EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6540
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Resolving Parameters in Generalized Sierpiński Networks Over Cycle of Length Five

K. Bharani Dharan[1], S. Radha[1,*]

[1] *Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Chennai, Tamil Nadu, India 600127*

Abstract. The fascinating family of fractal-based networks known as generalized Sierpiński networks has garnered significant interest across various research domains and practical applications. These graphs exhibit self-similar structures, making them particularly valuable in areas such as antenna structures, interconnection networks and the porous materials. Their recursive nature and hierarchical organization enhance their relevance in modeling complex systems and network structures. A key parameter for analyzing these graphs is the metric dimension, which represents the smallest set of reference vertices (or landmarks) needed to uniquely identify the distances between all other vertices in the graph. This parameter is crucial for network localization, efficient routing, and information retrieval. Beyond the standard metric dimension, other variations play important roles in different applications. The fault-tolerant metric dimension is essential in robust network design, ensuring that localization remains possible even if certain reference points fail. The edge metric dimension is widely used in network security and surveillance, where monitoring specific connections is more relevant than individual nodes. Meanwhile, the fault-tolerant edge metric dimension has applications in resilient communication networks, guaranteeing reliable identification of edges even under failure conditions. In this study, we specifically examine the metric, fault-tolerant metric, edge metric, and fault-tolerant edge metric dimensions of generalized Sierpiński networks over C_5. These findings provide deeper insights into their structural properties and distinguish them from traditional cycle networks, highlighting their potential in real-world applications.

2020 Mathematics Subject Classifications: 05C12, 94C15, 68M10

Key Words and Phrases: Generalized Sierpiński network; metric dimension; fault-tolerant metric dimension; edge metric dimension; fault-tolerant edge metric dimension

---

1. Introduction

Graph theory plays a crucial role in distributed parallel computing by providing a mathematical framework for modeling, analyzing, and optimizing system performance [1]. In

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6540

Email addresses: bharanidharan.k2022@vitstudent.ac.in (K. Bharani Dharan),
radha.s@vit.ac.in (S. Radha)

such systems, multiple processors collaborate to execute tasks efficiently, requiring well-structured communication and coordination, which can be effectively represented using graphs [2]. Various graph-theoretic techniques enhance different aspects of parallel computing, such as task scheduling, load balancing, fault tolerance, and network optimization [3]. Graph partitioning helps distribute workloads evenly, preventing bottlenecks, while interconnection network models, including hypercubes, meshes, and trees, assist in minimizing communication latency [4]. Connectivity and domination properties contribute to fault tolerance by ensuring alternative paths in case of node or link failures [5]. Graph-based methods like similarity and clustering also enhance security through anomaly detection in network traffic, helping mitigate unauthorized access and cyber threats [6]. Additionally, shortest path algorithms improve data transmission efficiency, reducing congestion in communication networks [7]. An effective approach particularly in designing scalable and robust parallel computing networks involves fractal-structured graphs, such as generalized Sierpiński networks and fractal cubic networks which exhibit self-similarity and hierarchical organization [8, 9]. These graphs offer several advantages, including seamless scalability, as their recursive nature allows for structured expansion without compromising efficiency [10]. Their hierarchical architecture enables optimized routing and communication, reducing congestion and improving data flow [11]. Moreover, their inherent redundancy enhances fault tolerance, ensuring system resilience even in the event of node or link failures. The structured nature of fractal graphs also facilitates balanced workload distribution, improving computational efficiency while simultaneously reducing energy consumption in communication, making them ideal for energy-efficient parallel computing architectures [12]. These networks have applications in various artificial and natural systems, including neuroscience [13], computer networks and musics [14]. They also assist in analyzing complex biological structures like bacterial growth patterns [15]. Among these structures, Sierpiński networks are particularly relevant to parallel computing, especially in designing high-performance computing clusters and supercomputers. Overall, graph theory, particularly through fractal-based structures, significantly contributes to optimizing distributed parallel computing by enhancing performance, scalability, security, and reliability, making it a key tool in high-performance computing system design [16].

The Sierpiński network is distinguished by its recursive hierarchical arrangement, where each level represents a smaller version of the overall structure. Because of these characteristics, they are seen to be a viable topology for systems that use parallel computing, especially in fields of study that are looking into new network architectures to improve fault tolerance, scalability, and efficiency [8].

Tower of Hanoi graphs with three pegs are a particular type of Sierpiński networks that occur when the base graph is a complete graph K_3. Recursive networks are created by extending Sierpiński graphs by appending an open connection to their extreme vertices. These networks have been used in Very Large Scale Integration (VLSI) designs since their initial introduction in 1988 as a foundation for message-passing architectures [17]. Sierpiński graphs were created in the late 1990s when Sierpiński labelings were applied to WK-recursive networks [18]. Deeper investigation of their characteristics has been made

possible by this labelling strategy.
In [19], a number of significant features of Sierpiński graphs have been discussed. These networks have been shown to have Hamiltonian characteristics and their geodesic distances between vertices have been calculated [20]. Additional metrics have been examined, such as median values, average eccentricity, and connectedness [21–23]. Furthermore, several Sierpiński graph topological descriptors have been studied [24], which has shed light on the shortest path structures in these networks [25].
Complete graphs serve as the basic building blocks for the construction of classical Sierpiński networks, as described in [18]. In [19], a more expansive category called as generalised Sierpiński networks was established and given the survey about their various graph theoretical properties. The concept of the Sierpiński product $G \otimes_f H$, where $f : V(G) \to V(H)$, generalizes the construction of Sierpiński-type graphs by embedding copies of H based on the structure of G. This construction and related metric properties have been rigorously studied in [26]. The study of metric dimension in recursive and fractal-based networks has drawn increasing attention due to its implications in network navigation and information retrieval [27–32]. In particular, the work presented in [33] investigates the metric dimension and its variants for generalized Sierpiński networks constructed over C_4, which are known to be spanning subgraphs of hypercubes. Their analysis showcases the interplay between structural properties, such as vertex twins, and the complexity of resolving sets. Motivated by these findings, we extend the investigation to the class of generalized Sierpiński networks over C_5, exploring how the odd cycle structure influences the metric dimension and its fault-tolerant variants.

2. Preliminaries

Let $\Gamma = (\mathbb{V}(\Gamma), \mathbb{E}(\Gamma))$ be a graph of order n. For $h \in \mathbb{N}$, the generalized Sierpiński graph [19], denoted as S^h_Γ, has the vertex set $\mathbb{V}(S^h_\Gamma) = \mathbb{V}(\Gamma)^h$. The notation for a vertex $g = (g_1, g_2, \dots, g_h)$ in $\mathbb{V}(S^h_\Gamma)$ is abbreviated as $g = g_1 g_2 \dots g_h$. Two vertices, $f = f_1 f_2 \dots f_h$ and $g = g_1 g_2 \dots g_h$, are adjacent if there exists an index $x \in \mathbb{Z}_{h+1} \setminus \{0\}$ such that for all $y \in \mathbb{Z}_{h+1} \setminus \{0\}$, the following properties hold:

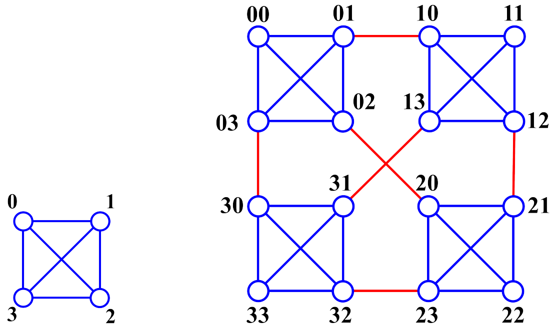
i) If $y < x$, then $f_y = g_y$;

ii) If $y = x$, then $f_x \neq g_x$ and $f_x g_x \in \mathbb{E}(G)$;

iii) If $y > x$, then $f_y = g_x$ and $f_y = g_x$.

For example, the graphs $S^1_{K_4}$ and $S^2_{K_4}$ are depicted in Figure 1. The graph S^h_Γ can be constructed recursively from a base graph Γ as follows. Let $S^1_\Gamma = \Gamma$. For each $h \geq 2$, consider n disjoint copies of the graph S^{h-1}_Γ. In the x^{th} copy, where $x \in \mathbb{Z}_n$, prefix each vertex label with the index x. These subgraphs are denoted by xS^{h-1}_Γ, for all $x \in \mathbb{Z}_n$. Furthermore, if two vertices f and g are adjacent in Γ, then in S^h_Γ, the vertex labeled fg^{h-1} is adjacent to the vertex labeled gf^{h-1}.

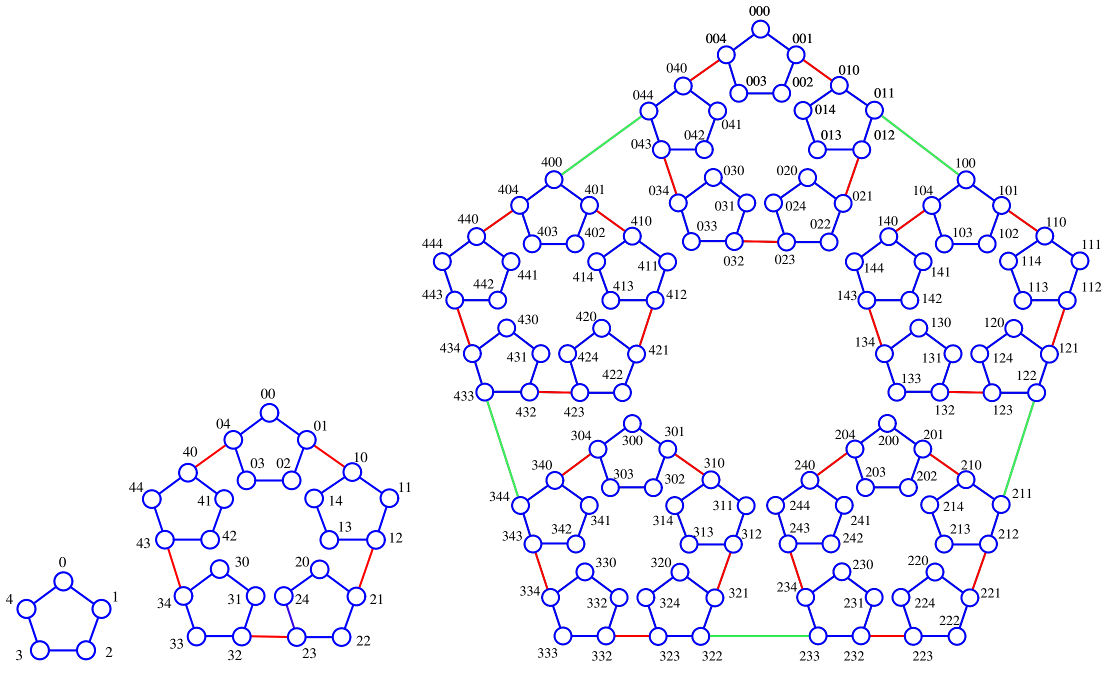
Generalized Sierpiński networks have been used to represent polymer networks [34]. Roman domination [35], strong metric dimension [36], and other graph-theoretic features



(a) (b)

Figure 1: (a)$S^1_{K_4} \cong K_4$ (b)$S^2_{K_4}$

such as chromatic number, vertex cover number and clique number for S^h_G are already determined. The topological indices of generalized Sierpiński networks are examined in [34]. In this work, we particularly analyze the family of $S^h_{C_5}$ and calculate its metric, fault-tolerant metric, edge metric and fault-tolerant edge metric dimensions. The graphs $S^1_{C_5}$, $S^2_{C_5}$ and $S^3_{C_5}$ are listed in the Figure 2.



(a) (b) (c)

Figure 2: (a)$S^1_{C_5}$ (b)$S^2_{C_5}$ (c)$S^3_{C_5}$

2.1. Metric basis and fault-tolerant basis

Networks are considered to be graphs with intriguing properties based on graph theory. The effectiveness of locating and differentiating each vertex of a graph using a small number of landmarks is measured by its metric dimension [37–39]. Location-based services, robotics and network design are just a few of the domains where metric dimension finds use [40, 41]. It aids in determining the best locations for sensors inside a network to guarantee accurate location identification [42]. Determining an exact value of this parameter of a graph is difficult task. There are effective methods for calculating this parameter for some graph types like trees. The problem of determining this parameter for directed graphs is NP-hard [29], bipartite graphs[43] and general graphs [37]. Despite the computational difficulties, this parameter is calculated for many graph networks, such as honeycomb [44], butterfly [43], Beneš [43], circulant graphs [45], generalized subdivision of prism [46], chemical structures [47], convex triangular networks [48], multistage interconnection networks [49] and Sierpiński [50]. A current assessment of the literature on metric dimension could be found in [51].

The distance $d_\Gamma(f,g)$ in a connected graph Γ is defined as the smallest number of edges required to travel from a vertex f to a vertex g. For a vertex u and an edge $e = fg$ (with $f, g \in \mathbb{V}(\Gamma)$), the distance from u to the edge e is defined as $d_\Gamma(u,e) = \min\{d_\Gamma(u,f), d_\Gamma(u,g)\}$. Given an ordered subset $\mathbb{Y} = \{y_1, y_2, \ldots, y_\ell\} \subseteq \mathbb{V}(\Gamma)$, the *representation* of a vertex $v \in \mathbb{V}(\Gamma)$ with respect to $\mathbb{Y}$ is the ℓ-tuple $r(v|\mathbb{Y}) = (d_\Gamma(v,y_1), d_\Gamma(v,y_2), \ldots, d_\Gamma(v,y_\ell))$.

A subset $\mathbb{Y}$ is called a *resolving set* if each vertex in Γ has a unique representation concerning $\mathbb{Y}$. The *metric dimension* of Γ, denoted by $\dim(\Gamma)$, is the minimum cardinality among all resolving sets in Γ.

For the Sierpiński network $S^2_{C_5}$, the two vertex subset $W = \{00, 20\}$ is enough to make other vertices to be resolved. We can easily verify that the representation of each vertex with respect to the set $\{00, 20\}$ are distint in the Figure 3 and the cardinality of resolving set cannot be 1 for $S^2_{C_5}$ as it is not isomorphic to a path. Then the set W is a metric basis for $S^2_{C_5}$. This implies $\dim(S^2_{C_5}) = 2$.

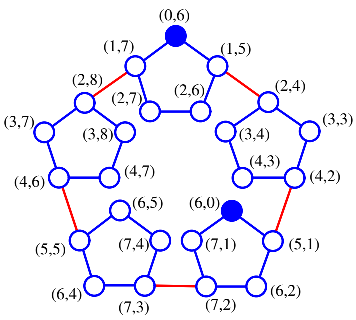


Figure 3: Metric representation of each vertex with respect to the set $W = \{00, 20\}$ in $S^2_{C_5}$

Many variations have been developed from this well-established concept in the literature [18, 52–56]. Fault-tolerant metric dimension is one of these new and highly motivated variants. The key idea in this version is that even if one vertex in the chosen set of vertices becomes flawed or unusable, the graph must still be resolved using that set. Let $\mathcal{F} \subseteq \mathbb{V}(\Gamma)$ is called a fault-tolerant resolving set, if we have $r(u|\mathcal{F} \setminus \{x\}) \neq r(v|\mathcal{F} \setminus \{x\})$, for every $x \in \mathcal{F}$ and $u, v \in \mathbb{V}(\Gamma)$(for every pair of distinct vertices $u, v \in \mathbb{V}(\Gamma)$, there exist at least two vertices $x, y \in \mathcal{F}$ such that $d_\Gamma(u, x) \neq d_\Gamma(v, x)$ and $d_\Gamma(u, y) \neq d_\Gamma(v, y)$). Among all such sets, one with the smallest possible size is called a *fault-tolerant metric basis.* The number of vertices in a fault-tolerant metric basis is known as the *fault-tolerant metric dimension* of Γ, and it is denoted by $\dim'(\Gamma)$. This idea was first presented in [57], and was further discussed in [5, 45, 58].

2.2. Edge metric and fault-tolerant edge metric basis

In parallel computing systems, an interconnection network comprising a structured configuration of processors and communication links, plays a vital role in facilitating data exchange among processors. Efficient fault detection within such networks requires the ability to distinguish between communication links. This can be accomplished by identifying a minimal set of vertices capable of uniquely determining every edge in the network graph. A subset $S \subseteq \mathbb{V}(\Gamma)$ is called an *edge resolving set* if, for every pair of distinct edges $e_1, e_2 \in \mathbb{E}(\Gamma)$, there exists a vertex $w \in S$ such that $d_\Gamma(w, e_1) \neq d_\Gamma(w, e_2)$. The smallest possible size of such a set is defined as the *edge metric dimension* of Γ, denoted by $\dim_{\mathbb{E}}(\Gamma)$.

Kelenc et al. [52] were the first to study this metric dimension variant and proved its NP-completeness. Since then, numerous research articles have explored this topic, including studies on the dimension of convex polytope graphs [59], web graphs, prism-related graphs, generalized Petersen graphs [60] and silicate networks [61]. Other works have examined this parameter for Erdős–Rényi random graphs [62], certain classes of planar graphs [63], and the identification of graph network with higher dimension [64, 65]. In [66], the metric and edge metric dimension of hypercubes were analyzed. Additionally, researchers have investigated graph operations such as join of graph networks, lexicographic product and corona product [67], as well as hierarchical products across various graph classes [68]. Recently, increasing attention has been given to identifying graphs where $\dim_{\mathbb{E}} < \dim$ [69, 70].

A subset $\mathcal{F} \subseteq \mathbb{V}(\Gamma)$ is said to be a *fault-tolerant edge resolving set* if, for every vertex $u \in \mathcal{F}$, the set $\mathcal{F} \setminus \{u\}$ remains an edge resolving set of Γ. Among all such subsets, one with the smallest possible size is called a *fault-tolerant edge metric basis.* The cardinality of a fault-tolerant edge metric basis is referred to as the *fault-tolerant edge metric dimension* of Γ, denoted by $\dim_{\mathbb{E}'}(\Gamma)$.

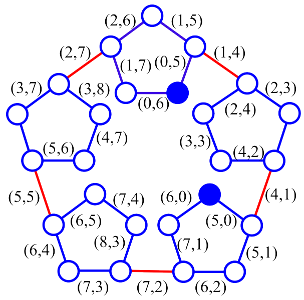


Figure 4: Edge representation of each edge with respect to the set $W = \{02, 20\}$ in $S^2_{C_5}$

From Figure 4, we can easily conclude that $\dim_{\mathbb{E}}(S^2_{C_5}) = 2$, and it is easy to verify that the fault-tolerant edge metric dimension of $S^2_{C_5}$ is equal to 4, where $\{02, 20, 03, 30\}$ is the fault-tolerant metric basis. In this paper, we use the following notations to represent various parameters: RS for metric resolving set, MB for metric basis, dim for metric dimension, FTRS for fault-tolerant metric resolving set, FTMB for fault-tolerant metric basis, $\dim'$ for fault-tolerant metric dimension, ERS for edge resolving set, EMB for edge metric basis, $\dim_{\mathbb{E}}$ for edge metric dimension, FTERS for fault-tolerant edge resolving set, FTEMB for fault-tolerant edge metric basis, and $\dim'_E$ for fault-tolerant edge metric dimension.

3. Main results

In this section, we present results pertaining to the exact values of both the $\dim(\Gamma)$ and the $\dim'(\Gamma)$s, as outlined in Subsection 3.1. Subsequently, Subsection 3.2 addresses the corresponding results for the $\dim_{\mathbb{E}}(\Gamma)$ and the $\dim'_{\mathbb{E}}(\Gamma)$.

To support the upcoming theorems, we introduce a particular subset of vertices that will be instrumental in the analysis. Let Γ' be an induced subgraph of $S^h_{C_5}$, and let W denote a resolving set for $S^h_{C_5}$ with $h \geq 3$. A vertex $v \in \mathbb{V}(\Gamma')$ is referred to as a *pseudo resolver* for Γ' if there exists a vertex $u \in W \setminus \mathbb{V}(\Gamma')$ such that, $\forall x \in \mathbb{V}(\Gamma')$, the equality $d_\Gamma(x, u) = d_\Gamma(x, v) + d_\Gamma(v, u)$ holds. This implies that replacing u in W with v results in a set that still resolves Γ', that is, $r(x|(W \setminus \{u\}) \cup \{v\}) \neq r(y|(W \setminus \{u\}) \cup \{v\})$ for all distinct $x, y \in \mathbb{V}(\Gamma')$. Figure 5 illustrates the collection of pseudo resolver vertices for each induced subgraph $iS^2_{C_5}$, where $i \in \mathbb{Z}_5$, within the graph $S^3_{C_5}$.

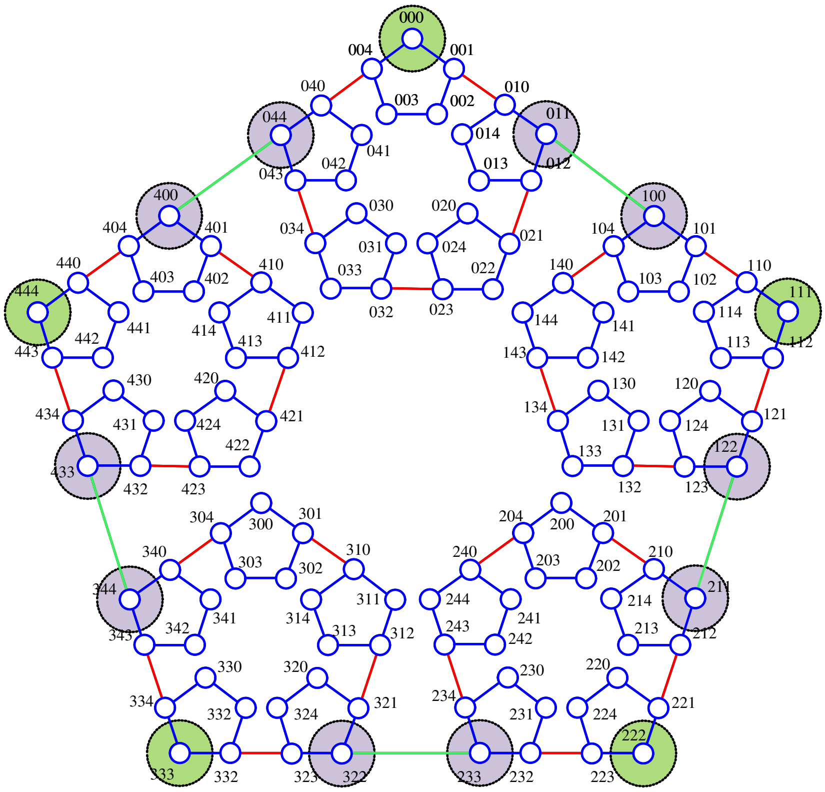


Figure 5: $S^3_{C_5}$ with vertices of the resolving set encircled with green color and pseudo resolvers for each $S^2_{C_5}$ encircled with violet color

3.1. Metric dimension of generalized Sierpiński networks over cycle of length five

We can easily say that $\dim(S^1_{C_n}) = 2$ as $S^1_{C_n}$ is isomorphic to C_n. Then, $\dim(S^2_{C_5}) = 2$ from the Section 2.1. Consider the graph $S^3_{C_5}$. It has exactly 5 induced subgraphs as $S^2_{C_5}$ the dim of those induced subgraphs is 2. Then let us consider the subset $W = \{[x]^3 : x \in \mathbb{Z}_5\}$ of $S^3_{C_5}$. Then for each induced subgraph $S^2_{C_5}$ there are exactly two pseudo resolvers in the subset $\{i_1 i_2^2 : i_2 \equiv (i_1 \pm 1) \text{mod } 5\} \subset \mathbb{V}(i_1 S^2_{C_5})$. These pseudo resolvers serve a crucial role: although they are not part of W, but due to their property (given in definition of pseudo resolvers), their relative positions with respect to the remaining vertices in W allow them to replace the resolvers that would belong to the subgraph $i_1 S^2_{C_5}$. This implies that the every pseudo resolvers of $i_2 S^2_{C_5}$ along with the vertices in $i_1 S^2_{C_5} \cap W$ are enough to resolve all the vertices of $i_1 S^2_{C_5}$, $i_1 \in \mathbb{Z}_5$. This implies W is enough to resolve all the vertices of $S^3_{C_5}$. So that, $\dim(S^3_{C_5}) \leq 5$. Suppose that $\dim(S^3_{C_5}) = 4$. By the pigeonhole principle, there exist at least one induced subgraph $S^2_{C_5}$ that do not contain any vertex from W. Then there exists at least two vertices that have the same representation, a contradiction. This implies that $\dim(S^3_{C_5}) \geq 5$. Then $\dim(S^3_{C_5}) = 5$. The following theorem gives the value of $\dim(S^h_{C_5})$ for any $h \geq 3$.

Theorem 1. *If* $h \geq 3$, *then* $\dim(S^h_{C_5}) = \frac{5(1+5^{h-3})}{2}$.

Proof. Let $W_3 = \{x^3 : x \in \mathbb{Z}_5\}$ and for $h \geq 4$,

$$W_h = \bigcup_{i=0}^{4} iW_{h-1} \setminus \{i((i-1)\text{mod } 5)^{h-1}, i((i+1)\text{mod } 5)^{h-1}\}.$$

We claim that W_h is a RS of $S^h_{C_5}$, $h \geq 4$ and proceed by induction on h. Already we know that the claim is true for $h = 3$. So it remains to verify that, W_{h+1} is RS of $S^{h+1}_{C_5}$, $h \geq 4$. Consider any two arbitrary vertices $u = u_1u_2 \ldots u_{h+1}$ and $v = v_1v_2 \ldots v_{h+1}$ of $S^{h+1}_{C_5}$.

Case 1: $u_1 - v_1 \equiv \pm 1 \text{ mod } 5$

Suppose there exists some vertices $x \in u_1S^h_{C_5}$ and $y \in v_1S^h_{C_5}$ such that $r(x|W_{h+1} \cap u_1S^h_{C_5}) = r(y|W_{h+1} \cap u_1S^h_{C_5})$ but there exists $t \in W_{h+1} \cap ((v_1+1) \text{ mod } 5)S^h_{C_5}$ such that $d_{S^{h+1}_{C_5}}(x,t) \neq d_{S^{h+1}_{C_5}}(y,t)$ when $u_1 < v_1$ or there exists $t \in W_{h+1} \cap ((u_1+1) \text{ mod } 5)S^h_{C_5}$ such that $d_{S^{h+1}_{C_5}}(x,t) \neq d_{S^{h+1}_{C_5}}(y,t)$ when $u_1 > v_1$. This implies that $r(x|W_{h+1}) \neq r(y|W_{h+1})$ for all $x \in u_1S^h_{C_5}$ and $y \in v_1S^h_{C_5}$. Then u and v have distinct representation with respect to W_{h+1} when $u_1 - v_1 \equiv \pm 1(\text{mod } 5)$.

Case 2: $u_1 - v_1 \not\equiv \pm 1(\text{mod } 5)$

In this case, we have $d_{S^{h+1}_{C_5}}(u,t) < d_{S^{h+1}_{C_5}}(v,t)$ for all $t \in W_{h+1} \cap u_1S^h_{C_5}$. Thus, u and v have distinct representation with respect to W_{h+1} when $u_1 < v_1$ and $u_1 - v_1 \not\equiv 1(\text{mod } 5)$.

Case 3: $u_1 = v_1$

We may assume that $u_1 = v_1 = 0$. By induction and the structure of $S^{h+1}_{C_5}$, the set $0W_h$ resolves all the vertices in $V(0S^h_{C_5})$. By the construction, $01^h, 04^h \in 0W_h$ but these vertices becomes pseudo resolvers for $0S^h_{C_5}$ as for every $x \in 0S^h_{C_5}$, we have $d_{S^{h+1}_{C_5}}(x,t) = d_{S^{h+1}_{C_5}}(x,01^h) + d_{S^{h+1}_{C_5}}(01^h,t)$, for $t \in W_{h+1} \cap 2S^h_{C_5}$ and $d_{S^{h+1}_{C_5}}(x,t) = d_{S^{h+1}_{C_5}}(x,04^h) + d_{S^{h+1}_{C_5}}(04^h,t)$, for $t \in W_{h+1} \cap 3S^h_{C_5}$. This implies that $r(u|W^{h+1}) \neq r(v|W^{h+1})$ for $u \neq v$ and $u_1 = v_1$. Thus, we conclude that W_{h+1} is a RS of $S^{h+1}_{C_5}$. For $h \geq 4$, $|W_h| = 5(|W_{h-1}| - 2)$, where $|W_3| = 5$. Solving this recurrence relation, we get $|W_h| = \frac{5(1+5^{h-3})}{2}$ for $h \geq 3$. Thus $\dim(S^h_{C_5}) \leq \frac{5(1+5^{h-3})}{2}$.

To prove that $\dim(S^h_{C_5}) \geq \frac{5(1+5^{h-3})}{2}$, let us assume that there exist a resolving set W, such that $|W| = \frac{5(1+5^{h-3})}{2} - 1$. Then by pigeonhole principle, there exist at least one induced subgraph $i_1i_2 \ldots i_{h-2}S^2_{C_5}$ with only two pseudo resolvers $u, v \in i_1i_2 \ldots i_{h-2}S^2_{C_5}$ such that $u_h = u_{h-1} = u_{h-2} \pm 1 \pmod 5$ and $\mathbb{V}(i_1i_2 \ldots i_{h-2}S^2_{C_5}) \cap W = \emptyset$. Then we have either

$$r(i_1 \ldots i_{h-2}(i_{h-2}-1)(i_{h-2}+2)|W) = r(i_1 \ldots i_{h-2}(i_{h-2}-2)(i_{h-2}-1)|W) \text{ or}$$

$$r(i_1 \ldots i_{h-2}(i_{h-2}+1)(i_{h-2}-2)|W) = r(i_1 \ldots i_{h-2}(i_{h-2}+2)(i_{h-2}+1)|W),$$

a contradiction. This leads to $\dim(S^h_{C_5}) \geq \frac{5(1+5^{h-3})}{2}$. Hence, $\dim(S^h_{C_5}) = \frac{5(1+5^{h-3})}{2}$ for $h \geq 3$.

Lemma 1. *Let* $R_3 = \{x((x+2)\text{mod } 5)^2 : x \in \mathbb{Z}_5\}$. *For* $h \geq 4$, $R_h = \bigcup_{i=0}^{4} iR_{h-1} \setminus \{i((i+1)\text{mod } 5)^{h-2}((i+3)\text{mod } 5), i((i-1)\text{mod } 5)^{h-2}((i-3)\text{mod } 5)\}$ *is a RS for* $S^h_{C_5}$.

Proof. The case $h = 3$ can be verified directly. Assume, as the induction hypothesis, that the claim holds for $h \geq 4$. We proceed to verify the claim for $h+1$. Let $p = p_1p_2 \dots p_{h+1}$ and $q = q_1q_2 \dots q_{h+1}$ be two arbitrary vertices of $S^{h+1}_{C_5}$.

Case 1: $p_1 - q_1 \equiv \pm 1 \text{ mod } 5$

Suppose there exists some vertices $x \in p_1S^h_{C_5}$ and $y \in q_1S^h_{C_5}$ such that $r(x|R_{h+1} \cap p_1S^h_{C_5}) = r(y|R_{h+1} \cap p_1S^h_{C_5})$ but there exists $t \in R_{h+1} \cap ((q_1+1) \text{ mod } 5)S^h_{C_5}$ such that $d_{S^h_{C_5}}(x,t) \neq d_{S^h_{C_5}}(y,t)$ when $p_1 < q_1$ or there exists $t \in R_{h+1} \cap ((p_1+1) \text{ mod } 5)S^h_{C_5}$ such that $d_{S^h_{C_5}}(x,t) \neq d_{S^h_{C_5}}(y,t)$ when $p_1 > q_1$. This implies that $r(x|R_{h+1}) \neq r(y|R_{h+1})$ for all $x \in p_1S^h_{C_5}$ and $y \in q_1S^h_{C_5}$. Then p and q have distinct representation with respect to R_{h+1} when $p_1 - q_1 \equiv \pm 1 (\text{mod } 5)$.

Case 2: $p_1 - q_1 \not\equiv \pm 1 (\text{mod } 5)$

In this case, we have $d_{S^{h+1}_{C_5}}(p,t) < d_{S^{h+1}_{C_5}}(q,t)$ for all $t \in R_{h+1} \cap p_1S^h_{C_5}$. So p and q have distinct representation with respect to R_{h+1} when $p_1 < q_1$ and $p_1 - q_1 \not\equiv 1 (\text{mod } 5)$.

Case 3: $p_1 = q_1$

Now assume that $p_1 = q_1$. We may assume that $p_1 = q_1 = 0$. By induction and the structure of $S^{h+1}_{C_5}$, the set $0R_h$ resolves all the vertices in $V(0S^h_{C_5})$. By the construction, $01^h, 04^h \in 0R_h$ but these vertices becomes pseudo resolvers for $0S^h_{C_5}$ as for every $x \in 0S^h_{C_5}$, we have $d_{S^{h+1}_{C_5}}(x,t) = d_{S^{h+1}_{C_5}}(x, 01^h) + d_{S^{h+1}_{C_5}}(01^h, t)$, for $t \in R_{h+1} \cap 2S^h_{C_5}$ and $d_{S^{h+1}_{C_5}}(x,t) = d_{S^{h+1}_{C_5}}(x, 04^h) + d_{S^{h+1}_{C_5}}(04^h, t)$, for $t \in R_{h+1} \cap 3S^h_{C_5}$. This implies that $r(p|R^{h+1}) \neq r(q|R^{h+1})$ for $p \neq q$ and $p_1 = q_1$. Thus, we conclude that R_{h+1} is a RS of $S^{h+1}_{C_5}$.

To prove the $\dim'$ of $S^h_{C_5}$ for $h \geq 3$, we defined a MB R_h in the Lemma 1, such that $R_h \cap W_h = \emptyset$. Then $W'_h = R_h \cup W_h$ is also a RS and W'_h is obviously a FTRS for $S^h_{C_5}$, when $h \geq 3$ and the proof for W'_h to be a FTB is given in the following theorem.

Theorem 2. *For* $h \geq 3$, $\dim'(S^h_{C_5}) = 5(1+5^{h-3})$.

Proof. We know that $W'_h = R_h \cup W_h$ is a FTRS for $S^h_{C_5}$, when $h \geq 3$. This implies that $\dim'(S^h_{C_5}) \leq 5(1+5^{h-3})$, for $h \geq 3$.

Now, we need to prove that $\dim'(S^h_{C_5}) \geq 5(1+5^{h-3})$. Suppose that there exist a FTRS W', such that $|W'| = 5(1+5^{h-3}) - 1$. Then by pigeonhole principle, there exists $i_1, i_2, \dots, i_{h-2} \in \mathbb{Z}_5$, such that $|i_1i_2 \dots i_{h-2}S^2_{C_5} \cap W'| = 1$ with two pseudo resolvers, that is not sufficient to fault-tolerantly resolve all the vertices of the induced subgraph $i_1i_2 \dots i_{h-2}S^2_{C_5}$, a contradiction. So, for $h \geq 3$, $\dim'(S^2_{C_h}) \geq 5(1+5^{h-3})$, implies $\dim'(S^2_{C_h}) = 5(1+5^{h-3})$.

3.2. Edge metric dimension of generalized Sierpiński networks over cycle of length five

We can easilty say that $\dim_{\mathbb{E}}(S^1_{C_5}) = 2$ as $S^1_{C_n}$ is isomorphic to C_n. For $S^2_{C_5}$, let $W_2 = \{02, 20\}$ be the subset of $\mathbb{V}(S^2_{C_5})$. From Figure 4, we can easily verify that each edges having unique representation with respect to W_2 and $\dim_{\mathbb{E}}(S^2_{C_5})$ cannot be less than 2 as it is not isomorphic to path graph.

For $S^3_{C_5}$, let $W_3 = \{022, 133, 244, 300, 411\}$ be the subset of $V(S^3_{C_5})$. Then each edge incident to $\mathbb{V}(0S^2_{C_5})$ is resolved by the vertices in $\{022, 133, 411\}$ where the vertices 011, 044 acts as an pseudo resolver for the edges incident with $\mathbb{V}(0S^2_{C_5})$. Now, suppose that there exists two edges e_{tu} $(t, u \in \mathbb{V}(0S^2_{C_5}))$ and e_{vw} $(v \in \mathbb{V}(iS^2_{C_5})$ or $w \in \mathbb{V}(iS^2_{C_5})$, $i > 0)$, such that $d_{S^3_{C_5}}(e_{tu}, 022) = d_{S^3_{C_5}}(e_{vw}, 022)$, then we have $d_{S^3_{C_5}}(e_{tu}, i(i+2)^2) \geq d_{S^3_{C_5}}(e_{vw}, i(i+2)^2)+1$. This implies that edges incident to the vertices in $\mathbb{V}(0S^2_{C_5})$ have distinct representation in $\mathbb{E}(S^3_{C_5})$. Thus, $\dim_{\mathbb{E}}(S^3_{C_5}) \leq 5$. Due to symmetrical property, all the edges in $S^3_{C_5}$ have unique representation with respect to W_3. This implies $\dim_{\mathbb{E}}(S^3_{C_5}) \leq 5$.

Suppose that there exists an ERS U with cardinality 4. Then by pigeonhole principle, there exists $i \in \mathbb{Z}_5$, such that $\mathbb{V}(iS^2_{C_5}) \cap U = \emptyset$. Then there exists two edges e_{tu} and e_{vw} $(t = i((i+2)\text{mod } 5)^2, u = i((i+2)\text{mod } 5)((i+1)\text{mod } 5), v = i((i+1)\text{mod } 5)((i-2)\text{mod } 5)$ and $w = i((i+1)\text{mod } 5)((i-1)\text{mod } 5))$ in an induced subgraph $iS^2_{C_5}$, whose edge representations with respect to U are same, which is a contradiction and implies that $\dim(S^3_{C_5}) \geq 5$. Hence $\dim(S^3_{C_5}) = 5$.

Theorem 3. *For $h \geq 3$, $\dim_{\mathbb{E}}(S^h_{C_5}) = 5^{h-2}$.*

Proof. Let $W_3 = \{x(x+2)^2 : x \in \mathbb{Z}_5\}$ and for $h \geq 4$, let

$$W_h = \bigcup_{i=0}^{4} iW_{h-1}$$

We claim that W_h is a ERS for $S^h_{C_5}$, $h \geq 4$ and prove this by induction on h. We know that the claim is true for $h = 3$. Thus, it needs to verify that W_{h+1}, $h \geq 4$, is a ERS of $S^{h+1}_{C_5}$. For any $e_{tu}, e_{vw} \in \mathbb{E}(S^{h+1}_{C_5})$, the following cases will occur.

Case 1: $t, u, v, w \in \mathbb{V}(0S^h_{C_5})$

By the construction of $S^{h+1}_{C_5}$ and ERS W_{h+1}, we have $r(e_{tu}|W_{h+1} \cap V(0S^h_{C_5})) \neq r(e_{vw}|W_{h+1} \cap V(0S^h_{C_5}))$ for all $t, u, v, w \in \mathbb{V}(0S^h_{C_5})$. This implies that no two edges incident only with the vertices in $\mathbb{V}(0S^h_{C_5})$ have same edge representation.

Case 2: $t, u \in \mathbb{V}(0S^h_{C_5})$ and $v, w \notin \mathbb{V}(0S^h_{C_5})$

In this case, there exists atleast one vertex $x \in W_{h+1} \cup 0S^h_{C_5}$ such that $d_{S^{h+1}_{C_5}}(e_{tu}, x) < d_{S^{h+1}_{C_5}}(e_{vw}, x)$. This implies that $r(e_{tu}|W_{h+1}) \neq r(e_{vw}|W_{h+1})$ for $t, u \in \mathbb{V}(0S^h_{C_5})$ and $v, w \notin \mathbb{V}(0S^h_{C_5})$.

Case 3: $t, u, v \in \mathbb{V}(0S^h_{C_5})$ and $w \notin \mathbb{V}(0S^h_{C_5})$

In this case, there exists some edges e_{tu} such that $r(e_{tu}|W_{h+1} \cup 0S^h_{C_5}) = r(e_{vw}|W_{h+1} \cup 0S^h_{C_5})$. But, there exists some $x \in W_{h+1} \setminus \mathbb{V}(0S^{h+1}_{C_5})$, we have $d_{S^{h+1}_{C_5}}(e_{tu}, x) > d_{S^{h+1}_{C_5}}(e_{vw}, x)$.

From the above cases, we can say that every edge incident to $\mathbb{V}(0S^h_{C_5})$ have distinct representations. Due to the symmetrical structure of $S^{h+1}_{C_5}$, all the edges in $S^{h+1}_{C_5}$ have distinct edge representations. This implies that W_{h+1} is an ERS of $S^{h+1}_{C_5}$. So that for $h \geq 4$, $|W_h| = 5(|W_{h-1}|)$. Solving this recurrence relation, we get $|W_h| = 5^{h-2}$ for $h \geq 4$, which implies that $\dim_{\mathbb{E}}(S^h_{C_5}) \leq 5^{h-2}$.

Now to prove that $\dim_{\mathbb{E}}(S^h_{C_5}) \geq 5^{h-2}$, let us suppose that there exist a resolving set W with cardinality less that 5^{h-2}. Let us assume that $\dim_{\mathbb{E}}(S^h_{C_5}) = 5^{h-2} - 1$. Then by pigeonhole principle, there exist at least one induced subgraph $i_1 i_2 \dots i_{h-2} S^2_{C_5}$ $(i_1, i_2, \dots, i_{h-2} \in \mathbb{Z}_5)$ with only two pseudo resolvers $u, v \in i_1 i_2 \dots i_{h-2} S^2_{C_5}$ such that $u_h = u_{h-1} = u_{h-2} \pm 1$ and $\mathbb{V}(i_1 i_2 \dots i_{h-2} S^2_{C_5}) \cap W = \emptyset$. Then there exists two edges $e_{tu}, e_{vw} \in \mathbb{E}(S^2_{C_5})$ $(t = i_1 \dots i_{h-2}((i_{h-2}+2) \bmod 5)^2, u = i_1 \dots i_{h-2}((i_{h-2}+2) \bmod 5)((i_{h-2}+1) \bmod 5), v = i_1 \dots i_{h-2}((i_{h-2}+1) \bmod 5)((i_{h-2}-2) \bmod 5)$ and $w = i_1 \dots i_{h-2}((i_{h-2}+1) \bmod 5)((i_{h-2}-1) \bmod 5))$ have the same representation with respect to W, a contradiction. This leads to $\dim_{\mathbb{E}}(S^h_{C_5}) \geq 5^{h-2}$. Hence, $\dim_{\mathbb{E}}(S^h_{C_5}) = 5^{h-2}$.

Lemma 2. *Let $U_3 = \{033, 144, 200, 311, 422\}$. For $h \geq 4$, $U_h = \bigcup_{i=0}^{4} iU_{h-1}$ is an EMB for $S^h_{C_5}$.*

The proof of the above lemma is similar to the proof of Theorem 3, due to the reflexive property of $S^h_{C_5}$.

To prove the $\dim_{\mathbb{E}'}$ of $S^h_{C_5}$ for $h \geq 3$, we defined a EMB U_h in the Lemma 2, such that $U_h \cap W_h = \emptyset$. Then $W'_h = U_h \cup W_h$ is also an ERS and W'_h is obviously a FTERS for $S^h_{C_5}$, when $h \geq 3$ and the proof for W'_h to be a FTEB is given in the following theorem.

Theorem 4. *For $h \geq 3$, $\dim_{\mathbb{E}'}(S^h_{C_5}) = 2 \cdot 5^{h-2}$.*

Proof. Let U_3 and W_3 be the distinct ERS of $S^3_{C_5}$, such that $U_3 \cap W_3 = \emptyset$. Obviously, the set $U_3 \cup W_3$ will be the FTERS for $S^3_{C_5}$, proving that $\dim_{\mathbb{E}'}(S^3_{C_5}) \leq 10$. To prove the sufficient part, let us assume that there exist a FTERS S, such that $|S| < 10$. Let us assume that $|S| = 9$. Then, by pigeonhole principle, there exist $i \in \mathbb{Z}_5$, such that $\mathbb{V}(iS^2_{C_5}) \cap W' < 2$ with only two pseudo resolvers $\{i(i+1)^2, i(i-1)^2\}$. When removing that one vertex in $\mathbb{V}(iS^2_{C_5}) \cap S$, there exists two edges incident to the vertices in $\mathbb{V}(iS^2_{C_5})$ that have the same edge representation, which implies that any subset W' with cardinality less than 10 cannot be a FTERS for $S^3_{C_5}$. For $h \geq 4$, let

$$W'_h = W_h \cup U_h,$$

where $W_h = \bigcup_{i=0}^{4} iW_{h-1}$ and $U_h = \bigcup_{i=0}^{4} iU_{h-1}$. We know that W_h and U_h are the ERS for $S^h_{C_5}$ for any $h \geq 2$ and $W_h \cap U_h = \emptyset$. Then W'_h be the FTERS for $S^h_{C_5}$, $\forall h \geq 4$. We have $|W'_h| = 5 \cdot |W'_{h-1}|$. Solving this recurrence relation, we get $|W'_h| = 2 \cdot 5^{h-2}$. This implies that $\dim_{\mathbb{E}'}(S^h_{C_5}) \leq 2 \cdot 5^{h-2}$.

Now we need to prove that $\dim_{\mathbb{E}'}(S^h_{C_5}) \geq 2 \cdot 5^{h-1}$. Let us assume the contrary that there exist a FTERS W'_h, such that $|W'_h| = 2 \cdot 5^{h-2} - 1$. Then, there exist $i_1, i_2, \dots, i_{h-2} \in \mathbb{Z}_5$ such that $\mathbb{V}(i_1i_2\dots i_{h-2}S^2_{C_5})\cap W'_h < 2$ with two pseudo resolvers $i_1i_2\dots i_{h-2}((i_{h-2}+1)\text{mod } 5)^2$ and $i_1i_2\dots i_{h-2}((i_{h-2}-1)\text{mod } 5)^2$, where these vertices are not enough to fault-tolerantly resolve the edges incident to the vertices in $\mathbb{V}(i_1i_2\dots i_{h-2}S^2_{C_5})$, a contradiction. So, we conclude that $\dim_{\mathbb{E}'}(S^h_{C_5}) \geq 2\cdot 5^{h-2}$, $\forall h \geq 3$. Then $\dim_{\mathbb{E}'}(S^h_{C_5}) = 2\cdot 5^{h-2}$.

4. Conclusion

In this study, we analyzed dim, dim′, $\dim_{\mathbb{E}}$, and $\dim_{\mathbb{E}'}$ of generalized Sierpiński networks over C_5. Our findings reveal that the fault-tolerant (edge) metric dimension is directly proportional to the (edge) metric dimension with proportionality constant 2, for this graph family. This relationship highlights the structural consistency of these graphs and provides a fundamental insight into their resolvability and robustness in network applications. The proportionality observed in our results has practical significance in fault-tolerant network design, where ensuring efficient localization despite failures is crucial. Additionally, these findings contribute to a deeper understanding of fractal-based graph structures, reinforcing their applicability in areas such as routing, surveillance, and resilient communication systems. Future work can explore whether similar proportional relationships hold for other families of generalized Sierpiński networks or different base structures.

References

[1] M.C. Chen. A design methodology for synthesizing parallel algorithms and architectueres. *Journal of Parallel and Distributed Computing*, 3(4):461–491, 1986.

[2] T.Y. Chen. Graph traversal techniques and the maximum flow problem in distributed computation. *IEEE Transactions on Software Engineering*, SE-9(4):504–512, 1983.

[3] R. Shang. Efficient task scheduling for large-scale graph data processing in cloud computing: A particle swarm optimization approach. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 122:135–148, 2024.

[4] H. Wang, W. Guo, and M. Zhang. Performance optimization of heterogenious computing for large-scale dynamic graph data. *The Journal of Supercomputing*, 81(41), 2025.

[5] H. Raza, S. Hayat, and X.-F. Pan. On the fault-tolerant metric dimension of certain interconnection networks. *Journal of Applied Mathematics and Computing*, 60:517–535, 2019.

[6] S. Lagraa, M. Husák, H. Seba, S. Vuppala, R. State, and M. Ouedraogo. A review on graph-based approaches for network security monitoring and botnet detection. *International Journal of Information Security*, 23:119–140, 2024.

[7] K.T. Min and N. Jeyanthi. Routing performance in wireless sensor networks: Determining shortest path algorithms effectiveness. *International Journal of Computer Networks & Communications*, 16(6), 2024.

[8] Y. Qi, Y. Dong, Z. Zhang, and Z. Zhang. Hitting times for random walks on Sierpiński networks and hierarchical graphs. *The Computer Journal*, 63(1):1385–1396, 2020.

[9] M. Arulperumjothi, S. Klavžar, and S. Prabhu. Redefining fractal cubic networks and determining their metric dimension and fault-tolerant metric dimension. *Applied Mathematics and Computation*, 452:128037, 2023.

[10] I. Javid, H. Benish, M. Imran, A. Khan, and Z. Ullah. On some bounds of the topological indices of generalized Sierpiński and extended Sierpiński networks. *Journal of Inequalities and Applications*, 2019(37), 2019.

[11] K. Liu and N. Abu-Ghazaleh. Virtual coordinates with backtracking for void traversal in geographic routing. In *ADHOC-NOW 2006, Lecture Notes in Computer Science*, volume 4104, pages 45–59, 2006.

[12] B. Hu, Z. Cao, and M. Zhou. Energy-minimized scheduling of real-time parallel workflows on heterogeneous distributed computing system. *IEEE Transactions on Services Computing*, 15(5):2766–2779, 2022.

[13] E. Fernández and H. F. Jelinek. Use of fractal theory in neuroscience: Methods, advantages, and potential problems. *Methods*, 24(4):309–321, 2001.

[14] O. López-Ortega and S.I. López-Popa. Fractals, fuzzy logic and expert systems to assist in the construction of musical pieces. *Expert Systems with Applications*, 39(15):11911–11923, 2012.

[15] M. Obert, P. Pfeifer, and M. Sernetz. Microbial growth patterns described by fractal geometry. *Journal of Bacteriology*, 172(3):1180–1185, 1990.

[16] Y. Zhao, K. Yoshigoe, J. Bian, M. Xie, and Y. Feng. A distributed graph-parallel computing system with lightweight communication overhead. *IEEE Transactions on Big Data*, 2(3):204–218, 2016.

[17] G.D. Vecchia and C. Sanges. A recursively scalable network vlsi implementation. *Future Generation Computer Systems*, 4(5):235–243, 1988.

[18] S. Klavžar and U. Milutinović. Graphs $s(n, k)$ and a variant of the tower of hanoi problem. *Czechoslovak Mathematical Journal*, 47(1):95–104, 1997.

[19] A.M. Hinz, S. Klavžar, and S.S. Zemljič. A survey and classification of Sierpiński-type graphs. *Discrete Applied Mathematics*, 217:565–600, 2017.

[20] J.A. Rodríguez-Velázquez, E.D. Rodríquez-Bazán, and A. Estrada-Moreno. On generalized Sierpiński networks. *Discussiones Mathematicae Graph Theory*, 37(3):547–560, 2017.

[21] K. Balakrishnan, M. Changat, A.M. Hinz, and D.S. Lekha. The median of Sierpiński networks. *Discrete Applied Mathematics*, 319:159–170, 2022.

[22] A.M. Hinz and D. Parisse. The average eccentricity of Sierpiński networks. *Graphs and Combinatorics*, 28(5):671–686, 2012.

[23] S. Klavžar and S.S. Zemljič. Connectivity and some other properties of generalized Sierpiński networks. *Applicable Analysis and Discrete Mathematics*, 12(2):401–412, 2018.

[24] M. Imran, Sabeel e Hafi, W. Gao, and M.R. Farahani. On the topological properties of Sierpiński networks. *Chaos, Solitons & Fractals*, 98:199–204, 2017.

[25] A.M. Hinz and C.H. auf der Heide. An efficient algorithm to determine all shortest

paths in Sierpiński networks. *Discrete Applied Mathematics*, 177:111–120, 2014.

[26] M.A. Henning, S. Klavžar, and I.G. Yero. Resolvability and convexity properties in the Sierpiński product of graphs. *Mediterranean Journal of Mathematics*, 21(3), 2024.

[27] S. Prabhu, T.J. Janany, and P. Manuel. An efficient way to represent the processors and their connections in omega networks. *Ain Shams Engineering Journal*, 16(3):103287, 2025.

[28] S. Prabhu, D. Jeba, P. Manuel, and A. Davoodi. Metric dimension of villarceau grids. *arXiv preprint*, 2024.

[29] B. Rajan, I. Rajasingh, J.A. Cynthia, and P. Manuel. Metric dimension of directed graphs. *International Journal of Computer Mathematics*, 91(7):1397–1406, 2014.

[30] P. Manuel and I. Rajasingh. Minimum metric dimension of silicate networks. *Ars Combinatoria*, 98:501–510, 2011.

[31] P. Manuel, B. Rajan, I. Rajasingh, and M.M. Chris. Landmarks in binary tree derived architectures. *Ars Combinatoria*, 99:473–486, 2011.

[32] P. Manuel, B. Rajan, I. Rajasingh, and M.C. Monica. Landmarks in torus networks. *Journal of Discrete Mathematical Sciences and Cryptography*, 9(2):263–271, 2006.

[33] S. Prabhu, T.J. Janany, and S. Klavžar. Metric dimensions of generalized Sierpiński networks over squares. *Applied Mathematics and Computation*, 505:129528, 2025.

[34] A. Estrada-Moreno and J.A. Rodríguez-Velázquez. On the general randić index of polymeric networks modelled by generalized Sierpiński networks. *Discrete Applied Mathematics*, 263:140–151, 2019.

[35] F. Ramezani, E.D. Rodríquex-Bazán, and J.A. Rodríguez-Velázquez. On the roman domination number of generalized Sierpiński networks. *Filomat*, 31(20):6515–6528, 2017.

[36] E. Estaji and J.A. Rodríguez-Velázquez. The strong metric dimension of generalized Sierpiński networks with pendant vertices. *Ars Mathematica Contemporanea*, 12:127–134, 2017.

[37] S. Khuller, J.B. Raghavachari, and A. Rosenfeld. Landmarks in graphs. *Discrete Applied Mathematics*, 70(3):217–229, 1996.

[38] G. Chartrand, L. Eroh, M.A. Johnson, and O.R. Oellermann. Resolvability in graphs and the metric dimension of a graph. *Discrete Applied Mathematics*, 105:99–113, 2000.

[39] G. Chartrand and P. Zhang. The theory and applications of resolvability in graphs: a survey. *Congressus Numerantium*, 160:47–68, 2003.

[40] F. Harary and R.A. Melter. On the metric dimension of a graph. *Ars Combinatoria*, 2:191–195, 1976.

[41] P.J. Slater. Leaves of trees. *Congressus Numerantium*, 14:549–559, 1975.

[42] C. Hernando, M. Mora, I.M. Pelayo, C. Seara, and D.R. Wood. Extremal graph theory for metric dimension and diameter. *Electronic Journal of Combinatorics*, 17:R30, 2010.

[43] P. Manuel, M. I. Abd-El-Barr, I. Rajasingh, and B. Rajan. An efficient representation of benes networks and its applications. *Journal of Discrete Algorithms*, 6(1):11–19, 2008.

[44] P. Manuel, B. Rajan, I. Rajasingh, and M.C. Monica. On minimum metric dimension of honeycomb networks. *Journal of Discrete Algorithms*, 6:20–27, 2008.

[45] M. Basak, L. Saha, G.K. Das, and K. Tiwary. Fault-tolerant metric dimension of circulant graphs $c(1, 2, 3)$. *Theoretical Computer Science*, 817:159–170, 2020.

[46] S. Kumar, S. Sharma, S.K. Sharma, and V.K. Bhat. On metric dimension of generalized subdivision prism graph. *Ain Shams Engineering Journal*, 16(8):103452, 2025.

[47] S. Prabhu, T. Flora, and M. Arulperumjothi. On independent resolving number of $TiO_2[m, n]$ nanotubes. *Journal of Intelligent & Fuzzy Systems*, 35(6):6421–6425, 2018.

[48] S. Prabhu, D.S.R. Jeba, M. Arulperumjothi, and S. Klavžar. Metric dimension of irregular convex triangular networks. *AKCE International Journal of Graphs and Combinatorics*, 21(3):225–231, 2023.

[49] S. Prabhu, V. Manimozhi, M. Arulperumjothi, and S. Klavžar. Twin vertices in fault-tolerant metric sets and fault-tolerant metric dimension of multistage interconnection networks. *Applied Mathematics and Computation*, 420:126897, 2022.

[50] S. Klavžar and S.S. Zemljič. On distances in Sierpiński networks: Almost-extreme vertices and metric dimension. *Applicable Analysis and Discrete Mathematics*, 7:72–82, 2013.

[51] R.C. Tillquist, R.M. Frongillo, and M.E. Lladser. Getting the lay of the land in discrete space: A survey of metric dimension and its application. *SIAM Review*, 65(4):919–962, 2023.

[52] A. Kelenc, N. Tratnik, and I.G. Yero. Uniquely identifying the edges of a graph: The edge metric dimension. *Discrete Applied Mathematics*, 251:204–220, 2018.

[53] S. Klavžar and D. Kuziak. Nonlocal metric dimension of graphs. *Bulletin of the Malaysian Mathematical Sciences Society*, 46:66, 2023.

[54] R.Umilasari, L. Susilowati, and S. Prabhu. On the dominant local metric dimension of corona product graphs. *IAENG International Journal of Applied Mathematics*, 52(4):38, 2022.

[55] R.Umilasari, L. Susilowati, Slamin, S. Prabhu, and O.J. Fadekemi. The local resolving dominating set of comb product graphs. *International Journal of Computer Science*, 51:115–120, 2024.

[56] V.J.A. Cynthia, M. Ramya, and S. Prabhu. Local metric dimension of certain classes of circulant networks. *Journal of Advanced Computational Intelligence and Intelligent Informatics*, 27(4):554–560, 2023.

[57] C. Hernando, M. Mora, P.J. Slater, and D.R. Wood. Fault-tolerant metric dimension of graphs. *Convexity in Discrete Structures*, 5:81–85, 2008.

[58] S. Prabhu, V. Manimozhi, A. Davoodi, and J.L.G. Guirao. Fault-tolerant basis of generalized fat trees and perfect binary tree. *The Journal of Supercomputing*, 80:15783–15798, 2024.

[59] Y. Zhang and S. Gao. On the edge metric dimension of convex polytopes and its related graphs. *Journal of Combinatorial Optimization*, 39:334–350, 2020.

[60] D.G.L. Wang, M.M.Y. Wang, and S. Zhang. Determining the edge metric dimension of the generalized petersen graph $p(n, 3)$. *Journal of Combinatorial Optimization*,

43:460–496, 2022.

[61] S. Prabhu and T.J. Janany. Edge metric dimension of silicate networks. *Communications in Combinatorics and Optimization*, 2024.

[62] N. Zublirina. Asymptotic behaviour of the edge metric dimension of the random graph. *Discussiones Mathematicae Graph Theory*, 41(2):589–599, 2021.

[63] C. Wei, M. Salman, S. Shalhzaib, M.U. Rehman, and J. Fang. Classes of planar graphs with constant edge metric dimension. *Complexity*, 2021:5599274, 2021.

[64] N. Zubrilina. On the edge dimension of a graph. *Discrete Mathematics*, 341:2083–2088, 2018.

[65] E. Zhu, A. Taranenko, Z. Shao, and J. Xu. On graphs with the maximum edge metric dimension. *Discrete Applied Mathematics*, 257:317–324, 2019.

[66] A. Kelenc, A.T.M. Toshi, R. Škrekovski, and I.G. Yero. On metric dimension of hypercubes. *Ars Mathematica Contemporanea*, 23(2):P2.08, 2022.

[67] I. Peterin and I.G. Yero. Edge metric dimension of some graph operations. *Bulletin of the Malaysian Mathematical Sciences Society*, 43:2465–2477, 2020.

[68] S. Klavžar and M. Tavakoli. Edge metric dimension via hierarchical product and integer linear programming. *Optimization Letters*, 15:1993–2003, 2021.

[69] M. Knor, S. Majstorović, A.T.M. Toshi, R. Škrekovski, and I.G. Yero. Graphs with the edge metric dimension smaller than the metric dimension. *Applied Mathematics and Computation*, 401:126076, 2021.

[70] M. Knor, R. Škrekovski, and I.G. Yero. A note on the metric and edge metric dimensions of 2-connected graphs. *Discrete Applied Mathematics*, 319:454–460, 2022.