EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
2025, Vol. 18, Issue 3, Article Number 6544
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Failed 2-Distance Zero Forcing Numbers of Some Graphs

Al-Fadzri P. Madjatul[1], Javier A. Hassan[1,2,*], Maria Andrea O. Bonsocan[3], Vergel T. Bilar[3]

[1] *Department of Mathematics, College of Arts and Sciences, MSU Tawi-Tawi College of Technology and Oceanography, Bongao, Tawi-Tawi, Philippines*
[2] *Department of Mathematics, College of Science, Korea University, Seoul, South Korea*
[3] *Department of Mathematics, Ateneo de Davao University, Davao City, Philippines*

Abstract. Let $G = (V(G), E(G))$ be a simple and undirected graph and let $x, y \in V(G)$. If x is a colored (active) vertex and exactly one hop neighbor y of x is uncolored (inactive), then y will become colored (active), and we call this process a 2-distance color change rule. A 2-distance zero forcing set N is a subset of $V(G)$ such that when the vertices in N are colored (active) and the remaining vertices outside N are uncolored (inactive) initially, then repeated application of a 2-distance color change rule, all vertices of G will become colored (active). Now, a set $M \subset V(G)$ is called a failed 2-distance zero forcing set of G if M fails to be a 2-distance zero forcing set of G. The failed 2-distance zero forcing number of a graph G, denoted by $F^2(G)$, is the maximum cardinality of a failed 2-distance zero forcing set of G. In this paper, we introduce the said parameter and study this on some special graphs and on the join of two graphs. Moreover, we characterize failed 2-distance zero forcing sets in the join of two graphs using a failed co-zero forcing concept. Finally, we derive some nice formulas for computing the said parameter on the join of any two graphs.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Failed 2-distance zero forcing set, failed 2-distance zero forcing number, co-zero forcing, failed co-zero forcing set, failed co-zero forcing number

---

1. Introduction

The concept of zero forcing has been explored over the past few years because of its application to minimum rank problems [1, 2]. The zero forcing process was initially

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v18i3.6544

Email addresses:
`alfadzrimadjatul@msutawi-tawi.edu.ph` (A. Madjatul)
`javierhassan@msutawi-tawi.edu.ph` (J. Hassan)
`maobonsocan@addu.edu.ph` (M.A. Bonsocan)
`vtbilar@addu.edu.ph` (V. Bilar)

proposed in [3], and has been further studied on different types of graphs. Some studies on this parameter can be found in [4–13].

In 2015, K. Fetcie, B. Jacob and D. Saavedra introduced a new graph parameter, the failed zero forcing number of a graph. They established bounds on the failed zero forcing number of a graph, both in general and for connected graph. Some studies on this parameter can be found in [14–17].

In a recent year, J. Hassan et al. introduced the concept of 2-distance zero forcing sets in a graph [18]. The said concept is another variant of zero forcing wherein its color change property differs from the standard color change rule. Particularly, the distance condition has extended to two. They had initially investigated the said concept on some special types of graphs, and presented some relationships with other parameters. In particular, they have shown that this new variant is incomparable with the usual zero forcing.

In this study, a new concept related to a 2-distance zero forcing in a graph is introduced and initially investigated on some families of graphs and on the join of any two graphs, and we call it a failed 2-distance zero forcing. This paper may give further insights and may develop new concepts that would benefit other researchers who are interested in the field of graph theory. Moreover, this parameter may offer interesting research topics in the future and may be applied to model a certain real-life scenarios.

2. Terminology and Notation

Let $G = (V(G), E(G))$ be a simple and undirected graph. The *distance* $d_G(u, v)$ in G of two vertices u, v is the length of a shortest *u-v path* in G. A vertex of a in G is a *hop neighbor* of a vertex b in G if $d_G(a, b) = 2$.

Let $G = (V(G), E(G))$ be a simple and undirected graph and let $x, y \in V(G)$. If x is a colored (active) vertex and exactly one hop neighbor y of x is uncolored (inactive), then y will become colored (active), and we call this process a 2-distance color change rule. A 2-distance zero forcing set N is a subset of vertices of G such that when the vertices in N are colored (active) and the remaining vertices outside N are uncolored (inactive) initially, then repeated application of a 2-distance color change rule, all vertices of G will become colored (active). The minimum cardinality of a 2-distance zero forcing set of G, denoted by $Z^2(G)$, is called the 2-distance zero forcing number of G.

Let G and H be any two graphs. The *join* of G and H, denoted by $G+H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set

$$E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

3. Results

We shall now define the failed 2-distance zero forcing in a graph and investigate this on some families of graphs and on the join of two graphs.

Definition 1. Let $G = (V(G), E(G))$ be a simple and undirected graph. Then $M \subset V(G)$ is called a failed 2-distance zero forcing set of G if M fails to be a 2-distance zero forcing set of G. The failed 2-distance zero forcing number of G, denoted by $F^2(G)$, is the maximum cardinality of a failed 2-distance zero forcing set in G.

4. Some Properties of Failed 2-Distance Zero Forcing in Some Special Graphs

Let's begin with the characterization of a failed 2-distance zero forcing sets in a complete graph as follows:

Theorem 1. Let n be a positive integer. Then $R \subset V(K_n)$ is a failed 2-distance zero forcing set in K_n if and only if $|R| \leq n-1$.

Proof. Suppose that $R \subset V(K_n)$ is a failed 2-distance zero forcing set of K_n. Then R is not a 2-distance zero forcing set of K_n. Thus, there exist $y \in V(K_n) \setminus R$ such that y cannot be 2-forced. That is, $R \neq V(K_n)$. Therefore, $|R| \neq n$, and so $|R| \leq n-1$.

Conversely, suppose that $|R| \leq n-1$. Then there is exist at least one vertex $u \in V(K_n)$ such that $u \notin R$. Suppose on the contrary that R is a 2-distance zero forcing set on K_n. Then there exist $y \in R$ such that $d_{K_n}(u, y) = 2$. However, this is a contradiction since the graph is complete. Thus, R is a failed 2-distance zero forcing set of K_n. □

The following result follows directly from Theorem 1.

Corollary 1. Let n be a positive integer. Then $F^2(k_n) = n-1$ for all $n \geq 1$

Now, let's study the behavior of a failed 2-distance zero forcing number in a path graph with order n, which is any positive integer.

Theorem 2. Let n be any positive integer. Then

$$F^2(P_n) = \begin{cases} n-1 \text{ , if } n = 1, 2, 3, \\ \lfloor \frac{n}{4} \rfloor - 1 \text{ , if } n \text{ is odd} \\ \frac{n}{2} \text{ , if } n \text{ is even.} \end{cases}$$

Proof. By Theorem 3.1.1, $F^2(P_1) = 0$ and $F^2(P_2) = 1$. For $n = 3$, let $P_3 = [v_1, v_2, v_3]$, and consider $S = \{v_1, v_3\}$. Then v_2 cannot be 2-forced by any vertex in S. Thus, S is a failed 2-distance zero forcing set of P_3. Obviously S is a maximum failed 2-distance zero forcing set of P_3 since $V(P_3)$ is a 2-distance zero forcing set of P_3. Therefore, $F^2(P_3) = 2$.

Let $P_n = [a_1, a_2, ..., a_n]$. For $n = 4$, consider $N = \{a_1, a_3\}$. Then a_2 and a_4 cannot be 2-forced by any vertex in N. It follows that N is a failed 2-distance zero forcing set of P_4. Thus, $F^2(P_4) \geq 2$. Notice that $N' = T \cup \{a_1, a_3\}$ where $T \subseteq \{a_2, a_4\}$, is a 2-distance zero forcing set of P_4. Therefore, $F^2(P_4) \geq 3$ is impossible, and so $F^2(P_4) = 2$.

Now, for $n \geq 6$ and even, consider

$$R = \{a_1, a_3, a_4, a_5, a_7, a_8, a_9, \ldots, a_{n-1}\}.$$

Then the vertices $a_2, a_6, ..., a_n$ cannot be 2-forced. Thus, R is a failed 2-distance zero forcing set of P_n. Now, if we add vertex a_2 to R, then repeatedly applying the 2-distance color change rule on R, all vertices outside R can now be 2-forced, which is a contradiction. Similarly, if we add vertices from $\{a_6, a_{10}, ..., a_n\}$ to R, then R is a 2-distance zero forcing set of P_n, a contradiction. Therefore, R is a maximum failed 2-distance zero forcing set of P_n.

For $n = 5$. Let $T = \{a_2, a_3, a_4\}$. Then a_1 and a_5 cannot be 2-forced. Hence, T is a failed 2-distance zero forcing set of P_5. Now, if we add a_1 to T. Then vertex a_5 can now be 2-forced by a vertex a_3, a contradiction. Thus, T is the maximum failed 2-distance zero forcing of P_5.

Assume that $n \geq 7$ and odd. Consider the following cases:
Case 1: For $n \in \{7, 11, 15, ...\}$, consider

$$Q = \{a_1, a_3, a_4, a_5, a_7, a_8, a_9, \ldots, a_{n-4}, a_{n-3}, a_{n-2}, a_n\}.$$

Then vertices in $V(P_n) \setminus Q$ cannot be 2-forced. It follows Q is a failed 2-distance zero forcing set. Now, if we add a_2 to Q, then repeatedly applying the 2-distance color change rule on Q, all vertices outside Q can now be 2-forced, which is also a contradiction. Similarly, if we add vertices from $\{a_6, a_{10}, ..., a_{n-1}\}$ to Q, then Q is a failed 2-distance zero forcing set of P_n, another contradiction. Thus, Q is a maximum failed 2-distance zero forcing set of P_n.

Case 2: For $n \in \{9, 13, 17, ...\}$, consider

$$Q' = \{a_1, a_3, a_4, a_5, a_7, a_8, a_9, \ldots, a_{n-6}, a_{n-5}, a_{n-4}, a_{n-2}, a_n\}.$$

Then vertices in $V(P_n) \setminus Q'$ cannot be 2-forced. It follows Q' is a failed 2-distance zero forcing set. Now, if we add a_2 to Q', then repeatedly applying the 2-distance color change rule on Q', all vertices outside Q' can now be 2-forced, a contradiction. Similarly, if we add vertices from $\{a_6, a_{10}, ..., a_{n-3}, a_{n-1}\}$ to Q', then Q' is a failed 2-distance zero forcing set of P_n, another contradiction. Thus, Q' is a maximum failed 2-distance zero forcing set of P_n.

Therefore,

$$F^2(P_n) = n - (\lfloor \frac{n}{4} \rfloor + 1) \text{ for } n \geq 4.$$

□

Now, we will characterize a failed 2-distance zero forcing set in star graph to derive the parameter's formula for the said graph.

Theorem 3. Let n be a positive integer and v be a dominating vertex of S_n. Then $S \subset V(S_n)$ is a failed 2-distance zero forcing set in S_n if and only if one of the following conditions holds:

(i) If $v \in S$, then at least two vertices in $\overline{K}_n$ must not be in S.

(ii) $v \notin S$, then $S \subseteq \overline{K}_n$.

Proof. Suppose that $S \subset V(S_n)$ is a failed- 2 distance zero forcing set of S_n. Let v be a dominating vertex of S_n such that $v \in S$. Assume that there is at most one vertex $w \in V(\overline{K}_n)$ such that $w \notin S$. Then $S = V(S_n) \setminus \{w\}= V(\overline{K}_n + K_1) \setminus \{w\}$ is a 2- distance zero forcing set of S_n since $d_{S_n}(w, y) = 2$ for some vertex $y \in V(S_n) \setminus \{w\}= S$. That is, w can be 2-forced by $y \in V(S_n)\setminus\{w\}=S$. However, this is a contradiction to our assumption that S is a failed 2-distance zero forcing Hence,(a) holds. Let $v \notin S$, then it is clearly that $S \subseteq V(\overline{K}_n)$. Thus, (b) also holds.

Conversely, if (a) holds. Then the remaining at least two vertices in $V(S_n) \setminus S$ cannot be 2-forced by S. Thus, S is a failed 2-distance zero forcing. Similarly, the same assertion follows when (b) holds. □

Since the failed 2-distance zero forcing number was defined with respect to the maximality of its corresponding set, the following result follows immediately from Theorem 3.

Corollary 2. Let n be a positive integer. Then $F^2(S_n) = n$ for all $n \geq 1$.

5. Failed 2-Distance Zero Forcing in the Join of two Graphs

We shall define the following definition to study the behavior of failed 2-distance zero forcing sets in the join of two graphs.

Definition 2. Let G be a simple and undirected graph. Then a co-color change rule is defined as follows: If a vertex $x \in V(G)$ is colored (active) and has exactly one non-neighbor y is uncolored (inactive), then y will become colored (active). In this case, we say that a vertex y is co-forced by a vertex x in G. A subset B of a vertex-set $V(G)$ of G is called a co-zero forcing set if repeatedly applying the co-color change rule on a set B, the whole vertex-set of G becomes colored (active). Moreover, a subset S of $V(G)$ is called a failed co-zero forcing set of G if S is not a co-zero forcing set of G. The maximum cardinality of a failed co-zero forcing set of G, denoted by $F_{co}(G)$, is called the failed co-zero forcing number of G.

We shall now characterize the failed 2-distance zero forcing sets in the join of two graphs as follows:

Theorem 4. Let G and H be graphs. Then Q is a failed 2-distance zero forcing set of G+H if and only if Q satisfies one of the following conditions.

(i) $Q \subseteq V(G)$.

(ii) $Q \subseteq V(H)$.

(iii) $Q = Q_G \cup Q_H$ such that Q_G or Q_H is failed a co-zero forcing set in G and H, respectively.

Proof. Suppose that Q is a failed 2- distance zero forcing set of $G+H$. Then $Q \neq V(G+H)$. Thus, either $Q \subseteq V(G)$, $Q \subseteq V(H)$ or $Q = Q_G \cup Q_H$, where $Q_G \subseteq V(G)$ and $Q_H \subseteq V(H)$, it follows that (a) and (b) hold. Assume that Q_G is a co-zero forcing set of G. Then by repeatedly applying the co-color change rule on Q_G, all other vertices in $V(G) \setminus Q_G$ will become colored. Suppose on the contrary that Q_H is a co-zero forcing set of H. Then by repeatedly applying the co-color change rule on Q_H, all other vertices in $V(H) \setminus Q_H$ will be 2-forced. It follows that Q is a 2-distance zero forcing set of $G+H$, which is a contradiction. Similarly, when Q_H is a co- zero forcing set of H , then Q_G must be a failed co-zero forcing set of G. Thus, (c) holds.

Conversely, if (a) holds. Then $V(H)$ cannot be 2-forced. Hence, $Q \subseteq V(G) \subseteq V(G+H)$ is a failed 2-distance zero forcing of $G + H$. Similarly, when (b) holds, then $Q \subseteq V(H) \subseteq V(G + H)$ is a failed 2-distance zero forcing of $G + H$. Now, suppose that (c) holds. If Q_G is failed a co-zero forcing set of G, then there exist $y \in V(G) \setminus Q_G$ such that y cannot be co-forced. It follows that $y \in V(G + H) \setminus Q$ cannot be 2-forced. Thus, Q is a failed 2-distance zero forcing set of G. Similarly, when Q_H is failed a co-zero forcing set of H, then Q is a failed 2-distance zero forcing of $G + H$. Moreover, Q is a failed 2-distance zero forcing set of $G + H$ whenever Q_G and Q_H are both failed co-zero forcing sets of G and H, respectively. □

The following result follows from Theorem 4.

Corollary 3. Let G and H be graphs. Then

$$F^2(G + H) = max\{|V(G)| + F_{co}(H), |V(H)| + F_{co}(G)\}.$$

6. Conclusion

The concept of failed 2-distance zero forcing in a graph has been introduced and investigated in this paper. Characterizations of failed 2-failed zero forcing sets in some special graphs and the join of any two graphs are formulated, and were used to derive some formulas of the parameter. Interested researchers may further study this concept on graphs which were not considered in this study. Providing real-life applications of the parameter and studying its complexity could also be an interesting cases to be considered by researchers.

7. Acknowledgements

The authors would like to thank Mindanao State University Tawi-Tawi College of Technology and Oceanography, Korea University, and Ateneo de Davao University for the support they had extended.

References

[1] F. Barioli, W. Barrett, S. M. Fallat, H. T. Hall, L. Hogben, B. Shader, P. van den Driessche, and H. van der Holst. Zero forcing parameters and minimum rank problems. *Linear Algebra and its Applications*, 433:401–411, 2010.

[2] S. M. Fallat and L. Hogben. *Minimum Rank, Maximum Nullity, and Zero Forcing Number of Graphs*. Handbook of Linear Algebra. CRC Press, Boca Raton, FL, 2nd edition, 2013.

[3] AIM Minimum Rank-Special Graphs Work Group. Zero forcing sets and the minimum rank of graphs. *Linear Algebra and its Applications*, 428:1628–1648, 2008.

[4] K. Benson, D. Ferrero, M. Flagg, V. Furst, L. Hogben, V. Vasilevska, and B. Wissman. Zero forcing and power domination for graphs products. *Australasian Journal of Combinatorics*, 70:221–235, 2018.

[5] A. Berliner, C. Bozeman, S. Butler, M. Catral, L. Hogben, B. Kroschel, J. C. H. Lin, N. Warnberg, and M. Young. Zero forcing propagation time on oriented graphs. *Discrete Applied Mathematics*, 224:45–59, 2017.

[6] R. Davila, T. Kalinowski, and S. Stephen. A lower bound on the zero forcing number. *Discrete Applied Mathematics*, 250:363–367, 2018.

[7] J. Ekstrand, C. Erickson, H. T. Hall, D. Hay, L. Hogben, R. Johnson, N. Kingsley, S. Osborne, T. Peters, J. Roat, A. Ross, D. D. Row, N. Warnberg, and M. Young. Positive semidefinite zero forcing. *Linear Algebra and its Applications*, 439:1862–1874, 2013.

[8] M. Gentner, L. D. Penso, D. Rautenbach, and U. S. Souza. Extremal values and bounds for the zero forcing number. *Discrete Applied Mathematics*, 214:196–200, 2016.

[9] J. Hassan, M. A. Bonsocan, M. Langamin, V. Bilar, S. D. Aming, and B. Amiruddin. Zero forcing domination in some graphs: Characterizations and derived formulas. *European Journal of Pure and Applied Mathematics*, 17(4):3772–3780, 2024.

[10] J. Hassan and L. S. Laja. Vertex cover zero forcing sets in graphs. *European Journal of Pure and Applied Mathematics*, 19(4):999–1003, 2024.

[11] J. Hassan, L. Laja, and Hounam B. Copel. 2-domination zero forcing in graphs. *International Journal of Mathematics and Computer Science*, 19(4):1065–1070, 2024.

[12] T. Kalinowski, N. Kamcev, and B. Sudakov. The zero forcing number of graphs. *SIAM Journal on Discrete Mathematics*, 33(1):95–115, 2019.

[13] J. Manditong, A. Tapeing, J. Hassan, A. R. Bakkang, N. H. Mohammad, and S. U. Kamdon. Some properties of zero forcing hop dominating sets in a graph. *European Journal of Applied Mathematics*, 17(1):324–337, 2014.

[14] A. Adams and B. Jacob. Failed zero forcing and critical sets on directed graphs. *Australasian Journal of Combinatorics*, 81(3):367–387, 2021.

[15] T. Ansill, B. Jacob, J. Penzellna, and D. Saavedra. Failed skew zero forcing on a graph. *Linear Algebra and its Applications*, 509:40–63, 2016.

[16] K. Fetcie, B. Jacob, and D. Saavedra. The failed zero-forcing number of a graph. *Involve*, 8:99–117, 2015.

[17] Y. Shitov. On the complexity of failed zero forcing. *Theoretical Computer Science*, 660:102–104, 2017.

[18] J. Hassan, L. T. Udtohan, and L. S. Laja. 2-distance zero forcing in graphs. *European Journal of Pure and Applied Mathematics*, 17(2):1283–1293, 2024.