}^{-1})$, and then by using (PCGM), we get $p^{-1}q \in c_{\tau(\varphi,\psi)}\left(\mathbb{F}^{-1} \odot \mathbb{G}\right)$ which is precisely statement (PCGM)′.

Lemma 13. *Let $\left(S, \cdot, \overline{c} = (c_\varphi)_{\varphi\in\Delta^+}\right)$ be a probabilistic convergence group under the largest triangle function τ, i.e., $\tau(\varphi,\varphi) = \varphi$. Then for all $p \in S$ and for all $\varphi \in \Delta^+$ the following holds:*

(a) $p \in c_\varphi(\mathbb{F}) \iff e \in c_\varphi([p^{-1}] \odot \mathbb{F}) \iff e \in c_\varphi\left(\mathbb{F} \odot [p^{-1}]\right), \forall \mathbb{F} \in \mathbb{F}(S)$ *(homogeneity);*

(b) $\mathfrak{N}_p^{\overline{c},\varphi} = [p] \odot \mathfrak{N}_e^{\overline{c},\varphi} = \mathfrak{N}_e^{\varphi} \odot [p]$*;*

(c) $\mathfrak{N}_e^{\overline{c},\varphi} = [p^{-1}] \odot \mathfrak{N}_p^{\overline{c},\varphi} = \mathfrak{N}_p^{\overline{c},\varphi} \odot [p^{-1}]$*;*

(d) $\left(\mathfrak{N}_p^{\overline{c},\varphi}\right)^{-1} = \mathfrak{N}_{p^{-1}}^{\overline{c},\varphi}$

Proof. (a) This is precisely homogeneity of probabilistic convergence group as shown in Lemma 4.4[3].
(b) We show here the first part, the other part follows similarly. For this, we have

$$[p] \odot \mathfrak{N}_e^{\overline{c},\varphi} \geq \mathfrak{N}_p^{\overline{c},\varphi} \odot \mathfrak{N}_e^{\overline{c},\varphi} \geq \mathfrak{N}_{pe}^{\overline{c},\tau(\varphi,\varphi)} = \mathfrak{N}_p^{\overline{c},\varphi}.$$

On the other hand, we have

$$[p] \odot \mathfrak{N}_e^{\overline{c},\varphi} = [p] \odot \mathfrak{N}_{p^{-1}p}^{\overline{c},\tau(\varphi,\varphi)} \leq [p] \odot \left(\mathfrak{N}_{p^{-1}}^{\overline{c},\varphi} \odot \mathfrak{N}_p^{\overline{c},\varphi}\right) \leq [p] \odot \left([p^{-1}] \odot \mathfrak{N}_p^{\overline{c},\varphi}\right)$$
$$= \left([p] \odot [p^{-1}]\right) \odot \mathfrak{N}_p^{\overline{c},\varphi} = [e] \odot \mathfrak{N}_p^{\overline{c},\varphi} = \mathfrak{N}_p^{\overline{c},\varphi}$$

(c) This follows almost the same way as in (b).
(d) Since the inversion mapping $\jmath \colon (S, \overline{c}) \longrightarrow (S, \overline{c}), p \longmapsto p^{-1}$ is a homeomorphism, one can easily obtain the result in this item.

Lemma 14. *Let $(S, \cdot)$ be a group and $\left(S, \mathfrak{N} = (\mathfrak{N}_p^{\varphi})_{(p,\varphi)\in S\times\Delta^+}\right)$ be a probabilistic neighborhood space and let τ be a triangle function. Define $p \in c_\varphi(\mathbb{F}) \iff \mathbb{F} \geq \mathfrak{N}_p^{\varphi}$ for each $\mathbb{F} \in \mathbb{F}(S)$, where $\mathfrak{N}_p^{\varphi} = \bigwedge_{p\in c_\varphi(\mathbb{F})} \mathbb{F}$.*
Then

(a) the mapping $\jmath \colon (S, \mathfrak{N}) \longrightarrow (S, \mathfrak{N})$ is continuous at $e \in S$ if and only if $e \in c_\varphi(\mathbb{F}) \Rightarrow e \in c_\varphi(\mathbb{F}^{-1})$ for all $\mathbb{F} \in \mathbb{F}(S)$ and for all $\varphi \in \Delta^+$.

(b) the mapping $m \colon (S \times S, \mathfrak{N} \times \mathfrak{N}) \longrightarrow (S, \mathfrak{N})$, $(p, q) \longmapsto pq$ is continuous at $(e, e) \in S \times S$ if and only if $e \in c_\varphi(\mathbb{F})$ and $e \in c_\psi(\mathbb{G}) \Rightarrow e \in c_{\tau(\varphi,\psi)}(\mathbb{F} \odot \mathbb{G})$, for all $\mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$ and for all $\varphi, \psi \in \Delta^+$.

Proof. (a) Note that the inversion mapping $\jmath \colon S \longrightarrow S, x \longmapsto x^{-1}$ is continuous $e \in S$ if and only if $\mathfrak{N}_{e^{-1}}^{\varphi} \leq (\mathfrak{N}_e^{\varphi})^{-1}$. Now for any $\mathbb{F} \in \mathbb{F}(S)$, if $e \in c_\varphi(\mathbb{F})$, then $\mathbb{F} \geq \mathfrak{N}_e^{\varphi}$. So, $\mathbb{F}^{-1} \geq (\mathfrak{N}_e^{\varphi})^{-1} \geq \mathfrak{N}_{e^{-1}}^{\varphi} = \mathfrak{N}_e^{\varphi}$ implying $\mathbb{F}^{-1} \geq \mathfrak{N}_e^{\varphi}$ which yields that $e \in c_\varphi(\mathbb{F}^{-1})$. Conversely, let $\mathbb{F} \in \mathbb{F}(S)$ and $\varphi \in \Delta^+$. Then

$$\mathfrak{N}_{e^{-1}}^{\varphi} = \bigwedge_{\jmath(e)\in c_\varphi(\mathbb{F}^{-1})} \mathbb{F}^{-1} \leq \bigwedge_{e\in c_\varphi(\mathbb{F})} \mathbb{F}^{-1} = (\mathfrak{N}_e^{\varphi})^{-1}.$$

This implies $\mathfrak{N}_{e^{-1}}^{\varphi} \leq (\mathfrak{N}_e^{\varphi})^{-1}$.

(b) Need to show that for a triangle function τ, $\mathfrak{N}_e^{\tau(\varphi,\psi)} \leq \mathfrak{N}_e^{\varphi} \odot \mathfrak{N}_e^{\psi}$ if and only if for any $\mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$ and for all $\varphi, \psi \in \Delta^+$, $e \in c_\varphi(\mathbb{F})$ and $e \in c_\psi(\mathbb{G})$ imply $e \in c_{\tau(\varphi,\psi)}(\mathbb{F} \odot \mathbb{G})$. First, for $\mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$ and $\varphi, \psi \in \Delta^+$, let $e \in c_\varphi(\mathbb{F})$ and $e \in c_\psi(\mathbb{G})$. Consequently, we have $\mathbb{F} \geq \mathfrak{N}_e^{\varphi}$ and $\mathbb{G} \geq \mathfrak{N}_e^{\psi}$. Thus, we have $\mathbb{F} \odot \mathbb{G} \geq \mathfrak{N}_e^{\varphi} \odot \mathfrak{N}_e^{\psi}$. Due to the assumption, we obtain: $\mathbb{F} \odot \mathbb{G} \geq \mathfrak{N}_e^{\tau(\varphi,\psi)}$. This implies $e \in c_{\tau(\varphi,\psi)}(\mathbb{F} \odot \mathbb{G})$, given the fact that $\mathbb{F} \odot \mathbb{G} \in \mathbb{F}(S)$. To prove the converse part, upon using the assumption, we have:

$$\mathfrak{N}_e^{\varphi} \odot \mathfrak{N}_e^{\psi} = \bigwedge_{e \in c_\varphi(\mathbb{F})} \mathbb{F} \odot \bigwedge_{e \in c_\psi(\mathbb{G})} \mathbb{G} = \bigwedge_{e \in c_\varphi(\mathbb{F})} \bigwedge_{e \in c_\psi(\mathbb{G})} \mathbb{F} \odot \mathbb{G} \geq \bigwedge_{e \in c_{\tau(\varphi,\psi)}(\mathbb{F} \odot \mathbb{G})} \mathbb{F} \odot \mathbb{G} = \bigwedge_{e \in c_{\tau(\varphi,\psi)(\mathbb{H})}} \mathbb{H} = \mathfrak{N}_e^{\tau(\varphi,\psi)}.$$

Lemma 15. *Let $(S, \bar{c})$ be a probabilistic convergence group under a triangle function τ, and $a \in S$. Then the left translation $\mathcal{L}_a \colon (S, \bar{c}) \longrightarrow (S, \bar{c}), p \longmapsto ap$ and the right translation $\mathcal{R}_a \colon (S, \bar{c}) \longrightarrow (S, \bar{c}), p \longmapsto pa$, the inversion $\jmath \colon (S, \bar{c}) \longrightarrow (S, \bar{c}), p \longmapsto p^{-1}$ are homeomorphisms; and the inner automorphism $\mathcal{I}_a \colon (S, \bar{c}) \longrightarrow (S, \bar{c}), p \longmapsto apa^{-1}$ is an isomorphism.*

Proof. Let $a \in S$, $\mathbb{F} \in \mathbb{F}(S)$ and $\varphi \in \Delta^+$. Since, in particular, $a \in c_{\epsilon_0}([a])$ by (PCS1), we have for any $p \in c_\varphi(\mathbb{F})$, annd (PCGM), $ap \in c_{\tau(\epsilon_0,\varphi)}([a] \odot \mathbb{F})$. Upon using the commutativity of τ and the fact that $\tau(\varphi, \epsilon_0) = \varphi$, we get $ap \in c_\varphi([a] \odot \mathbb{F})$ implying $\mathcal{L}_a(p) \in c_\varphi(\mathcal{L}_a(\mathbb{F}))$; this means that the left translation is continuous. Also, $\mathcal{L}_{a^{-1}}$ is continuous. Similarly, one can show that the right translation $\mathcal{R}_a$ and its inverse $\mathcal{R}_{a^{-1}}$ are continuous. Note that $\mathcal{I}_a = \mathcal{L}_a \circ \mathcal{R}_{a^{-1}}$, so their composition is continuous. We leave the details for the interested reader.

In view of the Lemma 3, Remark 5, and Lemma 4 – Lemma 6, we arrive at the following

Theorem 1. *The category **PNeighGrp** is isomorphic to the category **PPreTopGrp**.*

Proof. We only need to check (PCGM) and (PCGI). Recall that for any $\mathbb{F} \in \mathbb{F}(S)$, $p \in S$ and $\varphi \in \Delta^+$, we have $p \in c_\varphi(\mathbb{F}) \iff \mathbb{F} \geq \mathfrak{N}_p^{\varphi}$, whence $\mathfrak{N}_p^{\varphi} = \bigwedge_{p \in c_\varphi(\mathbb{F})} \mathbb{F}$. Now first, assume $\mathfrak{N}_{pq}^{\tau(\varphi,\psi)} \leq \mathfrak{N}_p^{\varphi} \odot \mathfrak{N}_q^{\psi}$. Let $p \in c_\varphi(\mathbb{F})$ and $q \in c_\psi(\mathbb{G})$ for any $\mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$. We need to show that $pq \in c_{\tau(\varphi,\psi)}(\mathbb{F} \odot \mathbb{G})$. It follows from the assumption that $\mathbb{F} \geq \mathfrak{N}_p^{\varphi}$ and $\mathbb{G} \geq \mathfrak{N}_q^{\psi}$. These imply $\mathbb{F} \odot \mathbb{G} \geq \mathfrak{N}_p^{\varphi} \odot \mathfrak{N}_q^{\psi} \geq \mathfrak{N}_{pq}^{\tau(\varphi,\psi)}$, i.e., $\mathbb{F} \odot \mathbb{G} \geq \mathfrak{N}_{pq}^{\tau(\varphi,\psi)}$ whence $\mathbb{F} \odot \mathbb{G} \in \mathbb{F}(S)$, and hence $pq \in c_{\tau(\varphi,\psi)}(\mathbb{F} \odot \mathbb{G})$. Similarly, one can show that if $p \in c_\varphi(\mathbb{F})$, then $p^{-1} \in c_\varphi(\mathbb{F}^{-1})$. For the converse part, let us first prove that for any $(s, t) \in S \times S$, $\varphi, \psi \in \Delta^+$ and $\mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$, $\mathfrak{N}_{st}^{\tau(\varphi,\psi)} \leq \mathfrak{N}_s^{\varphi} \odot \mathfrak{N}_t^{\psi}$. This follows from the compositions of the continuities of the mappings at the identity elements give rise to the continuity of $h \colon S \times S \longrightarrow S, (p, q) \longmapsto pq^{-1}$ at any point (s, t) in the following way:

$$S \times S \overset{\mathcal{L}_{s^{-1}} \times \mathcal{L}_{t^{-1}}}{\longrightarrow} S \times S \overset{m}{\longrightarrow} S \overset{\mathcal{I}_t}{\longrightarrow} S \overset{\mathcal{L}_{st^{-1}}}{\longrightarrow} S,\ (s, t) \longmapsto (e, e) \longmapsto e \longmapsto e \longmapsto st^{-1}.$$

The continuity at any $s \in S$ for the mapping $\jmath \colon S \longrightarrow S, s \longmapsto s^{-1}$ follows directly from the definition.

Lemma 16. *Let $(S, \cdot, \overline{c})$ be a probabilistic convergence group and $A \subseteq S$. Then the following are true:*

(a) if A is a subgroup of S, then $\mathbb{C}_\varphi(A)$ is also a subgroup of S;

(b) if N is a normal subgroup of the subgroup of A of S, then $\mathbb{C}_\varphi(N)$ is a normal subgroup of the subgroup $\mathbb{C}_\varphi(A)$.

Proof. (a) If $p, q \in A$, then $p^{-1}q \in A$ since A is a subgroup of S. Let $p, q \in \mathbb{C}_\varphi(A)$. Then there exists $\mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$ such that $p \in c_\varphi(\mathbb{F})$ and $q \in c_\psi(\mathbb{G})$ implying $A \in \mathbb{F}$ and $A \in \mathbb{G}$. By (PCGM)′, $p^{-1}q \in c_{\tau(\varphi,\psi)}\left(\mathbb{F}^{-1} \odot \mathbb{G}\right)$, whence $A^{-1} \cdot A \in \mathbb{F}^{-1} \odot \mathbb{G}$; since A is a subgroup of S meaning $A^{-1} \cdot A \subseteq A$, and $p^{-1}q \in A$, implying $A \in \mathbb{F}^{-1} \odot \mathbb{G}$. Thus, we have we obtained a filter $\mathbb{F}^{-1} \odot \mathbb{G} \in \mathbb{F}(S)$ such that $p^{-1}q \in c_{\tau(\varphi,\psi)}(\mathbb{F}^{-1} \odot \mathbb{G})$ and $A \in \mathbb{F}^{-1} \odot \mathbb{G}$, showing that $p^{-1}q \in \mathbb{C}_\varphi(A)$.
(b) Let $s \in \mathbb{C}_\varphi(A)$ and $p \in \mathbb{C}_\psi(N)$, we show that $sps^{-1} \in \mathbb{C}_{\tau(\varphi,\varphi)}(N)$. Then there are filters $\mathbb{F}$ and $\mathbb{G}$ such that $s \in c_\varphi(\mathbb{F})$ and $p \in c_\psi(\mathbb{G})$, whence $A \in \mathbb{F}$ and $N \in \mathbb{G}$. In view of the Lemma 15, it follows from the continuity of the inner automorphism $\mathcal{I}_s(p) = sps^{-1}$ which is also an isomorphism, upon using $\mathcal{I}_s(p) = sps^{-1}$, one obtains: $sps^{-1} \in c_{\tau(\varphi,\psi)}\left(\mathbb{F} \odot \mathbb{G} \odot \mathbb{F}^{-1}\right)$, whence the filter $\mathbb{F} \odot \mathbb{G} \odot \mathbb{F}^{-1}$ is generated by the set $\{F \cdot M \cdot F^{-1} \colon F \in \mathbb{F}, M \in \mathbb{G}\}$, where $F \subseteq A$ and $M \subseteq N$, since for any $a \in F$, and because of normality of N, we have $aNa^{-1} = N$, therefore, $sps^{-1} \in \mathbb{C}_{\tau(\varphi,\psi)}(N)$.

6. Probabilistic Cauchy structures and their relationships with other probabilistic convergence structures

In [23], a general theory of quantale-valued Cauchy tower spaces are introduced and among other interesting results, they proved that the quantale-valued tower limit space is a quantale-valued Cauchy tower space, and even proved that the category of quantale-valued limit tower groups gives rise to a quantale-valued Cauchy tower groups; furthermore, Cauchy completeness are also considered there in a full length. We just want to take a special case here which will be clear from the next section and what follows, for the details we refer to [23]. Furthermore, we note here that in [10], it is showed that **KT2ConvFCO**, the category of KT2 constant convergence spaces and continuous functions, and **Chy**, the category of Cauchy spaces and Cauchy maps are isomorphic. Also, it is proved there that **ConvFCO**, the category of constant convergence spaces and continuous functions and **FilPreChy**, the category of filter preCauchy spaces and Cauchy maps are isomorphic. Hence, the category **ConvFCO** is a link between the categories **FilPreChy** and **FCOConv**. Interested readers are referred to [10] for a detail study on these categories and their terminologies. One can also look into the paper [11] to see that in Hausdorff objects, it is proved that **CULim**, the category of completely uniform limit spaces and uniformly continuous functions and **KT2Lim**, the category of limit spaces and continuous functions are isomorphic; and deduce that every KT2 limit space induces the associated complete uniform limit space.

Definition 8. *[23]* A probabilistic Cauchy space *is a pair* $\left(S, \bar{\mathfrak{c}} = (\mathfrak{c}_\varphi)_{\varphi \in \Delta^+}\right)$, *where* $\mathfrak{c}_\varphi \subseteq \mathbb{F}(S)$ *is called the* probabilistic Cauchy structure *on* S *under a triangle function* τ *if the following conditions are satisfied:*

(PCHS1) $[p] \in \mathfrak{c}_\varphi$, $\forall p \in S$, $\forall \varphi \in \Delta^+$;

(PCHS2) if $\mathbb{F} \in \mathfrak{c}_\varphi$ *and* $\mathbb{F} \leq \mathbb{G}$ *implies* $\mathbb{G} \in \mathfrak{c}_\varphi$;

(PCHS3) $\forall \varphi, \psi \in \Delta^+$ *with* $\varphi \leq \psi$ *implies* $\mathfrak{c}_\psi \leq \mathfrak{c}_\varphi$;

(PCHS4) $\mathfrak{c}_{\epsilon_\infty} = \mathbb{F}(S)$;

(PCHS5) if $\mathbb{F} \in \mathfrak{c}_\varphi$ *and* $\mathbb{G} \in \mathfrak{c}_\psi$ *for* $\varphi, \psi \in \Delta^+$ *such that* $\mathbb{F} \vee \mathbb{G}$ *exists implies that* $\mathbb{F} \wedge \mathbb{G} \in \mathfrak{c}_{\tau(\varphi,\psi)}$.

A map $f\colon (S, \bar{\mathfrak{c}}) \longrightarrow (S', \bar{\mathfrak{c}}')$ *between probabilistic Cauchy spaces is called* probabilistic Cauchy-continuous *if* $\forall \varphi \in \Delta^+$, $\forall \mathbb{F} \in \mathbb{F}(S)$, $\mathbb{F} \in \mathfrak{c}_\varphi$ *implies* $f(\mathbb{F}) \in \mathfrak{c}'_\varphi$.
The category of probabilistic Cauchy spaces under a triangle function τ *and probabilistic Cauchy-continuous mappings is denoted by* ***PChy***.

Definition 9. *[23]* A probabilistic Cauchy space under t-norm $*$, *[26], is a pair* $\left(S, \overline{C} = (C_\alpha)_{\alpha \in [0,1]}\right)$, *where* S *is a set and* $(C_\alpha)_{\alpha \in [0,1]}$ *is a non-empty family of subsets of* $\mathbb{F}(S)$ *satisfying the following:*

(PCH1) $[p] \in C_\alpha$ *for all* $p \in S$ *and* $\alpha \in [0, 1]$;

(PCH2) $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$ *and* $\alpha \in [0, 1]$ *with* $\mathbb{F} \leq \mathbb{G}$ *and* $\mathbb{F} \in C_\alpha$ *implies* $\mathbb{G} \in C_\alpha$;

(PCH3) if $\alpha \leq \beta$, *then* $C_\beta \leq C_\alpha$;

(PCH4) $C_0 = \mathbb{F}(S)$;

(PCH5) if $\mathbb{F} \in C_\alpha$, $\mathbb{G} \in C_\beta$ *and* $\mathbb{F} \vee \mathbb{G}$ *exists, then* $\mathbb{F} \wedge \mathbb{G} \in C_{\alpha * \beta}$.

A map $f\colon \left(S, \overline{C}\right) \longrightarrow \left(S', \overline{C'}\right)$ *between probabilistic Cauchy spaces under* t*-norm* $*$ *is called* Cauchy-continuous *if and only if for all* $\mathbb{F} \in \mathbb{F}(S)$ *and for all* $\alpha \in [0, 1]$, $\mathbb{F} \in C_\alpha$ *implies* $f(\mathbb{F}) \in C'_\alpha$.
The category of all probabilistic Cauchy spaces under t*-norm* $*$ *and Cauchy-continuous mappings is called Richarson-Kent probabilistic convergence space and denoted by* ***RK-PChy***.

Example 1. *[22] Let* $\left(S, \overline{C} = (C_\alpha)_{\alpha \in [0,1]}\right) \in |\textbf{\textit{RK-PChy}}|$. *For* $\varphi \in \Delta^+$ *we denote* $\varphi(0^+) = \lim_{x \to 0^+} \varphi(x)$ *the right-hand limit of* φ *at* 0. *If we define* $\mathbb{F} \in \mathfrak{c}_\varphi^{\overline{C}}$ *iff* $\mathbb{F} \in C_{\varphi(0^+)}$, *then clearly* $\left(S, \bar{\mathfrak{c}}^{\overline{C}}\right) \in |\textbf{\textit{PChy}}|$. *Also, if* $f\colon \left(S, \overline{C} = (C_\alpha)_{\alpha \in [0,1]}\right) \longrightarrow \left(S', \overline{C'} = (C_\alpha)_{\alpha \in [0,1]}\right)$ *is a continuous function, then* $f\colon \left(S, \mathfrak{c}_\varphi^{\overline{C}}\right) \longrightarrow \left(S', \mathfrak{c}_\varphi^{\overline{C'}}\right)$ *is continuous. Thus we obtain an embedding functor* ***RK-PChy*** $\overset{\mathfrak{S}}{\longrightarrow}$ ***PChy***, $\overline{C} \mapsto \bar{\mathfrak{c}}^{\overline{C}}$; $f \mapsto f$. *Next let* $\left(S, \bar{\mathfrak{c}} = (\mathfrak{c}_\varphi)_{\varphi \in \Delta^+}\right) \in |\textbf{\textit{PChy}}|$. *Define* $\mathbb{F} \in C_\alpha^{\bar{\mathfrak{c}}}$ *iff* $\exists \varphi \in \Delta^+$ *such that* $\varphi(0^+) = \alpha$ *and* $\mathbb{F} \in \mathfrak{c}_\varphi$. *Then* $\left(S, C_\alpha^{\bar{\mathfrak{c}}}\right) \in$

$|$***RK-PChy***$|$. *Also, if* $f\colon \left(S,\bar{\mathfrak{c}}=(\mathfrak{c}_\varphi)_{\varphi\in\Delta^+}\right)\longrightarrow\left(S',\overline{\mathfrak{c}'}=(\mathfrak{c}'_\varphi)_{\varphi\in\Delta^+}\right)$ *is a continuous function, then* $f\colon \left(S,C_\alpha^{\bar{\mathfrak{c}}}\right)\longrightarrow\left(S',C_\alpha^{\overline{\mathfrak{c}'}}\right)$ *is a continuous function. Thus, one obtains the embedding functor* ***PChy*** $\xrightarrow{\mathfrak{T}}$ ***RK-PChy***, $\bar{\mathfrak{c}}\mapsto\overline{C}^{\bar{\mathfrak{c}}}$; $f\mapsto f$. *As* $\mathfrak{T}\circ\mathfrak{S}=id_{\textbf{\textit{RK-PChy}}}$ *and* $\mathfrak{S}\circ\mathfrak{T}\geq id_{\textbf{\textit{PChy}}}$, *we have that the category* ***RK-PChy*** *is a reflective subcategory of* ***PChy***.

Proposition 2. *The category* ***PChy*** *is a topological category.*

Proof. This follows from the Proposition 3.2[23].

We define for $\left(S,\bar{\mathfrak{c}}=(\mathfrak{c}_\varphi)_{\varphi\in\Delta^+}\right)\in|\mathbf{PChy}|$, the probabilistic convergence structure $\bar{c}^{\bar{\mathfrak{c}}}$ by

$$p\in c_\varphi^{\bar{\mathfrak{c}}}(\mathbb{F})\iff \mathbb{F}\wedge[p]\in\mathfrak{c}_\varphi,\ \forall\varphi\in\Delta^+,\text{ and }\mathbb{F}\in\mathbb{F}(S).$$

Lemma 17. *Let* $(S,\bar{\mathfrak{c}})\in|$***PChy***$|$ *under the largest triangle function* τ. *Then* $\left(S,\bar{c}^{\bar{\mathfrak{c}}}\right)\in$ $|$***PConv***$|$ *and satisfies the axiom (PCS5):*

$$\forall\varphi\in\Delta^+\ \textit{and}\ \forall\ \mathbb{F},\mathbb{G}\in\mathbb{F}(S),\ \ c_\varphi^{\bar{\mathfrak{c}}}(\mathbb{F})\cap c_\varphi^{\bar{\mathfrak{c}}}(\mathbb{G})\leq c_\varphi^{\bar{\mathfrak{c}}}\left(\mathbb{F}\wedge\mathbb{G}\right).$$

Proof. (PCS1) Since $[p]\in\mathfrak{c}_\varphi$ for all $\varphi\in\Delta^+$, and for all $p\in S$ we have $[p]=[p]\wedge[p]$ implies $p\in c_\varphi^{\bar{\mathfrak{c}}}([p])$.
(PCS2) Let $\mathbb{F},\mathbb{G}\in\mathbb{F}(S)$ with $\mathbb{F}\leq\mathbb{G}$, $\varphi\in\Delta^+$, and $p\in c_\varphi^{\bar{\mathfrak{c}}}(\mathbb{F})$. Then we have $\mathbb{F}\wedge[p]\in\mathfrak{c}_\varphi$. Since $\mathbb{F}\wedge[p]\leq\mathbb{G}\wedge[p]$, we have $\mathbb{G}\wedge[p]\in\mathfrak{c}_\varphi$ by (PCHS2), which implies that $p\in c_\varphi^{\bar{\mathfrak{c}}}(\mathbb{G})$.
(PCS3) Let $\varphi,\psi\in\Delta^+$ with $\varphi\leq\psi$ and $\mathbb{F}\in\mathbb{F}(S)$. If $p\in c_\psi^{\bar{\mathfrak{c}}}(\mathbb{F})$ implies $\mathbb{F}\wedge[p]\in\mathfrak{c}_\psi$. Then $\mathbb{F}\wedge[p]\in\mathfrak{c}_\varphi$ by (PCHS3), implying $p\in c_\varphi^{\bar{\mathfrak{c}}}(\mathbb{F})$.
(PCS4) Obvious.
(PCS5) Let $\varphi\in\Delta^+$ and $\mathbb{F},\mathbb{G}\in\mathbb{F}(S)$. If $p\in c_\varphi^{\bar{\mathfrak{c}}}(\mathbb{F})\cap c_\varphi^{\bar{\mathfrak{c}}}(\mathbb{F})$, then $\mathbb{F}\wedge[p]\in\mathfrak{c}_\varphi$ and $\mathbb{G}\wedge[p]\in\mathfrak{c}_\varphi$. This implies that $(\mathbb{F}\wedge\mathbb{G})\wedge[p]\in\mathfrak{c}_\varphi$ by (PCHS5), and hence $p\in c_\varphi^{\bar{\mathfrak{c}}}\left(\mathbb{F}\wedge\mathbb{G}\right)$.

Lemma 18. *Let* $(S,\bar{\mathfrak{c}})$, $\left(S,\overline{\mathfrak{c}'}\right)\in|$***PChy***$|$ *and* $f\colon (S,\bar{\mathfrak{c}})\longrightarrow\left(S',\overline{\mathfrak{c}'}\right)$ *be probabilistic Cauchy-continuous function. Then* $f\colon \left(S,c^{\bar{\mathfrak{c}}}\right)\longrightarrow\left(S',c^{\overline{\mathfrak{c}'}}\right)$ *is continuous.*

Proof. Let $p\in S$, $\varphi\in\Delta^+$ and $\mathbb{F}\in\mathbb{F}(S)$. If now $p\in c_\varphi^{\bar{\mathfrak{c}}}(\mathbb{F})$, then $\mathbb{F}\wedge[p]\in\mathfrak{c}_\varphi$ implying that $f\left(\mathbb{F}\wedge[p]\right)\in\mathfrak{c}'_\varphi$, implying $f(\mathbb{F})\wedge[f(p)]\in\mathfrak{c}'_\varphi$. Hence $f(p)\in c_\varphi^{\overline{\mathfrak{c}'}}(f(\mathbb{F}))$.

Then Lemma 17 and Lemma 18 yield the following

Corollary 2.

$$\mathfrak{F}:\begin{cases}\textbf{\textit{PChy}} & \longrightarrow & \textbf{\textit{PConv}}\\ (X,\bar{\mathfrak{c}}) & \longmapsto & \left(X,\bar{c}^{\bar{\mathfrak{c}}}\right)\\ f & \longmapsto & f\end{cases},$$

is a functor.

7. Probabilistic pre Cauchy groups and their relationship with probabilistic Cauchy groups

Definition 10. *Let* $(S,\cdot)$ *be a group.* A probabilistic pre-Cauchy group *is a triple* $\left(S,\cdot,\bar{\mathfrak{c}}=(\mathfrak{c}_\varphi)_{\varphi\in\Delta^+}\right)$ *fulfilling the following conditions:*

(PCHS1) $[p]\in\mathfrak{c}_\varphi$, $\forall p\in S$, $\forall\varphi\in\Delta^+$;

(PCHS2) if $\mathbb{F}\in\mathfrak{c}_\varphi$ *and* $\mathbb{F}\le\mathbb{G}\Rightarrow\mathbb{G}\in\mathfrak{c}_\varphi$;

(PCHS3) $\forall\varphi,\psi\in\Delta^+$ *with* $\varphi\le\psi\Rightarrow\mathfrak{c}_\psi\le\mathfrak{c}_\varphi$;

(PCHS4) $\mathfrak{c}_{\epsilon_\infty}=\mathbb{F}(S)$;

(PCHGM)′ $\mathbb{F}\in\mathfrak{c}_\varphi$ *and* $\mathbb{G}\in\mathfrak{c}_\psi\Rightarrow\mathbb{F}^{-1}\odot\mathbb{G}\in\mathfrak{c}_{\tau(\varphi,\psi)}$.

Let ***PPreChyGrp*** *denote the category consists of all probabilistic pre-Cauchy groups as objects and probabilistic Cauchy-continuous group-homomorphisms as morphisms.*

Proposition 3. *Let* $\left(S,\cdot,\bar{\mathfrak{c}}=(\mathfrak{c}_\varphi)_{\varphi\in\Delta^+}\right)\in|$***PPreChyGrp***$|$, *and* $\mathbb{F},\mathbb{G}\in\mathfrak{c}_\varphi$, *and* $\varphi\in\Delta^+$. *If* τ *is the largest triangle function, i.e.,* $\tau(\varphi,\varphi)=\varphi$, *then* $\mathbb{F}\wedge\mathbb{G}\in\mathfrak{c}_\varphi$ *if and only if* $e\in c_\varphi^{\bar{\mathfrak{c}}}\left(\mathbb{F}^{-1}\odot\mathbb{G}\right)$.

Proof. $\Rightarrow$: Let $\mathbb{F},\mathbb{G}\in\mathfrak{c}_\varphi$, and $\varphi\in\Delta^+$. Put $\mathbb{H}=\mathbb{F}\wedge\mathbb{G}$. Then $\mathbb{H}^{-1}\le\mathbb{F}^{-1}$; also, $\mathbb{H}\le\mathbb{G}$. So, $\mathbb{H}^{-1}\odot\mathbb{H}\le\mathbb{F}^{-1}\odot\mathbb{G}$, whence $\mathbb{F}^{-1}\odot\mathbb{G}\in\mathfrak{c}_\varphi$. By Lemma 2(a), $\mathbb{H}^{-1}\odot\mathbb{H}\le[e]$ and so, $\mathbb{H}^{-1}\odot\mathbb{H}\wedge[e]\in\mathfrak{c}_\varphi$ which yields that $e\in c_\varphi^{\bar{\mathfrak{c}}}\left(\mathbb{H}^{-1}\odot\mathbb{H}\right)$ and hence by (PCS2), $e\in c_\varphi^{\bar{\mathfrak{c}}}\left(\mathbb{F}^{-1}\odot\mathbb{G}\right)$.
$\Leftarrow$: If $e\in c_\varphi^{\bar{\mathfrak{c}}}\left(\mathbb{F}^{-1}\odot\mathbb{G}\right)$, then $\left(\mathbb{F}^{-1}\odot\mathbb{G}\right)\wedge[e]\in\mathfrak{c}_\varphi$. Since $\mathbb{F}\in\mathfrak{c}_\varphi$, $\mathbb{F}\odot\left(\left(\mathbb{F}^{-1}\odot\mathbb{G}\right)\wedge[e]\right)\in\mathfrak{c}_\varphi$. But in view of Lemma 2, $\left(\left(\mathbb{F}\odot\mathbb{F}^{-1}\right)\odot\mathbb{G}\wedge[e]\right)\right)=\left(\left(\mathbb{F}\odot\mathbb{F}^{-1}\right)\odot\mathbb{G}\right)\wedge(\mathbb{F}\odot[e])\le\mathbb{F}\wedge\mathbb{G}$. Thus we have $\mathbb{F}\wedge\mathbb{G}\in\mathfrak{c}_\varphi$.

Corollary 3. *If* $\left(X,\cdot,\bar{\mathfrak{c}}=(\mathfrak{c}_\varphi)_{\varphi\in\Delta^+}\right)$ *is a probabilistic pre-Cauchy group under a triangle function* τ, *then it is probabilistic Cauchy space.*

Proof. We only need to verify the axiom (PCHS5). For, let $\varphi\in\Delta^+$ and $\mathbb{F},\mathbb{G}\in\mathbb{F}(S)$. If $\mathbb{F}\in\mathfrak{c}_\varphi$, and $\mathbb{G}\in\mathfrak{c}_\psi$, then by (PCHGM)′, $\mathbb{F}^{-1}\odot\mathbb{G}\in\mathfrak{c}_{\tau(\varphi,\psi)}$, since $\mathbb{F}\in\mathfrak{c}_\varphi$, we have again, $\mathbb{F}\odot(\mathbb{F}^{-1}\odot\mathbb{G})\in\mathfrak{c}_{\tau(\varphi,\psi)}$. Then $\mathbb{F}\odot\mathbb{F}^{-1}\odot\mathbb{G}\le\mathbb{F}\wedge\mathbb{G}$ with $[e]\le\mathbb{F}$, whence $\mathbb{F}\wedge\mathbb{G}\in\mathfrak{c}_{\tau(\varphi,\psi)}$.

Proposition 4. ***PPreChyGrp*** *is a topological category with respect to the forgetful functor.*

Proof. Let $(S,\cdot)$ be a group and $f_j\colon S\longrightarrow S_j$ a group-homomorphism, and $\left(S_j,\cdot,(\mathfrak{c}_\varphi^j)_{\varphi\in\Delta^+}\right)$ be a family of probabilistic pre-Cauchy spaces. Let $\mathcal{S}=\left(f_j\colon S\longrightarrow\left(S_j,\cdot,\mathfrak{c}_\varphi^j\right)_\varphi\right)_{j\in J}$ be a source. Then due to Proposition 2, for any $\mathbb{F}\in\mathbb{F}(S)$, $\mathbb{F}\in\mathfrak{c}_\varphi\iff f_j(\mathbb{F})\in\mathfrak{c}_\varphi^j$, $\forall j\in J$,

$\forall\varphi \in \Delta^+$. Thus, the source has initial lift, giving that $(S, \cdot, \bar{\mathfrak{c}})$ is a probabilistic pre-Cauchy space. We only prove the condition (PCHGM)$'$. For, let $\mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$, $\varphi, \psi \in \Delta^+$, $\mathbb{F} \in \mathfrak{c}_\varphi$, and $\mathbb{G} \in \mathfrak{c}_\psi$. Since for each $j \in J$, $f_j(\mathbb{F})$, $f_j(\mathbb{G}) \in \mathbb{F}(S_j)$, we have $f_j(\mathbb{F}) \in \mathfrak{c}^j_\varphi$ and $f_j(\mathbb{G}) \in \mathfrak{c}^j_\psi$ implying that $(f_j(\mathbb{F}))^{-1} \odot f_j(\mathbb{G}) \in \mathfrak{c}^j_{\tau(\varphi,\psi)}$. Now using Lemma 2(l), we get $f_j\left(\mathbb{F}^{-1} \odot \mathbb{G}\right) = (f_j(\mathbb{F}))^{-1} \odot f_j(\mathbb{G}) \in \mathfrak{c}^j_{\tau(\varphi,\psi)}$, yields that $\mathbb{F}^{-1} \odot \mathbb{G} \in \mathfrak{c}_{\tau(\varphi,\psi)}$. The missing part follows by using definitions.

Definition 11. *Let $(S, \cdot)$ be a group and $\left(S, \bar{\mathfrak{c}} = (\mathfrak{c}_\varphi)_{\varphi\in\Delta^+}\right)$ a probabilistic Cauchy space. Then the triple $\left(S, \cdot, \bar{\mathfrak{c}} = (\mathfrak{c}_\varphi)_{\varphi\in\Delta^+}\right)$ is called a* probabilistic Cauchy group under a triangle function τ *if the following assertions are satisfied:*

(PCHGM) $\mathbb{F} \in \mathfrak{c}_\varphi$ *and* $\mathbb{G} \in \mathfrak{c}_\psi \Rightarrow \mathbb{F} \odot \mathbb{G} \in \mathfrak{c}_{\tau(\varphi,\psi)}$ $\forall\mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$ *and* $\forall\varphi, \psi \in \Delta^+$*;*

(PCHGI) $\mathbb{F} \in \mathfrak{c}_\varphi \Rightarrow \mathbb{F}^{-1} \in \mathfrak{c}_\varphi$, $\forall\mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$ *and* $\forall\varphi, \psi \in \Delta^+$.

*The category of all probabilistic Cauchy groups and probabilistic Cauchy-continuous group-homomorphisms is denoted by **PChyGrp**. Note that the category **PChyGrp** is a full subcategory of the category of **PPreChyGrp**.*

Lemma 19. *Let $\left(S, \cdot, \bar{\mathfrak{c}} = (\mathfrak{c}_\varphi)_{\varphi\in\Delta^+}\right)$ be a probabilistic Cauchy group under a triangle function τ. Then the condition (PCHGM)$'$ is equivalent to the conditions (PCHGM) and (PCHGI) as shown below:*

(PCHGM)$'$ $\forall\mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$, $\forall\varphi, \psi \in \Delta^+$, $\mathbb{F} \in \mathfrak{c}_\varphi$, $\mathbb{G} \in \mathfrak{c}_\psi \Rightarrow \mathbb{F}^{-1} \odot \mathbb{G} \in \mathfrak{c}_{\tau(\varphi,\psi)}$*;*

(PCHGM) $\forall\mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$, $\forall\varphi, \psi \in \Delta^+$, $\mathbb{F} \in \mathfrak{c}_\varphi$, $\mathbb{G} \in \mathfrak{c}_\psi \Rightarrow \mathbb{F} \odot \mathbb{G} \in \mathfrak{c}_{\tau(\varphi,\psi)}$*;*

(PCHGI) $\forall\mathbb{F} \in \mathbb{F}(S)$, $\forall\varphi \in \Delta^+$, $\mathbb{F} \in \mathfrak{c}_\varphi \Rightarrow \mathbb{F}^{-1} \in \mathfrak{c}_\varphi$.

Proof. (PCHGM)$'\Rightarrow$(PCHGM)+(PCHGI): First, we prove (PCHGI). Let $\mathbb{F} \in \mathbb{F}(S)$, $\varphi \in \Delta^+$ and $\mathbb{F} \in \mathfrak{c}_\varphi$. Since in particular, $[e] \in \mathfrak{c}_{\epsilon_0}$, we get by assumption that $\mathbb{F}^{-1} = \mathbb{F}^{-1} \odot [e] \in \mathfrak{c}_{\tau(\varphi,\epsilon_0)} = \mathfrak{c}_\varphi$ because of the fact that $\tau(\varphi, \epsilon_0) = \varphi$. Hence $\mathbb{F}^{-1} \in \mathfrak{c}_\varphi$. To prove the other part, let $\mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$, and $\varphi, \psi \in \Delta^+$. If now $\mathbb{F} \in \mathfrak{c}_\varphi$ and $\mathbb{G} \in \mathfrak{c}_\psi$, then by using (PCHGI) and the assumption, we get $\mathbb{F} \odot \mathbb{G} = (\mathbb{F}^{-1})^{-1} \odot \mathbb{G} \in \mathfrak{c}_{\tau(\varphi,\psi)}$, i.e., $\mathbb{F} \odot \mathbb{G} \in \mathfrak{c}_{\tau(\varphi,\psi)}$. Conversely, we prove (PCHGM)+(PCHGI)$\Rightarrow$ (PCHGM)$'$. If $\mathbb{F} \in \mathfrak{c}_\varphi$ and $\mathbb{G} \in \mathfrak{c}_\psi$, then because of (PCHGI), $\mathbb{F}^{-1} \in \mathfrak{c}_\varphi$ and hence by using (PCHGM), we have $\mathbb{F}^{-1} \odot \mathbb{G} \in \mathfrak{c}_{\tau(\varphi,\psi)}$.

Proposition 5. *Let $(S, \cdot, \bar{\mathfrak{c}}) \in |$**PChyGrp**$|$ under the a triangle function τ. Then $\left(S, \cdot, \bar{c}^{\bar{\mathfrak{c}}}\right) \in |$**PConvGrp**$|$.*

Proof. Let $\left(S, \cdot, \bar{\mathfrak{c}} = (\mathfrak{c}_\varphi)_{\varphi\in\Delta^+}\right)$ be a probabilistic Cauchy group. Define $p \in c^{\bar{\mathfrak{c}}}_\varphi(\mathbb{F}) \Longleftrightarrow [p] \wedge \mathbb{F} \in \mathfrak{c}_\varphi$, for $p \in S$, $\varphi \in \Delta^+$ and $\mathbb{F} \in \mathbb{F}(S)$. In the light of Lemma 17, we only need to prove

(PCHGM)$'$: Let $p, q \in S$ and $\mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$ with $\varphi, \psi \in \Delta^+$. If $p \in c^{\bar{\mathfrak{c}}}_\varphi(\mathbb{F})$ and $q \in c^{\bar{\mathfrak{c}}}_\psi(\mathbb{G})$, then $[p] \wedge \mathbb{F} \in \mathfrak{c}_{\varphi}$ and $[q] \wedge \mathbb{G} \in \mathfrak{c}_{\psi}$, whence $([p] \wedge \mathbb{F})^{-1} \in \mathfrak{c}_{\varphi}$. These together upon using Lemma 2(i) imply that $\left([p]^{-1} \wedge \mathbb{F}^{-1}\right) \odot ([q] \wedge \mathbb{G}) \in \mathfrak{c}_{\tau(\varphi,\psi)}$. Again using Lemma 2(j), we can simplify to: $[p^{-1}q] \wedge (\mathbb{F}^{-1} \odot \mathbb{G}) \geq \left([p]^{-1} \wedge \mathbb{F}^{-1}\right) \odot ([q] \wedge \mathbb{G}) \in \mathfrak{c}_{\tau(\varphi,\psi)}$. Hence we obtain $p^{-1}q \in c^{\bar{\mathfrak{c}}}_{\tau(\varphi,\psi)} \left(\mathbb{F}^{-1} \odot \mathbb{G}\right)$, which in the light of Lemma 17, we are done.

Lemma 20. *Let* $\left(S, \cdot, \bar{c} = (c_{\varphi})_{\varphi \in \Delta^+}\right) \in |\boldsymbol{PLimGrp}|$. *Then* $\left(S, \cdot, \bar{\mathfrak{c}}^{\bar{c}}\right) \in |\boldsymbol{PChyGrp}| \Longleftrightarrow$ $(\sharp)$*:* $\left(\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(S), \varphi \in \Delta^+, e \in c_{\varphi}\left(\mathbb{F}^{-1} \odot \mathbb{F}\right), e \in c_{\varphi}\left(\mathbb{F} \odot \mathbb{F}^{-1}\right), \textit{and } e \in c_{\varphi}(\mathbb{G})\right) \Longrightarrow$ $e \in c_{\varphi}\left(\mathbb{F} \odot \mathbb{G} \odot \mathbb{F}^{-1}\right)$.

Proof. Define

$$\mathbb{F} \in \mathfrak{c}_{\varphi} \Longleftrightarrow e \in c^{\bar{\mathfrak{c}}}_{\varphi}\left(\mathbb{F}^{-1} \odot \mathbb{F}\right) \textit{ and } e \in c^{\bar{\mathfrak{c}}}_{\varphi}\left(\mathbb{F} \odot \mathbb{F}^{-1}\right), \forall \mathbb{F} \in \mathbb{F}(S), \forall \varphi \in \Delta^+.$$

(PCHS1) Since in particular, $e \in c^{\bar{\mathfrak{c}}}_{\varphi}\left([p]^{-1} \odot [p]\right)$, we have $p \in c^{\bar{\mathfrak{c}}}_{\varphi}([p])$ and hence $[p] \in \mathfrak{c}_{\varphi}$.
(PCHS2) Let $\mathbb{F} \in \mathfrak{c}_{\varphi}$ with $\mathbb{F} \leq \mathbb{G}$. Then $e \in c^{\bar{\mathfrak{c}}}_{\varphi}\left(\mathbb{F}^{-1} \odot \mathbb{F}\right)$. Because of $\mathbb{F}^{-1} \odot \mathbb{F} \leq \mathbb{G}^{-1} \odot \mathbb{G}$, by using (PCS2), $e \in c^{\bar{\mathfrak{c}}}_{\varphi}\left(\mathbb{G}^{-1} \odot \mathbb{G}\right)$. Hence $\mathbb{G} \in \mathfrak{c}_{\varphi}$.
(PCHS3) Let $\varphi, \psi \in \Delta^+$ with $\varphi \leq \psi$. If $\mathbb{F} \in \mathfrak{c}_{\psi}$, then $e \in c^{\bar{\mathfrak{c}}}_{\psi}\left(\mathbb{F}^{-1} \odot \mathbb{F}\right)$. By (PCS3), $e \in c^{\bar{\mathfrak{c}}}_{\varphi}\left(\mathbb{F}^{-1} \odot \mathbb{F}\right)$ implies that $\mathbb{F} \in \mathfrak{c}_{\varphi}$.
(PCHS4) is obviously true.
(PCHS5) Let $\mathbb{F} \in \mathfrak{c}_{\varphi}$, then $\mathbb{F}^{-1} \in \mathfrak{c}_{\varphi}$. Consider $\mathbb{F}, \mathbb{G} \in \mathfrak{c}_{\varphi}$. Then by Lemma 2(g), $(\mathbb{F} \odot \mathbb{G}) \odot (\mathbb{F} \odot \mathbb{G})^{-1} = \mathbb{F} \odot \mathbb{G} \odot \mathbb{G}^{-1} \odot \mathbb{F}^{-1}$. But $e \in c_{\varphi}(\mathbb{G} \odot \mathbb{G}^{-1})$ since $\mathbb{G} \in \mathfrak{c}_{\varphi}$ and $e \in c_{\varphi}\Big(\mathbb{F} \odot \mathbb{G} \odot \mathbb{G}^{-1} \odot \mathbb{F}^{-1}\Big)$. Similarly, one obtains $e \in c_{\varphi}\Big((\mathbb{F} \odot \mathbb{G})^{-1} \odot (\mathbb{F} \odot \mathbb{G})\Big)$. Hence $\mathbb{F} \odot \mathbb{G} \in \mathfrak{c}_{\varphi}$. Conversely, assume that $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}(S)$, $e \in c_{\varphi}\left(\mathbb{F}^{-1} \odot \mathbb{F}\right)$, $e \in c_{\varphi}\left(\mathbb{F} \odot \mathbb{F}^{-1}\right))$, and $e \in c_{\varphi}(\mathbb{G})$. By definition $\mathbb{F}, \mathbb{G} \in \mathfrak{c}_{\varphi}$, so $\mathbb{F} \odot \mathbb{G} \in \mathfrak{c}_{\varphi}$. But this implies that $e \in c_{\varphi}\Big((\mathbb{F} \odot \mathbb{G}) \odot (\mathbb{F} \odot \mathbb{G})^{-1}\Big)$. If we assume $[e] \leq \mathbb{G}$, then by using Lemma 2, we obtain $\mathbb{F} \odot \mathbb{G} \odot \mathbb{G}^{-1} \odot \mathbb{F}^{-1} \leq \mathbb{F} \odot \mathbb{G} \odot \mathbb{F}^{-1}$ and hence $e \in c_{\varphi}\left(\mathbb{F} \odot \mathbb{G} \odot \mathbb{F}^{-1}\right)$.

Lemma 21. *Let* $(S, \cdot, \bar{c})$, $\left(S', \cdot, \overline{c'}\right) \in |\boldsymbol{PLimGrp}|$ *under the largest triangle function* τ, *and satisfies* $(\sharp)$. *If* $f \colon S \longrightarrow S'$ *a group homomorphism, then the following assertions are equivalent.*

(i) $f \colon (S, \bar{c}) \longrightarrow \left(S', \overline{c'}\right)$ *is continuous;*

(ii) $f \colon \left(S, \bar{\mathfrak{c}}^{\bar{c}}\right) \longrightarrow \left(S', \bar{\mathfrak{c}}^{\overline{c'}}\right)$ *is Cauchy-continuous.*

Proof. Assume (i) holds. Let $\mathbb{F} \in \mathfrak{c}^{\bar{c}}_{\varphi}$. Then $e \in c_{\varphi}\left(\mathbb{F}^{-1} \odot \mathbb{F}\right)$ implies that $e' = f(e) \in c_{\varphi}\left(f(\mathbb{F})^{-1} \odot f(\mathbb{F})\right)$ by Lemma 2(k),(l). Similarly, $e' \in c_{\varphi}\left(f(\mathbb{F}) \odot f(\mathbb{F})^{-1}\right)$. Hence $f(\mathbb{F}) \in \mathfrak{c}^{\bar{c}}_{\varphi}$. Conversely, assume (ii) holds. Then for any $p \in S$, $\varphi \in \Delta^+$ and $\mathbb{F} \in \mathbb{F}(S)$, if $p \in c_{\varphi}(\mathbb{F})$, then because of $p^{-1} \in c_{\varphi}(\mathbb{F}^{-1})$, $e = p^{-1}p \in c_{\tau(\varphi,\varphi)}(\mathbb{F}^{-1} \odot \mathbb{F}) = c_{\varphi}\left(\mathbb{F}^{-1} \odot \mathbb{F}\right)$ implies that $\mathbb{F} \in \mathfrak{c}_{\varphi}$. Then $f(\mathbb{F}) \in \mathfrak{c}'_{\varphi}$. By (PCHS5), we have $(f(\mathbb{F}) \wedge [f(p)]) \in \mathfrak{c}'_{\varphi}$.

This yields that $e' \in c_\varphi\Big(\big(f(\mathbb{F}) \wedge [f(p)]\big)^{-1} \odot \big(f(\mathbb{F}) \wedge [f(p)]\big)\Big)$. But it follows immediately that $\Big(\big(f(\mathbb{F}) \wedge [f(p)]\big)^{-1} \odot \big(f(\mathbb{F}) \wedge [f(p)]\big)\Big) \leq \big([f(p)]^{-1} \odot f(\mathbb{F})\big)$ which implies that $e' \in c_\varphi\left([f(p)]^{-1} \odot f(\mathbb{F})\right)$. Hence because of homogeneity, by Lemma 13(a), we get $f(p) \in c'_\varphi(f(\mathbb{F}))$.

Thus we get the following

Corollary 4.

$$\mathfrak{G} : \begin{cases} \boldsymbol{SNPLimGrp} & \longrightarrow & \boldsymbol{PChyGrp} \\ (S, \overline{c}) & \longmapsto & \left(S', \overline{\mathfrak{c}}^{\overline{c}}\right) \\ f & \longmapsto & f \end{cases},$$

is a functor, where ***SNPLimGrp*** *denotes the category of all objects (strongly normal probabilistic limit groups under the largest triangle function* τ*) satisfying condition* ($\sharp$) *and morphisms as continuous mappings between them.*

Remark 6. *This notion of* strongly normal *originally introduced by R. N. Ball [9] in relation to his notion of convergence structures and Cauchy structures for lattice order groups or so-called l-groups; some other authors also used this idea towards the construction of Cauchy completion, cf. [17] (see also, [23]).*

Proposition 6. *(i) Let* $\left(S, \cdot, \overline{\mathfrak{c}} = (\mathfrak{c}_\varphi)_{\varphi \in \Delta^+}\right)$ *be a probabilistic pre-Cauchy group, then* $\left(S, \cdot, c_\varphi^{\overline{\mathfrak{c}}}\right)$ *is a probabilistic limit group.*

(ii) If $(S, \cdot, \overline{c})$ *is a probabilistic limit group, then* $\left(S, \mathfrak{c}^{\overline{c}}\right)$ *is a probabilistic pre-Cauchy group if and only if* $(S, \cdot, \overline{c})$ *is probabilistic limit group satisfying* $\sharp$.

(iii) If $(S, \cdot, \overline{c})$ *is a probabilistic limit group and* $\left(S, \cdot, \mathfrak{c}^{\overline{c}}\right)$ *a probabilistic pre-Cauchy group, then* $\left(S, \cdot, \overline{\mathfrak{c}} = (\mathfrak{c}_\varphi)_{\varphi \in \Delta^+}\right)$ *is a probabilistic Cauchy group.*

Proof. We only prove item(i).
(i) Let $p \in c_\varphi^{\overline{\mathfrak{c}}}(\mathbb{F})$ and $q \in c_\psi^{\overline{\mathfrak{c}}}(\mathbb{G})$. Then as $[p] \wedge \mathbb{F} \in \mathfrak{c}_\varphi$ implying $([p] \wedge \mathbb{F})^{-1} \in \mathfrak{c}_\varphi$ which in turn upon using Lemma 2(g) implies that $\mathbb{F}^{-1} \wedge [p]^{-1} \in \mathfrak{c}_\varphi$ and $\mathbb{G} \wedge [q] \in \mathfrak{c}_\psi$. These together imply that $p^{-1}q \in c_{\tau(\varphi,\psi)}^{\overline{\mathfrak{c}}}(\mathbb{F}^{-1} \odot \mathbb{G})$. In fact, $\left(\mathbb{F}^{-1} \wedge [p]^{-1}\right) \odot (\mathbb{G} \wedge [q]) \in \mathfrak{c}_{\tau(\varphi,\psi)}$, and upon using Lemma 2(j) twice, one obtains: $\left(\mathbb{F}^{-1} \wedge [p]^{-1}\right) \odot (\mathbb{G} \wedge [q]) \leq \left(\mathbb{F}^{-1} \odot \mathbb{G}\right) \wedge ([p]^{-1} \odot [q]) = \left(\mathbb{F}^{-1} \odot \mathbb{G}\right) \wedge ([p^{-1}q])$, whence $\left(\mathbb{F}^{-1} \odot \mathbb{G}\right) \wedge [p^{-1}q] \in \mathfrak{c}_{\tau(\varphi,\psi)}$ showing that $p^{-1}q \in c_{\tau(\varphi,\psi)}^{\overline{\mathfrak{c}}}(\mathbb{F}^{-1} \odot \mathbb{G})$.

Theorem 2. *If* $\left(S, \cdot, \overline{\mathfrak{c}} = (\mathfrak{c}_\varphi)_{\varphi \in \Delta^+}\right) \in |\boldsymbol{PChyGrp}|$, *then* $\left(S, \overline{c}^{\overline{\mathfrak{c}}}\right) \in |\boldsymbol{SNPLimGrp}|$. *Conversely, if* $(S, \overline{c}) \in |\boldsymbol{SNPLimGrp}|$, *then* $\left(S, \mathfrak{c}^{\overline{c}}\right) \in |\boldsymbol{PChyGrp}|$. *Furthermore, there are functors*

$$\boldsymbol{PChyGrp} \underset{\mathfrak{G}}{\overset{\mathfrak{F}}{\rightleftarrows}} \boldsymbol{SNPLimGrp}$$

*such that $\mathfrak{F}\mathfrak{G} = id_{\boldsymbol{PChyGrp}}$ and $\mathfrak{S}\mathfrak{T} = id_{\boldsymbol{SNPLimGrp}}$. That is, the categories **PChyGrp** and **SNPLimGrp** are isomorphic under the largest triangle function τ, i.e., $\tau(\varphi,\varphi) = \varphi$.*

Proof. Observe that

$$\mathbf{PChyGrp} \xrightarrow{\mathfrak{F}} \mathbf{SNLimGrp} \xrightarrow{\mathfrak{G}} \mathbf{PChyGrp} \colon\ (S, \bar{\mathfrak{c}}) \longmapsto \left(S, \bar{c}^{\bar{\mathfrak{c}}}\right) \longmapsto (S, \bar{\mathfrak{c}})\,;$$

then one can check that $\mathfrak{G} \circ \mathfrak{F} = id_{\mathbf{PChyGrp}}$, i.e., $\mathfrak{c}^{c^{\bar{\mathfrak{c}}}} = \mathfrak{c}$. In fact, for any $\mathbb{F} \in \mathbb{F}(S)$ and $\varphi \in \Delta^+$, $\mathbb{F} \in \mathfrak{c}_\varphi^{c^{\bar{\mathfrak{c}}}} \iff e \in c_\varphi^{\bar{\mathfrak{c}}}(\mathbb{F}^{-1} \odot \mathbb{F}) \iff \mathbb{F} \in \mathfrak{c}_\varphi$; so $\mathfrak{c}^{c^{\bar{\mathfrak{c}}}} = \mathfrak{c}$. On the other hand,

$$\mathbf{SNLimGrp} \xrightarrow{\mathfrak{G}} \mathbf{PChyGrp} \xrightarrow{\mathfrak{F}} \mathbf{SNLimGrp} \colon\ (S, \bar{c}) \longmapsto \left(S, \mathfrak{c}^{\bar{c}}\right) \longmapsto (S, \bar{c})\,,$$

which yields that $\mathfrak{F} \circ \mathfrak{G} = id_{\mathbf{SNPLimGrp}}$, i.e., $c^{\mathfrak{c}^c} = c$. In fact, for any $p \in S$, $\mathbb{F} \in \mathbb{F}(S)$ and $\varphi \in \Delta^+$, let $p \in c_\varphi^{\mathfrak{c}^{\bar{c}}}(\mathbb{F})$. Then $\mathbb{F} \wedge [p] \in \mathfrak{c}_\varphi^{\bar{c}}$ and so $e \in c_\varphi\left((\mathbb{F} \wedge [p])^{-1} \wedge \odot(\mathbb{F} \wedge [p])\right)$. Upon using Lemma 2, then one obtains: $\left(\mathbb{F}^{-1} \wedge [p^{-1}]\right) \odot (\mathbb{F} \wedge [p]) \leq [p^{-1}] \wedge \mathbb{F}$. By (PCS2) and, then by homogeneity, one gets $p \in c_\varphi(\mathbb{F})$. Conversely, let $p \in c_\varphi(\mathbb{F})$, then $e \in c_\varphi\left([p]^{-1} \odot \mathbb{F}\right)$ and as because of $p^{-1} \in c_\varphi(\mathbb{F}^{-1})$, we have $e = p^{-1}p \in c_\varphi\left(\mathbb{F}^{-1} \odot \mathbb{F}\right)$, taking into account that $\tau(\varphi,\varphi) = \varphi$, i.e., because of our assumption of the largest triangle function τ. Now since we have by Lemma 2, $(\mathbb{F} \wedge [p])^{-1} \odot (\mathbb{F} \wedge [p]) = (\mathbb{F}^{-1} \odot \mathbb{F}) \wedge (\mathbb{F} \odot [p]) \wedge ([p]^{-1} \odot \mathbb{F}) \wedge [e]$ we get $e \in c_\varphi\left((\mathbb{F} \wedge [p])^{-1} \odot (\mathbb{F} \wedge [p])\right)$ which implies that $\mathbb{F} \wedge [p] \in \mathfrak{c}_\varphi$ and hence $p \in c_\varphi^{\mathfrak{c}^{\bar{c}}}(\mathbb{F})$. This end the proof.

Corollary 5. *If the group under consider is* Abelian*, then **PChyGrp** is isomorphic to **PLimGrp**.*

Proposition 7. *(a) If for each $\varphi \in \Delta^+$, $f \colon (S, \cdot, \mathfrak{c}_\varphi) \longrightarrow \left(S', \cdot, \mathfrak{c}'_\varphi\right)$ is a Cauchy-continuous mapping between probabilistic pre-Cauchy groups, then $f \colon \left(S, \cdot, c_\varphi^{\mathfrak{c}}\right) \longrightarrow \left(S', \cdot, c_\varphi^{\mathfrak{c}'}\right)$ is a continuous mapping between associated probabilistic convergence spaces;*

(b) Let $\left(S, \cdot, \bar{c} = (c_\varphi)_{\varphi \in \Delta^+}\right), \left(S', \cdot, \bar{c}' = (c'_\varphi)_{\varphi \in \Delta^+}\right) \in |\boldsymbol{SNPLimGrp}|$. If for each $\varphi \in \Delta^+$, $f \colon (S, \cdot, c_\varphi) \longrightarrow \left(S', \cdot, c'_\varphi\right)$ is a continuous group-homomorphism, then $f \colon \left(\mathfrak{c}_\varphi^{c}\right) \longrightarrow \left(S', \cdot, \mathfrak{c}_\varphi^{c'}\right)$ is Cauchy-continuous between probabilistic pre-Cauchy groups.

Proof. (a) Let $p \in S$, $\mathbb{F} \in \mathbb{F}(S)$, and $\varphi \in \Delta^+$. Now if $p \in c_\varphi^{\bar{\mathfrak{c}}}(\mathbb{F})$, then we have $[p] \wedge \mathbb{F} \in \mathfrak{c}$ and then by Cauchy-continuity, $f\left([p] \wedge \mathbb{F}\right) \in \mathfrak{c}'_\varphi$, and upon using Lemma 2 and (PCHS2), we have $f\left([p] \wedge \mathbb{F}\right) \leq f([p]) \wedge f(\mathbb{F}) = [f(p)] \wedge f(\mathbb{F}) \in \mathfrak{c}'_\varphi$, i.e., $[f(p)] \wedge f(\mathbb{F}) \in \mathfrak{c}'_\varphi$ which in turn yields that $f(p) \in c_\varphi^{\mathfrak{c}'}(f(\mathbb{F})$.
(b) Let $\mathbb{F} \in \mathfrak{c}^{\bar{c}}$. Then $e \in c_\varphi\left(\mathbb{F}^{-1} \odot \mathbb{F}\right)$ implies $e = f(e) \in c'_\varphi\left(f(\mathbb{F}^{-1} \odot \mathbb{F})\right) = c'_\varphi\left(f(\mathbb{F})^{-1} \odot f(\mathbb{F})\right)$ by using Lemma 2(k),(l). Hence $e \in c'_\varphi\left(f(\mathbb{F})^{-1} \odot f(\mathbb{F})\right)$ proving that $f(\mathbb{F}) \in \mathfrak{c}^{\overline{c'}}$.

8. Category of Probabilistic normed groups

Definition 12. *[3, 25] Let $(S,\cdot)$ be a group. A probabilistic metric $F\colon S\times S\longrightarrow \Delta^+$ under a triangle function τ is called* left-invariant *if $F(zx,zy)=F(x,y)$,* right-invariant *if $F(xz,yz)=F(x,y)$, and* invariant *if $F(zx,zy)=F(xz,yz)=F(x,y)$, for all $x,y,z\in S$.*

Definition 13. *[3] A triple $(S,\cdot,F)$ is called a* probabilistic metric group *if $(S,\cdot)$ is a group and (S,F) is a probabilistic metric space under the triangle function τ such the mappings $m : (S\times S, F\otimes F)\longrightarrow (S,F), (p,q)\mapsto pq$ and $\jmath : (S,F)\longrightarrow (S,F), p\mapsto p^{-1}$ are non-expansive, where the* product probabilistic metric *$F\otimes F$ on $S\times S$ is defined by [38]*

$$F\otimes F\left((p_1,p_2),(q_1,q_2)\right)=\tau\left(F_{p_1,q_1},F_{p_2,q_2}\right), \forall (p_1,p_2),(q_1,q_2)\in S\times S.$$

It is pointed out in [3] that a probabilistic metric groups are precisely groups equipped with invariant probabilistic metric. The category of probabilistic metric groups and non-expansive group-homomorphisms is denoted by **PMetGrp**.

Definition 14. *[3] A mapping $\nu\colon S\longrightarrow \Delta^+$ is called* probabilistic norm group on *S under a triangle function τ if for all $p,q,r\in S$:*
(PGN1) $\nu(p)=\epsilon_0$ if and only if $p=e$;
(PNG2) $\nu(p^{-1})=\nu(p)$;
(PNG3) $\tau(\nu(p),\nu(q))\leq \nu(pq)$.
We call the quadruple $(S,\cdot,\nu,\tau)$ of a group $(S,\cdot)$ and a probabilistic group norm ν on S a probabilistic normed group*.*

Let the category of probabilistic normed groups with abelian group norms as objects and norm-preserving mappings as morphisms be denoted by **PNormedGrp**.

Theorem 3. *[3, 12, 29] Let $(S,\cdot,\nu,\tau)$ be a probabilistic normed group. Then $F_{p,q}=\nu(pq^{-1})$ is a right-invariant probabilistic metric on S. Also, $\tilde{F}_{p,q}=F_{p^{-1},q^{-1}}=\nu(p^{-1}q)$ is a left-invariant probabilistic metric on S. Conversely, if F is a right invariant probabilistic metric on a group S and τ is a continuous triangle function, then for $\nu(x)=F_{e,p}=\tilde{F}_{e,p}$, the quadruple $(S,\cdot,\nu,\tau)$ is a probabilistic normed group. Consequently, probabilistic metric F is invariant of and only if $\nu(pq^{-1})=\nu(p^{-1}q)=\nu(q^{-1}p)$, it means that the probabilistic group norm ν is abelian.*
Furthermore, for a probabilistic norm group $(S,\cdot,\nu)$, the inversion $\jmath : (S,\cdot,\nu)\longrightarrow (S,\cdot,\tilde{F})$ is an isometry and hence a homeomorphism.

Let us denote the class of all left-invariant (resp. right-invariant, invariant) probabilistic metrics by **LtInvPMet** (resp. **RtInvPMet**, **InvPMet**) and class of all probabilistic group norms by **PGrpN**. Then in view of the content of the Theorem 5.3[6] and the Theorem 7, we have the following theorem.

Theorem 4. *Let $(S,\cdot)$ be a group. Then the following are true:*
*(a) There is a $1-1$ correspondence $\mathsf{T}\colon$ **LtInvPMet** $\longrightarrow$ **PGrpN**, $F\mapsto \mathsf{T}(F)$ defined by $\mathsf{T}(F)(p)=F(p,e)$, $x\in S$.*

The inverse transformation $\mathsf{T}^{-1}\colon \mathbf{PGrpN} \longrightarrow \mathbf{LtInvPMet}$, $\nu \mapsto \mathsf{T}^{-1}(\nu)$ *is defined by* $\mathsf{T}^{-1}(\nu)(p,q) = \nu(q^{-1}p)$, $p,q \in S$.
(b) There is a $1-1$ *correspondence* $\mathsf{S}\colon \mathbf{RtInvPMet} \longrightarrow \mathbf{PGrpN}$, $\tilde{F} \mapsto \mathsf{S}(\tilde{F})$ *defined by* $\mathsf{S}(\tilde{F})(p) = \tilde{F}(p,e)$, $p \in S$.
The inverse transformation $\mathsf{S}^{-1}\colon \mathsf{S}(\tilde{F}) \longrightarrow \mathbf{RtInvPMet}$, $\nu \mapsto \mathsf{S}^{-1}(\nu)$ *defined by* $\mathsf{S}^{-1}(\nu)(p,q) = \nu(pq^{-1})$, $p,q \in S$.
(c) The canonical $1-1$ *correspondence* $\mathsf{S}^{-1}\circ\mathsf{T}\colon \mathbf{InvPMet} \longrightarrow \mathbf{RtInvPMet}$, $F \mapsto \mathsf{S}^{-1}\circ\mathsf{T}(F)$ *given by*
$\mathsf{S}^{-1}\circ\mathsf{T}(F)(p,q) = F(p^{-1},q^{-1})$, $p,q \in S$.
The inverse transformation $\mathsf{T}^{-1}\circ\mathsf{S}\colon \mathbf{RtInvPMet} \longrightarrow \mathbf{LtInvPMet}$, $\tilde{F} \mapsto \mathsf{T}^{-1}\circ\mathsf{S}(\tilde{F})$ *given by*
$\mathsf{T}^{-1}\circ\mathsf{S}(\tilde{F})(p,q) = \tilde{F}(p^{-1},q^{-1})$, $p,q \in S$.

Proof. This follows from the Lemmas 6.2, 6.3 and 6.4 in [3]. See also [6].

Theorem 5. *Let F be a left-invariant, or right-invariant probabilistic metric on a group $(S,\cdot)$, and let ν^F be its probabilistic group norm. Then the following statements are equivalent:*
(i) F is invariant, that is both left and right invariant.
(ii) $F(p,q) = F(p^{-1},q^{-1})$, $p,q \in S$.
(iii) $\nu^F(pq) = \nu^F(qp)$, $p,q \in S$.
Conversely, a probabilistic metric on an abelian group is left-invariant if and only if it is right-invariant if and only if it is invariant.

Proof. Applying the above Theorem 3, one reach the conclusion.

Lemma 22. *Let* $(S,\cdot,\nu), (S',\cdot,\nu') \in |\boldsymbol{PNormedGrp}|$. *If* $f\colon (S,\nu) \longrightarrow (S',\nu')$ *is a normed-preserving group-homomorphism, then* $f\colon (S,F^{\nu}) \longrightarrow (S',F'^{\nu'})$ *is non-expansive. Conversely, if* $(S,\cdot,F), (S',\cdot,F') \in |\boldsymbol{PMetGrp}|$ *and* $f\colon (S,F) \longrightarrow (S',F')$ *is non-expansive, then* $f\colon (S,\nu'^{F}) \longrightarrow (S',\nu^{F'})$ *is normed-preserving group homomorphism.*

Proof. This is immediate from the observation that for any $p,q \in S$, $F^{\nu}(p,q) = \nu(p^{-1}q) \leq \nu'(f(p^{-1}q)) = \nu'((f(p))^{-1}f(q)) = F'^{\nu'}(f(p),f(q))$. For the converse part, we only need to see the following. For any $p,q \in S$, we have $\nu^F(p) \leq \nu^{F'}(f(p))$. So, we have, $\nu^F(p) = F(p,e) \leq F'(f(p),f(e)) = F'(f(p),e) = \nu'^{F'}(f(p))$.

Theorem 6. *If* $(S,\cdot,F) \in |\boldsymbol{PMetGrp}|$, *then* $\left(S,\cdot,\nu^F\right) \in |\boldsymbol{PNormedGrp}|$. *Conversely, if* $(S,\cdot,\nu) \in |\boldsymbol{PNormedGrp}|$, *then* $(S,\cdot,F^{\nu}) \in |\boldsymbol{PMetGrp}|$. *Furthermore, there are functors*

$$\boldsymbol{PMetGrp} \underset{\mathfrak{S}}{\overset{\mathfrak{T}}{\rightleftarrows}} \boldsymbol{PNormedGrp}$$

*such that $\mathfrak{T}\mathfrak{S} = id_{\boldsymbol{PNormedGrp}}$ and $\mathfrak{S}\mathfrak{T} = id_{\boldsymbol{PMetGrp}}$. That is, the categories **PMetGrp** and **PNormedGrp**.*

Proof. In view of the Lemmas 6.2, 6.3 and 6.4[3], we have the following functors:

$$\mathfrak{T} : \begin{cases} \mathbf{PMetGrp} & \longrightarrow & \mathbf{PNormedGrp} \\ (S, \cdot, F) & \longmapsto & \left(S, \cdot, \nu^F\right) \\ f & \longmapsto & f \end{cases},$$

and

$$\mathfrak{S} : \begin{cases} \mathbf{PNormedGrp} & \longrightarrow & \mathbf{PMetGrp} \\ (S, \cdot, \nu) & \longmapsto & (S, \cdot, F^\nu) \\ f & \longmapsto & f \end{cases}.$$

Then we have for any $(S, \cdot, \nu) \in |\mathbf{PNormedGrp}|$,

$$\mathfrak{T} \circ \mathfrak{S}\,(S, \cdot, \nu) = \mathfrak{T}\,(S, \cdot, F^\nu) = \left(S, \cdot, \nu^{F^\nu} = \nu\right) = id_{\mathbf{PNormedGrp}}\,(S, \cdot, \nu)\,.$$

In fact, for any $p \in S$, one obtains: $\nu^{F^\nu}(p) = F^\nu(p, e) = \nu(p^{-1}e) = \nu(p)$.
Next, for any $(S, \cdot, F) \in |\mathbf{PMetGrp}|$, we have:

$$\mathfrak{S} \circ \mathfrak{T}\,((S, \cdot, F)) = \mathfrak{S}\left(\left(S, \cdot, \nu^F\right)\right) = \left(S, \cdot, F^{\nu^F} = F\right) = (S, \cdot, F) = id_{\mathbf{PMetGrp}}\,(S, \cdot, F)\,.$$

In fact, for any $p, q \in R$, one obtains: $F^{\nu^F}(p, q) = \nu^F(p^{-1}q) = F(p^{-1}q, e) = F(p, q)$. Thus, we arrive at: $\mathfrak{T} \circ \mathfrak{S} = id_{\mathbf{PMetGrp}}$ and $\mathfrak{S} \circ \mathfrak{T} = id_{\mathbf{PMetGrp}}$. This ends the proof.

Theorem 7. *(Invariance of Probabilistic Norm Theorem)*
(a) The group norm ν is abelian (and probabilistic metric is invariant) if and only if

$$\tau(\nu(xa^{-1}), \nu(yb^{-1}) \leq \nu(xy(ab)^{-1}), \text{ for all } x, y, a, b;$$

(b) A probabilistic metric F on a group $(S, \cdot)$ is invariant if and only if

$$\tau\left(F(a, x), F(b, y)\right) \leq F(ab, xy);$$

in particular, this holds if S is abelian.
(c) The group norm is abelian if and only if the probabilistic norm is preserved under conjugacy (under automorphism).

Proof. (a) Let us assume that the group norm is abelian. Then we have

$$\nu(xy(ab)^{-1}) = \nu(xyb^{-1}.a^{-1}) = \nu(a^{-1}xyb^{-1}) \geq \tau\left(\nu(a^{-1}x), \nu(yb^{-1})\right) = \tau\left(\nu(xa^{-1}), \nu(b^{-1}y)\right).$$

For the converse part, we show the invariance by using $\nu(ba^{-1}) = \nu(a^{-1}b)$. In fact, it suffices to show that $\nu(yx^{-1}) \geq \nu(x^{-1}y)$. Take $x = a$ and $y = b$, then $\nu(ba^{-1}) \geq \nu(a^{-1}b)$, and taking $x = b^{-1}$, and $y = a^{-1}$, we get $\nu(a^{-1}b) \geq \nu(ba^{-1})$. Observe upon using the property of norm group that

$$\nu(yx^{-1}) = \nu(yy^{-1}xy^{-1}) \geq \tau\left(\nu(yy^{-1}), \nu(x^{-1}y)\right) = \tau\left(\nu(e), \nu(x^{-1}y)\right) = \tau\left(\epsilon_0, \nu(x^{-1}y)\right) = \nu(x^{-1}y).$$

(b) If F is invariant, then ν is abelian. Conversely, for a probabilistic metric F, let $\nu(x) = F(e, x)$. Then ν is a group norm since $\tau\left(F(e, x), F(e, y)\right) \leq F(ee, xy) = F(e, xy)$. Hence F is right-invariant, and $F(p, q) = \nu(pq^{-1})$. Now we show that the group norm is abelian since

$$\nu(xy(ab)^{-1}) = F(xy, ab) \geq \tau\left(F(x, a), F(y, b)\right) = \tau\left(\nu(xa^{-1}), \nu(yb^{-1})\right)$$

Hence F is also left-invariant.
(c) Suppose the probabilistic norm is abelian.. Then for any g, by the cyclic property $\nu(g^{-1}bg) = \nu(gg^{-1}b) = \nu(b)$. Conversely, if the probabilistic norm is preserved under automorphism, then we have invariance, since $\nu(ba^{-1}) = \nu(a^{-1}(ba^{-1}a)) = \nu(a^{-1}b)$.

9. conclusion

In this article, we studied various subcategories of the category of probabilistic convergence groups in a continuation of the previous works undertaken in these areas, cf. [3, 5]. In doing so, we introduced a category of probabilistic neighborhood spaces, **PNeigh**, and probabilistic neighborhood groups, **PNeighGrp**, showing that every probabilistic metric group is a probabilistic neighborhood group. Moreover, we talked about the category of probabilistic pre-Cauchy groups, **PPreChyGrp**, and its relation with an already known category of probabilistic Cauchy groups, **PChyGrp**, where it is observed that **PChyGrp** is a full subcategory of **PPreChyGrp**. Moreover, we discussed the fact that there exists a one-to-one correspondence between probabilistic metrics on groups and probabilistic group norms, here we showed that the category of probabilistic normed groups, **NormedGrp** is isomorphic to the category of probabilistic metric groups, **PMetGrp**. On the other hand, it is proved in [4] that the category **PMetGrp** is isomorphic to the category **PMetConvGrp** under sup-continuous triangle function τ, i.e., **PMetGrp**≅**PMetConvGrp** under sup-continuous triangle function τ. Question remains how to give a direct proof that the category **NormedGrp** is isomorphic to **PMetConvGrp**, i.e., **NormedGrp**≅ **PMetConvGrp**? While completing this paper, we further observe some open problems relating to this work, such as, (a) probabilistic norms on the class of homeomorphisms of groups in relation to Tardiff neighborhood system of probabilistic metrics, and their relationship with the work undertaken in [31]; (b) probabilistic neighborhood topological groups in relation to probabilistic normed groups; (c) although for a specific purpose, we brought into light the notion of probabilistic closure operator - a dual to probabilistic interior operator that studied in [22], we intend to discuss this probabilistic closure operator in relation to probabilistic approximation spaces. All of these open problems could be discussed in separate papers in near future. Finally, we would like to add here as already mentioned in Section 6 that in [10], it is shown that **KT2ConvFCO**, the category of KT2 constant convergence spaces and continuous functions, and **Chy**, the category of Cauchy spaces and Cauchy maps are isomorphic. Also, it is proved there that **ConvFCO**, the category of constant convergence spaces and continuous functions and **FilPreChy**, the category of filter preCauchy spaces and Cauchy maps are isomorphic. It would be interesting to see if one can study these new categories from the perspective of algebraic structures; for instance, search for group objects in these categories and their relationship.

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