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Completeness and Compactness on Hesitant Fuzzy Normed Linear Spaces

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Abstract. In this work, we examine the properties of completeness and compactness on hesitant fuzzy normed linear space, also address the same properties on intuitionistic hesitant fuzzy normed linear space in finite dimension using definitions, lemmas, and theorems. We also investigate the continuity of underlying t-norms and co-t-norm on finite-dimensional intuitionistic hesitant fuzzy normed linear space.

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1. Introduction

In 1965, Zadeh [1] invented fuzzy set theory. The search for fuzzy equivalents of classical theories has been intense since Zadeh's groundbreaking work. Additionally, other areas, fuzzy metric spaces along with fuzzy normed linear spaces have seen advancements, In [[2],[3]] Two kinds of fuzzy bounded linear operators—strong and weak—are developed in this study, along with the concept regarding boundedness of a linear operator out of one fuzzy normed linear space to another fuzzy normed linear space. A relationship between fuzzy boundedness and fuzzy continuity is examined. The concepts of fuzzy dual spaces and fuzzy bounded linear functionals are defined, establish Uniform Boundedness Principle, Closed Graph , Open Mapping and the Hahn-Banach Theorem, In [4] a fuzzy normed linear space, the terms "strongly and weakly fuzzy convergent sequence," are defined over this study. Fixed point theorems for fuzzy non-expansive mappings are established, along with the notions of uniformly convex fuzzy normed linear space, fuzzy normal structure,

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and fuzzy non-expansive mapping.
In [5] an introduction to fuzzy normed linear space is given. It has been demonstrated that fuzzy norms are equivalent up to fuzzy equivalency in a finite dimensional fuzzy normed linear space. It is demonstrated that fuzzy subspaces of a fuzzy normed linear space among finite dimensions must be full fuzzy normed linear spaces, [6] present the idea of a fuzzy metric space in this study. In a fuzzy metric space, The separation among two points, is a normal, convex, upper semicontinuous, non-negative fuzzy number. A few fixed point theorems are proved and the properties of fuzzy metric spaces are examined.
In [7] a few fuzzy topological vector space properties are examined. Additionally, for a fuzzy linear topology, necessary and sufficient criteria are shown for a family of fuzzy sets in vector space E to be the family of all neighborhoods of zero.As a generalized fuzzy set, Atanassov [8] developed the idea of intuitionistic fuzzy sets. Originating the concept of intuitionistic fuzzy metric space was J.H. Park [9] and researched a few fundamental characteristics. However, a significant addition to intuitionistic fuzzy topological spaces is made by Saadati Park [9]. They have also examined certain fundamental characteristics in intuitionistic fuzzy normed linear spaces and introduced the idea of such spaces. Many studies have been conducted on intuitive fuzzy sets, including those by T.K. Mandal and S.K. Samanta [10],[11].[12] N. Thillaigovindan et al. Vijayabalaji et al. [13] recently obtained some results and presented the idea of intuitionistic fuzzy n-normed linear space.
Bag et al., [10], introduced the concept of a fuzzy normed linear space, which T.K. Samanta et al. [14] examined. They stated an intuitionistic fuzzy normed linear space through a general context (using the t-norm $*$ along with the t-co-norm $\diamond$ correspondingly).
In finite dimensional intuitionistic fuzzy normed linear space, they mostly examined various outcomes. However, their findings rely on the intuitionistic fuzzy norm's decomposition theorem as part of a family of crisp norm pairings, because they have added requirements on the t-norm and t-conorm as $a_1 * a_1 = a_1$ and $a_1 \diamond a_1 = a_1, \forall a_1 \in 0, 1$, leading to $* = min$ and $\diamond = max$. The abstraction of the t-norm and t-conorm is thus practically lost. However, certain of the requirements involving the functions $N(x_1, t_1)$ and $M(x_1, t_1)$ in the definition taken into consideration , the relation $N(x_1, t_1) + M(x_1, t_1) \leq 1$. The following describes how the paper is structured: Section 2 describes, some preliminary the outcomes were presented , notion of hesitant fuzzy and intuitionistic hesitant fuzzy normed linear spaces. Section 3 and section 4 demonstrates some fundamental findings about completeness and compactness are demonstrated in finite dimension hesitant fuzzy normed linear space along with intuitionistic hesitant fuzzy normed linear space.

2. Preliminaries

This section provides pre-existing definitions of fuzzy sets, including some fundamental notions.

Definition 1. *[15]*
$* : [0,1] \times [0,1] \rightarrow [0,1]$ *in binary exists t−norm in case it meets these requirements listed below:*

[*a*] $*$ *is associative and commutative.*
[*b*] $a_{11} * 1 = a_{11}, \forall\ a_{11} \in [0,1]$.
[*c*] $a_{11}*b_{11} \leq c_{11}*d_{11}$ *whenever* $a_{11} \leq c_{11}$ *and* $b_{11} \leq d_{11}$ *for each* $a_{11}, b_{11}, c_{11}, d_{11} \in [0,1]$.
It can be described as the continuous $t-$*norm if* $*$ *is continuous.*

Definition 2. *[15]*
An operation that is binary $\diamond : [0,1] \times [0,1] \to [0,1]$ *is a* $t-$*co-norm when it meets the requirements listed below:*
[*a*] $\diamond$ *is associative and commutative.*
[*b*] $a_{11} \diamond 1 = a_{11}, \forall\ a_{11} \in [0,1]$.
[*c*] $a_{11}\diamond b_{11} \leq c_{11}\diamond d_{11}$ *whenever* $a_{11} \leq c_{11}$ *and* $b_{11} \leq d_{11}$ *for each* $a_{11}, b_{11}, c_{11}, d_{11} \in [0,1]$.
When $\diamond$ *is continuous, it has to be referred as continuous* $t-$*co-norm.*

Definition 3. *[16] Consider the nonempty set* V *combined with algebraic operations* $+, \centerdot$ *fulfil,* $(V,+)$ *is a group and regarding scalar multiplication,*

(i) $k_1(a_1 + b_1) = k_1 a_1 + k_1 b_1$

(ii) $(k_1 + l_1)a_1 = k_1 a_1 +_1 k_1 b$

(iii) $k_1(l a_1) = (k_1 l_1) a_1$

(iv) $1.a_1 = a_1, \forall a_1, b_1, c_1 \in V$ *and* $k_1, l_1 \in \mathrm{R}^*$

The triple $(V,+,\centerdot)$ *is referred to as vector space.*

Definition 4. *[17] The fuzzy set* N *in* $\mathrm{X} \times [0,\infty)$ *over a linear space* X *constitutes a fuzzy norm over* X *when it meets this criteria,*

(i) (FN1) $\mathrm{N}(x_1, 0) = 0, \forall x_1 \in X$

(ii) (FN2) $\mathrm{N}(x_1, t) = 1, \forall t > 0\ iff\ \ x_1 = 0.$

(iii) (FN3)$\mathrm{N}(\lambda x_1, t) = N\left(x_1, \frac{t}{|\lambda|}\right), \forall\ x_1 \in \mathsf{X}, \forall\ \mathsf{t} > 0, \forall\ \lambda \in K^*$,(K^* *is non negative real numbers)*

(iv) (FN4)$N(x_1 + y_1, t + s) \geq N(x_1, t) * N(y_1, s),\ \forall\ \ x_1, y_1 \in \mathsf{X}, \forall\ \ \mathsf{t}, \mathsf{s} > 0.$

(v) (FN5) $\forall x_1 \in \mathsf{X},\ \mathsf{N}(x_1, \bullet)$ *is left continuous along with* $\lim_{t\to\infty} \mathsf{N}(x_1, t) = 1$.

Thus,$(\mathsf{X}, \mathsf{N}, *)$ *known as fuzzy normed linear space.*

Definition 5. *[16]* ***Hesitant Fuzzy Normed Linear Space :*** *Given a vector space* V *through this field* $F, *$ *consists of t-norm, together with* $\mathcal{H} : \mathrm{V} \times [0,\infty) \to P[0,1]$ *exists as a hesitant fuzzy set with the subsequent characteristics,* $t_1, t_2 > 0$ *and* $\forall\ \mathrm{x}, \mathrm{y} \in V$

(i) $\mathcal{H}(x, 0) = \emptyset^*$ *(Empty set),* $\forall\ \mathrm{x} \in V$.

(ii) $\mathcal{H}(x, t) = U^*$ *(Full set),* $\forall t > 0$ *iff* $x = 0$.

(iii) $\mathcal{H}(\mu x, t) = \mathcal{H}(x, \frac{t}{|\mu|}), \forall x \in V, \forall t \geq 0, \forall \mu \in R^*$.

(iv) $\mathcal{H}(\mathrm{x}+\mathrm{y}, \mathrm{t}_1+\mathrm{t}_2) \supseteq \mathcal{H}(\mathrm{x}, \mathrm{t}_1) \cap \mathcal{H}(\mathrm{y}, \mathrm{t}_2), \forall \mathrm{x}, \mathrm{y} \in V, \forall\ \mathrm{t}_1, \mathrm{t}_2 \geq 0$.

(v) $\lim_{t\to\infty} \mathcal{H}(x, t) = U^*$.

Definition 6. *[16] A sequence* v_n *with a hesitant fuzzy normed linear space* $(\mathrm{V}, \mathcal{H})$*, known as converges towards* $v \in V$ *suppose every* $S^* \neq \emptyset^*$ *and* $t > 0$*,we could locate* N *using* $\mathcal{H}(v_n - v, t) \supset U^* \backslash S^*\ \forall\ n \geq \mathrm{N}$. *(or)* $\lim_{n\to\infty} \mathcal{H}(v_n - v, t) = U^*$.

Definition 7. *[16] A sequence* v_n *in a hesitant fuzzy normed space* $(V, \mathcal{H}, *)$ *is said to be a cauchy sequence if for all* $\emptyset^* \subset S^* \subset U^*, t > 0$ *there is number* N *with* $\mathcal{H}(v_m - v_n, t) \supset U^* \backslash S^*$ *for all* $m, n \geq \mathrm{N}$. *(or)* $\lim_{n\to\infty} \mathcal{H}(v_n - v, t) = U^*$.

3. Finite Dimensional Hesitant Fuzzy Normed Linear Space

Finite dimensional hesitant fuzzy normed linear spaces are defined, their completeness, and their compactness are examined in this section.

Definition 8. *Let* $(V, \mathcal{H}, *)$ *be a hesitant fuzzy normed linear space. If dimension of a vector space* V *is finite then it is called finite dimensional hesitant fuzzy normed linear space.*

Definition 9. *Given a hesitant fuzzy normed space* $(V, \mathcal{H}, *)$ *as well as a subset* W *of* V*,* $\overline{W}$ *signifies the closure of* W*, which corresponds to* $\bigcap\{\mathsf{W} \subseteq B : B \text{ is closed in } V\ \}$.

Lemma 1. *Consider* $(V, \mathcal{H}, *)$ *to become hesitant fuzzy normed space, along with* W *exists as a subset of* V. *A sequence* $\{w_n\}$ *within* W *converges towards* y*,if and only if* $y \in \overline{w}$.

Definition 10. *The completeness of a hesitant fuzzy normed space* $(V, \mathcal{H}, *)$ *is defined as the convergence of all cauchy sequences in* V *toward point in* V*.*

Definition 11. *The hesitant fuzzy normed space* $(V, \mathcal{H}, *)$ *is considered complete if all of the cauchy sequences in* V *converge toward point in* V*.*

Definition 12. *The sequence* $\{a_n\}_{n=1}^{\infty}$ *in* $\mathbb{R}$ *is said to be hesitant fuzzy bounded if there exists* $S^* \in P[0,1]$ *such that* $\mathcal{H}_{\mathbb{R}}(a_n, t) \supset U^* \backslash S^*, \forall t > 0$.

Theorem 1. *Let* $(\mathrm{V}, \mathcal{H}_V, *)$ *,*$(\mathbb{R}, \mathcal{H}_{\mathbb{R}}, *)$ *be two hesitant fuzzy normed linear spaces along with* $\{v_1, v_2 \dots v_n\}$ *be linearly independent set in* $(V, \mathcal{H}_V, *)$*.Then there is* $\emptyset^* \subset S_3^* \subset U^*$ *such that* $H_V[\ \beta_1 v_1 + \cdots + \beta_n v_n\ , t\ ] \subseteq S_3^* * \mathcal{H}_{\mathbb{R}}(\beta_j, t)$ *for some* $1 \leq j \leq n$.

Proof. If this isn't the case, we may discover a sequence $\{v_m\}$ in V where $v_m = \beta_{1m} v_1 +, \dots, +\beta_{1n} v_n$ so that $\lim_{n\to\infty} \mathcal{H}_V(v_m, t) = U^*$.
For every fixed j, we now have a sequence $\beta_{jm} = \{\beta_{j1} \dots, \beta_{jm}, \dots,)$ represents hesitant fuzzy bounded, because if the sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ is hesitant fuzzy approaches the limit $'a'$ then it is hesitant fuzzy bounded. Considering $\emptyset^* \subseteq \mathcal{H}_{\mathbb{R}}(\beta_{jm}, t) \subseteq U^*$, so $\{\beta_{jm}\}$

has a convergent subsequence. For every $1 \leq j \leq n.$, let β_j represent the limit of the subsequence $\{\beta_{jm}\}$. An analogous subsequence of scalars β_{jm} converges for β_j for every $1 \leq j \leq n.$, where $\{v_{jm}\}$ represents the corresponding subsequence of $\{v_m\}$. Now, put $v = \sum_{j=1}^{\infty} \beta_j v_j$ then $\{v_m\}$ has a subsequence $\{v_{j_m}\}$ converges to v, since $\{v_1, v_2, \ldots, v_n\}$ is linearly independen set so $v \neq 0$. Now $\{v_{j_m}\} \to v \implies$ The fuzzy continuity of the hesitant fuzzy norm is $\mathcal{H}_v(v_{j_m}, v)$. But ,$\mathcal{H}(v_m, 1) \to U^*$ by our assumption and $\{v_{j_m}\}$ is a subsequence of $\{v_m\}$. Thus $\mathcal{H}_v(v_{j_m}, v) \to U^*$. Hence $\mathcal{H}(v,t) = U^*$. So, $v = 0$. This contradicts $v \neq 0$.

Theorem 2. *Consider a hesitant fuzzy normed space $(V, \mathcal{H}_V, *)$. W is complete if it is a finite-dimensional subspace of V.*

Proof. Assume that the sequence $\{v_m\}$ is Cauchy in W. Assume *dim* W $= n$ and $B = \{w_1, w_2, \ldots, w_n\}$ become any basis to W. After that, every v_m is represented in a distinctive way as $v_m = \gamma_{1m} w_1 +, \ldots, + \gamma_{nm} w_n$. Given that the sequence $\{v_m\}$ is Cauchy, for any $\mathsf{m}, \mathsf{n} \geq N$. Now by theorem 1, we possess some $\emptyset^* \subset S_3^* \subset U^*$ such that
$U^* \backslash S_3^* \subset \mathcal{H}_v(v_m - v_n, t) = \mathcal{H}_v \left(\sum_{j=1}^{n} (\gamma_{jm} - \gamma_{jn}) w_j, t \,] \right) \subseteq S_3^* \, \cap \, \mathcal{H}_R(\gamma_{jm} - \gamma_{jn}, t)$
$\implies \mathcal{H}_R(\gamma_{jm} - \gamma_{jn}, t) \supset [U^* \cap S_5^*] \backslash S_3^*$.
The following demonstrates that $(\gamma_{jm}) = (\gamma_{j1}, \gamma_{j2}, \ldots,)$ is a Cauchy sequence in $\mathbb{R}$ or $\mathbb{C}$, thus $\gamma_{jm} \to \gamma_j$ for each $1 \leq j \leq n$.
Let $\mathcal{A} = \sum_{j=1} \gamma_j w_j$,clearly $\mathcal{A} \in W$. Also now for all $m > n$,
$\mathcal{H}_v(v_m - v, t) = \mathcal{H}_v \left(\sum_{j=1}^{n} (\gamma_{jm} - \gamma_{jn}) w_j, t \right)$
$\supseteq \mathcal{H}_{\mathrm{v}} \left(\mathrm{w}_1, \frac{t}{n|\gamma_{1m} - \gamma_1|} \right) \, \cap \, \mathcal{H}_v \left(\mathrm{w}_1, \frac{t}{n|\gamma_{2m} - \gamma_2|} \right) \, \cap, \ldots, \cap \, \mathcal{H}_v \left(\mathrm{w}_1, \frac{t}{n|\gamma_{nm} - \gamma_n|} \right)$
$\mathcal{H}_v(v_m - v, t) \supseteq (\mathrm{U}^* \backslash \mathrm{S}_{31}^*) \, \cap \, (\mathrm{U}^* \backslash \mathrm{S}_{32}^*) \cap, \ldots, (\mathrm{U}^* \backslash \mathrm{S}_{3n}^*)$.
Where $\mathcal{H}_{\mathrm{v}} \left(w_j, \frac{t}{n|\gamma_{jm} - \gamma_j|} \right) = (U^* \backslash S_{3j}^*)$for some $\emptyset^* \subset (\mathrm{U}^* \backslash S_{3j}^*) \subset \mathrm{U}^*, j = 1, 2, \ldots, n$.
Let $(\mathrm{U}^* \backslash \mathrm{S}_{31}^*) \cap (\mathrm{U}^* \backslash \mathrm{S}_{32}^*) \cap, \ldots, \cap \, (\mathrm{U}^* \backslash \mathrm{S}_{3n}^*) \supset (\mathrm{U}^* \backslash \mathrm{S}_4^*)$. So,$\mathcal{H}_v(v_m - v, t) \supset (\mathrm{U}^* \backslash \mathrm{S}_4^*)$ where $\mathrm{S}_4^* \in \mathcal{P}[0,1], \forall m > N$. Hence $v_m \to v$.

Theorem 3. *Fuzzy normed space $(V, \mathcal{H}, *)$ is compact if and only if each $\{v_n\}$ at* V *includes $\{v_{n_k}\}$ using $\{v_{n_k}\} \to v$.*

4. Finite Dimensional Intuitionistic hesitant Fuzzy Normed Linear Spaces

The completeness along with compactness features regarding intuitionistic hesitant fuzzy normed linear spaces with finite dimensions are examined in this section.

Definition 13. ***Intuitionistic Fuzzy Norm :*** *[18]*
V is a linear space throughout the field $\mathbb{F}$. Suppose $*$ constitute a continuous t-norm as well as $\diamond$ represent a continuous t-co-norm over V, an intuitionistic fuzzy norm is an object regarding the following form
$\{((\mathrm{x}_{11}, \mathrm{t}_{11}), N_{\mathrm{I}}(\mathrm{x}_{11}, \mathrm{t}_{11}), M_{\mathrm{I}}(\mathrm{x}_{11}, \mathrm{t}_{11})) : (\mathrm{x}_{11}, \mathrm{t}_{11}) \in \mathrm{V} \times \mathbb{R}^+\}$,*wherein $N_{\mathrm{I}}, M_{\mathrm{I}}$ have been fuzzy*

sets over$V \times \mathbb{R}^+$,N*indicates the degree of membership along with* M*indicates the degree of non-membership* $(\mathrm{x}_{11}, \mathrm{t}_{11}) \in \mathrm{V} \times \mathbb{R}^+$ *staisfying conditions listed below,*

(i) $N_{\mathrm{I}}(\mathrm{x}_{11}, \mathrm{t}_{11}) + M_{\mathrm{I}}(\mathrm{x}_{11}, t_{11}) \leq 1, \forall (\mathrm{x}_{11}, \mathrm{t}_{11}) \in V \times \mathbb{R}^+$.

(ii) $N_{\mathrm{I}}(\mathrm{x}_{11}, \mathrm{t}_{11}) > 0$.

(iii) $N_{\mathrm{I}}(\mathrm{x}_{11}, \mathrm{t}_{11}) = 1$ *iff* $\mathrm{x}_{11} = 0$.

(iv) $N_{\mathrm{I}}(c\mathrm{x}_{11}, \mathrm{t}_{11}) = N_{\mathrm{I}}(\mathrm{x}_{11}, \frac{t_{11}}{|c|}), c \neq 0, c \in \mathbb{F}$

(v) $N_{\mathrm{I}}(\mathrm{x}_{11}, s_{11}) * N_{\mathrm{I}}(y_{11}, t_{11}) \leq N_{\mathrm{I}}(\mathrm{x}_{11} + y_{11}, s_{11} + \mathrm{t}_{11})$.

(vi) $N_{\mathrm{I}}(\mathrm{x}_{11}, .)$ *is non-decreasing function of* $\mathbb{R}^+$ *and* $lim_{t_{11} \to \infty} N_{\mathrm{I}}(\mathrm{x}_{11}, \mathrm{t}_{11}) = 1$.

(vii) $M_{\mathrm{I}}(\mathrm{x}_{11}, \mathrm{t}_{11}) > 0$.

(viii) $M_{\mathrm{I}}(\mathrm{x}_{11}, \mathrm{t}_{11}) = 0$ *iff* $x_{11} = 0$.

(ix) $M_{\mathrm{I}}(c\mathrm{x}_{11}, \mathrm{t}_{11}) = M_{\mathrm{I}}(\mathrm{x}_{11}, \frac{t_{11}}{|c|}), c \neq 0, c \in \mathbb{F}$

(x) $M_{\mathrm{I}}(\mathrm{x}_{11}, s_{11}) \diamond M_{\mathrm{I}}(y_{11}, \mathrm{t}_{11}) \geq M_{\mathrm{I}}(\mathrm{x}_{11} + y_{11}, s_{11} + \mathrm{t}_{11})$.

(xi) $M_{\mathrm{I}}(\mathrm{x}_{11}, .)$ *is non-increasing function of* $\mathbb{R}^+$ *as well as* $lim_{t_{11} \to \infty} M_{\mathrm{I}}(\mathrm{x}_{11}, \mathrm{t}_{11}) = 0$.

According to this definition, the $5-$*tuple* $(V, M_{\mathrm{I}}, N_{\mathrm{I}}, *, \diamond)$ *constitutes an intuitionistic fuzzy normed linear space, whereas* $(M_{\mathrm{I}}, N_{\mathrm{I}})$ *represents an intuitionistic fuzzy norm.*

Definition 14. ***Intuitionistic Hesitant Fuzzy Set :*** *[19]*
When applied to $\mathcal{X}$*, the functions* $\mathfrak{h}$ *and* $\mathfrak{h}'$ *yield subsets regarding* $[0, 1]$*, which might be expressed mathematically* $E = \{(x, \mathfrak{h}(x), H'(x))/x \in \mathcal{X}\}$*. This consists of intuitioistic hesitant fuzzy set on* $\mathcal{X}$*, wherein sets of some values in* $[0, 1]$ *are represented by* $\mathfrak{h}(x), \mathfrak{h}'(x)$*, The elements* $x \in \mathcal{X}$ *that represent the membership along with non-membership degrees of the set* E*, Suppose that* $max(\mathfrak{h}(x)) + min(\mathfrak{h}'(x)) \leq 1$ *along with* $min(\mathfrak{h}(x)) + max(\mathfrak{h}'(x)) \leq 1$*, because* $(\mathfrak{h}(x), \mathfrak{h}'(x))$ *is an intuitionistic hesitant fuzzy element.*

Definition 15. ***Intuitionistic Hesitant Fuzzy Norm :*** *[19]*
Over the field $\mathbb{F}$, V *represents a linear space. Let* $*$ *represent a continuous t-norm, as well as* $\diamond$ *represent a continuous t-co-norm and an item of the following type is an intuitionistic hesitant fuzzy norm on V*
$\{\mathcal{H}_{IHF} = ((\mathrm{x}_{11}, \mathrm{t}_{11}), \mathcal{N}_I(x_{11}, \mathrm{t}_{11}), \mathcal{M}_I(\mathrm{x}_{11}, \mathrm{t}_{11})) : (\mathrm{x}_{11}, \mathrm{t}_{11}) \in \mathrm{V} \times \mathbb{R}^+\}$,*wherein* $\mathcal{N}_I, \mathcal{M}_I$ *are fuzzy sets over*$\mathrm{V} \times \mathbb{R}^+$,$\mathcal{N}_I$ *indicates the degree of membership along with* $\mathcal{M}_I$ *denote the degree of non-membership* $(\mathrm{x}_{11}, \mathrm{t}_{11}) \in \mathrm{V} \times \mathbb{R}^+$ *meeting the requirements listed here,*

(i) $\mathcal{N}_I(\mathrm{x}_{11}, \mathrm{t}_{11}) \cup \mathcal{M}_I(\mathrm{x}_{11}, \mathrm{t}_{11}) \subseteq U^*, \forall (\mathrm{x}_{11}, \mathrm{t}_{11}) \in V \times \mathbb{R}^+$.

(ii) $\mathcal{N}_I(\mathrm{x}_{11}, \mathrm{t}_{11}) \neq \emptyset^*$.

(iii) $\mathcal{N}_I(\mathrm{x}_{11}, \mathrm{t}_{11}) = U^*$ *iff* $\mathrm{x}_{11} = 0$.

(iv) $\mathcal{N}_I(c\mathrm{x}_{11}, \mathrm{t}_{11}) = \mathcal{N}_I\left(\mathrm{x}_{11}, \frac{t_{11}}{|c|}\right), c \neq 0, c \in \mathbb{F}$

(v) $\mathcal{N}_I(\mathrm{x}_{11}, s_{11}) * \mathcal{N}_I(y_{11}, \mathrm{t}_{11}) \subseteq \mathcal{N}_I(\mathrm{x}_{11} + y_{11}, s_{11} + \mathrm{t}_{11})$.

(vi) $\mathcal{N}_I(\mathrm{x}_{11}, \cdot)$ *is non-decreasing function of* $\mathbb{R}^+$ *and* $lim_{t_{11} \to \infty} \mathcal{N}_I(\mathrm{x}_{11}, \mathrm{t}_{11}) = U^*$.

(vii) $\mathcal{M}_I(\mathrm{x}_{11}, \mathrm{t}_{11}) \neq \emptyset^*$.

(viii) $\mathcal{M}_I(\mathrm{x}_{11}, \mathrm{t}_{11}) = \emptyset^*$ *iff* $\mathrm{x}_{11} = 0$.

(ix) $\mathcal{M}_I(c\mathrm{x}_{11}, \mathrm{t}_{11}) = \mathcal{M}_I\left(\mathrm{x}_{11}, \frac{t_{11}}{|c|}\right), c \neq 0, c \in \mathbb{F}$

(x) $\mathcal{M}_I(\mathrm{x}_{11}, s_{11}) \diamond \mathcal{M}_I(y_{11}, \mathrm{t}_{11}) \supseteq \mathcal{M}_I(\mathrm{x}_{11} + y_{11}, s_{11} + \mathrm{t}_{11})$.

(xi) $\mathcal{M}_I(\mathrm{x}_{11}, \cdot)$ *is non-increasing function of* $\mathbb{R}^+$ *as well as* $lim_{t_{11} \to \infty} \mathcal{M}_I(\mathrm{x}_{11}, \mathrm{t}_{11}) = \emptyset^*$.

Definition 16. ***Intuitionistic Hesitant Fuzzy Normed Linear Space:*** *[19]*
Assuming that $\mathcal{H}_{IHF}$ *represents an Intuitionistic Hesitant Fuzzy Norm over* V *on* $\mathbb{F}$, *afterwards* $(V, \mathcal{H}_{IHF})$ *is an intuitionistic hesitant fuzzy normed linear space or IHFNLS.*

Example: [19] Consider the normed linear space $(V = \mathbb{R}, \|\ \cdot\ \|)$, wherein $\| x \| = |x|, \forall x \in \mathbb{R}$. Describe for all $\mathrm{S}_1^*, \mathrm{S}_2^* \in \mathcal{P}[0,1], \mathrm{S}_1^* * \mathrm{S}_2^* = \mathrm{S}_1^* \cap \mathrm{S}_2^*$ and $\mathrm{S}_1^* \diamond \mathrm{S}_2^* = \mathrm{S}_1^* \cup \mathrm{S}_2^*$. Also define

$$\mathcal{N}_I(x_1, t_1) = \begin{cases} \emptyset^* & \text{if } \ \mathrm{t}_1 = 0 \ \textit{and} \ \ \forall \ \mathrm{x}_1 > 0 \in V, \\ U^* & \text{if } \ \mathrm{x}_1 = 0 \ \textit{and} \ \ \forall \ \ \mathrm{t}_1 > 0, \\ S^* & \text{otherwise } \ S^* \in \mathcal{P}[0,1], \text{ Where } S^* \text{ is an arbitrary subset of } \mathcal{P}[0,1]. \end{cases}$$

and

$$\mathcal{M}_I(x_1, t_1) = \begin{cases} U^* & \text{if } \ t_1 = 0 \ \textit{and} \ \forall \ \ x_1 > 0 \in V, \\ \emptyset^* & \text{if } \ x_1 = 0 \ \textit{and} \ \forall \ \ t_1 > 0, \\ S^* & \text{otherwise } \ S^* \in \mathcal{P}[0,1], \text{ Where } S^* \text{ is an arbitrary subset of } \mathcal{P}[0,1]. \end{cases}$$

Definition 17. *Let* $(V, \mathcal{H}_{IHF})$ *be an intuitionistic hesitant fuzzy normed linear space. If dimension of a vector space* V *is finite then it is called finite dimensional intuitionistic hesitant fuzzy normed linear space.*

Definition 18. *[19] If given* $S_1^* \neq \emptyset^*, t > 0, \emptyset^* \subset S_1^* \subset U^*, \exists\ n_0 \in \mathbb{N}$ *so that,* $N_{IHF}(\mathrm{x}_n - \mathrm{x}_1, t_1) \supset U^* \backslash S_1^*$ *and* $M_{IHF}(\mathrm{x}_n - \mathrm{x}_1, t_1) \subset S_1^*, \ \ \forall \ \ \mathrm{n} \geq n_0$.

Theorem 4. *[19] In an IHFNLS* $(V, \mathcal{H}_{IHF})$, *a sequence* $\{x_n\}_n$ *converges to* $x_1 \in V$ *if and only if* $lim_{n \to \infty}\ \mathcal{N}_{IHF}(\mathrm{x}_n - \mathrm{x}_1, t_1) = U^*$ *along with* $lim_{n \to \infty}\ \mathcal{M}_{IHF}(\mathrm{x}_n - \mathrm{x}_1, t_1) = \emptyset^*$.

Theorem 5. *[19] In an IHFNLS* $(V, \mathcal{H}_{IHF})$, *a sequence* $\{x_n\}_n$ *has a unique limit if it is convergent.*

Lemma 2. *Consider an intuitionistic hesitant fuzzy normed linear space $(V, \mathcal{H}_{IHF})$, where $\{x_1, x_2, \ldots, x_n\}$ is a set of vectors in V, that are linearly independent along with the underlying t-norm $*$ along with t-co–norm $\diamond$ are continuous at $(0,0)$ and $(1,1)$ respectively. Then there exists $h_1, h_2 > 0$ and there exists $S_1^*, S_2^* \in \mathcal{P}[0,1]$ so that for any collection of scalars, $\{\delta_1, \delta_2, \ldots, \delta_n\}$,*

$$\mathcal{N}_{IHF}\left\{\delta_1 x_1 + \delta_2 x_2 + \ldots, +\delta_n x_n,\ h_1 \sum_{k=1}^{n} |\delta_k|\right\} \subset U^* \backslash S_1^* \tag{1}$$

$$\mathcal{M}_{IHF}\left\{\delta_1 x_1 + \delta_2 x_2 + \ldots, +\delta_n x_n,\ h_2 \sum_{k=1}^{n} |\delta_k|\right\} \supset S_2^* \tag{2}$$

Proof. Let $T = |\delta_1| + |\delta_2| + \cdots + |\delta_n|$. If $T = 0$, then $\delta_k = 0, \forall k = 1, 2, \ldots n$ and the relation, $\mathcal{N}_{\mathrm{IHF}}\{\delta_1 \mathrm{x}_1 + \delta_2 \mathrm{x}_2 + \ldots, +\delta_n \mathrm{x}_n,\ h_1 \sum_{k=1}^{\mathrm{n}} |\delta_{\mathrm{k}}|\} \subset \mathrm{U}^* \backslash \mathrm{S}_1^*$ is true for any $h > 0$,and $S^* \in \mathcal{P}[0,1]$.
Then, we assume that $\mathrm{T} > 0$. Then (1) is equivalent to

$$\mathcal{N}_{IHF}\{\omega_1 x_1 + \omega_2 x_2 + \ldots, +\omega_n x_n,\ h_1\} \subset U^* \backslash S_1^* \tag{3}$$

for any scalars $\omega' s$ with $\sum_{k=1}^{n} |\omega_k| = 1$ and for some $h_1 > 0$ and $S^* \in \mathcal{P}[0,1]$.
Assume (3) is not true, if at all possible. Consequently, for each $h > 0$ and $S_1^* \in \mathcal{P}[0,1]$, there will be a collection regarding scalars $\{\omega_1, \omega_2, \ldots \omega_n\}$ using $\sum_{k=1}^{n} |\omega_k| = 1$ for which, $\mathcal{N}_{IHF}\{\omega_1 x_1 + \omega_2 x_2 + \ldots, +\omega_n x_n,\ h\} \supseteq U^* \backslash S^*$. Then for $h = \{\frac{1}{m}\}, m = 1, 2, \ldots,$ there will be a collection regarding scalars $\{\omega_1^m, \omega_2^m, \ldots, \omega_n^m\}$ with $\sum_{k=1}^{n} |\omega_k^m| = 1$ so that $\mathcal{N}_{\mathrm{IHF}}(\mathrm{y}_m, \frac{1}{m}) \supset U^* \backslash \{\frac{1}{m}\}$ where $\mathrm{y}_m = \omega_1^m \mathrm{x}_1 + \omega_2^m \mathrm{x}_2 +, \ldots, \omega_n^m \mathrm{x}_n$. since, $\sum_{k=1}^{n} |\omega_k^m| = 1$, we have $0 \leq |\omega_k^m| \leq 1$ for $k = 1, 2, \ldots n$. Consequently, ω_1^m has a convergent subsequence since the sequence $\{\omega_k^m\}$ is confined for each fixed k. Let ω_1 represent the subsequence's limit, as well as allow $\{y_{1_m}\}$ represent the equivalent subsequence regarding $\{y_m\}$. The equivalent subsequence of scalars $\{\omega_2^m\}$ converges to ω_2. for the subsequence $\{y_{1_m}\}$, according to the same argument. Following this procedure, we get a subsequence after n steps, $\{y_{n_m}\}$ whereas $y_{1_m} = \sum_{k=1}^{n} \eta_k^m x_k$ with $\sum_{k=1}^{n} |\eta_k^m| = 1$, and $\eta_k^m \to \omega_k$ as $m \to \infty$.
Let $y = \eta_1 x_1 + \cdots + \eta_k x_k$ Now we show that $\lim_{m\to\infty} \mathcal{N}_{IHF}(\mathsf{y}_{n,m} - \mathsf{y}, t) = U^*, \forall\ \ \mathsf{t} > 0$.
We possess $\mathcal{N}_{IHF}(y_{n,\mathrm{m}} - \mathrm{y}, t) = \mathcal{N}_{IHF}(\sum_{k=1}^{n} (\eta_k^m - \omega_k) x_k, t)$
$\supseteq \mathcal{N}_{IHF}(x_1, \frac{t}{n|\eta_1^m - \omega_1|}) * \cdots * \mathcal{N}_{IHF}(x_1, \frac{t}{n|\eta_n^m - \omega_n|})$
So, $\lim_{m\to\infty} \mathcal{N}_{IHF}(y_{n,m} - y, t) \supseteq \lim_{m\to\infty} \mathcal{N}_{IHF}(x_1, \frac{t}{n|\eta_1^m - \omega_1|}) * \cdots * \lim_{m\to\infty} \mathcal{N}_{IHF}(x_1, \frac{t}{n|\eta_n^m - \omega_n|})$
$\implies \lim_{m\to\infty} \mathcal{N}_{IHF}(y_{n,m} - y, t) \supseteq U^* * \cdots * U^*$ (through the t-norm $*$'s continuity at (1,1))

$$\lim_{\mathrm{m}\to\infty} \mathcal{N}_{IHF}(y_{n,m} - \mathrm{y}, \mathrm{t}) = U^*, \forall t > 0. \tag{4}$$

Select m so that $\frac{1}{m} < l$. for $l > 0$. We have
$\lim_{m\to\infty} \mathcal{N}_{IHF}(y_{n,m}, l) = \lim_{m\to\infty} \mathcal{N}_{IHF}(y_{n,m} + 0, \frac{1}{m} + l - \frac{1}{m}) \supseteq \lim_{m\to\infty} \mathcal{N}_{IHF}(y_{n,m}, \frac{1}{m}) * \lim_{m\to\infty} \mathcal{N}_{IHF}(0, l - \frac{1}{m})$
$\supseteq (U^* \backslash \frac{1}{m}) * U^* = U^* \backslash \frac{1}{m} \implies \lim_{m\to\infty} \mathcal{N}_{IHF}(y_{n,m}, l) \supseteq U^*$

$$\implies \lim_{m\to\infty} \mathcal{N}_{IHF}(y_{n,m}, l) = U^* \tag{5}$$

Now, $\lim_{m\to\infty}\mathcal{N}_{IHF}(y,2l) = \lim_{m\to\infty}\mathcal{N}_{IHF}(y-y_{n,m}+y_{n,m},l+l)$
$\supseteq \lim_{m\to\infty}\mathcal{N}_{IHF}(\mathrm{y}-y_{\mathrm{n,m}},l) * \lim_{\mathrm{m}\to\infty}\mathcal{N}_{IHF}(\mathrm{y}_{\mathrm{n,m}},l)$
$\implies \lim_{m\to\infty}\mathcal{N}_{IHF}(y,2l) \supseteq U^* * U^*$ (through the t-norm $*$'s continuity at (1,1))
$\implies \lim_{m\to\infty}\mathcal{N}_{IHF}(y,2l) = U^* * U^* = U^*$. (By (4) and (5)) This is because $l>0$ becomes random. In addition, given that $\sum_{k=1}^{n}|\omega_k^m| = 1$,the linear independence of the vectors $\{x_1,x_2,\ldots x_n\}$ is established. Consequently, $y = \omega_1x_1+\omega_2x_2+\ldots,+\omega_nx_n \neq 0$.The result is a contradiction. We now demonstrate the relationship. $\mathcal{M}_{IHF}\{\delta_1x_1+\delta_2x_2+\ldots,+\delta_nx_n,\ h_2\sum_{k=1}^{n}|\delta_k|\} \supset S_2^*$. If $T=0$, then $\delta_k=0, \forall k=1,2,\ldots n$ and the relation, $\mathcal{M}_{IHF}\{\delta_1x_1+\delta_2x_2+\ldots,+\delta_nx_n,\ h_2\sum_{k=1}^{n}|\delta_k|\} \supset S_2^*$ remains true for every $h>0$, and $S^* \in \mathcal{P}[0,1]$.
Then, assuming that $T>0$, (2) is equal to

$$\mathcal{M}_{IHF}\{\omega_1x_1+\omega_2x_2+\ldots,+\omega_nx_n,\ h_2\} \supset S_2^* \tag{6}$$

for any scalars $\omega's$ with $\sum_{k=1}^{n}|\omega_k| = 1$. and for some $h_2>0$ and $S_2^* \in \mathcal{P}[0,1]$.
If at all feasible, assume that (6) is not true. Consequently, for every $h>0$ and $S^* \in \mathcal{P}[0,1]$, There is such a collection regarding scalars $\{\omega_1,\omega_2,\ldots\omega_n\}$along side$\sum_{k=1}^{n}|\omega_k| = 1$ that,
$\mathcal{M}_{IHF}\{\omega_1x_1+\omega_2x_2+\ldots,+\omega_nx_n,\ h\} \subseteq S^*$. Then for $h=\{\frac{1}{m}\}, m=1,2,\ldots,$ A collection of scalars $\{\eta_1^m,\eta_2^m,\ldots,\eta_n^m\}$ exists. with $\sum_{k=1}^{n}|\eta_k^m| = 1$ so that $\mathcal{M}_{IHF}(z_m,\frac{1}{m}) \subseteq \{\frac{1}{m}\}$ where $z_m = \eta_1^mx_1+,\eta_2^mx_2+,\ldots,\eta_nx_n^m$. since, $\sum_{k=1}^{n}|\eta_k^m| = 1$, we have $0 \leq |\eta_k^m| \leq 1$ regarding $k=1,2,\ldots n$. Afterwards, using the similar justification as before, we obtain a subsequence $\{z_{n_m}\}$ where $z_{n_m} = \sum_{\mathrm{k}=1}^{n}\xi_k^mx_k$ with $\sum_{k=1}^{n}|\xi_k^m| = 1$, and $\xi_k^m \to \xi_k$ as $m\to\infty$. Thus $\sum_{k=1}^{n}|\xi_k| = 1$.
Let $\mathsf{z} = \xi_1x_1+\cdots+\xi_kx_k$.Then we have

$$\lim_{m\to\infty}\mathcal{M}_{IHF}(z_{n,m}-z,t) = \emptyset^*, \forall t>0. \tag{7}$$

In this case, $l>0$, select m so that $\frac{1}{m}<l$. We now possess
$\lim_{m\to\infty}\mathcal{M}_{IHF}(z_{n,m},l) = \lim_{m\to\infty}\mathcal{M}_{IHF}(z_{n,m}+0,\frac{1}{m}+l-\frac{1}{m})$
$\subseteq \lim_{m\to\infty}\mathcal{M}_{IHF}(z_{n,m},\frac{1}{m})\ \lim_{m\to\infty}\mathcal{M}_{IHF}(0,l-\frac{1}{m}) \subseteq \emptyset^* \diamond \emptyset^* = \emptyset^*$
$\implies \lim_{m\to\infty}\mathcal{M}_{IHF}(z_{n,m},l) \subseteq \emptyset^*$

$$\implies \lim_{m\to\infty}\mathcal{M}_{IHF}(z_{n,m},l) = \emptyset^* \tag{8}$$

Now, $\lim_{m\to\infty}\mathcal{M}_{IHF}(z,2l) = \lim_{m\to\infty}\mathcal{M}_{IHF}(z-z_{n,m}+z_{n,m},l+l)$
$\subseteq \lim_{m\to\infty}\mathcal{M}_{IHF}(z-z_{n,m},l) \diamond \lim_{m\to\infty}\mathcal{M}_{IHF}(z_{n,m},l)$
$\implies \lim_{m\to\infty}\mathcal{M}_{IHF}(z,2l) \subseteq \emptyset^* \diamond \emptyset^*$ (according to $t$co–norm $\diamond$'s continuity at (0,0))
$\implies \lim_{m\to\infty}\mathcal{M}_{IHF}(z,2l) = \emptyset^* \diamond \emptyset^* = \emptyset^*$. (By (7) and (8))
Assuming that $l>0$ is random. Therefore, $z=0$. once again because $\sum_{k=1}^{n}|\xi_k^m| = 1$ along with $\{x_1,x_2,\ldots x_n\}$ is a collection of vectors that are linearly independent. So $z = \xi_1x_1+\xi_2x_2+\ldots,+\xi_nx_n \neq 0$.Consequently, This leads to a contradiction, This brings the lemma to an end.

Theorem 6. *All finite-dimensional IHFnNLS* $(\mathsf{V}, \mathcal{H}_{IHF})$ *are complete if the underlying t-norm* $*$ *at* $(1,1)$ *together with t-co-norm* $\diamond$ *at* $(0,0)$ *are continuous.*

Proof. Assume that $dim\mathrm{V} = k$ and that $(V, \mathcal{H}_{IHF})$ is an intuitionistic hesitant fuzzy normed linear space. Consider $\{\mathrm{x}_n\}$ to become a cauchy sequence in V and $\{\mathsf{e}_1, \mathsf{e}_2, \dots, \mathsf{e}_k\}$ providing a basis of V. Let $x_n = \omega_1^n e_1 + \omega_2^n e_2 +, \dots, \omega_k^n e_k$. where $\omega_1^n, \omega_2^n, \dots, \omega_k^n$ are appropriate scalars. Thus,

$$\lim_{\mathrm{m,n}\to\infty} \mathcal{N}_{IHF}(\mathsf{x}_{\mathrm{m}} - \mathsf{x}_{\mathrm{n}}, \mathsf{t}) = U^*, \forall \ \ \mathrm{t} > 0 \tag{9}$$

along with

$$\lim_{\mathsf{m,n}\to\infty} \mathcal{N}_{IHF}(\mathsf{x}_{\mathrm{m}} - \mathsf{x}_{\mathrm{n}}, \mathsf{t}) = \emptyset^*, \forall \ \ \mathrm{t} > 0 \tag{10}$$

It is evident out of lemma (2) that ,there exists $h_1, h_2 > 0$ and $S_1^*, S_2^* \in \mathcal{P}[0,1]$ such that

$$\mathcal{N}_{IHF}\left(\sum_{i=1}^{k}(\omega_i^m - \omega_i^n)e_i, h_1 \sum_{i=1}^{k}(|\omega_i^m - \omega_i^n|)\right) \subset U^* \backslash S_1^* \tag{11}$$

$$\mathcal{M}_{IHF}\left(\sum_{i=1}^{k}(\omega_i^m - \omega_i^n)e_i, h_2 \sum_{i=1}^{k}(|\omega_i^m - \omega_i^n|)\right) \supset S_2^* \tag{12}$$

Again for $U^* \supset S_1^* \supset \emptyset^*$ out of (9), Consequently, a positive integer n_0 exists with regard to

$$\mathcal{N}_{IHF}\left(\sum_{i=1}^{k}(\omega_i^m - \omega_i^n)e_i, t\right) \supset U^* \backslash S_1^*, \forall\ m, n \geq n_0. \tag{13}$$

and for $U^* \supset S_2^* \supset \emptyset^*$ out of (10), consequently, a positive integer exists. m_0 such that

$$\mathcal{M}_{IHF}\left(\sum_{i=1}^{k}(\omega_i^m - \omega_i^n)e_i, t\right) \subset S_2^*, \forall\ m, n \geq m_0. \tag{14}$$

Now from (11) and (13) we have, $\mathcal{N}_{IHF}\left(\sum_{i=1}^{\mathrm{k}}(\omega_i^m - \omega_i^n)e_i, t\right) \supset U^* \backslash S_1^*$
$\supset \mathcal{N}_{IHF}\left(\sum_{\mathrm{i}=1}^{\mathrm{k}}(\omega_i^m - \omega_i^n)e_i, h_1 \sum_{i=1}^{\mathrm{k}}(|\omega_i^m - \omega_i^n|)\right) \ \forall \ \ \mathrm{m, n} \geq n_0$.
$\Longrightarrow \ \mathsf{h}_1 \sum_{i=1}^{k}(|\omega_i^m - \omega_i^n|) < \mathrm{t}, \forall\ \mathrm{m,n} \geq n_0$ (Given that $\mathcal{N}_I(x, \centerdot)$ is non decreasing in t) $\Longrightarrow \ \sum_{\mathsf{i}=1}^{\mathsf{k}}(|\omega_i^{\mathsf{m}} - \omega_i^{\mathsf{n}}|) < \frac{\mathsf{t}}{\mathsf{h}_1}, \forall\ \mathsf{m}, \mathsf{n} \geq n_0 \ \Longrightarrow \ |\omega_i^{\mathsf{m}} - \omega_i^{\mathsf{n}}| < \frac{\mathrm{t}}{\mathrm{h}_1}, \forall\ \mathsf{m}, \mathsf{n} \geq n_0$ alon with $\mathsf{i} = 1, 2, \dots k$. Given that $t > 0$ is random, based on previously mentioned, we possess $\lim_{m,n\to\infty} |\omega_i^m - \omega_i^n| = 0$ for $\mathsf{i} = 1, 2 \dots, k$. $\Longrightarrow$ $\{\omega_i^n\}$ this constitutes a cauchy sequence of scalars for every $\mathsf{i} = 1, 2 \dots, k$.. Thus, every sequence $\{\omega_i^n\}$ converges. Let $\lim_{n\to\infty} \omega_i^n = \omega_i$ regarding $i = 1, 2 \dots, k$. along with $x = \sum_{i=1}^{k} \omega_i e_i$. Obviously, $\mathrm{x} \in V$. Afterwards
$\forall \ \mathrm{t} > 0$, $\mathcal{N}_{IHF}(\mathsf{x}_n - \mathsf{x}, \mathrm{t}) = \mathcal{N}_{IHF}\left(\sum_{\mathsf{i}=1}^{\mathsf{k}} \omega_i^n e_i - \sum_{i=1}^{k} \omega_i e_i, \mathsf{t}\right) = \mathcal{N}_{IHF}\left(\sum_{\mathsf{i}=1}^{\mathsf{k}}(\omega_i^n - \omega_i)e_i, \mathsf{t}\right)$
That is

$$\mathcal{N}_{IHF}(\mathsf{x}_n - \mathsf{x}, \mathsf{t}) \supseteq \mathcal{N}_{IHF}\left(\mathsf{e}_1, \frac{t}{k|\omega_1^n - \omega_1|}\right) * \cdots * \mathcal{N}_{IHF}\left(e_k, \frac{t}{k|\omega_k^n - \omega_k|}\right) \tag{15}$$

when $n \to \infty$, then $\frac{t}{k|\omega_i^n - \omega_i|} \to \infty$ (Since $\omega_i^n \to \omega_i$) for $\mathsf{i} = 1, 2 \ldots, \mathsf{k}$. and $\mathsf{t} > 0$. Utilizing the t-norm $*$ continuity at (1,1), we derive from (15)
$\lim_{n\to\infty} \mathcal{N}_{IHF}(\mathrm{x}_n - \mathrm{x}, t) \supseteq U^* * \cdots * U^*, \forall \ \mathrm{t} > 0.$

$$\implies \lim_{\mathrm{n}\to\infty} \mathcal{N}_{IHF}(\mathrm{x}_n - \mathrm{x}, \mathsf{t}) = U^*, \ \forall \ \mathrm{t} > 0. \tag{16}$$

Now from (12) and (14) we have, $\mathcal{M}_{IHF}\left(\sum_{\mathsf{i}=1}^{\mathsf{k}}(\omega_i^m - \omega_i^n)e_i, t\right) \subset S_2^*$
$\subset \mathcal{M}_{IHF}\left(\sum_{\mathsf{i}=1}^{k}(\omega_i^m - \omega_i^n)e_i, h_2 \sum_{\mathsf{i}=1}^{\mathsf{k}}(|\omega_i^{\mathsf{m}} - \omega_i^{\mathsf{n}}|)\right) \ \forall \ \ \mathsf{m}, n \geq n_0.$
$\implies$ $\mathrm{h}_2 \sum_{i=1}^{k}(|\omega_i^m - \omega_i^n|) < \mathsf{t}, \forall \ \mathsf{m}, \mathsf{n} \geq n_0$ (Because $\mathcal{M}_I(x, \cdot)$ has non increasing in t) $\implies$ $\sum_{\mathsf{i}=1}^{\mathsf{k}}(|\omega_i^m - \omega_i^n|) < \frac{\mathsf{t}}{\mathsf{h}_2}, \forall \ \mathsf{m}, \mathsf{n} \geq m_0 \implies |\omega_i^{\mathsf{m}} - \omega_i^{\mathsf{n}}| < \frac{t}{h_2}, \forall \ m, n \geq m_0$ in addition $\mathsf{i} = 1, 2, \ldots k$. Given that $\mathsf{t} > 0$ is random, we can derive $\lim_{m,n\to\infty} |\omega_i^m - \omega_i^n| = 0$ over $\mathsf{i} = 1, 2 \ldots, k$. for every $\mathsf{i} = 1, 2 \ldots, k$. is a cauchy sequence of scalars. Thus, every sequence $\{\omega_i^n\}$ converges. Let $\lim_{n\to\infty} \omega_i^n = \omega_i$ for $\mathsf{i} = 1, 2 \ldots, k$. together with $x = \sum_{i=1}^{k} \omega_i e_i$. Evidently $x \in V$. Afterwards $\forall \ \ t > 0$,
$\mathcal{M}_{IHF}(\mathrm{x}_n - \mathrm{x}, t) = \mathcal{M}_{IHF}\left(\sum_{i=1}^{k} \omega_i^n e_i - \sum_{i=1}^{k} \omega_i e_i, t\right) = \mathcal{M}_{IHF}\left(\sum_{i=1}^{k}(\omega_i^n - \omega_i)e_i, t\right)$
That is

$$\mathcal{M}_{IHF}(x_n - x, t) \supseteq \mathcal{M}_{IHF}\left(e_1, \frac{t}{k|\omega_1^n - \omega_1|}\right) \diamond \cdots \diamond \mathcal{M}_{IHF}\left(e_k, \frac{t}{k|\omega_k^n - \omega_k|}\right) \tag{17}$$

when $n \to \infty$, then $\frac{t}{k|\omega_i^n - \omega_i|} \to \infty$ (Since $\omega_i^n \to \omega_i$) for $\mathsf{i} = 1, 2 \ldots, \mathsf{k}$. and $\mathsf{t} > 0$. Applying the t-co-norm's continuity $\diamond$ at (0,0), we obtain from (17).
$\lim_{n\to\infty} \mathcal{M}_{IHF}(\mathrm{x}_n - \mathrm{x}, t) \supseteq \emptyset^* \diamond \cdots \diamond \emptyset^*, \forall \ t > 0.$

$$\implies \lim_{\mathrm{n}\to\infty} \mathcal{M}_{IHF}(x_n - \mathrm{x}, \mathsf{t}) = \emptyset^*, \ \ \forall \ \mathrm{t} > 0. \tag{18}$$

We obtain $x_n \to x$ as $n \to \infty$. by (16) and (18). Consequently $(V, \mathcal{H}_{IHF})$ is complete.

Definition 19. *Consider the IHFNLS $(V, \mathcal{H}_{IHF})$ and $X \subset V$. For every S^*, X is considered to be bounded, $\emptyset^* \subset S^* \subset U^*, \exists t_1, t_2 > 0$ such that $\mathcal{N}_{IHF}(x, t_1) \supset U^* \backslash S^*$ and $\mathcal{M}_{IHF}(x, t_2) \subset S^*, \forall x \in X$.*

Theorem 7. *A subset X is compact if and only if it is closed and bounded in the finite dimensional IHFNLS $(V, \mathcal{H}_{IHF})$, where both the t-co-norm $\diamond$ along with the underlying t-norm $*$ are continuous at (0,0) together with (1,1) respectively.*

Proof. We start by assuming that X is compact. We need for prove it X is bounded alongwith closed. Take $x \in$ X. Then $\lim_{n\to\infty} x_n = x$. indicates that there is a sequence $\{x_n\}$ in X. There is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges to a point in X because X is compact. Afterwards, $\{x_n\} \to x$, and since $x \in X$, X is closed. If at all possible, assume that X is unbounded. Then there exists a S_0^* with $\emptyset^* \subset S_0^* \subset U^*$ so that for any positive number n, there exists $x_0 \in X$ in a manner that,$\mathcal{N}_{IHF}(x_n, n) \subseteq U^* \backslash S_0^*$ or

$\mathcal{M}_{IHF}(x_n, n) \supseteq S_0^*$. So there exists a subsequence of $\{x_n\}$ whereby minimum of one of the relationships have

$$\mathcal{N}_{IHF}(\mathsf{x}_{n_k}, \mathsf{n}_k) \subseteq U^* \backslash S_0^*, \quad \forall \ \mathsf{n} \in \mathbb{N}. \tag{19}$$

$$\mathcal{M}_{IHF}(\mathsf{x}_{n_k}, \mathsf{n}_k) \supseteq S_0^*, \quad \forall \ \mathsf{n} \in \mathbb{N}. \tag{20}$$

possesses. Initially, we consider that $\mathcal{N}_{IHF}(x_{n_k}, n_k) \subseteq U^* \backslash S_0^*, \quad \forall n \in \mathbb{N}$ holds. Now for $t > 0$,
$U^* \backslash S_0^* \supseteq \mathcal{N}_{IHF}(\mathsf{x}_{\mathsf{n}_k}, \mathsf{n}_k) = \mathcal{N}_{IHF}(\mathrm{x}_{\mathrm{n}_k} - \mathsf{x} + \mathrm{x}, \mathsf{n}_k - \mathsf{t} + \mathsf{t})$ where $t > 0$.
$\implies U^* \backslash S_0^* \supseteq \mathcal{N}_{IHF}(\mathsf{x}_{\mathsf{n}_k}, \mathsf{n}_k) = \mathcal{N}_{IHF}(\mathrm{x}_{\mathrm{n}_k} - x, \mathsf{t}) * \mathcal{N}_{IHF}(\mathsf{x}, \mathsf{n}_k - \mathsf{t})$
$\implies U^* \backslash S_0^* \supseteq \lim_{k \to \infty} \mathcal{N}_{IHF}(\mathrm{x}_{\mathsf{n}_k} - \mathrm{x}, \mathsf{t}) * \lim_{k \to \infty} \mathcal{N}_{IHF}(\mathrm{x}, \mathrm{n}_k - \mathsf{t})$
$\implies U^* \backslash S_0^* \supseteq U^* * U^* = U^*$. (Applying the t-norm's continuity at (1,1)) $\implies U^* \backslash S_0^* \supseteq U^*$ which is a contradiction. In case $\mathcal{M}_{IHF}(x_{n_k}, n_k) \supseteq S_0^*, \ \forall n \in \mathbb{N}$ possess , examining the function $\mathcal{M}_{IHF}(x, t)$ and continuing as previously mentioned, We get a contradiction, Hence X is bounded.

Conversly, Assuming that X is bounded and closed, we must demonstrate that X is compact. Consider $dim \mathsf{V} = n$ along with $\{\mathrm{e}_1, \mathsf{e}_2, \dots \mathrm{e}_n\}$ as a basis of V respectively. Select $\{x_k\}$ as a sequence in Xwhile assume $x_{\mathsf{k}} = \omega_1^{(\mathsf{k})} e_1 + \dots + \omega_n^{(\mathsf{k})} e_n$ here $\omega_1^{(\mathsf{k})}, \dots \omega_n^{(\mathsf{k})}$ are scalars. Now, according to lemma (1), there is $h_1, h_2 > 0$ and there exists $S_1^*, S_2^* \in \mathcal{P}[0,1]$ such that

$$\mathcal{N}_{IHF}\left(\sum_{i=1}^{n} \omega_i^{(k)} e_i, h_1 \sum_{i=1}^{n} |\omega_i^{(k)}|\right) \subset U^* \backslash S_1^* \tag{21}$$

and

$$\mathcal{M}_{IHF}\left(\sum_{i=1}^{n} \omega_i^{(k)} e_i, h_2 \sum_{i=1}^{n} |\omega_i^{(k)}|\right) \supset S_2^* \tag{22}$$

Again since X is bounded, for $S_1^* \in \mathcal{P}[0,1], \exists \ t_1 > 0$, so that
$\mathcal{N}_{IHF}(\mathsf{x}, t_1) \supset U^* \backslash S_1^*$ and there exists $t_2 > 0$, so that $\mathcal{M}_{IHF}(\mathsf{x}, t_2) \subset S_1^*, \forall \ \mathsf{x} \in X$. So,

$$\mathcal{N}_{IHF}\left(\sum_{i=1}^{n} \omega_i^{(k)} e_i, t_1\right) \supset U^* \backslash S_1^* \tag{23}$$

and

$$\mathcal{M}_{IHF}\left(\sum_{i=1}^{n} \omega_i^{(k)} e_i, t_1\right) \subset S_2^* \tag{24}$$

from (21) and (23) we get ,
$\mathcal{N}_{IHF}\left(\sum_{i=1}^{n} \omega_i^{(k)} e_i, h_1 \sum_{i=1}^{n} |\omega_i^{(k)}|\right) \subset U^* \backslash S_1^* \subset \mathcal{N}_{IHF}\left(\sum_{i=1}^{n} \omega_i^{(k)} e_i, t_1\right)$
$\implies \mathcal{N}_{IHF}\left(\sum_{i=1}^{n} \omega_i^{(k)} e_i, h_1 \sum_{i=1}^{n} |\omega_i^{(k)}|\right) \subset \mathcal{N}_{IHF}\left(\sum_{i=1}^{n} \omega_i^{(k)} e_i, t_1\right)$
$\implies h_1 \sum_{i=1}^{n} |\omega_i^{(k)}| < \mathrm{t}_1$ (Given that $\mathcal{N}_{IHF}(x, \centerdot)$ is non- decreasing)
$\implies |\omega_i^{(k)}| \leq \frac{t_1}{h_1}$ where $i = 1, 2 \dots n$. and $k = 1, 2 \dots$ Thus, for every $\{\omega_i^{(k)}\}$It is bounded $(i = 1, 2, \dots n)$. As a result of repeatedly applying the Bolzano -Weierstrass theorem, every sequence in $\{\omega_i^{(k)}\}$ possesses a subsequence that converges $\{\omega_i^{(\mathrm{k}_l)}\}, \forall \ \mathrm{i} = 1, 2, \dots n$. Assume

that $x_{k_l} = \omega_i^{(k_l)} e_1 + \cdots + \omega_i^{(k_l)} e_n$ and that $\{\omega_1^{(k_l)}\}, \{\omega_2^{(k_l)}\}, \ldots \{\omega_n^{(k_l)}\}$ are all convergent. Consider $\omega_i = \lim_{l\to\infty} \omega_i^{(k_l)}, i = 1, 2 \ldots n.$ and $x = \omega_1 e_1 + \omega_2 e_2 1 + \cdots + \omega_n e_n$. Currently, for t > 0,
we possess $\mathcal{N}_{IHF}(x_{k_l} l - x, t) = \mathcal{N}_{IHF}\left(\sum_{i=1}^{n}(\omega_i^{(k_l)} - \omega_i)e_i, t\right)$
$\supseteq \mathcal{N}_{IHF}\left(e_1, \frac{t}{|\omega_1^{(k_l)} - \omega_1|}\right) * \cdots * \mathcal{N}_{IHF}\left(e_n, \frac{t}{|\omega_n^{(k_l)} - \omega_n|}\right)$
$\implies \lim_{l\to\infty} \mathcal{N}_{IHF}(x_{k_l} - x, t) \supseteq U^* * \cdots * U^*.(\omega_n^{(k_l)} \to \omega_i$ as $l \to \infty)$(With the use of contuity of t- norm $*$ at $(1, 1)$).

$$\implies \lim_{l\to\infty} \mathcal{N}_{IHF}(x_{k_l} - \mathrm{x}, \mathrm{t}) = U^* \tag{25}$$

Currently, for $\mathrm{t} > 0$, we possess
$\mathcal{M}_{IHF}(x_{k_l} - x, \mathrm{t}) = \mathcal{M}_{IHF}\left(\sum_{i=1}^{n}(\omega_i^{(k_l)} - \omega_i)e_i, t\right)$
$\subseteq \mathcal{M}_{IHF}\left(e_1, \frac{t}{|\omega_1^{(k_l)} - \omega_1|}\right) \diamond \cdots \diamond \mathcal{M}_{IHF}\left(e_n, \frac{t}{|\omega_n^{(k_l)} - \omega_n|}\right)$
$\implies \lim_{l\to\infty} \mathcal{M}_{IHF}(x_{k_l} - x, t) \supseteq \emptyset^* \diamond \cdots \diamond \emptyset^*.(\omega_n^{(k_l)} \to \omega_i \text{as} l \to \infty)$(using the contuity of t-co- norm $\diamond$ at (0,0)).

$$\implies \lim_{l\to\infty} \mathcal{N}_{IHF}(x_{k_l} - x, \mathrm{t}) = \emptyset^* \tag{26}$$

According to (25) and (26), $x(k_l) \to x$. Given that X is closed, $x \in X$ implies that X is compact. Thus, the proof is completed.

5. Conclusion

We extended the foundational work on fuzzy normed linear space, initially proposed by Samanta et al., to the domain of hesitant fuzzy normed linear space and also examine the properties of completeness and compactness on hesitant fuzzy normed linear space, also address the same properties on intuitionistic hesitant fuzzy normed linear space in finite dimension using definitions, lemmas, and theorems. We also investigate the continuity of underlying t-norms and co-t-norm on finite-dimensional intuitionistic hesitant fuzzy normed linear space. In this context, there is room for more work (If possible in practical scenario).

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