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On Pairs of Disjoint Hop Dominating Sets in Graphs

Viralou Abrille B. Besana[1,2,*], Ferdinand P. Jamil[1,2], Sergio R. Canoy, Jr.[1,2]

[1] *Department of Mathematics and Statistics, College of Science and Mathematics, Mindanao State University - Iligan Institute of Technology, 9200 Iligan City, Philippines*
[2] *Center for Mathematical and Theoretical Physical Sciences, Premier Research Institute of Science and Mathematics, Mindanao State University - Iligan Institute of Technology, 9200 Iligan City, Philippines*

Abstract. A set S of vertices of a graph G is a hop dominating set of G if for every $v \in V(G) \setminus S$, v is at distance 2 from a vertex in S. The minimum cardinality $\gamma_h(G)$ of a hop dominating set is the hop domination number of G. Any hop dominating set of cardinality $\gamma_h(G)$ is a γ_h-set. A pair (S,T) of sets of vertices of G is a disjoint hop dominating pair if $S \cap T = \varnothing$ and both S and T are hop dominating sets of G. In particular, if S is a γ_h-set, then T is an inverse hop dominating set of G. The minimum sum $|S| + |T|$ among all pairs (S,T) of disjoint hop dominating sets of G is the disjoint hop domination number, denoted by $\gamma_{hh}(G)$. The minimum cardinality of an inverse hop dominating set of G is the inverse hop domination number of G, denoted by $\widetilde{\gamma}_h(G)$.
In this paper, we initiate the study of inverse hop domination and disjoint hop domination. Interestingly, for every pair of positive integers m and n with $2 \leq m \leq n$, there exists a connected graph G for which $\gamma_h(G) = m$ and $\widetilde{\gamma}_h(G) = n$. Also, for each positive integer $n \geq 4$, there exists a connected graph G for which $\gamma_h(G) + \widetilde{\gamma}_h(G) - \gamma_{hh}(G) = n$. Here we investigate these new concepts for some specific graphs including the join, corona and lexicographic product of graphs.

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Key Words and Phrases: Hop domination, inverse hop domination, disjoint hop domination

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1. Introduction

All throughout this paper, we consider only graphs which are simple, finite and undirected. Given a graph $G = (V(G), E(G))$, we call $V(G)$ the *vertex set* of G and $E(G)$ its *edge set*. The cardinality $|V(G)|$ of $V(G)$ is the *order* of G. All terminologies used here which are not defined are adapted from [1].

Let G and H be disjoint graphs. The *join* $G + H$ of G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* $G \circ H$ of G and H is the graph obtained by taking one copy of G and $|V(G)|$ copies of H,

*Corresponding author.
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Email addresses: viralouabrille.besana@g.msuiit.edu.ph (V.A Besana),
ferdinand.jamil@g.msuiit.edu.ph (F. Jamil),sergio.canoy@g.msuiit.edu.ph (S. Canoy Jr.)

and then joining each i^{th} vertex of G to every vertex in the i^{th} copy of H. In particular, we call $G \circ K_1$ the corona of G, and write $cor(G) = G \circ K_1$. The *lexicographic product* $G[H]$ of G and H is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u,v)(u',v') \in E(G[H])$ if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$. In any of these graphs, G and H are referred to as their basic component graphs.

Vertices u and v of a graph G are *neighbors* if $uv \in E(G)$. The *open neighborhood* of v refers to the set $N_G(v)$ consisting of all neighbors of v. The *degree* of v refers to the cardinality $|N_G(v)|$ of the open neighborhood of v. Vertex v is *isolated* if the degree of v is 0. The *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. Customarily, for $S \subseteq V(G)$, $N_G(S) = \cup_{v \in S} N_G(v)$ and $N_G[S] = \cup_{v \in S} N_G[v]$. A subset $S \subseteq V(G)$ is a *dominating set* of G if $N_G[S] = V(G)$. In case $N_G(S) = V(G)$, then S is a *total dominating set* of G. The minimum cardinality $\gamma(G)$ of a dominating set of G is the *domination number* of G, and the minimum cardinality $\gamma_t(G)$ of a total dominating set is the *total domination number of* G. A dominating set of cardinality $\gamma(G)$ is called a γ-*set* of G. Similarly, a γ_t-*set* is a total dominating set of cardinality $\gamma_t(G)$. The reader is referred to [2–6] for the history, fundamental concepts and some of the recent developments in domination in graphs as well as its various applications.

A set $S \subseteq V(G)$ is a $(1,2)^*$-*dominating set* (resp. $(1,2)^*$-*total dominating set*) of G if it is a dominating (resp. total dominating) set of G and for each $x \in V(G)\backslash S$ there exists $z \in S$ such that $d_G(x,z) = 2$. The smallest cardinality of a $(1,2)^*$-dominating (resp. $(1,2)^*$-total dominating) set of G, denoted by $\gamma^*_{1,2}(G)$ (resp. $\gamma^{*t}_{1,2}(G)$), is the $(1,2)^*$-*domination number* (resp. $(1,2)^*$-*total domination number*) of G. Any $(1,2)^*$-dominating (resp. $(1,2)^*$-total dominating) set of G of cardinality $\gamma^*_{1,2}(G)$ (resp. $\gamma^{*t}_{1,2}(G)$) is a $\gamma^*_{1,2}$-*set* (resp. $\gamma^{*t}_{1,2}$-*set*) of G. Both $(1,2)^*$-domination and $(1,2)^*$-total domination are introduced and studied in [7].

A set $S \subseteq V(G)$ is a *point-wise non-dominating set* of G if for each $v \in V(G) \setminus S$, there exists $u \in S$ such that $v \notin N_G(u)$. The smallest cardinality of a point-wise non-dominating set of G, denoted by $pnd(G)$, is called the *point-wise non-domination number* of G. A dominating set S which is also a point-wise non-dominating set of G is called a *dominating point-wise non-dominating set* of G. The smallest cardinality of a dominating point-wise non-dominating set of G will be denoted by $\gamma_{pnd}(G)$. Any point-wise non- dominating (resp. dominating point-wise non-dominating) set S of G of cardinality $|S| = pnd(G)$ (resp. $|S| = \gamma_{pnd}(G)$), is called a *pnd-set* (resp. γ_{pnd}-*set*) of G. Point-wise non-dominating sets and dominating point-wise non-dominating sets are discussed in [7].

Let G be a graph without isolated vertices. A subset $S \subseteq V(G)$ is an *inverse dominating set* of G if $V(G) \setminus S$ contains a γ-set of G. A minimum cardinality of an inverse dominating set of G is the *inverse domination number* of G, and is denoted by $\tilde{\gamma}(G)$. Motivated by C. Berge[2], inverse domination of a graph was introduced by V.R. Kulli and S.C. Sigarkanti [8] in 1991, and studied further in [9–12]. It may be noted that P.G. Bhat and S.R. Bhat in [9] made mention of its application in an Information Retrieval System.

For a graph G with no isolated vertex, any pair of subsets S and D of $V(G)$ is called *dd*-pair if S and D are disjoint dominating sets of G. The symbol $\gamma\gamma(G)$ is the smallest sum $|S| + |D|$ for all *dd*-pairs (S,D) of G. Since an inverse dominating set together with its associated γ-set constitute a *dd*-pair, $\gamma\gamma(G) \leq \gamma(G) + \tilde{\gamma}(G)$. Disjoint dominating sets are studied extensively in [11–13].

A subset $S \subseteq V(G)$ of a connected graph G is a *hop dominating set* (resp. *total hop dominating set*) of G if for each $v \in V(G) \setminus S$ (resp. $v \in V(G)$), there exists $u \in S$ for which $d_G(u, v) = 2$. The minimum cardinality of a hop dominating set (resp. total hop dominating set) is called the *hop domination number* (resp. *total hop domination number*) of G, and is denoted by $\gamma_h(G)$ (resp. $\gamma_{th}(G)$). Any hop dominating set (resp. total hop dominating set) of cardinality $\gamma_h(G)$ (resp. $\gamma_{th}(G)$) is called γ_h*-set* (resp. γ_{th}*-set*) of G. Using the symbol $HD(G)$ to denote the family of all hop dominating sets of G, more precisely, $\gamma_h(G) = \min\{|S| : S \in HD(G)\}$. Hop domination was introduced by C. Natarajan C and S.K. Ayyaswamy [14] in 2015, and is investigated further in [7, 15–19].

For a vertex v of a connected graph G, $N_G(v, 2) = \{u \in V(G) : d_G(u, v) = 2\}$, and for $S \subseteq V(G)$, $N_G(S, 2) = \cup_{v\in S} N_G(v, 2)$ and $N_G[S, 2] = N_G(S, 2) \cup S$. Precisely, S is a hop dominating set (resp. total hop dominating set) if and only if $N_G[S, 2] = V(G)$ (resp. $N_G(S, 2) = V(G)$).

The relevance of hop domination is very well illustrated by the relatively well-known application cited in [20] which, for our purpose, can be rephrased as follows : A factory wants to set up a quality assurance team where some employees evaluate their co-workers. To keep costs low and evaluators anonymous, the number of evaluators is kept as small as possible and evaluators should not be direct friends or enemies of the workers they assess to avoid bias. A social network can be modelled by a graph G with vertices representing the workers where two workers are adjacent in G whenever they are either friends or enemies of each other. In this graph, evaluators are not connected to the people they evaluate, but instead are connected to the friends or enemies of those people. In hop domination, every worker is evaluated by someone who is two steps away in the social network. This method ensures privacy, fairness and efficient evaluation.

The present study is motivated by the situation where the management considers the possibility that the quality assurance team might fail to deliver the desired output, and reserves another separate team (composed of evaluators who are not members of the first team) that can perform the same evaluation job. It deals mainly with the following two more likely approaches:

- The second team will proceed only after the management found that the first team's evaluation result is a failure; or
- the second team will perform its task simultaneously with the first team.

2. Some existing results in hop domination

The following existing results are useful in the present study.

Proposition 1. [14] (*i*) *For a complete graph* K_n*,* $\gamma_h(K_n) = n$*.*

(*ii*) *For a complete bipartite graph* $K_{m,n}$*,* $\gamma_h(K_{m,n}) = 2$*.*

(*iii*) *For a path P_n on n vertices,*

$$\gamma_h(P_n) = \begin{cases} 2r, & \text{if } n = 6r; \\ 2r+1, & \text{if } n = 6r+1; \\ 2r+2, & \text{if } n = 6r+s; 2 \leq s \leq 5. \end{cases}$$

(*iv*) *For a cycle C_n of length n,*

$$\gamma_h(C_n) = \begin{cases} 2r, & \text{if } n = 6r; \\ 2r+1, & \text{if } n = 6r+1; \\ 2r+2, & \text{if } n = 6r+s; 2 \leq s \leq 5. \end{cases}$$

(*v*) *For the Petersen graph P, $\gamma_h(P) = 2$.*

Proposition 2. [7] *Let G be a graph of order n. Then $1 \leq pnd(G) \leq n$. Moreover,*

(*i*) *$pnd(G) = n$ if and only if $G = K_n$.*

(*ii*) *$pnd(G) = 1$ if and only if G has an isolated vertex.*

(*iii*) *$pnd(G) = 2$ if and only if G has no isolated vertex and there exist distinct vertices a and b of G such that $N_G(a) \cap N_G(b) = \varnothing$.*

Theorem 1. [7] *Let G and H be any two graphs. A set $S \subseteq V(G+H)$ is a hop dominating set of $G+H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are point-wise non-dominating sets of G and H, respectively.*

Theorem 2. [7] *Let G and H be any two graphs. A set $C \subseteq V(G \circ H)$ is a hop dominating set of $G \circ H$ if and only if*

$$C = A \cup \left(\cup_{v \in V(G) \cap N_G(A)} S_v\right) \cup \left(\cup_{w \in V(G) \setminus N_G(A)} E_w\right),$$

where

(*i*) *$A \subseteq V(G)$ such that for each $w \in V(G) \setminus A$, there exists $x \in A$ with $d_G(x, w) = 2$ or there exists $y \in V(G) \cap N_G(w)$ with $V(H^y) \cap C \neq \varnothing$;*

(*ii*) *$S_v \subseteq V(H^v)$ for each $v \in V(G) \cap N_G(A)$; and*

(*iii*) *$E_w \subseteq V(H^w)$ is a point-wise non-dominating set of H^w for each $w \in V(G) \setminus N_G(A)$.*

Theorem 3. [7]*Let G be a nontrivial connected graph and let H be any graph. Then*

(*i*) *$\gamma_h(G \circ H) \leq \min\{\gamma^{*t}_{1,2}(G), [1 + pnd(H)]\gamma(G)\}$.*

(*ii*) *$\gamma_h(G \circ H) = 2$ if $\gamma^{*t}_{1,2}(G) = 2$.*

(*iii*) $\gamma_h(G \circ H) = 2$ *if* $\gamma(G) = 1$ *and* H *has an isolated vertex.*

Theorem 4. [7] *Let* G *and* H *be non-trivial connected graphs. A subset* $C = \cup_{x\in S}(\{x\} \times T_x)$ *of* $V(G[H])$ *is a hop dominating set of* $G[H]$ *if and only if the following conditions hold:*

(*i*) S *is a hop dominating set of* G*;*

(*ii*) T_x *is a point-wise non-dominating set of* H *for each* $x \in S$ *with* $|N_G(x, 2) \cap S| = 0$.

Corollary 1. [7] *Let* G *and* H *be non-trivial connected graphs of orders* m *and* n*, respectively. Then*

(*i*) $\gamma_h(G[H]) = \rho_H(G)$ *if* $\gamma(G) = 1$*, where* $\rho_H(G) = \min\{|S \cap N_G(S, 2)| + pnd(H)|S \setminus N_G(S, 2)| : S \text{ is a hop dominating set of } G\}$;

(*ii*) $\gamma_h(G[H]) = \gamma_{th}(G)$ *if* $\gamma(G) \neq 1$*; and*

(*iii*) $\gamma_h(G[H]) = m[pnd(H)]$ *if* $G = K_m$.

3. Results

By an *ntc* graph we mean a nontrivial connected graph. For vertices u and v of an *ntc* graph G, a *u-v geodesic* is any shortest path in G joining u and v. The length of a u-v geodesic is the *distance* between u and v, and is denoted by $d_G(u, v)$. The *eccentricity* of v refers to the quantity $e(v) = \max\{d_G(u, v) : v \in V(G)\}$. The *diameter* and *radius* of G are defined, respectively, as $diam(G) = \max\{e(v) : v \in V(G)\}$ and $r(G) = \min\{e(v) : v \in V(G)\}$.

Proposition 3. *Let* G *be an ntc graph with* $r(G) \geq 2$*. For each* γ_h*-set* $S \subseteq V(G)$*,* $V(G) \setminus S$ *is a hop dominating set of* G*.*

Proof: Let $S \subseteq V(G)$ be a γ_h-set of G. Suppose, in the contrary, that there exists $u \in S$ for which $d_G(u, v) \neq 2$ for all $v \in V(G) \setminus S$. Since $r(G) \geq 2$, there exists $v \in V(G)$ such that $d_G(u, v) = 2$. The previous statement implies that $v \in S$. Put $S^* = S \setminus \{u\}$. Then S^* is a hop dominating set of G, a contradiction since $|S^*| < |S| = \gamma_h(G)$. ■

In what follows, $\mathscr{G}$ is the family of all *ntc* graphs G such that $r(G) \geq 2$.

3.1. Inverse hop domination

Let $G \in \mathscr{G}$. A subset $S \subseteq V(G)$ is an *inverse hop dominating set* provided S is a hop dominating set and $V(G) \setminus S$ contains a γ_h-set of G. The minimum cardinality of an inverse hop dominating set is called the *inverse hop domination number* of G, and is denoted by $\widetilde{\gamma}_h(G)$.

Clearly, for $G \in \mathscr{G}$ of order n,

$$2 \leq \gamma_h(G) \leq \widetilde{\gamma}_h(G) \leq n - \gamma_h(G) \leq n - 2. \tag{1}$$

Theorem 5. *Let $G \in \mathscr{G}$ of order n. Then*

(i) $\widetilde{\gamma}_h(G) = 2$ *if and only if* $\gamma_h(G) = 2$ *and G has two disjoint γ_h-sets.*

(ii) $\widetilde{\gamma}_h(G) = n - 2$ *if and only if* $\gamma_h(G) = 2$ *and for every γ_h-set $\{u, v\}$ of G, $d_G(x, y) \neq 2$ for all $x, y \in V(G) \setminus \{u, v\}$.*

Proof: For (i), from Inequality 1, if $\widetilde{\gamma}_h(G) = 2$, then $\gamma_h(G) = 2$ and the conclusion follows. The converse is clear.

Suppose that $\widetilde{\gamma}_h(G) = n - 2$. Then Inequality 1 implies that $\gamma_h(G) = 2$. Let $\{u, v\}$ be a γ_h-set of G. Then $S = V(G) \setminus \{u, v\}$ is a $\widetilde{\gamma}_h$-set of G. Let $x, y \in S$ with $d_G(x, y) = 2$. First, if $u, v \notin N_G(x, 2)$, then $S \setminus \{x\}$ is a hop dominating set of G, a contradiction. Next, if $u, v \in N_G(x, 2)$, then $S \setminus \{y\}$ is a hop dominating set of G, a contradiction. Now assume that $u \in N_G(x, 2)$ and $v \notin N_G(x, 2)$, and let $[x, z, u]$ be a x-u geodesic in G. Suppose that $z \neq v$. Then $d_G(z, v) = 2$ and $S \setminus \{y\}$ is a hop dominating set of G, a contradiction. Suppose that $z = v$. If $[x, v, y]$ is a x-y geodesic in G, then $S \setminus \{y\}$ is a hop dominating set of G, a contradiction. Suppose not, and let $[x, w, y]$ be a geodesic in G. If $wv \in E(G)$, then $S \setminus \{w\}$ is a hop dominating set of G. If $wv \notin E(G)$, then $S \setminus \{y\}$ is a hop dominating set of G, a contradiction. The above contradictions imply that $d_G(x, y) \neq 2$ for all $x, y \in V(G) \setminus \{u, v\}$.

Conversely, suppose that $\gamma_h(G) = 2$, and let $\{u, v\}$ be a γ_h-set of G. If $S = V(G) \setminus \{u, v\}$ is not a $\widetilde{\gamma}_h$-set of G, then there exists $x \in S$ such that $S \setminus \{x\}$ is a hop dominating set of G. This means that, in particular, there exists $y \in S \setminus \{x\}$ such that $d_G(x, y) = 2$, contrary to the hypothesis. ■

Observe that for graph G_1 in Figure 1, $\{u, v\}$ in particular, is a γ_h-set and $x, z \in V(G_1) \setminus \{u, v\}$ with $d_G(x, z) = 2$. By Theorem 5(ii), $\widetilde{\gamma}_h(G_1) < 3$. Since $\{x, y\}$ is a γ_h-set, $\widetilde{\gamma}_h(G_1) = 2$ as also affirmed by statement (i).

For G_2 in Figure 1, $\{u, v\}$ and $\{v, w\}$ are the only γ_h-sets of G_2. Both γ_h-sets satisfy the conditions in Theorem 5(ii). Thus, $\widetilde{\gamma}_h(G_2) = n - 2 = 6 - 2 = 4$.

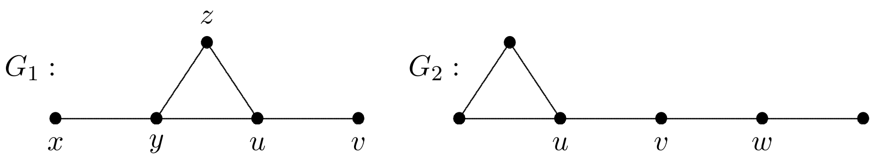


Figure 1: Examples of graphs described in Theorem 5

In view of Proposition 1, the following observations hold.

Observation 1. (i) *For a complete multipartite graph $G = K_{r_1, r_2, \ldots, r_n}$ with $2 \leq r_1 \leq r_2 \leq \cdots \leq r_n$, $\widetilde{\gamma}_h(G) = n$.*

(ii) *For a path P_n on $n \geq 4$ vertices,*

$$\widetilde{\gamma}_h(P_n) = \begin{cases} 2r + 3, & \text{if } n = 6r + 5; \\ 2r + 2, & \text{if } n = 6r + s; 0 \leq s \leq 4. \end{cases}$$

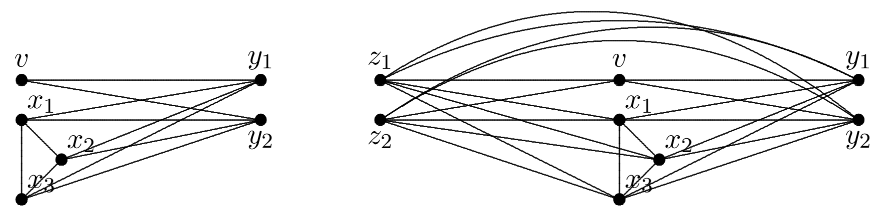
(*iii*) *For a cycle C_n of length $n \geq 4$,*

$$\widetilde{\gamma}_h(C_n) = \begin{cases} r, & \text{if } n = 3r; \\ 2r+1, & \text{if } n = 6r+1; \\ 2r+2, & \text{if } n = 6r+s, s = 2,4,5. \end{cases}$$

(*iv*) *For the Petersen graph P, $\widetilde{\gamma}_h(P) = 2$.*

Theorem 6. *For every pair of positive integers m and n with $2 \leq m \leq n$, there exists $G \in \mathscr{G}$ for which $\gamma_h(G) = m$ and $\widetilde{\gamma}_h(G) = n$.*

Proof: If $m = n$, then we take $G = K_{r_1,r_2,\ldots,r_n}$ with $2 \leq r_1 \leq r_2 \leq \cdots \leq r_n$. Assume that $m < n$, and write $n = m + k$, where $k \geq 1$. If $m = 2$, then take the graph $G = (K_1 \cup K_{k+1}) + \overline{K_2}$ (see graph G_1 in Figure 2 when $k = 2$). If $V(K_1) = \{v\}$ and $V(\overline{K_2}) = \{y_1, y_2\}$, then $\{v, y_1\}$ and $V(K_{k+1}) \cup \{y_2\}$ are, respectively, a γ_h-set and a $\widetilde{\gamma}_h$-set of G. Suppose that $m \geq 3$. Then we take the graph $G = (K_1 \cup K_{k+1}) + K_{r_1,r_2,\ldots,r_{m-1}}$, where $r_1 = r_2 = \cdots = r_{m-1} = 2$ (see graph G_2 in Figure 2 for $m = 3$ and $k = 2$). Put $V(K_1) = \{v\}$ and let $U_{r_j} = \{y^1_{r_j}, y^2_{r_j}\}$ $(j = 1, 2, \ldots, m-1)$ be the partite sets of $K_{r_1,r_2,\ldots,r_{m-1}}$. Then $\{v, y^1_{r_j} : j = 1, 2, \ldots, m-1\}$ and $V(K_{k+1}) \cup \{y^2_{r_j} : j = 1, 2, \ldots, m-1\}$ are, respectively, a γ_h-set and a $\widetilde{\gamma}_h$-set of G. In any case, $\gamma_h(G) = m$ and $\widetilde{\gamma}_h(G) = (m-1) + (k+1) = n$.■



$G_1 = (K_1 \cup K_3) + \overline{K_2}$ $\qquad$ $G_2 = (K_1 \cup K_3) + K_{2,2}$

Figure 2: Examples of graphs described in the proof of Theorem 6

Corollary 2. *The difference $\widetilde{\gamma}_h(G) - \gamma_h(G)$ can be made arbitrarily large.*

3.2. Disjoint hop domination

For $G \in \mathscr{G}$, Proposition 3 guarantees the existence in G of hop dominating sets A and B with $A \cap B = \varnothing$. Denote by $PHD(G)$ the family of all pairs (A, B) where A and B are disjoint hop dominating sets of G. We define

$$\gamma_{hh}(G) = \min\{|A| + |B| : (A, B) \in PHD(G)\}.$$

Any pair $(A, B) \in PHD(G)$ with $|A| + |B| = \gamma_{hh}(G)$ is called γ_{hh}*-pair* of G.

It should be noted that for $(A,B) \in PHD(G)$, any of A and B need not be a γ_h-set of G. For all $G \in \mathscr{G}$ of order n,

$$2\gamma_h(G) \le \gamma_{hh}(G) \le \gamma_h(G) + \widetilde{\gamma}_h(G) \le n. \qquad (2)$$

If G is any of the graphs G_1 and G_2 in Figure 2, then $\gamma_{hh}(G) = \gamma_h(G) + \widetilde{\gamma}_h(G) = |V(G)|$. Consider the graph G in Figure 3, the sets $\{a_1, b_1, c_1\}$ and $\{a_2, a_3, b_2, b_3, c_2, c_3\}$ are a γ_h-set and a $\widetilde{\gamma}_h$-set, respectively, of G. While the sets $\{a_1, a_2, b_3, c_3\}$ and $\{b_1, c_1, b_2, a_3\}$ constitute a γ_{hh}-pair of G. For this G, $2\gamma_h(G) < \gamma_{hh}(G) < \gamma_h(G) + \widetilde{\gamma}_h(G)$.

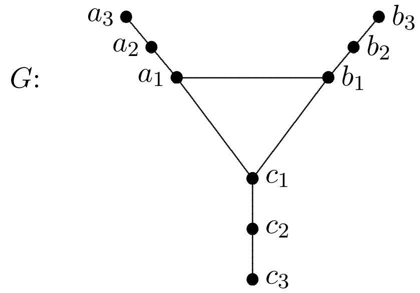


Figure 3: Graph G with $2\gamma_h(G) < \gamma_{hh}(G) < \gamma_h(G) + \widetilde{\gamma}_h(G)$

Proposition 4. *For all $G \in \mathscr{G}$, if*

$$\widetilde{\gamma}_h(G) \le 1 + \gamma_h(G),$$

then

$$\gamma_{hh}(G) = \gamma_h(G) + \widetilde{\gamma}_h(G), \qquad (3)$$

but not conversely. In particular,

- (i) *$\gamma_{hh}(G) = 2\gamma_h(G)$ for any of the following graphs G: the complete multipartite graph, cycle C_n and the Petersen graph described in Proposition 1.*
- (ii) *For a path P_n on $n \ge 4$ vertices,*

$$\gamma_{hh}(P_n) = \begin{cases} 4r+4, & \text{if } n = 6r+s, 2 \le s \le 4; \\ 4r+2, & \text{if } n = 6r; \\ 4r+3, & \text{if } n = 6r+1; \\ 4r+5, & \text{if } n = 6r+5. \end{cases}$$

Proof: Equation 3 is clear if $\widetilde{\gamma}_h(G) = \gamma_h(G)$. Assume $\widetilde{\gamma}_h(G) = 1 + \gamma_h(G)$, and let $(A,B) \in PHD(G)$. If $|A|+|B| < 1+2\gamma_h(G)$, then $|A| = |B| = \gamma_h(G)$. Consequently, $\widetilde{\gamma}_h(G) = \gamma_h(G)$, a contradiction. Since (A,B) is arbitrary, $\gamma_h(G)+\widetilde{\gamma}_h(G) = 1+2\gamma_h(G) \le \gamma_{hh}(G)$. Equation 2 yields the desired equality.

Revisit the graph $G = (K_1 \cup K_{k+1}) + K_{r_1,r_2,\ldots,r_{m-1}}$, where $r_1 = r_2 = \cdots = r_{m-1} = 2$, in Theorem 6. As shown, $\gamma_{hh}(G) = 2m + k = \gamma_h(G) + \widetilde{\gamma}_h(G)$. However, if $k \geq 2$, then $\widetilde{\gamma}_h(G) > 1 + \gamma_h(G)$.

The rest of the proof follows from Proposition 1 and Observation 1. ■

Proposition 5. *For each positive integer $n \geq 4$, there exists $G \in \mathscr{G}$ for which $\gamma_h(G) + \widetilde{\gamma}_h(G) - \gamma_{hh}(G) = n$.*

Proof: Let G be the graph given in Figure 4 which is obtained from the complete graph K_4 (with vertices $\{u, v, w, z\}$) by adding to K_4 three copies of the join $K_1 + C_4$ through the vertices u, v and w and then adding the join $\langle z \rangle + K_{n-2}$. Let $V(K_{n-2}) = \{x_1, x_2, \ldots, x_{n-2}\}$

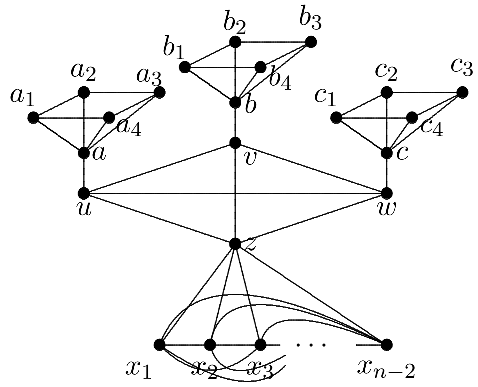


Figure 4: A graph G with $\gamma_{hh}(G) < \gamma_h(G) + \widetilde{\gamma}_h(G)$

and let the copies of $K_1 + C_4$ be given by the vertices $\{a, a_1, a_2, a_3, a_4\}$, $\{b, b_1, b_2, b_3, b_4\}$ and $\{c, c_1, c_2, c_3, c_4\}$ with $au, vb, wc \in E(G)$. Then $\{u, v, w, z\}$ is a γ_h-set of G and $\{a, a_1, a_2\} \cup \{b, b_1, b_2\} \cup \{c, c_1, c_2\} \cup \{x_1, x_2, \ldots, x_{n-2}\}$ is a $\widetilde{\gamma}_h$-set of G. On the other hand, the sets $\{w, c, b_3, b_4, c_3, c_4\}$ and $\{u, v, z, c_1, c_2\}$ constitute a γ_{hh}-pair of G. Thus, $\gamma_h(G) + \widetilde{\gamma}_h(G) - \gamma_{hh}(G) = 4 + (7 + n) - 11 = n$. ■

Corollary 3. *The quantity $\gamma_h(G) + \widetilde{\gamma}_h(G) - \gamma_{hh}(G)$ can be made arbitrarily large.*

3.3. In the join of graphs

A proof similar to that of Proposition 3 establishes the following lemma.

Lemma 1. *Let $G \in \mathscr{G}$. If $S \subseteq V(G)$ is a pnd-set of G, then $V(G) \setminus S$ contains a point-wise non-dominating set of G.*

Lemma 1 makes sense to the following definition. Let $G \in \mathscr{G}$. A subset $S \subseteq V(G)$ is an *inverse point-wise non-dominating set* of G if there exists a *pnd*-set D of G for which $S \cap D = \varnothing$. The minimum cardinality of an inverse point-wise non-dominating set of G is denoted by *ipnd*(G). Any inverse point-wise non-dominating set of G of cardinality *ipnd*(G) is called *ipnd-set* of G.

Theorem 7. *Let $G, H \in \mathscr{G}$ and $S \subseteq V(G+H)$. Then S is an inverse hop dominating set of $G+H$ if and only if $S = S_G \cup S_H$, where S_G and S_H are inverse point-wise non-dominating sets of G and H, respectively.*

Proof: Note first that $G + H \in \mathscr{G}$. Assume that S is an inverse hop dominating set of $G + H$, and let $D \subseteq V(G + H)$ be a γ_h-set of $G + H$ such that $S \cap D = \varnothing$. By Theorem 1, $S = S_G \cup S_H$ and $D = D_G \cup D_H$, where S_G and D_G are point-wise non-dominating sets of G and S_H and D_H are point-wise non-dominating sets of H. Moreover, D_G and D_H are *pnd*-sets of G and H, respectively. Thus, S_G and S_H are inverse point-wise non-dominating sets of G and H, respectively.

Conversely, suppose that $S = S_G \cup S_H$, where $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$ are inverse point-wise non-dominating sets of G and H, respectively. Then, there exist *pnd*-sets $D_G \subseteq V(G)$ and $D_H \subseteq V(H)$, such that $S_G \cap D_G = \varnothing$ and $S_H \cap D_H = \varnothing$. By Theorem 1, both S and $D = D_G \cup D_H$ are hop dominating sets of $G + H$. Using the same theorem, it is straightforward to show that D is a γ_h-set of $G + H$. Since $S \cap D = \varnothing$, S is an inverse hop dominating set of $G + H$. ■

Corollary 4. *For all $G, H \in \mathscr{G}$,*

$$\widetilde{\gamma}_h(G + H) = ipnd(G) + ipnd(H). \tag{4}$$

Given $G \in \mathscr{G}$, we use the symbol $PPND(G)$ to denote the family of all pairs (A, B), where $A, B \subseteq V(G)$ are disjoint point-wise non-dominating sets of G. By Lemma 1, $PPND(G) \neq \varnothing$. We define

$$ppnd(G) = \min\{|A| + |B| : (A, B) \in PPND(G)\}.$$

Any pair $(A, B) \in PPND(G)$ for which $|A| + |B| = ppnd(G)$ is called *ppnd-pair* of G.

Theorem 8. *Let $G, H \in \mathscr{G}$, and let $A, B \subseteq V(G + H)$. Then $(A, B) \in PHD(G + H)$ if and only if $A = A_G \cup A_H$ and $B = B_G \cup B_H$, where $(A_G, B_G) \in PPND(G)$ and $(A_H, B_H) \in PPND(H)$.*

Proof: Assume $(A, B) \in PHD(G+H)$. By Theorem 1 and since $A \cap B = \varnothing$, $A = A_G \cup A_H$ and $B = B_G \cup B_H$, where $(A_G, B_G) \in PPND(G)$ and $(A_H, B_H) \in PPND(H)$.

Conversely, if $A = A_G \cup A_H$ and $B = B_G \cup B_H$, where $(A_G, B_G) \in PPND(G)$ and $(A_H, B_H) \in PPND(H)$, then A and B are hop dominating sets of $G + H$ by Theorem 1. Moreover, since $A \cap B = \varnothing$, $(A, B) \in PHD(G + H)$. ■

Corollary 5. *For all $G, H \in \mathscr{G}$,*

$$\gamma_{hh}(G + H) = ppnd(G) + ppnd(H). \tag{5}$$

3.4. In the corona of graphs

Statements (ii) and (iii) of Theorem 3 assert that under some conditions, the value of $\gamma_h(G \circ H)$ is attainable by the value of $\gamma^{*t}_{1,2}(G)$ or $[1 + pnd(H)]\gamma(G)$. Moreover, It is shown in [7] that a strict inequality in statement (i) is also attainable.

For a $(1, 2)$-total dominating set A of a graph G, we write

$$\Gamma(A) = \{v \in A : N_G(v) \setminus A \neq \varnothing\}.$$

For each $v \in \Gamma(A)$, choose exactly one $u_v \in N_G(v) \setminus A$, and define

$$A^{\circ} = \{u_v : v \in \Gamma(A)\}.$$

Clearly, $A \cap A^{\circ} = \varnothing$.

Proposition 6. *Let G be an ntc graph of order n and let H be any graph. Then*

(i) $\gamma_{hh}(G \circ H) \leq [1 + pnd(H)]\, \gamma\gamma(G)$, *and this bound is sharp.*

(ii) *If* $\gamma_h(G \circ H) = \gamma^{*t}_{1,2}(G)$, *then*

$$\begin{aligned}\widetilde{\gamma}_h(G \circ H) \quad \leq \quad & \min\{|A^{\circ}| + |A^{\circ} \cap N_G(A^{\circ})| + \\ & [n - |N_G(A^{\circ})|]pnd(H) : A \text{ is a } \gamma^{*t}_{1,2}\text{- set of } G\},\end{aligned}$$

and equality is attained for star graphs G on $n \geq 3$ vertices.

(iii) *If* $\gamma_h(G \circ H) = [1 + pnd(H)]\gamma(G)$, *then*

$$\widetilde{\gamma}_h(G \circ H) \leq [1 + pnd(H)]\widetilde{\gamma}(G).$$

In particular, if $\gamma(G) = \widetilde{\gamma}(G)$, *then* $\gamma_h(G \circ H) = \widetilde{\gamma}_h(G \circ H)$.

Proof: Let (A, B) be a $\gamma\gamma$-pair of G. For each $v \in A$, let $S_v \subseteq V(H^v)$ be a *pnd*-set of H^v. Similarly, for each $v \in B$, let $T_v \subseteq V(H^v)$ be a *pnd*-set of H^v. Define $S = A \cup (\cup_{v \in A} S_v)$ and $T = B \cup (\cup_{v \in B} T_v)$. Let $x \in V(G \circ H) \setminus S$ and let $v \in V(G)$ for which $x \in V(H^v + v)$. If $x = v$, then since A is a dominating set and $v \notin A$, there exists $u \in A$ such that $uv \in E(G)$. Pick $y \in S_u$. Then $y \in S$ and $d_{G \circ H}(x, y) = 2$. On the other hand, if $x \neq v$, then since S_v is a *pnd*-set of H^v and $x \in V(H^v) \setminus S_v$, there exists $y \in S_v$ for which $xy \notin E(H^v)$. This means that $y \in S$ and $d_{G \circ H}(x, y) = 2$. Accordingly, S is a hop dominating set of $G \circ H$. Similarly, T is a hop dominating set of $G \circ H$. Since $S \cap T = \varnothing$, $(S, T) \in PHD(G \circ H)$. Therefore, $\gamma\gamma(G \circ H) \leq |S| + |T| = [1 + pnd(H)]\, \gamma\gamma(G)$. In particular, if H has an isolated vertex, then $\gamma_{hh}(P_4 \circ H) = 4 = 2\gamma\gamma(G)$. This proves (i).

To prove (ii), let $A \subseteq V(G \circ H)$ be a $\gamma^{*t}_{1,2}$-set of G. Then A is a γ_h-set of $G \circ H$. For each $v \in A^{\circ} \cap N_G(A^{\circ})$, let $S_v \subseteq V(H^v)$ be singleton. For each $v \in V(G) \setminus N_G(A^{\circ})$, let $T_v \subseteq V(H^v)$ be a *pnd*-set of H^v. Define

$$C = A^{\circ} \cup \left(\cup_{v \in A^{\circ} \cap N_G(A^{\circ})} S_v\right) \cup \left(\cup_{v \in V(G) \setminus N_G(A^{\circ})} T_v\right).$$

Clearly, $C \cap A = \varnothing$. We claim that C is a hop dominating set of $G \circ H$. Let $v \in V(G) \setminus A^\circ$. We consider the following cases:

Case 1: $v \in A$

Since A is a total dominating set of G, $N_G(v) \neq \varnothing$. Moreover, because $v \notin A^\circ$, $N_G(v) \subseteq A$. Pick $w \in A \cap N_G(v)$. First, suppose that $w \in \Gamma(A)$ and $y = u_w \in A^\circ$. Since $y \notin N_G(v)$, $d_G(v, y) = 2$. Next, suppose that $w \notin \Gamma(A)$. Then $w \notin N_G(A^\circ)$ and T_w is a pnd-set of H^w. Pick $y \in T_w$. Then $y \in V(H^w) \cap C$.

Case 2: $v \notin A$

Since A is a dominating set of G, there exists $w \in A \cap N_G(v)$. Since $v \in N_G(w) \setminus A$, $w \in \Gamma(A)$ and there exists $y = u_w \in A^\circ$. If $vy \notin E(G)$, then $d_G(v, y) = 2$. Suppose that $vy \in E(G)$. If $y \in N_G(A^\circ)$, then $V(H^y) \cap C = S_y \neq \varnothing$. If $y \notin N_G(A^\circ)$, then $V(H^y) \cap C = T_y \neq \varnothing$.

By Theorem 2, C is a (inverse) hop dominating set of $G \circ H$. Thus,

$$\widetilde{\gamma}_h(G \circ H) \leq |C| = |A^\circ| + |A^\circ \cap N_G(A^\circ)| + [n - |N_G(A^\circ)|]pnd(H).$$

In particular, if G is the star graph $K_{1,n-1}$ $(n \geq 3)$, then $\gamma_h(G \circ H) = 2$ and $\widetilde{\gamma}_h(G \circ H) = 1 + (n-1)pnd(H)$, for any graph H. Any $\gamma^{*t}_{1,2}$-set A contains the central vertex, $|A^\circ| = 1$ and $A^\circ \cap N_G(A^\circ) = \varnothing$. Thus, $|A^\circ| + |A^\circ \cap N_G(A^\circ)| + [n - |N_G(A^\circ)|]pnd(H) = 1 + (n-1)pnd(H)$.

Finally, to prove (iii), let $B \subseteq V(G)$ be $\widetilde{\gamma}$-set of G and let $A \subseteq V(G)$ be a γ-set of G for which $A \cap B = \varnothing$. For each $v \in A$, let $S_v \subseteq V(H^v)$ be a pnd-set of H^v. Similarly, for each $v \in B$, let $T_v \subseteq V(H^v)$ be a pnd-set of H^v. Define $S = A \cup (\cup_{v \in A} S_v)$ and $T = B \cup (\cup_{v \in B} T_v)$. As shown in the proof of statement (i), $(S, T) \in PHD(G \circ H)$. Moreover, since $|S| = [1 + pnd(H)]\gamma(G)$, T is an inverse hop dominating set of $G \circ H$. Therefore,

$$\widetilde{\gamma}_h(G \circ H) \leq |T| = [1 + pnd(H)]\widetilde{\gamma}(G).$$

■

The corona $G \circ H$, where $pnd(H) \geq 2$ and G is the graph in Figure 5, shows that strict inequality may be attained in Proposition 6(ii). Here $A = \{z, w\}$ is the unique γ_h-set of $G \circ H$ and $\widetilde{\gamma}_h(G \circ H) = 2 + pnd(H)$. Now, choose $A^\circ = \{y\}$. Then $|A^\circ| + |A^\circ \cap N_G(A^\circ)| + [4 - |N_G(A^\circ)|]pnd(H) = 1 + 2pnd(H) > \widetilde{\gamma}_h(G \circ H)$.

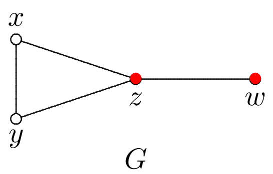


Figure 5: Graph G for illustration of Proposition 6(ii)

Let H be a graph with isolated vertex. For $n \geq 2$, $\tilde{\gamma}_h(K_{1,n} \circ H) = n + 1 < 2n = [1 + pnd(H)]\tilde{\gamma}(K_{1,n})$. This means that inequality in Proposition 6(iii) is also attainable.

3.5. In the lexicographic product of graphs

A subset $S \subseteq V(G)$ is a *ρ_H-set of* G if S is a hop dominating set of G with $\rho_H(G) = |S \cap N_G(S,2)| + pnd(H)|S \setminus N_G(S,2)|$.

Theorem 9. *Let G and H be ntc graphs with $\gamma(G) \neq 1$. Then*

$$\widetilde{\gamma}_h(G[H]) = \gamma_{th}(G).$$

Proof: Since $\gamma(G) \neq 1$, G admits a total hop dominating set. Let $S \subseteq V(G)$ be a γ_{th}-set of G, and let $u, v \in V(H)$ with $u \neq v$. By Theorem 4, $C_1 = S \times \{u\}$ and $C_2 = S \times \{v\}$ are hop dominating sets of $G[H]$. Since $|C_1| = \gamma_{th}(G)$, C_1 is a γ_h-set of $G[H]$ by Corollary 1. Consequently, C_2 is an inverse hop dominating set of $G[H]$. Thus, $\gamma_{th}(G) = \gamma_h(G[H]) \leq \widetilde{\gamma}(G[H]) \leq |C_2| = \gamma_{th}(G)$. ■

Theorem 10. *Let G and H be ntc graphs with $\gamma(G) = 1$ and $H \in \mathscr{G}$. Let $C = \cup_{x \in S} (\{x\} \times T_x) \subseteq V(G[H])$ with $T_x \neq V(H)$ for $x \in S$. Then C is an inverse hop dominating set of $G[H]$ if and only if each of the following holds:*

(i) S is a hop dominating set of G;

(ii) T_x is a point-wise non-dominating set of H for all $x \in S \setminus N_G(S,2)$;

(iii) There exists a ρ_H-set S^* of G such that for each $x \in (S \cap S^*) \setminus N_G(S^*,2)$, $V(H) \setminus T_x$ admits a pnd-set of H. More particularly, for each $x \in (S \cap S^*) \setminus (N_G(S,2) \cup N_G(S^*,2))$, T_x is an inverse point-wise non-dominating set of H.

Proof: First, assume that C is an inverse hop dominating set of $G[H]$. By Theorem 4, both (i) and (ii) hold for S. Since C is an inverse hop dominating set, there exists a γ_h-set $C^* = \cup_{x \in S^*} (\{x\} \times T^*_x)$ of $G[H]$ for which $C \subseteq V(G[H]) \setminus C^*$. By Corollary 1, S^* is a ρ_H-set of G and T^*_x is a *pnd*-set of H for each $x \in S^* \setminus N_G(S^*,2)$. Let $x \in (S \cap S^*) \setminus N_G(S^*,2)$. Because $C \cap C^* = \varnothing$, $T^*_x \subseteq V(H) \setminus T_x$. More particularly, if $x \in (S \cap S^*) \setminus (N_G(S,2) \cup N_G(S^*,2))$, then T_x is a point-wise non-dominating set of H. Further, since $T_x \subseteq V(H) \setminus T^*_x$, T_x is an inverse point-wise non-dominating set of H. This proves (iii).

Conversely, suppose that C satisfies all conditions (i), (ii) and (iii). Then, by Theorem 4, C is a hop dominating set of $G[H]$. We construct a γ_h-set $C^* = \cup_{x \in S^*} (\{x\} \times T^*_x)$ for which $C \subseteq V(G[H]) \setminus C^*$ as follows: Let $x \in S^*$.

Case 1: Suppose that $x \in S$. If $x \in N_G(S^*,2)$, then we take $T^*_x = \{y\}$, where $y \in V(H) \setminus T_x$. If $x \notin N_G(S^*,2)$, then as provided by condition (iii), we take a *pnd*-set T^*_x of H with which $T_x \subseteq V(H) \setminus T^*_x$.

Case 2: Suppose that $x \notin S$. If $x \in N_G(S^*,2)$, then choose $T^*_x = \{y\}$ for any $y \in V(H)$. If $x \notin N_G(S^*,2)$, then we choose any *pnd*-set T^*_x of H.

Define $C^* = \cup_{x\in S^*}(\{x\}\times T_x^*)$. Then C^* is a hop dominating set of $G[H]$ by Theorem 4. Moreover, $C\cap C^* = \varnothing$ and

$$\begin{aligned}|C^*| &= \sum_{x\in S^*\cap N_G(S^*,2)} |T_x^*| + \sum_{x\in S^*\setminus N_G(S^*,2)} |T_x^*| \\ &= |S^*\cap N_G(S^*,2)| + pnd(H)|S^*\setminus N_G(S^*,2)| \\ &= \rho_H(G).\end{aligned}$$

Therefore, C is an inverse hop dominating set of $G[H]$. ■

Corollary 6. *If G and H are ntc graphs with $\gamma(G) = 1$ and $H \in \mathscr{G}$, then*

$$\widetilde{\gamma}_h(G[H]) \leq \min\{|S\cap N_G(S,2)| + ipnd(H)|S\setminus N_G(S,2)| : S\in HD(G)\}.$$

Proof: Put $\widetilde{\rho}_H(G) = \min\{|S\cap N_G(S,2)| + ipnd(H)|S\setminus N_G(S,2)| : S\in HD(G)\}$. Let $S\subseteq V(G)$ be a ρ_H-set of G, $y\in V(H)$ and $A\subseteq V(H)$ an inverse point-wise non-dominating sets of H. Define $C = \cup_{x\in S}(\{x\}\times T_x)$, where $T_x = \{y\}$ for all $x\in S\cap N_G(S,2)$ and $T_x = A$ for all $x\in S\setminus N_G(S,2)$. By Theorem 10, C is an inverse hop dominating set of $G[H]$. Thus,

$$\widetilde{\gamma}_h(G[H]) \leq |C| = |S\cap N_G(S,2)| + ipnd(H)|S\setminus N_G(S,2)|.$$

Since S is arbitrary, $\widetilde{\gamma}_h(G[H]) \leq \widetilde{\rho}_H(G)$. ■

Corollary 7. *For all $H\in\mathscr{G}$ and $m\geq 2$,*

$$\widetilde{\gamma}_h(K_m[H]) = m\cdot ipnd(H).$$

Proof: Note first that $S = V(K_m)$ is the unique hop dominating set of K_m and $N_{K_m}(S,2) = \varnothing$. Thus, Corollary 6 yields $\gamma_h(K_m[H]) \leq m\cdot ipnd(H)$.

Now, let $C\subseteq V(K_m)$ be a $\widetilde{\gamma}_h$-set of $K_m[H]$. By Theorem 10, $C = \cup_{x\in V(K_m)}(\{x\}\times T_x)$, where $T_x\subseteq V(H)$ is an inverse point-wise non-dominating set of H for each $x\in V(K_m)$. Thus,

$$\widetilde{\gamma}_h(K_m[H]) = \sum_{x\in V(K_m)} |T_x| \geq m\cdot ipnd(H).$$

■

Equality in Corollary 6 can be attained even with a noncomplete G. Consider, for example, $G = P_3 = [x_1, x_2, x_3]$. Then G has only three distinct hop dominating sets, namely $S_1 = \{x_1, x_2\}$, $S_2 = \{x_2, x_3\}$ and $S_3 = V(G)$. In view of Proposition 2, for any graph $H\in\mathscr{G}$, S_3 is the unique ρ_H-set of G. Thus, $\widetilde{\gamma}_h(G[H]) = \widetilde{\rho}_H(G) = 2 + ipnd(H)$.

Proposition 7. *Let G and H be ntc graphs with $\gamma(G)\neq 1$. Then*

$$\gamma_{hh}(G[H]) = 2\gamma_{th}(G).$$

Proof: Applying Corollary 1 and Theorem 9, we have

$$2\gamma_{th}(G) = 2\gamma_h(G[H]) \leq \gamma_{hh}(G[H]) \leq \gamma_h(G[H]) + \widetilde{\gamma}_h(G[H]) = 2\gamma_{th}(G).$$

■

The following follows immediately from Theorem 4.

Theorem 11. *Let G and H be ntc graphs with $\gamma(G) = 1$ and $H \in \mathscr{G}$. Let $C = \cup_{x\in S}(\{x\} \times T_x), C^* = \cup_{x\in S^*}(\{x\} \times T_x^*) \subseteq V(G[H])$ with $T_x \neq V(H)$ for $x \in S$ and $T_x^* \neq V(H)$ for all $x \in S^*$. Then $(C, C^*) \in PHD(G[H])$ if and only if each of the following holds:*

(i) Both S and S^* satisfy the conditions (i) and (ii) of Theorem 4; and

(ii) $T_x \cap T_x^* = \varnothing$ for all $x \in S \cap S^*$. More particularly, $(T_x, T_x^*) \in PPND(H)$ for all $x \in (S \cap S^*) \setminus (N_G(S, 2) \cup N_G(S^*, 2))$.

Corollary 8. *For all graphs $H \in \mathscr{G}$ and $m \geq 2$,*

$$\gamma_{hh}(K_m[H]) = m \cdot ppnd(H).$$

Proof: Put $S = V(K_m)$ and let $C = \cup_{x\in S}(\{x\} \times T_x), C^* = \cup_{x\in S}(\{x\} \times T_x^*) \in V(K_m[H])$ such that (T_x, T_x^*) is a *ppnd*-pair of H for each $x \in S$. Then $(C, C^*) \in PHD(K_m[H])$ by Theorem 4. Thus,

$$\gamma_{hh}(K_m[H]) \leq |C| + |C^*| = \sum_{x\in V(K_m)} (|T_x| + |T_x^*|) = m \cdot ppnd(H).$$

Now let (C, C^*) be a γ_{hh}-pair of $K_m[H]$. By Theorem 11(i), $C = \cup_{x\in S}(\{x\} \times T_x)$ and $C^* = \cup_{x\in S^*}(\{x\} \times T_x^*)$ for some hop dominating sets S and S^* of K_m with T_x a point-wise non-dominating sets of H for each $x \in S \setminus N_{K_m}(S, 2)$ and T_x^* a point-wise non-dominating set of H fo all $x \in S^* \setminus N_{K_m}(S^*, 2)$. Since $V(K_m)$ is the unique hop dominating set of K_m, $S = S^* = V(K_m)$ and $N_{K_m}(S, 2) = N_{K_m}(S^*, 2) = \varnothing$. Further, by Theorem 11$(ii)$, $(T_x, T_x^*) \in PPND(H)$. Thus,

$$\gamma_{hh}(K_m[H]) = |C| + |C^*| = \sum_{x\in V(K_m)} (|T_x| + |T_x^*|) \geq m \cdot ppnd(H).$$

■

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References

[1] F. Buckley and F. Harary. *Distance in Graphs.* Addison-Wesley, Redwood City, CA, 1990.

[2] C. Berge. *The theory of Graphs and its Applications.* Wiley, New York, 1962.

[3] E. Cockayne and S. Hedetniemi. Towards a theory of domination in graphs. *Networks*, 7(3):247–261, 1977.

[4] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater. *Fundamentals of Domination in Graphs.* Marcel Dekker, Inc., New York, 1998.

[5] F.P. Jamil and H.N. Maglanque. Cost effective domination in the join, corona and composition of graphs. *European Journal of Pure and Applied Mathematics*, 12(3):978–998, 2019.

[6] O. Ore. *Theory of Graphs*, volume 38 of *American Mathematical Society Colloquium Publications.* American Mathematical Society, Providence, RI, 1962.

[7] S.R Canoy Jr., R.V. Mollejon, and J.G. Canoy. Hop dominating sets in graphs under binary operations. *European Journal of Pure and Applied Mathematics*, 12(4):1455–1463, 2019.

[8] V.R. Kulli and S.C. Sigarkanti. Inverse domination in graphs. *National Academy Science Letters*, 14:473–475, 1991.

[9] P.G. Bhat and S.R. Bhat. Inverse independence number of a graph. *International Journal of Computer Applications*, 42(5), 2012.

[10] G.S. Domke, J.E. Dunbar, and L.R. Markus. The inverse domination number of a graph. *Ars Combinatoria*, 72:149–160, 2004.

[11] E.M. Kiunisala and F.P. Jamil. Inverse domination numbers and disjoint domination numbers of graphs under some binary operations. *Applied Mathematical Sciences*, 8(107):5303–5315, 2014.

[12] E.M. Kiunisala and F.P. Jamil. On pairs of disjoint dominating sets in a graph. *International Journal of Mathematical Analysis*, 10(13):623–637, 2016.

[13] S.M. Hedetniemi, S.T. Hedetniemi, R.C. Laskar, L. Markus, and P.J. Slater. Disjoint dominating sets in graphs. In *Proceedings of the International Conference on Discrete Mathematics*, volume 7 of *Ramanujan Mathematical Society Lecture Notes Series*, pages 87–100. 2008.

[14] C. Natarajan and S.K. Ayyaswamy. Hop domination in graphs – ii. *Versita*, 23(2):187–199, 2015.

[15] M.A. Bonsocan and F.P. Jamil. Transversal hop domination in graphs. *European Journal of Pure and Applied Mathematics*, 16(1):192–206, 2023.

[16] M.A. Henning, S. Pal, and D. Pradhan. Algorithm and hardness results on hop domination in graphs. *Information Processing Letters*, 153:105872, 2020.

[17] M.A. Henning and N.J. Rad. On 2-step and hop dominating sets in graphs. *Graphs and Combinatorics*, 33:913–927, 2017.

[18] C. Natarajan, S.K. Ayyaswamy, and G. Sathiamoorthy. A note on hop domination number of some special families of graphs. *International Journal of Pure and Applied Mathematics*, 119(12):14165–14171, 2018.

[19] S. Shanmugavelan and C. Natarajan. On hop domination number of some generalized graph structures. *Ural Mathematical Journal*, 7(2):121–135, 2021.

[20] W. Desormeaux, T. W. Haynes, and M. A. Henning. A note on non-dominating set partitions in graphs. *Discussiones Mathematicae Graph Theory*, 36:1043–1050, 2016.