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An MPP Monounary Algebra Induced by an Endomorphism of the Direct Product of Two Chains

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Abstract. A finite modular lattice **A** is said to be *MPP* if there is an endomorphism, called an *MPP endomorphism*, whose the pre-period is equal to the length of **A**. A monounary algebra (A, f) is said to be *MPP* if f is an MPP endomorphism of a lattice **A**, called an MPP corresponding lattice to (A, f). In this work, we show all MPP monounary algebras induced by endomorphisms of the direct products of two chains.

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Key Words and Phrases: Pre-period, Monounary algebra, Chain, Endomorphism

1. Introduction

One of universal algebras which play important roles to simplify many problems in computer science is a monounary algebra. It is often considered as a special type of automata (see e.g. in [1, 2]). The advantage of monounary algebras is their easy visualization; especially, they can be represented as planar directed graphs. The important theories of unary and monounary algebras are shown in many monographs; for instance, [3]. Moreover, monounary algebras have closed relationships with all algebras via their endomorphisms.

A *monounary algebra* is a set A equipped with a unary operation $f : A \to A$ and it is denoted by $\underline{A} = (A, f)$. Denote f^0 is the identity map on A and $f^n = f \circ f^{n-1}$ for all $n \in \mathbb{N}$. A monounary algebra $\underline{A}$ is said to be *connected* if for each $a, b \in A$, there exist nonnegative integers n, m such that $f^n(a) = f^m(b)$. An element $a \in A$ is called a *cyclic* if $f^n(a) = a$ for some $n \in \mathbb{N}$. The *height* of an element $x \in A$, denoted by $\mathrm{ht}(x)$, is the least non-negative integer i such that $f^i(x)$ is a cyclic element. The *height* of the finite monounary algebra $\underline{A}$ is defined by

$$\mathrm{ht}(\underline{A}) := \max \{\mathrm{ht}(x) \mid x \in A\}.$$

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In other words, $\mathrm{ht}(\underline{A})$ is the least non-negative integer $\lambda(f)$ satisfying $f^{\lambda(f)}(A) = f^{\lambda(f)+1}(A)$ and it is known as the *pre-period* of f (see [4]).

For any algebra $\mathbf{A}$, we can study the monounary algebra (A, f) induced by an endomorphism f of $\mathbf{A}$. Besides, an endomorphism is studied in any category and it is relevant to solve many problems in algebraic structures, relational structures and graphs; reader may look in [5–9]. If $\mathbf{A}$ is finite, one can see that $|A| - 1$ is an upper bound of $\lambda(f)$; so, it is interesting to study the least upper bound as follows. The *pre-period* of algebra $\mathbf{A}$ is

$$\lambda(\mathbf{A}) = \sup \{\lambda(f) \mid f \text{ is an endomorphism of } \mathbf{A}\}.$$

In [10, 11], the authors focused on a finite lattice and showed that for a finite modular lattice $\mathbf{L}$, its pre-period is less than or equal to the length of $\mathbf{L}$ where the *length* $\ell(\mathbf{L})$ of $\mathbf{L}$ is defined by $|C| - 1$ for the longest chain C in $\mathbf{L}$. A finite modular lattice $\mathbf{A}$ is said to be *MPP* if there is an endomorphism f (called an *MPP endomorphism*) whose $\lambda(f) = \ell(\mathbf{A})$. A monounary algebra (A, f) is said to be *MPP* if there is an MPP endomorphism g of a lattice $\mathbf{B}$ such that (A, f) is isomorphic to (B, g). Such the lattice $\mathbf{B}$ is called an *MPP corresponding lattice* to (A, f).

In the present work, we will show that all monounary algebras induced by an MPP endomorphism of the direct products of two chains (studied in [10]) are isomorphic to the monounary algebras (A_n, f_n) and (B_n, g_n) shown in the figures 1, 2 and 3.

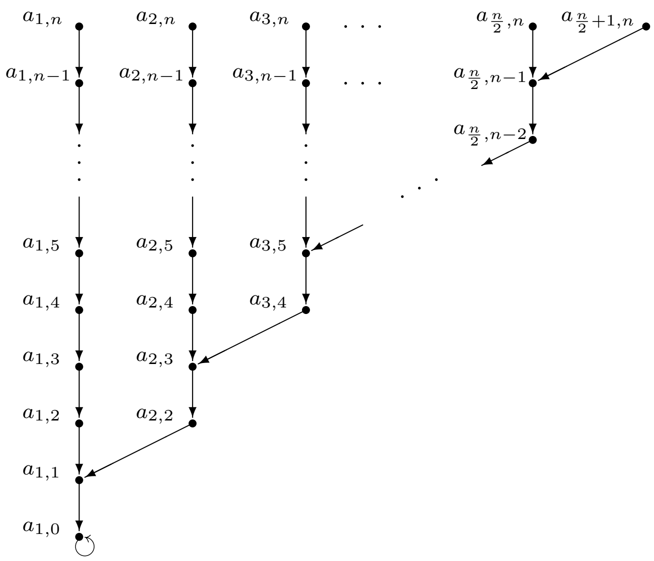

Figure 1: The graph of (A_n, f_n) where n is an even natural number.

2. Basic concepts

We denote the top and bottom of a lattice $\mathbf{A}$ by $1_{\mathbf{A}}$ and $0_{\mathbf{A}}$ (shortly, 1 and 0), respectively. A unary operation f on a lattice $\mathbf{A} = (A; \vee, \wedge)$ is said to be an *endomorphism* of

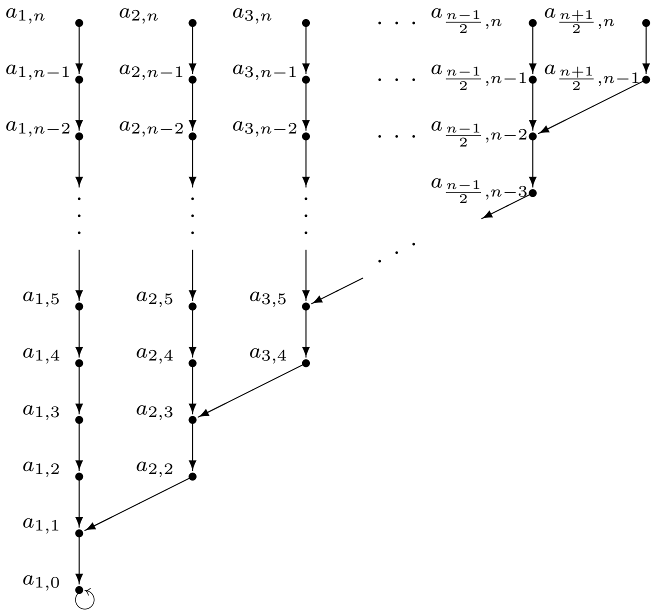

Figure 2: The graph of (A_n, f_n) where n is an odd natural number.

A if $f(a \vee b) = f(a) \vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)$ for all $a, b \in A$.

The results in [11, Corollary 6] imply the following theorem.

Theorem 1. *[11] Let* **A** *be a finite modular lattice and f be an endomorphism of* **A**. *Then f is MPP if and only if f satisfies either*

$$0 = f^{\lambda(f)}(1) \prec f^{\lambda(f)-1}(1) \prec \ldots \prec f(1) \prec 1 \tag{1}$$

or

$$0 \prec f(0) \prec \ldots \prec f^{\lambda(f)-1}(0) \prec f^{\lambda(f)}(0) = 1. \tag{2}$$

Corollary 1. *Let f be an MPP endomorphism of a finite modular lattice* **A**.

(i) If f satisfies the condition (1), then $f(0) = 0$,

$$\mathrm{ht}(x) = \min\{n \in \mathbb{N} \cup \{0\} \mid f^n(x) = 0\}$$

for all $x \in A$ and $\mathrm{ht}(1) = \ell(\mathbf{A})$.

(ii) If f satisfies the condition (2), then $f(1) = 1$,

$$\mathrm{ht}(x) = \min\{n \in \mathbb{N} \cup \{0\} \mid f^n(x) = 1\}$$

for all $x \in A$ and $\mathrm{ht}(0) = \ell(\mathbf{A})$.

We denote the m-element chain by $\mathbf{C}_m = \{\overline{1} \prec \overline{2} \prec \ldots \prec \overline{m}\}$ for $m \in \mathbb{N}$. For convenient, let $\overline{a} = \overline{1}$ and $\overline{b} = \overline{m}$ in $\mathbf{C}_m$ for all $a \leq 1$ and $b \geq m$.

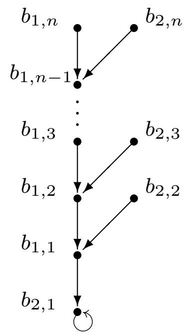

Figure 3: The graph of (B_n, g_n) for $n \geq 2$.

Theorem 2. *[10] For each $m \in \mathbb{N}$, the unary operation $\varphi_{m\times 2}$ on $C_m \times C_2$ defined by*

$$\varphi_{m\times 2}(\bar{i}, \bar{j}) = \begin{cases} (\overline{i-1}, \bar{2}) & \text{if } i > 1, \\ (\bar{1}, \bar{1}) & \text{if } i = 1. \end{cases}$$

is an MPP endomorphism of $\mathbf{C}_m \times \mathbf{C}_2$ fixing the bottom.

By Theorem 1, the operations (seen in [10]) in the following theorem are MPP endomorphisms.

Theorem 3. *For each $m \in \mathbb{N}$, the operations $\zeta_{(m,m-1)} : C_m^2 \to C_m^2$ and $\zeta_{(m-1,m)} : C_m^2 \to C_m^2$ defined by*

$$\zeta_{(m,m-1)}(\bar{i}, \bar{j}) = (\bar{j}, \overline{i-1})$$

and

$$\zeta_{(m-1,m)}(\bar{i}, \bar{j}) = (\overline{j-1}, \bar{i})$$

are MPP endomorphisms of $\mathbf{C}_m^2$ fixing the bottom.

Lemma 1. *[10] For each $m, n \geq 3$, if f is an MPP endomorphism of $\mathbf{C}_m \times \mathbf{C}_n$ fixing the bottom, then either*

(i) $f^{2k}(\overline{m}, \overline{n}) = (\overline{m-k}, \overline{n-k})$ *and* $f^{2k+1}(\overline{m}, \overline{n}) = (\overline{m-k}, \overline{n-(k+1)})$ *for all* $0 \leq k \leq \min\{m-1, n-2\}$, *or*

(ii) $f^{2k}(\overline{m}, \overline{n}) = (\overline{m-k}, \overline{n-k})$ *and* $f^{2k+1}(\overline{m}, \overline{n}) = (\overline{m-(k+1)}, \overline{n-k})$ *for all* $0 \leq k \leq \min\{m-2, n-1\}$.

Theorem 4. *[10] Let $m, n \in \mathbb{N}$. Then*

$$\mathbf{C}_m \times \mathbf{C}_n \text{ is MPP if and only if either } m \leq 2,\ n \leq 2 \text{ or } |m-n| \leq 1.$$

3. All MPP endomorphisms of product of two chains

It is well-known that if an algebra $\mathbf{A}$ is isomorphic to an algebra $\mathbf{B}$ under ϕ, the monoid $\mathrm{End}(\mathbf{A})$ of all endomorphisms of $\mathbf{A}$ is isomorphic to $\mathrm{End}(\mathbf{B})$ under the isomorphism Φ defined by $\Phi(f) = \phi \circ f \circ \phi^{-1}$ for all $f \in \mathrm{End}(\mathbf{A})$. Observe that the pre-period is invariant under Φ.

Proposition 1. *Let $\phi : \mathbf{A} \to \mathbf{B}$ be an isomorphism between finite algebras $\mathbf{A}$ and $\mathbf{B}$ and $f \in \mathrm{End}(\mathbf{A})$. Then*

(i) $\phi \circ f \circ \phi^{-1} \in \mathrm{End}(\mathbf{B})$,

(ii) $\lambda(f) = \lambda(\phi \circ f \circ \phi^{-1})$, *and*

(iii) ϕ *is an isomorphism from* (A, f) *to* $\left(B, \phi \circ f \circ \phi^{-1}\right)$.

Proof. Let $g = \phi \circ f \circ \phi^{-1}$. Since ϕ, f and ϕ^{-1} are homomorphisms, so is g. Hence, $\phi \circ f \circ \phi^{-1} \in \mathrm{End}(\mathbf{B})$. Moreover,

$$\begin{aligned} g^{\lambda(f)}(B) &= \phi \circ f^{\lambda(f)} \circ \phi^{-1}(B) \\ &= \phi \circ f^{\lambda(f)}(A) \\ &= \phi \circ f^{\lambda(f)+1}(A) \\ &= \phi \circ f^{\lambda(f)+1} \circ \phi^{-1}(B) \\ &= g^{\lambda(f)+1}(B). \end{aligned}$$

So, $\lambda(g) \leq \lambda(f)$. Similarly, $\lambda(g) \geq \lambda(f)$. Thus $\lambda(g) = \lambda(f)$. Since

$$\phi \circ f = \phi \circ f \circ \phi^{-1} \circ \phi = g \circ \circ \phi,$$

ϕ is an isomorphism from (A, f) to $\left(B, \phi \circ f \circ \phi^{-1}\right)$.

Remark 1. *Let $m, n \in \mathbb{N}$. Then*

(i) $\phi : \mathbf{C}_m \times \mathbf{C}_n \to (\mathbf{C}_m \times \mathbf{C}_n)^\partial$ *defined by*

$$\phi(\bar{i}, \bar{j}) = (\overline{m - i + 1}, \overline{n - j + 1})$$

is an isomorphism where $(\mathbf{C}_m \times \mathbf{C}_n)^\partial$ *is the dual of* $\mathbf{C}_m \times \mathbf{C}_n$.

(ii) $\psi : \mathbf{C}_m \times \mathbf{C}_n \to \mathbf{C}_n \times \mathbf{C}_m$ *defined by*

$$\psi(\overline{x}, \overline{y}) = (\overline{y}, \overline{x})$$

is an isomorphism.

For each $f \in \mathrm{End}(\mathbf{C}_m \times \mathbf{C}_n)$, we denote

$$f^{\partial} := \phi \circ f \circ \phi^{-1}$$

and

$$f^{\smile} := \psi \circ f \circ \psi^{-1}.$$

One can see that for each $f \in \mathrm{End}(\mathbf{C}_m \times \mathbf{C}_n)$, $f^{\smile} \in \mathrm{End}(\mathbf{C}_n \times \mathbf{C}_m)$ and f fixes the bottom if and only if f^{∂} fixes the top. By Theorem 4 and Proposition 1, we will focus on MPP endomorphisms of $\mathbf{C}_m \times \mathbf{C}_2$, $\mathbf{C}_m \times \mathbf{C}_m$ and $\mathbf{C}_m \times \mathbf{C}_{m-1}$ fixing the bottom for $m \in \mathbb{N} \setminus \{1\}$.

Lemma 2. *Let $m \geq 3$ and f be an MPP endomorphism of $\mathbf{C}_m \times \mathbf{C}_2$ fixing the bottom. Then*

(i) $f(\overline{m}, \overline{2}) = (\overline{m-1}, \overline{2})$;

(ii) *if* $f(\overline{m-1}, \overline{2}) = (\overline{m-1}, \overline{1})$, *then* $m = 3$;

(iii) *if there is $t < m-1$ such that $f(\overline{i}, \overline{2}) = (\overline{i-1}, \overline{2})$ for all $i > t$ and $f(\overline{t}, \overline{2}) = (\overline{t}, \overline{1})$, then $t = 1$ and $f(\overline{j}, \overline{1}) = (\overline{j-1}, \overline{2})$ for all $j > t$.*

Proof. (i) Assume that $f(\overline{m}, \overline{2}) = (\overline{m}, \overline{1})$. Then $f(\overline{m}, \overline{1}) = (\overline{m-1}, \overline{1})$. Since $f(\overline{1}, \overline{2}) \leq f(\overline{m}, \overline{2}) = (\overline{m}, \overline{1})$, we get $f(\overline{1}, \overline{2}) = (\overline{k}, \overline{1})$ for some $1 \leq k \leq m$. Since

$$(\overline{1}, \overline{1}) = f(\overline{1}, \overline{1}) = f(\overline{1}, \overline{2}) \wedge f(\overline{m}, \overline{1}) = (\overline{k}, \overline{1}) \wedge (\overline{m-1}, \overline{1}),$$

we get $k = 1$. So,

$$(\overline{m}, \overline{1}) = f(\overline{m}, \overline{2}) = f(\overline{1}, \overline{2}) \vee f(\overline{m}, \overline{1}) = (\overline{1}, \overline{1}) \vee (\overline{m-1}, \overline{1}) = (\overline{m-1}, \overline{1}),$$

a contradiction. By Theorem 1, $f(\overline{m}, \overline{2}) = (\overline{m-1}, \overline{2})$.
(ii) Suppose that $f(\overline{m-1}, \overline{2}) = (\overline{m-1}, \overline{1})$. By Theorem 1, $f(\overline{m-1}, \overline{1}) = (\overline{m-2}, \overline{1})$. Since $f(\overline{m-2}, \overline{2}) \leq f(\overline{m-1}, \overline{2}) = (\overline{m-1}, \overline{1})$, we get $f(\overline{m-2}, \overline{2}) = (\overline{k}, \overline{1})$ for some $1 \leq k \leq m-1$. Since

$$(\overline{m-1}, \overline{1}) = f(\overline{m-1}, \overline{2}) = f(\overline{m-1}, \overline{1}) \vee f(\overline{m-2}, \overline{2}) = (\overline{m-2}, \overline{1}) \vee (\overline{k}, \overline{1}),$$

we get $k = m-1$. Since

$$f(\overline{m-2}, \overline{1}) = f(\overline{m-1}, \overline{1}) \wedge f(\overline{m-2}, \overline{2}) = (\overline{m-2}, \overline{1}) \wedge (\overline{m-1}, \overline{1}) = (\overline{m-2}, \overline{1})$$

and $(\overline{1}, \overline{1})$ is the unique fixed point, $m-2 = 1$; that is, $m = 3$.
(iii) Suppose that there is $t < m-1$ such that $f(\overline{i}, \overline{2}) = (\overline{i-1}, \overline{2})$ for all $i > t$ and $f(\overline{t}, \overline{2}) = (\overline{t}, \overline{1})$ and let $j > t$. Then

$$(\overline{j-1}, \overline{2}) = f(\overline{j}, \overline{2}) = f(\overline{t}, \overline{2}) \vee f(\overline{j}, \overline{1}) = (\overline{t}, \overline{1}) \vee f(\overline{j}, \overline{1}).$$

For $j > t+1$, we have $\overline{j-1} > \bar{t}$ which implies by the property of chain that $f(\bar{j},\bar{1}) = (\overline{j-1},\bar{2})$ and

$$f(\overline{t+1},\bar{1}) = f(\overline{t+2},\bar{1}) \wedge f(\overline{t+1},\bar{2}) = (\overline{t+1},\bar{2}) \wedge (\bar{t},\bar{2}) = (\bar{t},\bar{2}).$$

Hence,

$$f(\bar{t},\bar{1}) = f(\bar{t},\bar{2}) \wedge f(\overline{t+1},\bar{1}) = (\bar{t},\bar{1}) \wedge (\bar{t},\bar{2}) = (\bar{t},\bar{1}).$$

Since $(\bar{1},\bar{1})$ is the unique fixed point, $t = 1$.

Theorem 5. *Let $m \in \mathbb{N}$ with $m \geq 2$.*

(i) For $m \leq 3$, $\zeta_{(m,m-1)} \restriction_{C_m \times C_2}$ and $\varphi_{m\times 2}$ are all MPP endomorphisms of $\mathbf{C}_m \times \mathbf{C}_2$ fixing the bottom.

(ii) For $m > 3$, $\varphi_{m\times 2}$ is the unique MPP endomorphism of $\mathbf{C}_m \times \mathbf{C}_2$ fixing the bottom.

(iii) For $m \geq 3$, $\zeta_{(m,m-1)}$ and $\zeta_{(m-1,m)}$ are all MPP endomorphisms of $\mathbf{C}_m \times \mathbf{C}_m$ fixing the bottom.

(iv) For $m > 3$, $\zeta_{(m,m-1)} \restriction_{C_m \times C_{m-1}}$ is the unique MPP endomorphism of $\mathbf{C}_m \times \mathbf{C}_{m-1}$ fixing the bottom.

Proof. (i) Let f be an MPP endomorphism of $\mathbf{C}_2 \times \mathbf{C}_2$ fixing the bottom. Then $f(\bar{2},\bar{2}) = (\bar{2},\bar{1})$ or $f(\bar{2},\bar{2}) = (\bar{1},\bar{2})$.
Case $f(\bar{2},\bar{2}) = (\bar{2},\bar{1})$. Then $f(\bar{2},\bar{1}) = (\bar{1},\bar{1})$. Since

$$(\bar{2},\bar{1}) = f(\bar{2},\bar{2}) = f(\bar{2},\bar{1}) \vee f(\bar{1},\bar{2}) = (\bar{1},\bar{1}) \vee f(\bar{1},\bar{2}),$$

$f(\bar{1},\bar{2}) = (\bar{2},\bar{1})$ which implies that $f = \zeta_{(2,1)}$.
Case $f(\bar{2},\bar{2}) = (\bar{1},\bar{2})$. Similarly, $f = \zeta_{(1,2)}(= \varphi_{2\times 2})$.
In any cases, we are done for $m = 2$.

Let f be an MPP endomorphism of $\mathbf{C}_3 \times \mathbf{C}_2$ fixing the bottom. By Lemma 2 (i), $f(\bar{3},\bar{2}) = (\bar{2},\bar{2})$; and so, $f(\bar{3},\bar{1}) \leq (\bar{2},\bar{2})$. Thus $f \restriction_{C_2\times C_2}$ is an MPP endomorphism of $\mathbf{C}_2 \times \mathbf{C}_2$ fixing the bottom.
Case $f \restriction_{C_2\times C_2} = \zeta_{(2,1)}$. Thus $f(\bar{2},\bar{2}) = (\bar{2},\bar{1})$ and $f(\bar{2},\bar{1}) = (\bar{1},\bar{1})$. Since

$$(\bar{1},\bar{1}) = f(\bar{2},\bar{1}) = f(\bar{2},\bar{2}) \wedge f(\bar{3},\bar{1}) = (\bar{2},\bar{1}) \wedge f(\bar{3},\bar{1})$$

and

$$(\bar{2},\bar{2}) = f(\bar{3},\bar{2}) = f(\bar{2},\bar{2}) \vee f(\bar{3},\bar{1}) = (\bar{2},\bar{1}) \vee f(\bar{3},\bar{1}),$$

we get $f(\bar{3},\bar{1}) = (\bar{1},\bar{2})$. So, $f = \zeta_{(3,2)} \restriction_{C_3\times C_2}$.
Case $f \restriction_{C_2\times C_2} = \zeta_{(1,2)}$. Thus $f(\bar{2},\bar{2}) = (\bar{1},\bar{2})$ and $f(\bar{2},\bar{1}) = (\bar{1},\bar{2})$. Since

$$(\bar{1},\bar{2}) = f(\bar{2},\bar{1}) = f(\bar{2},\bar{2}) \wedge f(\bar{3},\bar{1}) = (\bar{1},\bar{2}) \wedge f(\bar{3},\bar{1})$$

and

$$(\bar{2},\bar{2}) = f(\bar{3},\bar{2}) = f(\bar{2},\bar{2}) \vee f(\bar{3},\bar{1}) = (\bar{1},\bar{2}) \vee f(\bar{3},\bar{1}),$$

we get $f(\overline{3},\overline{1}) = (\overline{2},\overline{2})$. So, $f = \varphi_{3\times 2}$.
(*ii*) The proof follows directly from Lemma 2 (*ii*) and (*iii*).
(*iii*) We will prove by the induction under the cardinality of the chain. By (*i*), this statement is true for $m = 2$. Let $m \geq 3$ and f be an MPP endomorphism of $\mathbf{C}_m \times \mathbf{C}_m$ fixing the bottom. By Lemma 1, we may assume that

$$f^{2k}(\overline{m},\overline{m}) = (\overline{m-k},\overline{m-k}) \text{ and } f^{2k+1}(\overline{m},\overline{m}) = (\overline{m-k},\overline{m-(k+1)})\text{.......}(*)$$

for all $0 \leq k \leq m-2$. Then for each $0 \leq k \leq m-2$ and $1 \leq i \leq m-k$,

$$\begin{aligned}(\overline{m-k},\overline{m-(k+1)}) &= f(\overline{m-k},\overline{m-k})\\ &= f(\overline{m-k},\overline{m-(k+1)}) \vee f(\overline{i},\overline{m-k})\\ &= (\overline{m-(k+1)},\overline{m-(k+1)}) \vee f(\overline{i},\overline{m-k})\end{aligned}$$

which implies that

$$f(\overline{i},\overline{m-k}) = (\overline{m-k},\overline{j}) \text{ for some } 1 \leq j \leq m-(k+1) \tag{3}$$

and

$$\begin{aligned}f(\overline{i},\overline{m-(k+1)}) &= f(\overline{m-k},\overline{m-(k+1)}) \wedge f(\overline{i},\overline{m-k})\\ &= (\overline{m-(k+1)},\overline{m-(k+1)}) \wedge (\overline{m-k},\overline{j})\\ &= (\overline{m-(k+1)},\overline{j});\end{aligned}$$

and for $i = m-(k+1)$, we get by $(*)$ that $j = m-(k+2)$. Thus

$$f(\overline{m-(k+1)},\overline{m-k}) = (\overline{m-k},\overline{m-(k+2)}). \tag{4}$$

Besides,

$$\begin{aligned}(\overline{m-k},\overline{m-(k+1)}) &= f(\overline{m-k},\overline{m-k})\\ &= f(\overline{m-k},\overline{i}) \vee f(\overline{m-(k+1)},\overline{m-k})\\ &= f(\overline{m-k},\overline{i}) \vee (\overline{m-k},\overline{m-(k+2)})\end{aligned}$$

implies

$$f(\overline{m-k},\overline{i}) = (\overline{j},\overline{m-(k+1)}) \text{ for some } 1 \leq j \leq m-k. \tag{5}$$

By equations (3) and (5) ($k = 1$), $C_{m-1} \times C_{m-1}$ is closed under f. By the induction hypothesis, $f \restriction_{C_{m-1}\times C_{m-1}}$ is either $\zeta_{(m-1,m-2)}$ or $\zeta_{(m-2,m-1)}$. By the condition $(*)$, $f \restriction_{C_{m-1}\times C_{m-1}} = \zeta_{(m-1,m-2)}$; that is, $f(\overline{r},\overline{s}) = (\overline{s},\overline{r-1})$ for all $1 \leq r,s \leq m-1$. For each $1 \leq i \leq m-1$,

$$(\overline{m-1},\overline{i-1}) = f(\overline{i},\overline{m-1}) = f(\overline{i},\overline{m}) \wedge f(\overline{m-1},\overline{m-1}) = f(\overline{i},\overline{m}) \wedge (\overline{m-1},\overline{m-2})$$

implies by the equation (3) that $f(\overline{i},\overline{m}) = (\overline{m},\overline{i-1})$. For each $1 \leq i \leq m-1$,

$$(\bar{i},\overline{m-2}) = f(\overline{m-1},\bar{i}) = f(\overline{m},\bar{i}) \wedge f(\overline{m-1},\overline{m-1}) = f(\overline{m},\bar{i}) \wedge (\overline{m-1},\overline{m-2})$$

implies by the equation (5) that $f(\overline{m},\bar{i}) = (\bar{i},\overline{m-1})$. Hence, $f = \zeta_{(m,m-1)}$.

(iv) Let $m > 3$ and f be an MPP endomorphism of $\mathbf{C}_m \times \mathbf{C}_{m-1}$ fixing the bottom. Suppose that $f(\overline{m},\overline{m-1}) = (\overline{m},\overline{m-2})$. By Lemma 1, $f^{2(m-3)}(\overline{m},\overline{m-1}) = (\overline{3},\overline{2})$ and $f^{2(m-3)+1}(\overline{m},\overline{m-1}) = (\overline{3},\overline{1})$. Again by Theorem 1, $f^{2(m-2)}(\overline{m},\overline{m-1}) = (\overline{2},\overline{1})$ and $f^{2(m-3)+1}(\overline{m},\overline{m-1}) = (\overline{1},\overline{1})$. Since

$$(\overline{3},\overline{1}) = f(\overline{3},\overline{2}) = f(\overline{3},\overline{1}) \vee f(\overline{2},\overline{2}) = (\overline{2},\overline{1}) \vee f(\overline{2},\overline{2}),$$

we get $f(\overline{2},\overline{2}) = (\overline{3},\overline{1})$. Since

$$(\overline{1},\overline{1}) = f(\overline{2},\overline{1}) = f(\overline{3},\overline{1}) \wedge f(\overline{2},\overline{2}) = (\overline{2},\overline{1}) \wedge (\overline{3},\overline{1}) = (\overline{2},\overline{1}),$$

we get $\overline{2} = \overline{1}$, a contradiction. By Theorem 1, $f(\overline{m},\overline{m-1}) = (\overline{m-1},\overline{m-1})$. By Lemma 1, $f^{2k}(\overline{m},\overline{m-1}) = (\overline{m-k},\overline{m-1-k})$ and $f^{2k+1}(\overline{m},\overline{m-1}) = (\overline{m-(k+1)},\overline{m-1-k})$ for all $0 \le k \le m-2$. By the same arguments of proving (*iii*), we get $f = \zeta_{(m,m-1)} \downharpoonleft_{C_m \times C_{m-1}}$.

Example 1. *All MPP endomorphisms (fixing the bottom) of* $\mathbf{C}_2 \times \mathbf{C}_2$*,* $\mathbf{C}_3 \times \mathbf{C}_2$*,* $\mathbf{C}_3 \times \mathbf{C}_3$ *and* $\mathbf{C}_4 \times \mathbf{C}_3$ *are shown in the figure 4, 5, 6 and 7, respectively.*

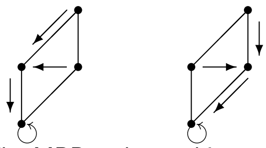

Figure 4: The MPP endomorphisms of $\mathbf{C}_2 \times \mathbf{C}_2$.

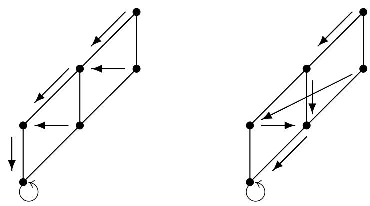

Figure 5: The MPP endomorphisms of $\mathbf{C}_3 \times \mathbf{C}_2$.

Corollary 2. *Let* $m \in \mathbb{N}$ *with* $m \ge 2$*.*

(i) For $m \le 3$*,* $\zeta_{(m,m-1)} \downharpoonleft_{C_m \times C_2}$*,* $\varphi_{m\times 2}$*,* $\zeta_{(m,m-1)} \downharpoonleft^{\partial}_{C_m \times C_2}$ *and* $\varphi^{\partial}_{m\times 2}$ *are all MPP endomorphisms of* $\mathbf{C}_m \times \mathbf{C}_2$*.*

(ii) For $m > 3$*,* $\varphi_{m\times 2}$ *and* $\varphi^{\partial}_{m\times 2}$ *are all MPP endomorphisms of* $\mathbf{C}_m \times \mathbf{C}_2$*.*

(iii) For $m \ge 3$*,* $\zeta_{(m,m-1)}$*,* $\zeta_{(m-1,m)}$*,* $\zeta^{\partial}_{(m,m-1)}$ *and* $\zeta^{\partial}_{(m-1,m)}$ *are all MPP endomorphisms of* $\mathbf{C}_m \times \mathbf{C}_m$*.*

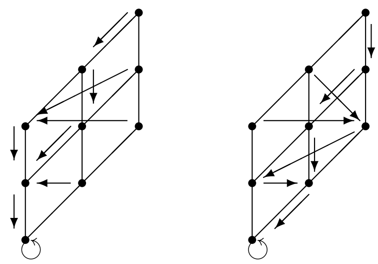

Figure 6: The MPP endomorphisms of $\mathbf{C}_3 \times \mathbf{C}_3$.

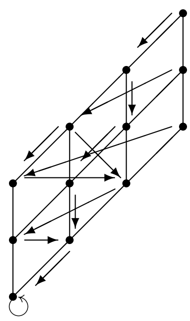

Figure 7: The MPP endomorphism of $\mathbf{C}_4 \times \mathbf{C}_3$.

(iv) *For* $m > 3$, $\zeta_{(m,m-1)} \restriction_{C_m \times C_{m-1}}$ *and* $\zeta_{(m,m-1)} \restriction^{\partial}_{C_m \times C_{m-1}}$ *are all MPP endomorphisms of* $\mathbf{C}_m \times \mathbf{C}_{m-1}$.

For each $n \in \mathbb{N}$, we define monounary algebras (A_n, f_n) and (B_n, g_n) by

$$A_n = \{a_{c,h} \mid 2c - 2 \leq h \leq n \text{ for some } c \in \mathbb{N} \text{ and } h \in \mathbb{N}_0\},$$

$$B_n = \{b_{c,h} \mid c \in \{1, 2\} \text{ and } h \in \{1, \dots, n\}\},$$

$$f_n(a_{c,h}) = \begin{cases} a_{c,h-1} & \text{if } 2c - 2 < h, \\ a_{c-1,h-1} & \text{if } 2c - 2 = h, \\ a_{1,0} & \text{if } c = 1 \text{ and } h = 0, \end{cases}$$

and

$$g_n(b_{c,h}) = \begin{cases} b_{c,h-1} & \text{if } c = 1 \text{ and } h \neq 1, \\ b_{c-1,h-1} & \text{if } c = 2 \text{ and } h \neq 1, \\ b_{2,1} & \text{if } h = 1. \end{cases}$$

One can observe that

$$f_n^{-1}(\{a_{c,h}\}) = \begin{cases} \emptyset & \text{if } h = n, \\ \{a_{c,h+1}\} & \text{if } 2c-1 < h < n, \\ \{a_{c,h+1}, a_{c+1,h+1}\} & \text{if } 2c-1 = h, \\ \{a_{1,0}, a_{1,1}\} & \text{if } h = 0 \end{cases}$$

for all $a_{c,h} \in A_n$ and

$$g_n^{-1}(\{b_{c,h}\}) = \begin{cases} \emptyset & \text{if either } h = n \text{ or } c = 2 \text{ and } h \neq 1, \\ \{b_{1,1}, b_{2,1}\} & \text{if } c = 2 \text{ and } h = 1, \\ \{b_{c,h+1}, a_{c+1,h+1}\} & \text{if } c = 1 \text{ and } h \neq n \end{cases}$$

for all $b_{c,h} \in B_n$. For each $n \in \mathbb{N}$, let $\zeta_n = \zeta_{(m+1,m)}$ if $n = 2m$ and let $\zeta_n = \zeta_{(m+1,m)} \restriction_{C_{m+1} \times C_m}$ if $n = 2m-1$.

Theorem 6. *All monounary algebras induced by an MPP endomorphism of the direct of two chains are isomorphic to either (A_n, f_n) or (B_n, g_n) for some $n \in \mathbb{N}$.*

Proof. By Theorem 4, 5, and Proposition 1, it suffices to show that (A_n, f_n) is isomorphic to $\left(C_{\lceil \frac{n+2}{2} \rceil} \times C_{\lceil \frac{n+1}{2} \rceil}, \zeta_n\right)$ and (B_n, g_n) is isomorphic to $(C_n \times C_2, \varphi_{n\times 2})$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$.

Firstly, we will show that $\phi_n : A_n \to C_{\lceil \frac{n+2}{2} \rceil} \times C_{\lceil \frac{n+1}{2} \rceil}$ defined by

$$\phi_n(a_{c,h}) = \begin{cases} (\overline{\frac{h}{2} + 2 - c}, \overline{\frac{h}{2} + 1}) & \text{if } h \in \mathbb{E}, \\ \\ (\overline{\frac{h+3}{2}}, \overline{\frac{h+3}{2} - c}) & \text{if } h \in \mathbb{O} \end{cases}$$

is an isomorphism. Let $a_{c,h} \in A_n$.

Case 1: $h \in \mathbb{E}$. If $a_{c,h} = a_{1,0}$, then

$$\phi_n(f_n(a_{1,0})) = \phi_n(a_{1,0}) = (\overline{1}, \overline{1}) = \zeta_n(\overline{1}, \overline{1}) = \zeta_n(\phi_n(a_{1,0})).$$

If $2c - 2 < h$, then $\frac{h}{2} + 2 - c > 1$ and

$$\phi_n(f_n(a_{c,h})) = \phi_n(a_{c,h-1}) = (\overline{\frac{h+2}{2}}, \overline{\frac{h+2}{2} - c}) = \zeta_n(\overline{\frac{h}{2} + 2 - c}, \overline{\frac{h}{2} + 1}) = \zeta_n(\phi_n(a_{c,h})).$$

If $2c - 2 = h$, then

$$\begin{aligned} \phi_n(f_n(a_{c,h})) = \phi_n(a_{c-1,h-1}) &= (\overline{\frac{h+2}{2}}, \overline{\frac{h+2}{2} - c + 1}) \\ &= (\overline{c}, \overline{1}) = \zeta_n(\overline{1}, \overline{c}) = \zeta_n(\overline{\frac{h}{2} + 2 - c}, \overline{\frac{h}{2} + 1}) = \zeta_n(\phi_n(a_{c,h})). \end{aligned}$$

Case 2: $h \in \mathbb{O}$. Then $2c-2 < h$ and $h \neq 0$. Hence, $\frac{h+3}{2} > c + \frac{1}{2} > 1$ and

$$\phi_n(f_n(a_{c,h})) = \phi_n(a_{c,h-1}) = (\overline{\frac{h-1}{2}+2-c}, \overline{\frac{h-1}{2}+1}) = \zeta_n(\overline{\frac{h+3}{2}}, \overline{\frac{h+3}{2}-c}) = \zeta_n(\phi_n(a_{c,h})).$$

Finally, we will show that $\psi_n : B_n \to C_n \times C_2$ defined by

$$\psi_n(b_{c,h}) = \begin{cases} (\overline{h}, \overline{2}) & \text{if } c = 1, \\ (\overline{h}, \overline{1}) & \text{if } c = 2 \end{cases}$$

is an isomorphism. Let $b_{c,h} \in B_n$. Then $\psi_n(b_{c,1}) \in \{(\overline{1}, \overline{1}), (\overline{1}, \overline{2})\}$ which implies that

$$\psi_n(g_n(b_{c,1})) = \psi_n(b_{2,1}) = (\overline{1}, \overline{1}) = \varphi_{n\times 2}(\psi_n(b_{c,1}))$$

and for $h \geq 2$

$$\psi_n(g_n(b_{1,h})) = \psi_n(b_{1,h-1}) = (\overline{h-1}, \overline{2}) = \varphi_{n\times 2}(\overline{h}, \overline{2}) = \varphi_{n\times 2}(\psi_n(b_{1,h}))$$

and

$$\psi_n(g_n(b_{2,h})) = \psi_n(b_{1,h-1}) = (\overline{h-1}, \overline{2}) = \varphi_{n\times 2}(\overline{h}, \overline{1}) = \varphi_{n\times 2}(\psi_n(b_{2,h})).$$

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